AP CALCULUS PROBLEM SET 13 DIFFERENTIAL EQUATIONS

(74-7)

1. The rate of change in the number of bacteria in a culture is proportional to the number present. In a certain laboratory experiment, a culture has 10 000 bacteria initially, 20 000 at time t_1, and 100 000 at time $(t_1 + 10)$ minutes

(a) In terms of t only, find the number of bacteria in the culture at any time t minutes, $t \geq 0$

(b) How many bacteria were there after 20 minutes?

(c) How many minutes had elapsed when the 20 000 bacteria were observed?

(2013-6)

2. Consider the differential equation $\frac{dy}{dx} = e^y(3x^2 - 6x)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through (1, 0).

(a) Write an equation for the line tangent to the graph of f at the point (1, 0). Use the tangent line to approximate $f(1.2)$.

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through (1, 0).

(93-6)

3. Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k.

(a) If $P(0) = 500$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 700$, find k.

(c) Find $\lim_{t\to\infty} P(t)$.

(97-6)

4. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t > 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2v - 32$, with initial condition $v(0) = -50$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

(2000-6)

5. Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

(b) Find the domain and range of the function f found in part (a).

(2011-5)

6. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

OPTIONAL

(89-6)

7. Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t. Initially there were 1 000 000 gallons of oil in the well, and 6 years later there were 500 000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50 000 gallons remaining.

(a) Write an equation for y, the amount of oil remaining in the well at any time t.

(b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining ?

(c) In order not to lose money, at what time t should oil no longer be pumped from the well?

(2010-6)

8. Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1+3x^2y^2\right)$.

Let $y = f(x)$ be a particular solution to the differential equation with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$.
Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

(83-5)

9. At time $t = 0$, a jogger is running at a velocity of 300 meters per minute. The jogger is slowing down with a negative acceleration that is directly proportional to time t. This brings the jogger to a stop in 10 minutes.

(a) Write an expression for the velocity of the jogger at time t.

(b) What is the total distance travelled by the jogger in that 10-minute interval?

(92-6)

10. At time t, $t \geq 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius.

At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2.

(The volume V of a sphere with radius r *is* $V = \frac{4}{3}\pi r^3$)

(a) Find the radius of the sphere as a function of t.

(b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$?