Miscellaneous Free-Response Practice Exercises

CHAPTER 12

These problems provide further practice for both parts of Section II of the examination. Solutions begin on page 498.

Part A. Directions: A graphing calculator is required for some of these problems. See instructions on page 4.

x	2.5	3.2	3.5	4.0	4.6	5.0
$f(x)$	7.6	5.7	4.2	3.1	2.2	1.5

1. A function *f* is continuous, differentiable, and strictly decreasing on the interval [2.5,5]; some values of *f* are shown in the table above.

(a) Estimate $f'(4.0)$ and $f'(4.8)$.
(b) What does the table suggest may be true of the concavity of *f*? Explain.
(c) Estimate $\int_{2.5}^{5} f(x)dx$ with a Riemann sum using left endpoints.
(d) Set up (but do not evaluate) a Riemann sum that estimates the volume of the solid formed when *f* is rotated around the *x*-axis.

2. The equation of the tangent line to the curve $x^2y - x = y^3 - 8$ at the point (0,2) is $12y + x = 24$.

(a) Given that the point $(0.3, y_0)$ is on the curve, find y_0 approximately, using the tangent line.
(b) Find the true value of y_0.
(c) What can you conclude about the curve near $x = 0$ from your answers to parts (a) and (b)?

3. A differentiable function *f* defined on $-7 < x < 7$ has $f(0) = 0$ and $f'(x) = 2x \sin x - e^{-x^2} + 1$. (Note: The following questions refer to *f*, not to f'.)

(a) Describe the symmetry of *f*.
(b) On what intervals is *f* decreasing?
(c) For what values of *x* does *f* have a relative maximum? Justify your answer.
(d) How many points of inflection does *f* have? Justify your answer.

4. Let C represent the piece of the curve $\sqrt[3]{64-16x^2}$ that lies in the first quadrant. Let S be the region bounded by C and the coordinate axes.

   (a) Find the slope of the line tangent to C at $y = 1$.
   (b) Find the area of S.
   (c) Find the volume generated when S is rotated about the x-axis.

5. Let R be the point on the curve of $y = x - x^2$ such that the line OR (where O is the origin) divides the area bounded by the curve and the x-axis into two regions of equal area. Set up (but do not solve) an integral to find the x-coordinate of R.

6. Suppose $f'' = \sin(2^x)$ for $-1 < x < 3.2$.

   (a) On what intervals is the graph of f concave downward? Justify your answer.
   (b) Find the x-coordinates of all relative minima of f'.
   (c) How many points of inflection does the graph of f' have? Justify your answer.

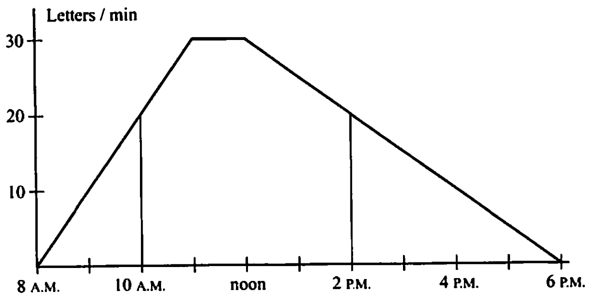
7. Let $f(x) = \cos x$ and $g(x) = x^2 - 1$.

   (a) Find the coordinates of any points of intersection of f and g.
   (b) Find the area bounded by f and g.

8. (a) In order to investigate mail-handling efficiency, each hour one morning a local post office checked the rate (letters/min) at which an employee was sorting mail. Use the results shown in the table to estimate the total number of letters he may have sorted that morning.

Time	8	9	10	11	12
Letters/min	10	12	8	9	11

   (b) Hoping to speed things up a bit, the post office tested a sorting machine that can process mail at the constant rate of 20 letters per minute. The graph shows the rate at which letters arrived at the post office and were dumped into this sorter.



   (i) When did letters start to pile up?
   (ii) When was the pile the biggest?
   (iii) How big was it then?
   (iv) At about what time did the pile vanish?

9. Let R represent the region bounded by $y = \sin x$ and $y = x^4$. Find:

   (a) the area of R;

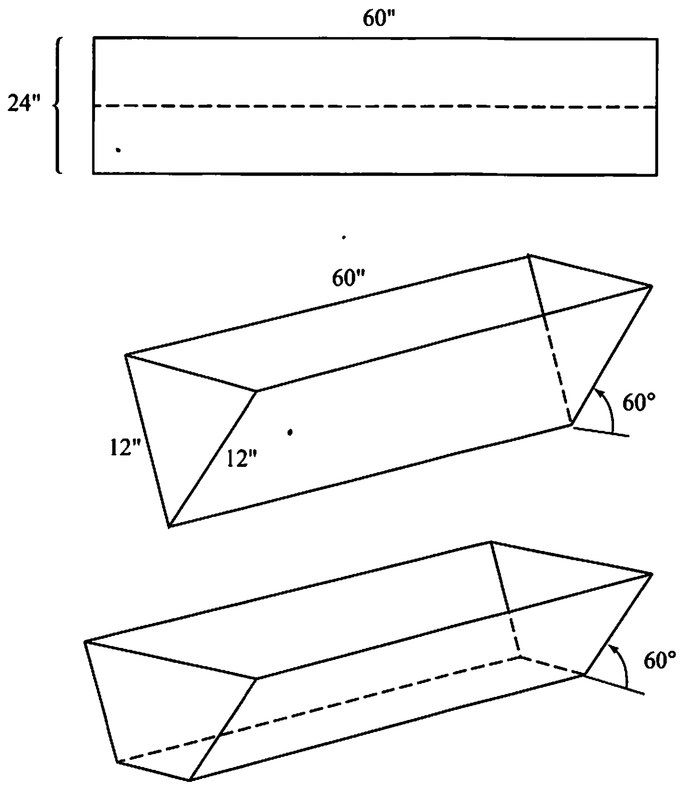
   (b) the volume of the solid whose base is R, if all cross sections perpendicular to the x-axis are isosceles triangles with height 3;

   (c) the volume of the solid formed when R is rotated around the x-axis.

10. The town of East Newton has a water tower whose tank is an ellipsoid, formed by rotating an ellipse about its minor axis. Since the tank is 20 feet tall and 50 feet wide, the equation of the ellipse is $\frac{x^2}{625}+\frac{y^2}{100}=1$.

(a) If there are 7.48 gallons of water per cubic foot, what is the capacity of this tank to the nearest thousand gallons?

(b) East Newton imposes water rationing whenever the tank is only one-quarter full. Write an equation to find the depth of the water in the tank when rationing becomes necessary? (Do not solve.)



Note: Scales are different on the three figures.

11. The sides of a watering trough are made by folding a sheet of metal 24 inches wide and 5 feet (60 inches) long at an angle of 60°, as shown in the figure above. Ends are added, and then the trough is filled with water.

(a) If water pours into the trough at the rate of 600 cubic inches per minute, how fast is the water level rising when the water is 4 inches deep?

(b) Suppose, instead, the sheet of metal is folded twice, keeping the sides of equal height and inclined at an angle of 60°, as shown. Where should the folds be in order to maximize the volume of the trough? Justify your answer.

BC ONLY

12. (a) Using your calculator, verify that

$$\left(4\tan^{-1}(1/5)\right)-\left(\tan^{-1}(1/239)\right)\approx\frac{\pi}{4}.$$

(b) Use the Taylor polynomial of degree 7 about 0,

$$\tan^{-1}x\approx x-x^3/3+x^5/5-x^7/7,$$

to approximate $\tan^{-1}1/5$, and the polynomial of degree 1 to approximate $\tan^{-1}1/239$.

(c) Use part (b) to evaluate the expression in (a).

(d) Explain how the approximation for $\pi/4$ given here compares with that obtained using $\pi/4=\tan^{-1}1$.

13. (a) Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\ln(n+1)}$ converges.

(b) How many terms of the series are needed to get a partial sum within 0.1 of the sum of the whole series?

(c) Tell whether the series $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{n\ln n}$ is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

14. Given $\frac{dy}{dt}=ky(10-y)$ with $y=2$ at $t=0$ and $y=5$ at $t=2$:

(a) Find k.
(b) Express y as a function of t.
(c) For what value of t will $y=8$?
(d) Describe the long-range behavior of y.

15. An object P is in motion in the first quadrant along the parabola $y=18-2x^2$ in such a way that at t sec the x-value of its position is $x=\frac{1}{2}t$.

(a) Where is P when $t=4$?
(b) What is the vertical component of its velocity there?
(c) At what rate is its distance from the origin changing then?
(d) When does it hit the x-axis?
(e) How far did it travel altogether?

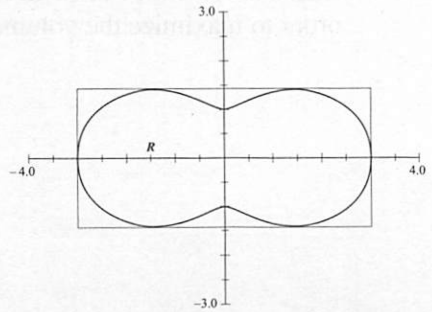
16. A particle moves in the xy-plane in such a way that at any time $t\geq 0$ its position is given by $x(t)=4\arctan t,\ y(t)=\frac{12t}{t^2+1}$.

(a) Sketch the path of the particle, indicating the direction of motion.
(b) At what time t does the particle reach its highest point? Justify.
(c) Find the coordinates of that highest point, and sketch the velocity vector there.
(d) Describe the long-term behavior of the particle.

17. Let $\boldsymbol{R}$ be the region bounded by the curve $r=2+\cos 2\theta$, as shown.

(a) Find the dimensions of the smallest rectangle that contains $\boldsymbol{R}$ and has sides parallel to the x- and y-axes.

(b) Find the area of $\boldsymbol{R}$.

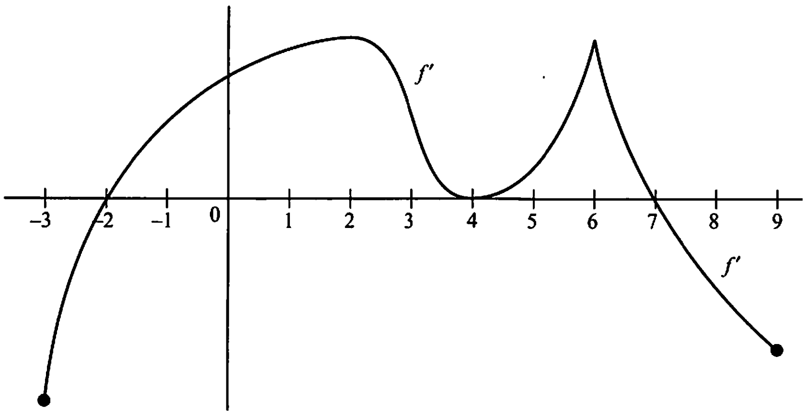


Part B. Directions: Answer these questions *wtihout* using your calculator.

18. Draw a graph of $y = f(x)$, given that f satisfies all the following conditions:

(1) $f'(-1) = f'(1) = 0$.
(2) If $x < -1, f'(x) > 0$ but $f'' < 0$.
(3) If $-1 < x < 0, f'(x) > 0$ and $f'' > 0$.
(4) If $0 < x < 1, f'(x) > 0$ but $f'' < 0$.
(5) If $x > 1, f'(x) < 0$ and $f'' < 0$.

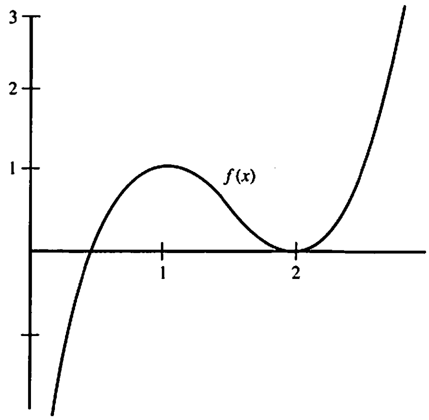
19. The figure below shows the graph of f', the derivative of f, with domain $-3 \le x \le 9$. The graph of f' has horizontal tangents at $x = 2$ and $x = 4$, and a corner at $x = 6$.



(a) Is f continuous? Explain.
(b) Find all values of x at which f attains a relative minimum. Justify.
(c) Find all values of x at which f attains a relative maximum. Justify.
(d) At what value of x does f attain its absolute maximum? Justify.
(e) Find all values of x at which the graph of f has a point of inflection. Justify.

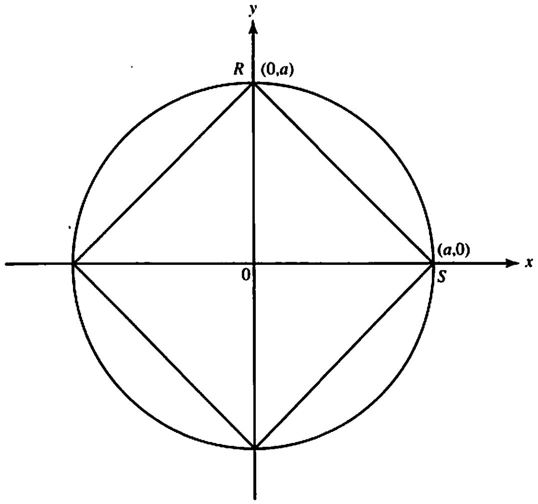
20. Find the area of the largest rectangle (with sides parallel to the coordinate axes) that can be inscribed in the region bounded by the graphs of $f(x) = 8 - 2x^2$ and $g(x) = x^2 - 4$.

21. Given the graph of $f(x)$, sketch the graph of $f'(x)$.



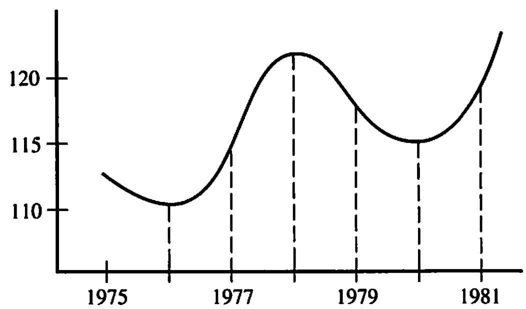
22. A cube is contracting so that its surface area decreases at the constant rate of 72 in.^{2}/sec. Determine how fast the volume is changing at the instant when the surface area is 54 ft^2.

23. A square is inscribed in a circle of radius a as shown in the diagram. Find the volume obtained if the region outside the square but inside the circle is rotated about a diagonal of the square.



24. (a) Sketch the region in the first quadrant bounded above by the line $y = x + 4$, below by the line $y = 4 - x$, and to the right by the parabola $y = x^2 + 2$.
(b) Find the area of this region.

25. The graph shown below is based roughly on data from the U.S. Department of Agriculture.



Food Production in South Asia
(Index was 100 during 1969–71)

(a) During which intervals did food production decrease in South Asia?
(b) During which intervals did the rate of change of food production increase?
(c) During which intervals did the increase in food production accelerate?

26. A particle moves along a straight line so that its acceleration at any time t is given in terms of its velocity v by $a = -2v$.

(a) Find v in terms of t if $v = 20$ when $t = 0$.
(b) Find the distance the particle travels while v changes from $v = 20$ to $v = 5$.

27. Let R represent the region bounded above by the parabola $y = 27 - x^2$ and below by the x-axis. Isosceles triangle AOB is inscribed in region R with its vertex at the origin O and its base $\overline{AB}$ parallel to the x-axis. Find the maximum possible area for such a triangle.

BC ONLY

28. (a) Find the Maclaurin series for $f(x) = \ln(1 + x)$.
(b) What is the radius of convergence of the series in (a)?
(c) Use the first five terms in (a) to approximate $\ln(1.2)$.
(d) Estimate the error in (c), justifying your answer.

29. A cycloid is given parametrically by $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

(a) Find the slope of the curve at the point where $\theta = \dfrac{2\pi}{3}$.

(b) Find the equation of the tangent to the cycloid at the point where $\theta = \dfrac{2\pi}{3}$.

30. Find the area of the region enclosed by both the polar curves $r = 4\sin\theta$ and $r = 4\cos\theta$.

31. (a) Find the 4th degree Taylor polynomial about 0 for $\cos x$.

(b) Use part (a) to evaluate $\int_0^1 \cos x\, dx$.

(c) Estimate the error in (b), justifying your answer.

32. A particle moves on the curve of $y^3 = 2x + 1$ so that its distance from the x-axis is increasing at the constant rate of 2 units/sec. When $t = 0$, the particle is at (0,1).

(a) Find a pair of parametric equations $x = x(t)$ and $y = y(t)$ that describe the motion of the particle for nonnegative t.
(b) Find $|\mathbf{a}|$, the magnitude of the particle's acceleration, when $t = 1$.

33. Find the area of the region that the polar curves $r = 2 - \cos\theta$ and $r = 3\cos\theta$ enclose in common.

Answers Explained

Part A

1. (a) $f'(4.0) \approx \dfrac{f(4.6)-f(4.0)}{4.6-4.0} = \dfrac{2.2-3.1}{0.6} = -1.5.$

$f'(4.8) \approx \dfrac{f(5)-f(4.6)}{5-4.6} = \dfrac{1.5-2.2}{0.4} = -1.75.$

(b) It appears that the rate of change of f, while negative, is increasing. This implies that the graph of f is concave upward.

(c) $L = 7.6(0.7) + 5.7(0.3) + 4.2(0.5) + 3.1(0.6) + 2.2(0.4) = 11.87.$

(d) Using disks $\Delta V = \pi r^2 \Delta x$. One possible answer uses the left endpoints of the subintervals as values of r:

$V \approx \pi(7.6)^2(0.7) + \pi(5.7)^2(0.3) + \pi(4.2)^2(0.5) + \pi(3.1)^2(0.6) + \pi(2.2)^2(0.4)$

2. (a) $12y_0 + 0.3 = 24$ yields $y_0 \approx 1.975.$

(b) Replace x by 0.3 in the equation of the curve:

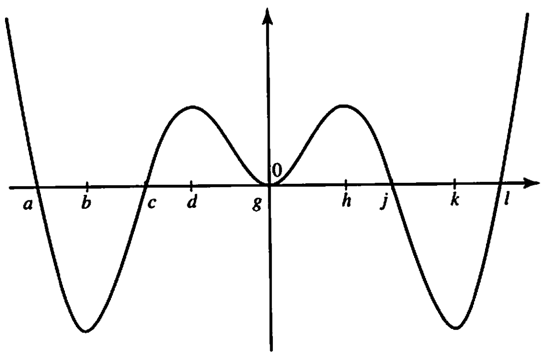
$$(0.3)^2 y_0 - (0.3) = y_0^3 - 8 \text{ or}$$

$$y_0^3 - 0.09y_0 - 7.7 = 0.$$

The calculator's solution to three decimal places is $y_0 = 1.990.$

(c) Since the true value of y_0 at $x = 0.3$ exceeds the approximation, conclude that the given curve is concave up near $x = 0$. (Therefore, it is above the line tangent at $x = 0$.)

3. Graph $f'(x) = 2x \sin x - e^{(-x^2)} + 1$ in $[-7, 7] \times [-10, 10]$.



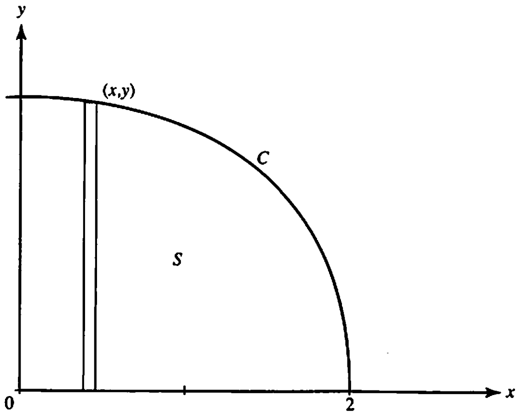
(a) Since f' is even and f contains (0, 0), f is odd and its graph is symmetric about the origin.

(b) Since f is decreasing when $f' < 0$, f decreases on the intervals (a, c) and (j, l). Use the calculator to solve $f'(x) = 0$. Conclude that f decreases on $-6.202 < x < -3.294$ and (symmetrically) on $3.294 < x < 6.202$.

(c) f has a relative maximum at $x = q$ if $f'(q) = 0$ and if f changes from increasing ($f' > 0$) to decreasing ($f' < 0$) at q. There are two relative maxima here:
at $x = a = -6.202$ and at $x = j = 3.294$.

(d) f has a point of inflection when the graph of f changes its concavity; that is, when f' changes from increasing to decreasing, as it does at points d and h, or when f' changes from decreasing to increasing, as it does at points b, g, and k. So there are five points of inflection altogether.

4. In the graph below, C is the piece of the curve lying in the first quadrant. S is the region bounded by the curve C and the coordinate axes.



(a) Graph $y = \sqrt[3]{(64-16x^2)}$ in $[0,3] \times [0,5]$. Since you want dy/dx, the slope of the tangent, where $y = 1$, use the calculator to solve

$$\sqrt[3]{64-16x^2} = 1$$

(storing the answer at B). Then evaluate the slope of the tangent to C at $y = 1$:

$$f'(B) \approx -21.182.$$

(b) Since $\Delta A = y\Delta x$, $A = \int_0^2 y\,dx \approx 6.730$.

(c) When S is rotated about the x-axis, its volume can be obtained using disks:

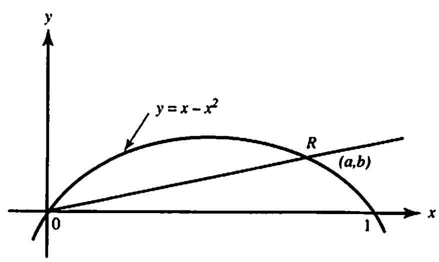
$$\Delta V = \pi R^2 \Delta x = \pi y^2\,\Delta x,$$

$$V = \pi \int_0^2 y^2\,dx$$

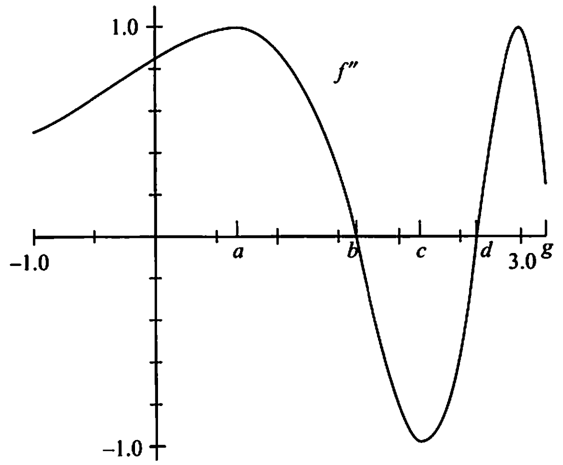
$$= \pi \int_0^2 \left(\sqrt[3]{64-16x^2}\right)^2 dx \approx 74.310.$$

5. See the figure, where R is the point (a, b), and seek a such that

$$\int_0^a \left(x - x^2 - \frac{b}{a} \cdot x\right) dx = \frac{1}{2}\int_0^1 \left(x - x^2\right) dx.$$



6. Graph $y = \sin 2^x$ in $[-1, 3.2] \times [-1, 1]$. Note that $y = f''$.

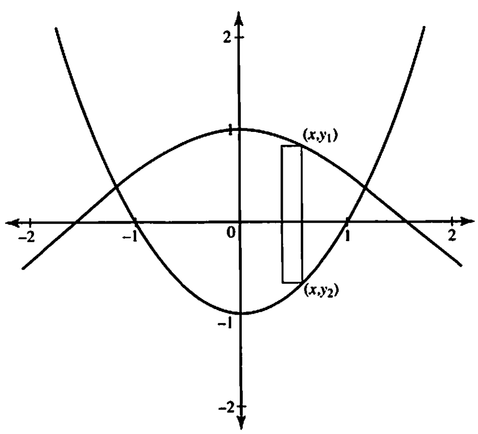


(a) The graph of f is concave downward where f'' is negative, namely, on (b, d). Use the calculator to solve $\sin 2^x = 0$, obtaining $b = 1.651$ and $d = 2.651$. The answer to (a) is therefore $1.651 < x < 2.651$.

(b) f' has a relative minimum at $x = d$, because f'' equals 0 at d, is less than 0 on (b, d), and is greater than 0 on (d, g). Thus f' has a relative minimum (from part a) at $x = 2.651$.

(c) The graph of f' has a point of inflection wherever its second derivative f''' changes from positive to negative or vice versa. This is equivalent to f'' changing from increasing to decreasing (as at a and g) or vice versa (as at c). Therefore, the graph of f' has three points of inflection on $[-1, 3.2]$.

7. Graph $f(x) = \cos x$ and $g(x) = x^2 - 1$ in $[-2, 2] \times [-2, 2]$. Here, $y_1 = f$ and $y_2 = g$.



(a) Solve $\cos x = x^2 - 1$ to find the two points of intersection: (1.177,0.384) and (−1.177,0.384).

(b) Since $\Delta A = (y_1 - y_2)\,\Delta x = [f(x) - g(x)]\,\Delta x$, the area A bounded by the two curves is

$$A = 2\int_0^{1.777} (y_1 - y_2)\,dx,$$

$$= 2\int_0^{1.777} (\cos x - (x^2 - 1))\,dx$$

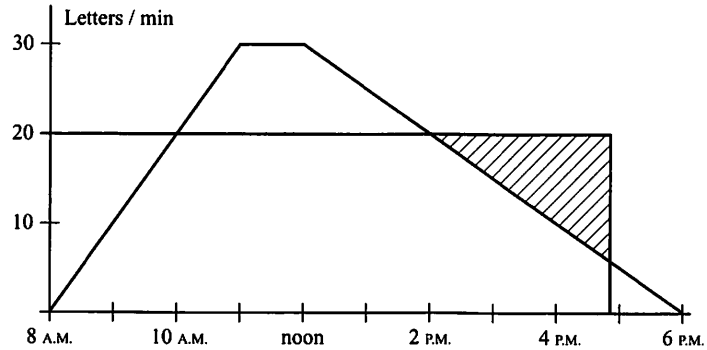
$$\approx 3.114.$$

8. (a) Use the Trapezoid Rule, with $h = 60$ min:

$$\frac{h}{2}\left(y_0 + 2y_0 + 2y_0 + 2y_0 + y_5\right) =$$

$$\frac{60}{2}\left(10 + 2 \cdot 12 + 2 \cdot 8 + 2 \cdot 9 + 11\right) = 2370 \text{ letters.}$$

(b) Draw a horizontal line at $y = 20$ (as shown on the graph below), representing the rate at which letters are processed then.



(i) Letters began to pile up when they arrived at a rate greater than that at which they were being processed, that is, at $t = 10$ A.M.

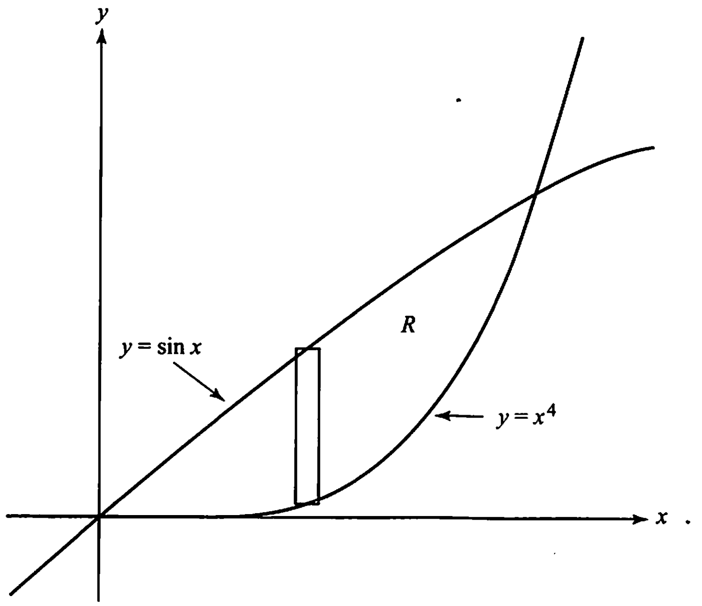
(ii) The pile was largest when the letters stopped piling up, at $t = 2$ P.M.

(iii) The number of letters in the pile is represented by the area of the small trapezoid above the horizontal line: $\frac{1}{2}(4 \cdot 60 + 1 \cdot 60)(10) = 1500$.

(iv) The pile began to diminish after 2 P.M., when letters were processed at a rate faster than they arrived, and vanished when the area of the shaded triangle represented 1500 letters. At 5 P.M. this area is

$\frac{1}{2}(3 \cdot 60)(15) = 1350$ letters, so the pile vanished shortly after 5 P.M.

9. Draw a vertical element of area as shown below.

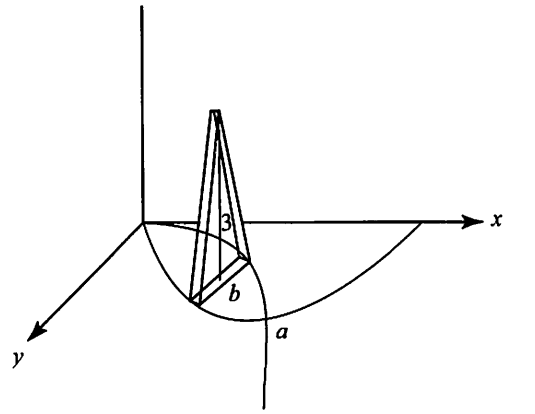


(a) Let a represent the x-value of the positive point of intersection of $y = x^4$ and $y = \sin x$. Solving $a^4 = \sin a$ with the calculator, we find $a = 0.9496$.

$$\Delta A = (y_{\text{top}} - y_{\text{bottom}})\Delta x = (\sin x - x^4)\Delta x,$$

$$A = \int_0^a (\sin x - x^4)\,dx \simeq 0.264.$$

(b) Elements of volume are triangular prisms with height $h = 3$ and base $b = (\sin x - x^4)$, as shown below.



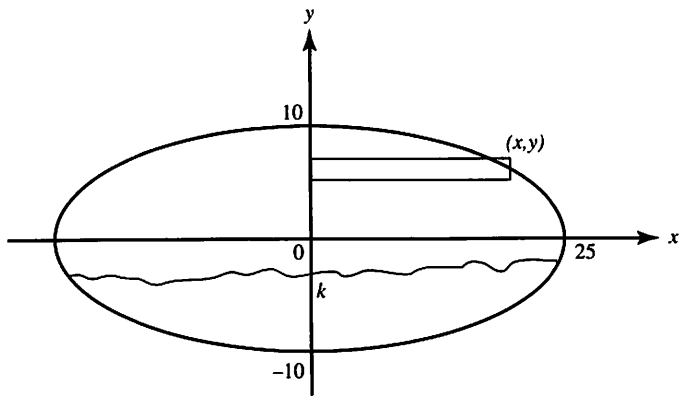
$$\Delta V = \frac{1}{2}(\sin x - x^4)(3)\Delta x,$$

$$V = \frac{3}{2}\int_0^a (\sin x - x^4)\,dx = 0.395.$$

(c) When R is rotated around the x-axis, the element generates washers. If r_1 and r_2 are the radii of the larger and smaller disks, respectively, then

$$\Delta V = \pi(r_1^2 - r_2^2)\Delta x = \pi\left((\sin x)^2 - (x^4)^2\right)\Delta x,$$

$$V = \pi\int_0^a (\sin^2 x - x^8)\,dx = 0.529.$$



10. The figure above shows an elliptical cross section of the tank. Its equation is

$$\frac{x^2}{625} + \frac{y^2}{100} = 1.$$

(a) The volume of the tank, using disks, is $V = 2\pi \int_0^{10} x^2 dy$, where the ellipse's symmetry about the x-axis has been exploited. The equation of the ellipse is equivalent to $x^2 = 6.25(100 - y^2)$, so

$$V = 12.5\pi \int_0^{10} \left(100 - y^2\right) dy.$$

Use the calculator to evaluate this integral, storing the answer as V to have it available for part (b).

The capacity of the tank is $7.48V$, or 196,000 gal of water, rounded to the nearest 1000 gal.

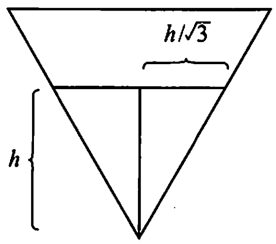
(b) Let k be the y-coordinate of the water level when the tank is one-fourth full.

Then

$$6.25\pi \int_{-10}^{k} \left(100 - y^2\right) dy = \frac{V}{4}$$

and the depth of the water is $k + 10$.

11. (a) Let h represent the depth of the water, as shown.



Then h is the altitude of an equilateral triangle, and the base $b = \frac{2h}{\sqrt{3}}$.
The volume of water is

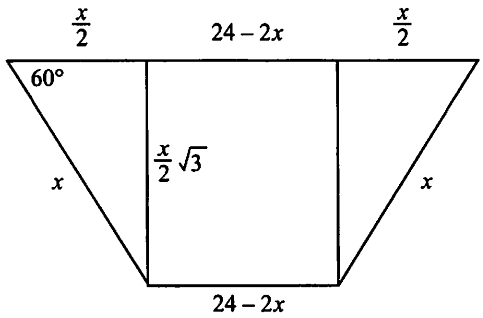
$$V = \frac{1}{2}\left(\frac{2h}{\sqrt{3}}\right)h \cdot 60 = \frac{60h^2}{\sqrt{3}} \text{ in.}^3$$

Now $\frac{dV}{dt} = \frac{120}{\sqrt{3}} h \frac{dh}{dt}$, and it is given that $\frac{dV}{dt} = 600$. Thus, when $h = 4$,

$$600 = \frac{120}{\sqrt{3}} 4 \frac{dh}{dt}, \text{ and } \frac{dh}{dt} = \frac{5\sqrt{3}}{4} \text{ in/min.}$$

(b) Let x represent the length of one of the sides, as shown.

The bases of the trapezoid are $24 - 2x$ and $24 - 2x + 2\frac{x}{2}$, and the height is $\frac{x}{2}\sqrt{3}$.



The volume of the trough (in in.^{3}) is given by

$$V = \frac{(24-2x)+(24-x)}{2} \cdot \frac{x}{2}\sqrt{3}\times 60 = 15\sqrt{3}\left(48x-3x^2\right) \qquad (0<x<12),$$

$V' = 15\sqrt{3}(48-6x) = 0$ when $x = 8$.

Since $V'' = 15\sqrt{3}(-6) < 0$, the maximum volume is attained by folding the metal 8 inches from the edges.

12. (a) Both $\pi/4$ and the expression in brackets yield 0.7853981634, which is accurate to ten decimal places.

(b) $\tan^{-1}\frac{1}{5} = \frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 - \frac{1}{7}\left(\frac{1}{5}\right)^7 = 0.197396.$

$$\tan^{-1}\frac{1}{239} = \frac{1}{239} = 0.004184.$$

(c) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = 0.7854$; this agrees with the value of $\frac{\pi}{4}$ to four decimal places.

(d) The series

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

converges *very* slowly. Example 56, page 438, evaluated the sum of 60 terms of the series for π (which equals $4\tan^{-1}1$). To four decimal places, we get $\pi = 3.1249$, which yields 0.7812 for $\pi/4$—not accurate even to two decimal places.

13. (a) The given series is alternating. Since $\lim_{n\to\infty} \ln(n+1) = \infty$, $\lim_{n\to\infty}\frac{1}{\ln(n+1)} = 0$.

Since $\ln x$ is an increasing function,

$$\ln(n+1) > \ln n \quad \text{and} \quad \frac{1}{\ln(n+1)} < \frac{1}{\ln n}.$$

The series therefore converges.

(b) Since the series converges by the Alternating Series Test, the error in using the first n terms for the sum of the whole series is less than the absolute value of the $(n+1)$st term. Thus the error is less than $\frac{1}{\ln(n+1)}$. Solve for n using $\frac{1}{\ln(n+1)} < 0.1$:

$$\begin{aligned}\ln(n+1) &> 10,\\ (n+1) &> e^{10},\\ n &> e^{10} - 1 > 22{,}025.\end{aligned}$$

The given series converges very slowly!

(c) The series $\sum_{2}^{\infty}(-1)^n \frac{1}{n \ln n}$ is conditionally convergent. The given alternating series converges since the nth term approaches 0 and $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$. However, the *nonnegative* series diverges by the Integral Test, since

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_2^b = \infty.$$

14. (a) Solve by separation of variables:

$$\frac{dy}{y(10-y)} = k\,dt,$$

$$\frac{1}{10}\int\left(\frac{1}{y} + \frac{1}{10-y}\right)dy = \int k\,dt,$$

$$\frac{1}{10}\ln\left(\frac{y}{10-y}\right) = kt + C,$$

$$\ln\left(\frac{10-y}{y}\right) = -10\,(kt + C).$$

Let $c = e^{-10C}$; then

$$\frac{10-y}{y} = ce^{-10kt}.$$

Now use initial condition $y = 2$ at $t = 0$:

$$\frac{8}{2} = ce^0 \text{ so } c = 4;$$

and the other condition, $y = 5$ at $t = 2$, gives

$$\frac{5}{5} = 4e^{-20k} \text{ or } k = \frac{1}{10}\ln 2.$$

(b) Since $c = 4$ and $k = \frac{1}{10}\ln 2$, then $\frac{10-y}{y} = 4e^{-10\left(\frac{1}{10}\ln 2\right)t}$.

Solving for y yields $y = \frac{10}{1+4\cdot 2^{-t}}$.

(c) $8 = \dfrac{10}{1+4\cdot 2^{-t}}$. means $1 + 4 \cdot 2^{-t} = 1.25$, so $t = 4$.

(d) $\lim_{t\to\infty} \dfrac{10}{1+4\cdot 2^{-t}} = 10$, so the value of y approaches 10.

15. (a) Since $x = \frac{1}{2}t$, $x(4) = \frac{1}{2}(4) = 2$. Since $y = 18 - 2 \cdot 2^2 = 10$, P is at (2,10).

(b) Since $y = 18 - 2x^2$, $\dfrac{dy}{dt} = -4x\dfrac{dx}{dt}$. Since $x = \frac{1}{2}t$, $\dfrac{dx}{dt} = \dfrac{1}{2}$. Therefore

$$\frac{dy}{dt} = -4x\frac{dx}{dt} = -4\cdot 2\cdot \frac{1}{2} = -4 \text{ unit/sec.}$$

(c) Let D = the object's distance from the origin. Then

$$D^2 = x^2 + y^2, \text{ and at } (2,10)\ D = \sqrt{104}.$$

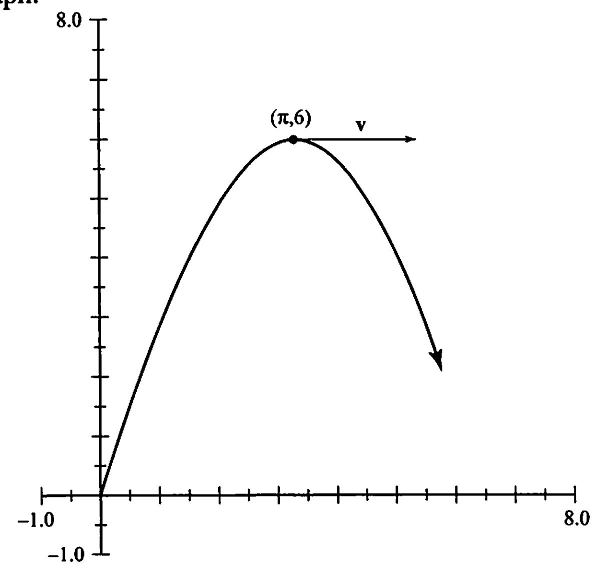
$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$
$$2\sqrt{104}\frac{dD}{dt} = 2\cdot 2\cdot \frac{1}{2} + 2\cdot 10(-4),$$
$$\frac{dD}{dt} = \frac{-78}{2\sqrt{104}} = -3.824 \text{ unit/sec.}$$

(d) The object hits the x-axis when $y = 18 - 2x^2 = 0$, or $x = 3$. Since $x = \frac{1}{2}t = 3, t = 6$.

(e) The length of the arc of $y = 18 - 2x^2$ for $0 \le x \le 3$ is given by

$$L = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1+(-4x)^2}\, dx = 18.460 \text{ units.}$$

16. (a) See graph.



(b) You want to maximize $y(t) = \dfrac{12t}{t^2+1}$.

$$y'(t) = \frac{(t^2+1)(12) - 12t(2t)}{(t^2+1)^2} = \frac{12(1-t)(1+t)}{(t^2+1)^2}.$$

See signs analysis.

t	0		1	
y		Inc.		Dec.
y'		+		−

The maximum y occurs when $t = 1$, because y changes from increasing to decreasing there.

(c) Since $x(1) = 4\arctan 1 = \pi$ and $y(1) = \dfrac{12}{1+1} = 6$, the coordinates of the highest point are $(\pi, 6)$.

Since $x'(t) = \dfrac{4}{1+t^2}$ and $y'(t) = \dfrac{12(1-t^2)}{(t^2+1)^2}$, so $\mathbf{v}(1) = \langle 2,0 \rangle$. This vector is shown on the graph.

(d) $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} 4\arctan t = 4\left(\dfrac{\pi}{2}\right) = 2\pi$, and $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} \dfrac{12t}{t^2+1} = 0$. Thus the particle approaches the point $(2\pi, 0)$.

17. (a) To find the smallest rectangle with sides parallel to the x- and y-axes, you need a rectangle formed by vertical and horizontal tangents as shown in the figure. The vertical tangents are at the x-intercepts, $x = \pm 3$. The horizontal tangents are at the points where y (not r) is a maximum. You need, therefore, to maximize

$$y = r\sin\theta = (2 + \cos 2\theta)\sin\theta,$$

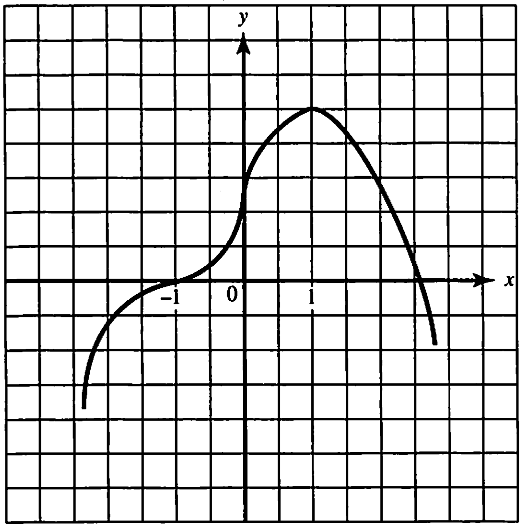
$$\frac{dy}{d\theta} = (2 + \cos 2\theta)\cos\theta + \sin\theta(-2\sin 2\theta).$$

Use the calculator to find that $\dfrac{dy}{d\theta} = 0$ when $\theta = 0.7854$. Therefore, $y = 1.414$, so the desired rectangle has dimensions 6×2.828.

(b) Since the polar formula for the area is $\dfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$, the area of R (enclosed by r) is $4 \cdot \dfrac{1}{2}\int_0^{\pi/2} r^2\,d\theta$, which is 14.137.

Part B

18. The graph shown below satisfies all five conditions. So do many others!



19. (a) f' is defined for all x in the interval. Since f is therefore differentiable, it must also be continuous.

(b) Because $f'(2) = 0$ and f' changes from negative to positive there, f has a relative minimum at $x = 2$. To the left of $x = 9$, f' is negative, so f is decreasing as it approaches that endpoint and reaches another relative minimum there.

(c) Because f' is negative to the right of $x = -3$, f decreases from its left endpoint, indicating a relative max there. Also, $f'(2) = 0$ and f' changes from positive to negative there, so f has a relative minimum at $x = 7$.

(d) Note that $f(7) - f(-3) = \int_{-3}^{7} f'(x)\,dx$. Since there is more area above the x-axis than below the x-axis on $[-3,7]$, the integral is positive and $f(7) - f(-3) > 0$. This implies that $f(7) > f(-3)$, and that the absolute maximum occurs at $x = 7$.

(e) At $x = 2$ and also at $x = 6$, f' changes from increasing to decreasing, indicating that f changes from concave upward to concave downward at each. At $x = 4$, f' changes from decreasing to increasing, indicating that f changes from concave downward to concave upward there. Hence the graph of f has points of inflection at $x = 2$, 4, and 6.

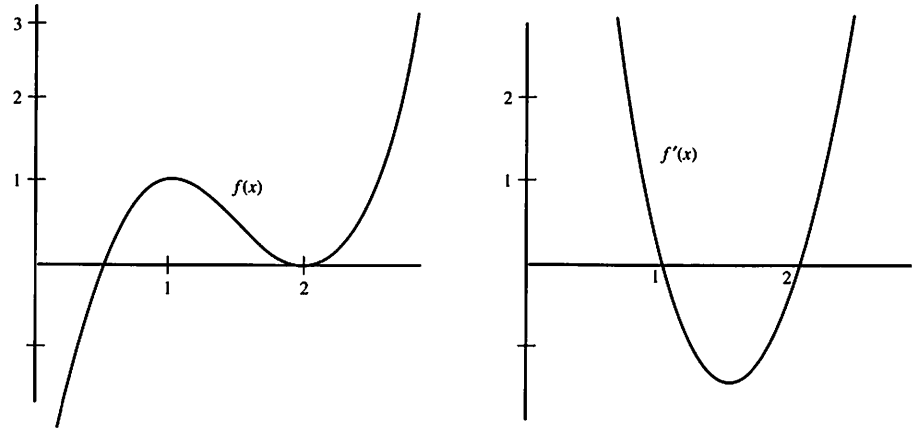
20. Draw a sketch of the region bounded above by $y_1 = 8 - 2x^2$ and below by $y_2 = x^2 - 4$, and inscribe a rectangle in this region as described in the question. If (x, y_1) and (x, y_2) are the vertices of the rectangle in quadrants I and IV, respectively, then the area

$$A = 2x\,(y_1 - y_2) = 2x(12 - 3x^2), \quad \text{or} \quad A(x) = 24x - 6x^3.$$

Then $A'(x) = 24 - 18x^2 = 6(4 - 3x^2)$, which equals 0 when $x = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Check to verify that $A''(x) < 0$ at this point. This assures that this value of x yields maximum area, which is given by $\frac{4\sqrt{3}}{3} \times 8$.

21. The graph of $f'(x)$ is shown here.

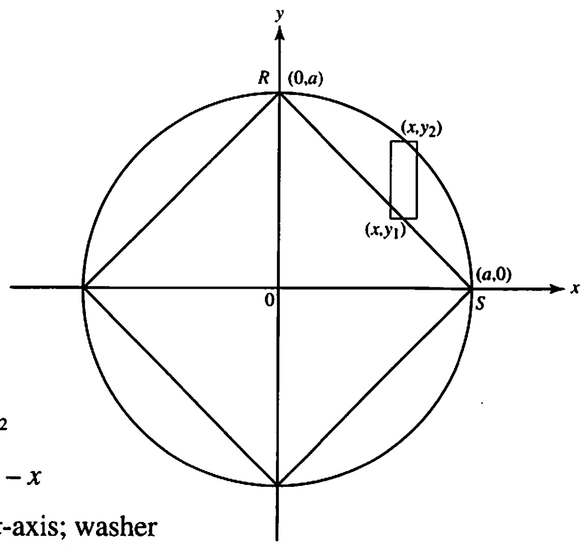


22. The rate of change in volume when the surface area is 54 ft^3 is $-\frac{3}{8}$ ft^3/sec.

23. See the figure. The equation of the circle is $x^2 + y^2 = a^2$; the equation of RS is $y = a - x$. If y_2 is an ordinate of the circle and y_1 of the line, then,

$$\Delta V = \pi y_2{}^2 \Delta x - \pi y_1{}^2 \Delta x,$$

$$V = 2\pi \int_0^a [(a^2 - x^2) - (a - x)^2]dx = \frac{2}{3}\pi a^3.$$



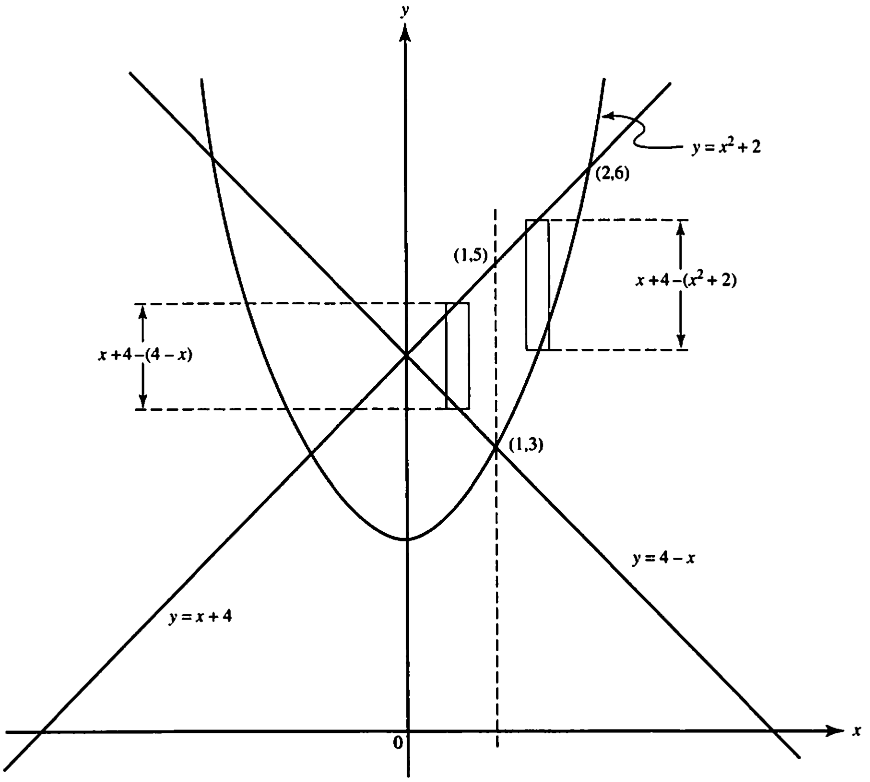
$x^2 + y_2^2 = a^2$

$y_1 = a - x$

About the x-axis; washer

$\Delta V = \pi y_2^2\, \Delta x - \pi y_1^2\, \Delta x$

24. (a) The region is sketched in the figure. The pertinent points of intersection are labeled.



(b) The required area consists of two parts. The area of the triangle is represented by $\int_1^2 [(x+4)-(4-x)]\,dx$ and is equal to 1, while the area of the region bounded at the left by $x = 1$, above by $y = x + 4$, and at the right by the parabola is represented by $\int_1^2 [(x+4)-(x^2+2)]\,dx$. This equals

$$\int_1^2 (x+2-x^2)\,dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_1^2 = \frac{7}{6}.$$

The required area, thus, equals $2\frac{1}{6}$ or $\frac{13}{6}$.

25. (a) 1975 to 1976 and 1978 to 1980.
(b) 1975 to 1977 and 1979 to 1981.
(c) 1976 to 1977 and 1980 to 1981.

26. (a) Since $a = \frac{dv}{dt} = -2v$, then, separating variables, $\frac{dv}{v} = -2dt$. Integrating gives

$$\ln v = -2t + C, \qquad (1)$$

and, since $v = 20$ when $t = 0$, $C = \ln 20$. Then (1) becomes $\ln \frac{v}{20} = -2t$ or, solving for v,

$$v = 20e^{-2t}. \qquad (2)$$

(b) Note that $v > 0$ for all t. Let s be the required distance traveled (as v decreases from 20 to 5); then

$$s = \int_{v=20}^{v=5} 20e^{-2t}\,dt = \int_{t=20}^{\ln=2} 20e^{-2t}\,dt, \qquad (3)$$

where, when $v = 20$, $t = 0$. Also, when $v = 5$, use (2) to get $\frac{1}{4} = e^{-2t}$ or $-\ln 4 = -2t$. So $t = \ln 2$. Evaluating s in (3) gives

$$s = -10e^{-2t}\Big|_0^{\ln 2} = -10\left(\frac{1}{4} - 1\right) = \frac{15}{2}.$$

27. Let (x,y) be the point in the first quadrant where the line parallel to the x-axis meets the parabola. The area of the triangle is given by

$$A = xy = x(27 - x^2) = 27x - x^3 \text{ for } 0 \le x \le 3\sqrt{3}.$$

Then $A'(x) = 27 - 3x^2 = 3(3 + x)(3 - x)$, and $A'(x) = 0$ at $x = 3$.

Since A' changes from positive to negative at $x = 3$, the area reaches its maximum there.

The maximum area is $A(3) = 3(27 - 3^2) = 54$.

28. (a) Let $f(x) = \ln(1 + x)$. Then $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$, and $f^{(4)}(x) = -\frac{3!}{(1+x)^4}$, $f^{(5)}(x) = \frac{4!}{(1+x)^5}$. At $x = 0$, $f(0) = 0, f'(0) = 1, f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -(3!)$, and $f^{(5)}(0) = 4!$. So

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

(b) Using the Ratio Test, you know that the series converges when

$\lim_{x\to\infty}\left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| < 1$, that is, when $|x| < 1$, or $-1 < x < 1$. Thus, the radius of convergence is 1.

(c) $\ln(1.2) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5}.$

(d) Since the series converges by the Alternating Series Test, the error in the answer for (c) is less absolutely than $\frac{(0.2)^6}{6}$.

29. From the equations for x and y,

$$dx = (1 - \cos\theta)\,d\theta \quad \text{and} \quad dy = \sin\theta\,d\theta.$$

(a) The slope at any point is given by $\frac{dy}{dx}$, which here is $\frac{\sin\theta}{1-\cos\theta}$. When $\theta = \frac{2\pi}{3}$, the slope is $\frac{\sqrt{3}}{3}$.

(b) When $\theta = \frac{2\pi}{3}$, $x = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ and $y = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$. The equation of the tangent is $9y - 3\sqrt{3}\cdot x = 18 - 2\pi\sqrt{3}$.

30. Both curves are circles with centers at, respectively, (2, 0) and $\left(2, \frac{\pi}{2}\right)$; the circles intersect at $\left(2\sqrt{2}, \frac{\pi}{4}\right)$. The common area is given by

$$\int_0^{\pi/4} (4\sin\theta)^2\,d\theta \text{ or } \int_{\pi/4}^{\pi/2} (4\cos\theta)^2\,d\theta.$$

The answer is $2(\pi - 2)$.

31. (a) For $f(x) = \cos x, f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, f^{(4)}(x) = \cos x,$ $f^{(5)}(x) = -\sin x, f^{(6)}(x) = -\cos x$. The Taylor polynomial of order 4 about 0 is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

Note that the next term of the alternating Maclaurin series for $\cos x$ is $-\frac{x^6}{6!}$.

(b) $\int_0^1 \cos x\,dx = x - \frac{x^3}{3\cdot 2!} + \frac{x^5}{5\cdot 4!}\Big|_0^1 = 1 - \frac{1}{6} + \frac{1}{120}$.

(c) The error in (b), convergent by the Alternating Series Test, is less absolutely than the first term dropped:

$$\text{error} < \int_0^1 \frac{x^6}{6!}\,dx = \frac{x^7}{7!}\Big|_0^1 = \frac{1}{7!}.$$

32. (a) Since $\frac{dy}{dt} = 2$, $y = 2t + 1$ and $x = 4t^3 + 6t^2 + 3t$.

(b) Since $\frac{d^2y}{dt^2} = 0$ and $\frac{d^2x}{dt^2} = 24t + 12$, then, when $t = 1$, $|\mathbf{a}| = 36$.

33. See the figure. The required area A is twice the sum of the following areas: that of the limaçon from 0 to $\frac{\pi}{3}$, and that of the circle from $\frac{\pi}{3}$ to $\frac{\pi}{2}$. Thus

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3}(2-\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2\,d\theta\right]$$

$$= \frac{9\pi}{4} - 3\sqrt{3}\,.$$