Chapter Summary

In this chapter, we have reviewed definite integrals, starting with the Fundamental Theorem of Calculus. We've looked at techniques for evaluating definite integrals algebraically, numerically, and graphically. We've reviewed Riemann sums, including the left, right, and midpoint approximations as well as the trapezoid rule. We have also looked at the average value of a function.

This chapter also reviewed integrals based on parametrically defined functions, a BC Calculus topic.

Practice Exercises

Part A. Directions: Answer these questions *without* using your calculator.

1. $\int_{-1}^{1} (x^2 - x - 1)\, dx =$

(A) $\frac{2}{3}$ (B) 0 (C) $-\frac{4}{3}$ (D) -2 (E) -1

2. $\int_{1}^{2} \frac{3x-1}{3x}\, dx =$

(A) $\frac{3}{4}$ (B) $1 - \frac{1}{3}\ln 2$ (C) $1 - \ln 2$ (D) $-\frac{1}{3}\ln 2$ (E) 1

3. $\int_{0}^{3} \frac{dt}{\sqrt{4-t}} =$

(A) 1 (B) -2 (C) 4 (D) -1 (E) 2

4. $\int_{-1}^{0} \sqrt{3u+4}\, du =$

(A) 2 (B) $\frac{14}{9}$ (C) $\frac{14}{3}$ (D) 6 (E) $\frac{7}{2}$

5. $\int_{2}^{3} \frac{dy}{2y-3} =$

(A) $\ln 3$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\frac{16}{9}$ (D) $\ln\sqrt{3}$ (E) $\sqrt{3} - 1$

6. $\int_{0}^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\, dx =$

(A) 1 (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) -1 (E) 2

7. $\int_0^1 (2t-1)^3\,dt =$

(A) $\frac{1}{4}$ (B) 6 (C) $\frac{1}{2}$ (D) 0 (E) 4

8. $\int_4^9 \frac{2+x}{2\sqrt{x}}\,dx =$

(A) $\frac{25}{3}$ (B) $\frac{41}{3}$ (C) $\frac{100}{3}$ (D) $\frac{5}{3}$ (E) $\frac{1}{3}$

9. $\int_{-3}^3 \frac{dx}{9+x^2} =$

(A) $\frac{\pi}{2}$ (B) 0 (C) $\frac{\pi}{6}$ (D) $-\frac{\pi}{2}$ (E) $\frac{\pi}{3}$

10. $\int_0^1 e^{-x}\,dx =$

(A) $\frac{1}{e}-1$ (B) $1-e$ (C) $-\frac{1}{e}$ (D) $1-\frac{1}{e}$ (E) $\frac{1}{e}$

11. $\int_0^1 xe^{x^2}\,dx =$

(A) $e-1$ (B) $\frac{1}{2}(e-1)$ (C) $2(e-1)$ (D) $\frac{e}{2}$ (E) $\frac{e}{2}-1$

12. $\int_0^{\pi/4} \sin 2\theta\,d\theta =$

(A) 2 (B) $\frac{1}{2}$ (C) –1 (D) $-\frac{1}{2}$ (E) –2

13. $\int_1^2 \frac{dz}{3-z} =$

(A) $-\ln 2$ (B) $\frac{3}{4}$ (C) $2(\sqrt{2}-1)$ (D) $\frac{1}{2}\ln 2$ (E) $\ln 2$

14. If we let $x = 2\sin\theta$, then $\int_1^2 \frac{\sqrt{4-x^2}}{x}\,dx$ is equivalent to

(A) $2\int_0^2 \frac{\cos^2\theta}{\sin\theta}\,d\theta$ (B) $\int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin\theta}\,d\theta$ (C) $2\int_{\pi/6}^{\pi/2} \frac{\cos^2\theta}{\sin\theta}\,d\theta$

(D) $\int_1^2 \frac{\cos\theta}{\sin\theta}\,d\theta$ (E) none of these

15. $\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta =$

(A) $-\frac{2}{3}$ **(B)** $\frac{1}{3}$ **(C)** 1 **(D)** $\frac{2}{3}$ **(E)** 0

16. $\int_1^{e} \frac{\ln x}{x} \, dx =$

(A) $\frac{1}{2}$ **(B)** $\frac{1}{2}(e^2 - 1)$ **(C)** 0 **(D)** 1 **(E)** $e - 1$

BC ONLY

17. $\int_0^{1} xe^x \, dx =$

(A) -1 **(B)** $e + 1$ **(C)** 1 **(D)** $e - 1$ **(E)** $\frac{1}{2}(e - 1)$

18. $\int_0^{\pi/6} \frac{\cos\theta}{1 + 2\sin\theta} \, d\theta =$

(A) $\ln 2$ **(B)** $\frac{3}{8}$ **(C)** $-\frac{1}{2}\ln 2$ **(D)** $\frac{3}{2}$ **(E)** $\ln\sqrt{2}$

19. $\int_{\sqrt{2}}^{2} \frac{u}{u^2 - 1} \, du =$

(A) $\ln\sqrt{3}$ **(B)** $\frac{8}{9}$ **(C)** $\ln\frac{3}{2}$ **(D)** $\ln 3$ **(E)** $1 - \sqrt{3}$

20. $\int_{\sqrt{2}}^{2} \frac{u\,du}{(u^2 - 1)^2} =$

(A) $-\frac{1}{3}$ **(B)** $-\frac{2}{3}$ **(C)** $\frac{2}{3}$ **(D)** -1 **(E)** $\frac{1}{3}$

21. $\int_{\pi/12}^{\pi/4} \frac{\cos 2x\,dx}{\sin^2 2x} =$

(A) $-\frac{1}{4}$ **(B)** 1 **(C)** $\frac{1}{2}$ **(D)** $-\frac{1}{2}$ **(E)** -1

22. $\int_0^{1} \frac{e^{-x} + 1}{e^{-x}} \, dx =$

(A) e **(B)** $2 + e$ **(C)** $\frac{1}{e}$ **(D)** $1 + e$ **(E)** $e - 1$

23. $\int_0^1 \frac{e^x}{e^x+1}\,dx =$

(A) $\ln 2$ **(B)** e **(C)** $1+e$ **(D)** $-\ln 2$ **(E)** $\ln \frac{e+1}{2}$

24. If we let $x = \tan\theta$, then $\int_1^{\sqrt{3}} \sqrt{1+x^2}\,dx$ is equivalent to

(A) $\int_{\pi/4}^{\pi/3} \sec\theta\,d\theta$ **(B)** $\int_1^{\sqrt{3}} \sec^3\theta\,d\theta$ **(C)** $\int_{\pi/4}^{\pi/3} \sec^3\theta\,d\theta$

(D) $\int_{\pi/4}^{\pi/3} \sec^2\theta\tan\theta\,d\theta$ **(E)** $\int_1^{\sqrt{3}} \sec\theta\,d\theta$

25. If the substitution $u = \sqrt{x+1}$ is used, then $\int_0^3 \frac{dx}{x\sqrt{x+1}}$ is equivalent to

(A) $\int_1^2 \frac{du}{u^2-1}$ **(B)** $\int_1^2 \frac{2\,du}{u^2-1}$ **(C)** $2\int_0^3 \frac{du}{(u-1)(u+1)}$

(D) $2\int_1^2 \frac{du}{u(u^2-1)}$ **(E)** $2\int_0^3 \frac{du}{u(u-1)}$

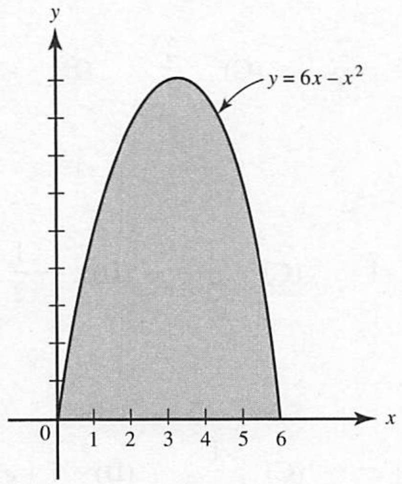
26. The table shows some values of continuous function f and its first derivative. Evaluate $\int_8^0 f'(x)\,dx$.

x	$f(x)$	$f'(x)$
0	11	3
2	15	2
4	16	–1
6	12	–3
8	7	0

(A) –1/2 **(B)** –3/8 **(C)** 3
(D) 4 **(E)** none of these

27. Using $M(3)$, we find that the approximate area of the shaded region below is

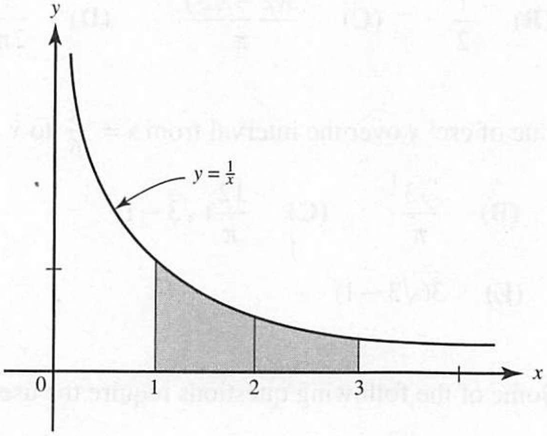
(A) 9 **(B)** 19 **(C)** 36 **(D)** 38 **(E)** 54



28. The graph of a continuous function f passes through the points (4,2), (6,6), (7,5), and (10,8). Using trapezoids, we estimate that $\int_4^{10} f(x)dx \approx$

(A) 25 **(B)** 30 **(C)** 32 **(D)** 33 **(E)** 41

29. The area of the shaded region in the figure is equal exactly to ln 3. If we approximate ln 3 using $L(2)$ and $R(2)$, which inequality follows?



(A) $\frac{1}{2} < \int_1^2 \frac{1}{x}\,dx < 1$ **(B)** $\frac{1}{3} < \int_1^3 \frac{1}{x}\,dx < 2$ **(C)** $\frac{1}{2} < \int_0^2 \frac{1}{x}\,dx < 2$

(D) $\frac{1}{3} < \int_2^3 \frac{1}{x}\,dx < \frac{1}{2}$ **(E)** $\frac{5}{6} < \int_1^3 \frac{1}{x}\,dx < \frac{3}{2}$

30. Let $A = \int_0^1 \cos x\,dx$. We estimate A using the L, R, and T approximations with $n = 100$ subintervals. Which is true?

(A) $L < A < T < R$
(B) $L < T < A < R$
(C) $R < A < T < L$
(D) $R < T < A < L$
(E) The order cannot be determined.

31. $\int_{-1}^{3} |x|\,dx =$

(A) $\frac{7}{2}$ **(B)** 4 **(C)** $\frac{9}{2}$ **(D)** 5 **(E)** $\frac{11}{2}$

32. $\int_{-3}^{2} |x+1|\,dx =$

(A) $\frac{5}{2}$ **(B)** $\frac{7}{2}$ **(C)** 5 **(D)** $\frac{11}{2}$ **(E)** $\frac{13}{2}$

33. The average value of $y = \sqrt{64 - x^2}$ on its domain is

(A) 2 **(B)** 4 **(C)** 2π **(D)** 4π **(E)** none of these

34. The average value of $\cos x$ over the interval $\frac{\pi}{3} \leqq x \leqq \frac{\pi}{2}$ is

(A) $\frac{3}{\pi}$ (B) $\frac{1}{2}$ (C) $\frac{3(2-\sqrt{3})}{\pi}$ (D) $\frac{3}{2\pi}$ (E) $\frac{2}{3\pi}$

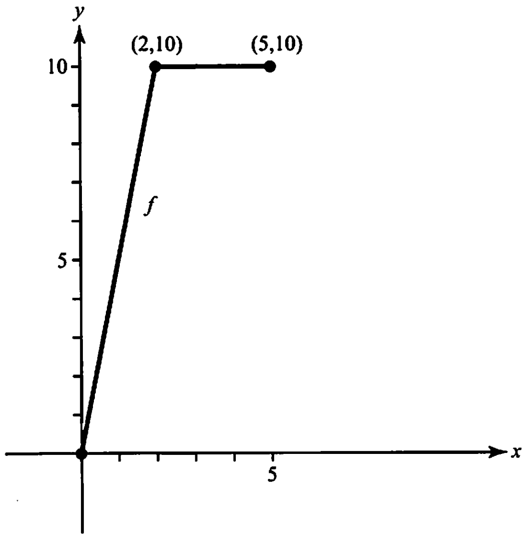
35. The average value of $\csc^2 x$ over the interval from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{4}$ is

(A) $\frac{3\sqrt{3}}{\pi}$ (B) $\frac{\sqrt{3}}{\pi}$ (C) $\frac{12}{\pi}(\sqrt{3}-1)$

(D) $3\sqrt{3}$ (E) $3(\sqrt{3}-1)$

Part B. Directions: Some of the following questions require the use of a graphing calculator.

36. Find the average value of function f, as shown in the graph below, on the interval [0,5].



(A) 2 (B) 4 (C) 5 (D) 7 (E) 8

37. The integral $\int_{-4}^{4} \sqrt{16-x^2}\, dx$ gives the area of

(A) a circle of radius 4
(B) a semicircle of radius 4
(C) a quadrant of a circle of radius 4
(D) an ellipse whose semimajor axis is 4
(E) none of these

38. $\int_0^{\pi/4} \sqrt{1-\cos 2\alpha}\, d\alpha =$

(A) 0.25 (B) 0.414 (C) 1.000 (D) 1.414 (E) 2.000

39. If $f(x)$ is continuous on the interval $a \le x \le b$ and $a < c < b$, then $\int_c^b f(x)\,dx$ is equal to

(A) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (B) $\int_a^c f(x)\,dx - \int_a^b f(x)\,dx$

(C) $\int_c^a f(x)\,dx + \int_b^a f(x)\,dx$ (D) $\int_a^b f(x)\,dx - \int_a^c f(x)\,dx$

(E) $\int_a^c f(x)\,dx - \int_b^c f(x)\,dx$

40. If $f(x)$ is continuous on $a \le x \le b$, then

(A) $\int_a^b f(x)\,dx = f(b) - f(a)$ (B) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(C) $\int_a^b f(x)\,dx \geqq 0$ (D) $\frac{d}{dx}\int_a^x f(t)\,dt = f'(x)$

(E) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x) - f(a)$

41. If $f(x)$ is continuous on the interval $a \leqq x \leqq b$, if this interval is partitioned into n equal subintervals of length Δx, and if x_k is a number in the kth subinterval, then $\lim_{n\to\infty} \sum_1^n f(x_k)\ \Delta x$ is equal to

(A) $f(b) - f(a)$

(B) $F(x) + C$, where $\frac{dF(x)}{dx} = f(x)$ and C is an arbitrary constant

(C) $\int_a^b f(x)\,dx$

(D) $F(b - a)$, where $\frac{dF(x)}{dx} = f(x)$

(E) none of these

42. If $F'(x) = G'(x)$ for all x, then

(A) $\int_a^b F'(x)\,dx = \int_a^b G'(x)\,dx$ (B) $\int F(x)\,dx = \int G(x)\,dx$

(C) $\int_a^b F(x)\,dx = \int_a^b G(x)\,dx$ (D) $\int F(x)\,dx = \int G(x)\,dx + C$

(E) $F(x) = G(x)$ for all x.

43. If $f(x)$ is continuous on the closed interval $[a,b]$, then there exists at least one number c, $a < c < b$, such that $\int_a^b f(x)\,dx$ is equal to

(A) $\dfrac{f(c)}{b-a}$ (B) $f'(c)(b-a)$ (C) $f(c)(b-a)$

(D) $\dfrac{f'(c)}{b-a}$ (E) $f(c)[f(b)-f(a)]$

44. If $f(x)$ is continuous on the closed interval $[a,b]$ and k is a constant, then $\int_a^b kf(x)\,dx$ is equal to

(A) $k(b-a)$ (B) $k[f(b)-f(a)]$ (C) $kF(b-a)$, where $\dfrac{dF(x)}{dx} = f(x)$

(D) $k\int_a^b f(x)\,dx$ (E) $\left.\dfrac{[kf(x)]^2}{2}\right]_a^b$

45. $\dfrac{d}{dt}\int_0^t \sqrt{x^3+1}\,dx =$

(A) $\sqrt{t^3+1}$ (B) $\dfrac{\sqrt{t^3+1}}{3t^2}$ (C) $\dfrac{2}{3}(t^3+1)(\sqrt{t^3+1}-1)$

(D) $3x^2\sqrt{x^3+1}$ (E) none of these

46. If $F(u) = \int_1^u (2-x^2)^3\,dx$, then $F'(u)$ is equal to

(A) $-6u(2-u^2)^2$ (B) $\dfrac{(2-u^2)^4}{4} - \dfrac{1}{4}$ (C) $(2-u^2)^3 - 1$

(D) $(2-u^2)^3$ (E) $-2u(2-u^2)^3$

47. $\dfrac{d}{dx}\int_{\pi/2}^{x^2} \sqrt{\sin t}\,dt =$

(A) $\sqrt{\sin t^2}$ (B) $2x\sqrt{\sin x^2} - 1$ (C) $\dfrac{2}{3}(\sin^{3/2} x^2 - 1)$

(D) $\sqrt{\sin x^2} - 1$ (E) $2x\sqrt{\sin x^2}$

48. If $x = 4\cos\theta$ and $y = 3\sin\theta$, then $\int_2^4 xy\,dx$ is equivalent to

(A) $48\int_{\pi/3}^{0} \sin\theta\cos^2\theta\,d\theta$ (B) $48\int_2^4 \sin^2\theta\cos\theta\,d\theta$

(C) $36\int_2^4 \sin\theta\cos^2\theta\,d\theta$ (D) $-48\int_0^{\pi/3} \sin\theta\cos^2\theta\,d\theta$

(E) $48\int_0^{\pi/3} \sin^2\theta\cos\theta\,d\theta$

BC ONLY

49. A curve is defined by the parametric equations $x = 2a\tan\theta$ and $y = 2a\cos^2\theta$, where $0 \leqq \theta \leqq \pi$. Then the definite integral $\pi\int_0^{2a} y^2\,dx$ is equivalent to

(A) $4\pi a^2\int_0^{\pi/4}\cos^4\theta\,d\theta$ **(B)** $8\pi a^3\int_{\pi/2}^{\pi}\cos^2\theta\,d\theta$ **(C)** $8\pi a^3\int_0^{\pi/4}\cos^2\theta\,d\theta$

(D) $8\pi a^3\int_0^{2a}\cos^2\theta\,d\theta$ **(E)** $8\pi a^3\int_0^{\pi/4}\sin\theta\cos^2\theta\,d\theta$

50. A curve is given parametrically by $x = 1 - \cos t$ and $y = t - \sin t$, where $0 \leqq t \leqq \pi$. Then $\int_0^{3/2} y\,dx$ is equivalent to

(A) $\int_0^{3/2}\sin t(t - \sin t)\,dt$ **(B)** $\int_{2\pi/3}^{\pi}\sin t(t - \sin t)\,dt$

(C) $\int_0^{2\pi/3}(t - \sin t)\,dt$ **(D)** $\int_0^{2\pi/3}\sin t(t - \sin t)\,dt$

(E) $\int_0^{3/2}(t - \sin t)\,dt$

51. When $\int_0^1 \sqrt{1+x^2}\,dx$ is estimated using $n = 5$ subintervals of equal width, which is (are) true?

I. $L(5) = \left(1 + \sqrt{1+0.2^2} + \sqrt{1+0.4^2} + \sqrt{1+0.6^2} + \sqrt{1+0.8^2}\right)$

II. $M(5) = \left(\sqrt{1+0.1^2} + \sqrt{1+0.3^2} + \sqrt{1+0.5^2} + \sqrt{1+0.7^2} + \sqrt{1+0.9^2}\right)\cdot(0.2)$

III. $T(5) = \frac{0.2}{2}\left(1 + 2\sqrt{1+0.2^2} + 2\sqrt{1+0.4^2} + 2\sqrt{1+0.6^2} + 2\sqrt{1+0.8^2} + \sqrt{2}\right)$

(A) II only
(B) III only
(C) I and II only
(D) I and III only
(E) II and III only

52. Find the value of x at which the function $y = x^2$ reaches its average value on the interval [0,10].

(A) 4.642 **(B)** 5 **(C)** 5.313 **(D)** 5.774 **(E)** 7.071

53. The average value of $f(x) = \begin{cases} x^3, & x < 2 \\ 4x, & x \geq 2 \end{cases}$ on the interval $0 \leq x \leq 5$ is

(A) 8 **(B)** 9.2 **(C)** 16 **(D)** 23
(E) undefined because f is not differentiable on this interval

Answer Key

1. C	12. B	23. E	34. C	45. A
2. B	13. E	24. C	35. C	46. D
3. E	14. C	25. B	36. E	47. E
4. B	15. D	26. D	37. B	48. E
5. D	16. A	27. D	38. B	49. C
6. A	17. C	28. D	39. D	50. D
7. D	18. E	29. E	40. B	51. E
8. A	19. A	30. D	41. C	52. D
9. C	20. E	31. D	42. A	53. B
10. D	21. C	32. E	43. C	
11. B	22. A	33. C	44. D	

Answers Explained

1. (C) The integral is equal to

$$\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 - x\right)\Big|_{-1}^{1} = -\frac{7}{6} - \frac{1}{6}.$$

2. (B) Rewrite as $\int_1^2 \left(1 - \frac{1}{3}\cdot\frac{1}{x}\right)dx$. This equals

$$\left(x - \frac{1}{3}\ln x\right)\Big|_1^2 = 2 - \frac{1}{3}\ln 2 - 1.$$

3. (E) Rewrite as

$$-\int_0^3 (4-t)^{-1/2}(-1\,dt) = -2\sqrt{4-t}\,\Big|_0^3 = -2(1-2).$$

4. (B) This integral equals

$$\frac{1}{3}\int_{-1}^{0} (3u+4)^{1/2}\cdot 3\,du = \frac{1}{3}\cdot\frac{2}{3}(3u+4)^{3/2}\Big|_{-1}^{0}$$
$$= \frac{2}{9}(4^{3/2} - 1^{3/2}).$$

5. (D) $$\frac{1}{2}\int_2^3 \frac{2\,dy}{2y-3} = \frac{1}{2}\ln(2y-3)\Big|_2^3 = \frac{1}{2}(\ln 3 - \ln 1)$$

6. (A) Rewrite as

$$-\frac{1}{2}\int_0^{\sqrt{3}} (4-x^2)^{-1/2}(-2x\,dx) = -\frac{1}{2}\cdot 2\sqrt{4-x^2}\,\Big|_0^{\sqrt{3}} = -(1-2).$$

7. (D) $$\frac{1}{2}\int_0^1 (2t-1)^3(2dt) = \frac{1}{2}\cdot\frac{(2t-1)^4}{4}\Big|_0^1 = \frac{1}{2}\left(\frac{(2\cdot 1-1)^4}{4} - \frac{(2\cdot 0-1)^4}{4}\right)$$

8. **(A)** Divide:

$$\int_4^9 \left(x^{-1/2} + \frac{1}{2}x^{1/2}\right)dx = \left(2x^{1/2} + \frac{1}{2}\cdot\frac{2}{3}x^{3/2}\right)\Big|_4^9$$

$$= \left(2\cdot 3 + \frac{1}{3}\cdot 27\right) - \left(2\cdot 2 + \frac{1}{3}\cdot 8\right).$$

9. **(C)** $\frac{1}{9}\int_{-3}^{3} \frac{dx}{1+\frac{x^2}{9}} = 3\cdot\frac{1}{9}\int_{-3}^{3}\frac{\frac{1}{3}dx}{1+\left(\frac{x}{3}\right)^2} = \frac{1}{3}\tan^{-1}\frac{x}{3}\Big|_{-3}^{3} = \frac{1}{3}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right).$

10. **(D)** You get $-e^{-x}\Big|_0^1 = -(e^{-1} - 1).$

11. **(B)** $\frac{1}{2}\int_0^1 e^{x^2}(2x\,dx) = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}(e-1).$

12. **(B)** Evaluate $-\frac{1}{2}\cos 2\theta\Big|_0^{\pi/4}$, which equals $-\frac{1}{2}(0-1).$

13. **(E)** $-\int_1^2 \frac{-dx}{3-z} = -\ln(3-z)\Big|_1^2 = -(\ln 1 - \ln 2).$

14. **(C)** If $x = 2\sin\theta$, $\sqrt{4-x^2} = 2\cos\theta$, $dx = 2\cos\theta\, d\theta$. When $x = 1$, $\theta = \frac{\pi}{6}$;
when $x = 2$, $\theta = \frac{\pi}{2}$. The integral is equivalent to $\int_{\pi/6}^{\pi/2} \frac{(2\cos\theta)(2\cos\theta)\,d\theta}{2\sin\theta}$.

15. **(D)** Evaluate $-\int_0^{\pi}\cos^2\theta(-\sin\theta\, d\theta)$. This equals $-\frac{1}{3}\cos^3\theta\Big|_0^{\pi} = -\frac{1}{3}(-1-1).$

16. **(A)** $\int_1^e (\ln x)\left(\frac{1}{x}dx\right) = \frac{1}{2}\ln^2 x\Big|_1^e = \frac{1}{2}(1-0).$

17. **(C)** Use the Parts Formula with $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. The result is

$$\left(xe^x - \int e^x dx\right)\Big|_0^1 = (xe^x - e^x)\Big|_0^1 = (e-e) - (0-1).$$

18. **(E)** $\frac{1}{2}\int_0^{\pi/6}\frac{2\cos\theta\, d\theta}{1+2\sin\theta} = \frac{1}{2}\ln(1+2\sin\theta)\Big|_0^{\pi/6}$ and get $\frac{1}{2}(\ln(1+1) - \ln 1).$

19. **(A)** Evaluate the integral $\frac{1}{2}\int_{\sqrt{2}}^{2}\frac{2u}{u^2-1}du$. It equals

$$\frac{1}{2}\ln(u^2-1)\Big|_{\sqrt{2}}^{2} \quad \text{or} \quad \frac{1}{2}(\ln 3 - \ln 1).$$

20. **(E)** Evaluate $\frac{1}{2}\int_{\sqrt{2}}^{2}(u^2-1)^{-2}\cdot 2u\,du$ and get

$$-\frac{1}{2(u^2-1)}\Bigg|_{\sqrt{2}}^{2} \quad\text{or}\quad -\frac{1}{2}\left(\frac{1}{3}-\frac{1}{1}\right).$$

Compare with Question 19.

21. **(C)** Rewrite:

$$\frac{1}{2}\int_{\pi/12}^{\pi/4}\sin^{-2}2x\cos 2x\,(2\,dx) = -\frac{1}{2}\cdot\frac{1}{\sin 2x}\Bigg|_{\pi/12}^{\pi/4}$$

$$= -\frac{1}{2}\left(\frac{1}{1}-\frac{1}{1/2}\right).$$

22. **(A)** The integral is equivalent to

$$\int_0^1(1+e^x)\,dx = (x+e^x)\Big|_0^1 = (1+e)-1.$$

23. **(E)** Evaluate $\ln(e^x+1)\Big|_0^1$, getting $\ln(e+1)-\ln 2$.

24. **(C)** Note that $dx=\sec^2\theta\,d\theta$ and that $\sqrt{1+\tan^2\theta}=\sec\theta$. Be sure to express the limits as values of θ: $1=\tan\theta$ yields $\theta=\frac{\pi}{4}$; $\sqrt{3}=\tan\theta$ yields $\theta=\frac{\pi}{3}$.

25. **(B)** If $u=\sqrt{x+1}$, then $u^2=x+1$, and $2u\,du=dx$. When you substitute for the limits, you get $2\int_1^2\frac{u\,du}{u(u^2-1)}$. Since $u\neq 0$ on its interval of integration, you may divide numerator and denominator by it.

26. **(D)** $\int_8^0 f'(x)\,dx = f(0)-f(8) = 11-7 = 4$

27. **(D)** On [0,6] with $n=3$, $\Delta x=2$. Heights of rectangles at $x=1$, 3, and 5 are 5, 9, and 5, respectively; $M(3)=(5+9+5)(2)$.

28. **(D)** $\int_4^{10} f(x)\,dx \approx \left(\frac{2+6}{2}\right)\cdot 2+\left(\frac{6+5}{2}\right)\cdot 1+\left(\frac{5+8}{2}\right)\cdot 3 \approx 33$

29. **(E)** For $L(2)$ use the circumscribed rectangles:

$$1\cdot 1+\frac{1}{2}\cdot 1=\frac{3}{2};$$

for $R(2)$ use the inscribed rectangles:

$$\frac{1}{2}\cdot 1+\frac{1}{3}\cdot 1=\frac{5}{6}.$$

30. (D) On $[0,1]$ $f(x) = \cos x$ is decreasing, so $R < L$. Furthermore, f is concave downward, so $T < A$.

31. (D) Rewrite the integral to evaluate it, using the fact that x changes sign at 0. The result is

$$\int_{-1}^{0} (-x)\,dx + \int_{0}^{3} x\,dx = -\frac{x^2}{2}\bigg|_{-1}^{0} + \frac{x^2}{2}\bigg|_{0}^{3}.$$

Draw a sketch of $y = |x|$, and verify that the area over $-1 \leqq x \leqq 3$ equals 5.

32. (E) Since $x + 1$ changes sign at $x = -1$, $|x + 1| = -(x + 1)$ if $x < -1$ but equals $x + 1$ if $x \geqq -1$. The given integral is therefore equivalent to

$$\int_{-3}^{-1} -(x+1)\,dx + \int_{-1}^{2} (x+1)\,dx = -\frac{(x+1)^2}{2}\bigg|_{-3}^{-1} + \frac{(x+1)^2}{2}\bigg|_{-1}^{2}$$

$$= -\frac{1}{2}(0-4) + \frac{1}{2}(9-0).$$

Draw a sketch of $y = |x + 1|$, and verify that the area over $-3 \leqq x \leqq 2$ is $\frac{13}{2}$.

33. (C) Because $y = \sqrt{64 - x^2}$ is a semicircle of radius 8, its area is 32π. The domain is $[-8,8]$, or 16 units wide. Hence the average height of the function is $\frac{32\pi}{16}$.

34. (C) The average value is equal to $\frac{1}{\pi/2 - \pi/3}\int_{\pi/3}^{\pi/2} \cos x\,dx.$

35. (C) The average value is equal to $\frac{1}{\pi/4 - \pi/6}\int_{\pi/6}^{\pi/4} \csc^2 x\,dx.$

36. (E) The average value is $\frac{1}{5-0}\int_0^5 f(x)dx$, where the integral represents the area of a trapezoid. That area is $\frac{1}{2}(5 + 3)\cdot 10 = 40$ making the average value $\frac{1}{5}(40)$.

37. (B) Since $x^2 + y^2 = 16$ is a circle, the given integral equals the area of a semicircle of radius 4.

38. (B) Use a graphing calculator.

39. (D) Note that the integral from a to b is the sum of the two integrals from a to c and from c to b.

40. (B) In (A), we'd need to use the antiderivative of f to evaluate the definite integral. In (C) if $f(x) < 0$, the definite integral would be negative. In (D) and (E), the correct derivative of the definite integral would be $f(x)$.

41. (C) This is the definition of the definite integral given on page 249. As antiderivatives of F' and G', F and G may differ by a constant.

42. (A) Find examples of functions F and G that show that (B), (C), and (D) are false.

43. (C) This is the Mean Value Theorem for Integrals (page 250).

44. (D) This is theorem (2) on page 250. Prove by counterexamples that (A), (B), (C), and (D) are false.

45. (A) This is a restatement of the Fundamental Theorem. In theorem (1) on page 250, interchange t and x.

46. (D) Apply theorem (1) on page 250, noting that

$$F'(u) = \frac{d}{du}\int_a^u f(x)\,dx = f(u).$$

47. (E) Let $y = \int_{\pi/2}^{x^2} \sqrt{\sin t}\,dt$ and $u = x^2$; then

$$y = \int_{\pi/2}^{u} \sqrt{\sin t}\,dt$$

By the Chain Rule, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sqrt{\sin u}\cdot 2x$, where theorem (1) on page 250 is used to find $\frac{dy}{du}$. Replace u by x^2.

48. (E) Since $dx = -4\sin\theta\,d\theta$, you get the new integral $-48\int_{\pi/3}^{0}\sin^2\theta\cos\theta\,d\theta$. Use theorem (4) on page 250 to get the correct answer.

49. (C) Since $dx = 2a\sec^2\theta\,d\theta$, you get $8\pi a^3\int_0^{\pi/4}\cos^4\theta\sec^2\theta\,d\theta$. Use the fact that $\cos^2\theta\sec^2\theta = 1$.

50. (D) Use the facts that $dx = \sin t\,dt$, that $t = 0$ when $x = 0$, and that $t = \frac{2\pi}{3}$ when $x = \frac{3}{2}$.

51. (E) The expression for $L(5)$ does not multiply the heights of the rectangles by $\Delta x = 0.2$.

52. (D) The average value is $\frac{1}{10-0}\int_0^{10} x^2\,dx = \frac{1}{10}\cdot\frac{x^3}{3}\Big|_0^{10} = \frac{100}{3}$. Solve $x^2 = \frac{100}{3}$.

53. (B) $\frac{1}{5-0}\int_0^5 f(x)\,dx = \frac{1}{5}\left(\int_0^2 x^3\,dx + \int_2^5 4x\,dx\right) = \frac{1}{5}\left(\frac{x^4}{4}\Big|_0^2 + 2x^2\Big|_2^5\right) = \frac{1}{5}(4+42)$