AP Calculus
3.1 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What is the original limit definition of a derivative?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

2. What is the alternative definition of a derivative?

$$f'(x) = \lim_{c\to x} \frac{f(c)-f(x)}{c-x}$$

3. Use the original definition of the derivative to find the derivative of each function at the indicated point.

a) $f(x)=\frac{1}{x}$ at $a=2$

$$\frac{f(2+h)-f(2)}{h} = \frac{\frac{1}{2+h}-\frac{1}{2}}{h}$$

$$= \frac{\frac{2-(2+h)}{2(2+h)}}{h} = -\frac{1}{2(2+h)}$$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = -\frac{1}{4} \qquad f'(2)=-\frac{1}{4}$$

b) $h(x)=3-x^2$ at $a=-2$.

$$\frac{h(-2+h)-h(-2)}{h} = \frac{3-(-2+h)^2-(3-(-2)^2)}{h}$$

$$= \frac{3-4+4h-h^2-3+4}{h} = 4-h \xrightarrow[h\to 0]{} 4$$

$$h'(-2)=4$$

c) $g(x)=\sqrt{x+1}$ at $a=3$

$$\frac{g(3+h)-g(3)}{h} = \frac{\sqrt{3+h+1}-\sqrt{3+1}}{h}$$

$$= \frac{\sqrt{4+h}-2}{h} \times \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$= \frac{1}{\sqrt{4+h}+2} \xrightarrow[h\to 0]{} \frac{1}{4} \qquad g'(3)=\frac{1}{4}$$

d) $f(x)=x-x^3$ at $a=-1$

$$\frac{f(-1+h)-f(-1)}{h} = \frac{(-1+h)-(-1+h)^3-(-1-(-1)^3)}{h}$$

$$= \frac{h+1-3h+3h^2-h^3-1}{h} = -h^2+3h-2 \xrightarrow[h\to 0]{} -2$$

$$f'(-1)=-2$$

4. Repeat question 3a – 3c using the alternative definition of the derivative.

a) $$\frac{f(2)-f(c)}{2-c} = \frac{\frac{1}{2}-\frac{1}{c}}{2-c} = \frac{\frac{c-2}{2c}}{2-c}$$

$$= -\frac{1}{2c} \xrightarrow[c\to 2]{} -\frac{1}{4}$$

b) $$\frac{h(-2)-h(c)}{-2-c} = \frac{3-4-3+c^2}{-2-c}$$

$$= \frac{-4+c^2}{-2-c}$$

$$= \frac{(c+2)(c-2)}{-(c+2)}$$

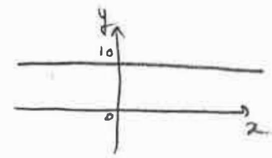
$$= -c+2 \xrightarrow[c\to -2]{} 4$$

c) $$\frac{g(3)-g(c)}{3-c} = \frac{2-\sqrt{c+1}}{3-c} \times \frac{2+\sqrt{c+1}}{2+\sqrt{c+1}}$$

$$= \frac{4-(c+1)}{(3-c)(2+\sqrt{c+1})}$$

$$= \frac{1}{2+\sqrt{c+1}} \xrightarrow[c\to 3]{} \frac{1}{4}$$

5. Consider the function $g(x)=10$.



a) Using what you know about the graph of $g(x)$, what does $g'(6)=$ **0**

b) Use the alternative definition AND the original definition of the derivative to verify your answer for $g'(6)$. YES … use BOTH definitions ☺

$$\frac{g(6+h)-g(6)}{h} = \frac{10-10}{h} = 0 \qquad h \neq 0$$

$$\frac{g(6)-g(c)}{6-c} = \frac{10-10}{6-c} = 0 \qquad 6 \neq c$$

$$g'(6)=0$$

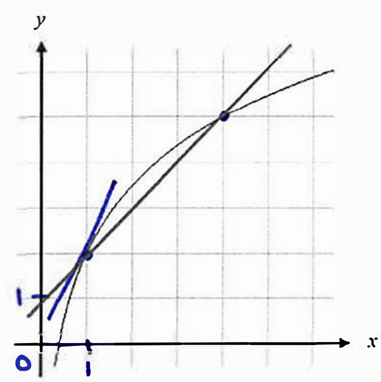
6. If $f(2)=3$ and $f'(2)=5$, find an equation of the tangent line at the point where $x=2$.

point $(2;3)$
slope : 5

$$y-3=5(x-2)$$

$$\boxed{y=5x-7}$$

7. Use the figure to the right to answer the following questions:



a) What is $f(1)$ and $f(4)$?

$f(1)=2 \qquad f(4)=5$

b) What is the geometric interpretation of $\frac{f(4)-f(1)}{4-1}$?

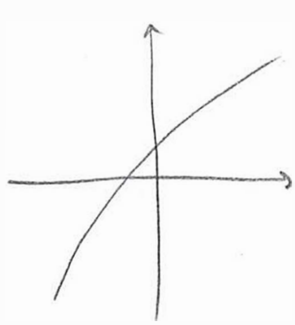
~~slope of the secant line.~~ average rate of change

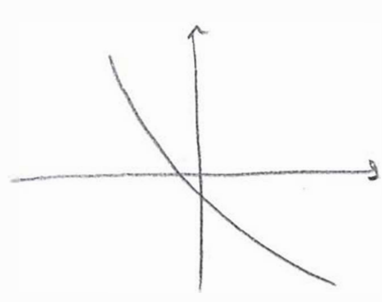
c) Using the geometric interpretation of each expression, insert the proper inequality symbol (< or >).

i) $\frac{f(4)-f(1)}{4-1} \boxed{>} \frac{f(4)-f(3)}{4-3}$ ii) $\frac{f(4)-f(1)}{4-1} \boxed{<} f'(1)$

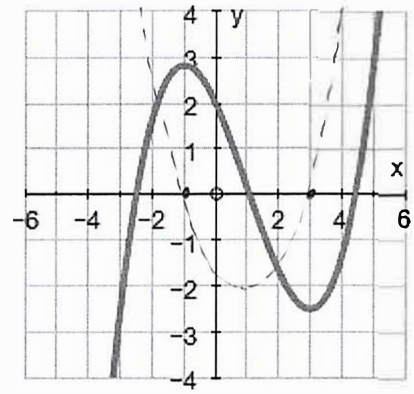
(comparing the slopes …)

8. Sketch a function whose derivative is ALWAYS negative and another whose derivative is ALWAYS positive.





9. Use the graph of $f(x)$ shown to the right.



a) Where is $f'(x)=0$? Explain.

$x=-1$ and $x=3$ (horizontal tangent lines)

b) Where is $f'(x)>0$? Explain.

over $(-\infty;-1)$ and over $(3;+\infty)$

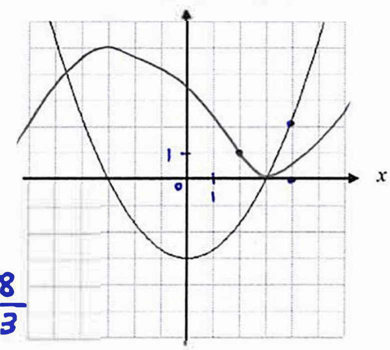
c) Where is $f'(x)<0$? Explain.

over $(-1;3)$

d) On the same graph, draw a possible sketch of $f'(x)$.

10. The figure to the right shows the graph of $g'(x)$.

The graph of $\underline{g'(x)}$



a) What does $g'(0)=$? … How about $g'(3)$?

$g'(0)=-3$ $g'(3)=0$

c) From the graph, $g'(1)=-\frac{8}{3}$. What does this tell us about the graph of g?

the slope of the tangent line of g at 1 is $-\frac{8}{3}$
(therefore, it's decreasing at 1)

d) From the graph, $g'(4)=\frac{7}{3}$. What does this tell us about the graph of g?

———————————————— at 4 is $\frac{7}{3}$
(therefore, it's increasing at 4)

e) Is $g(6)-g(4)$ positive or negative (those are g values not g')? Explain.

$g'(x)$ is positive over $[4;6]$. Therefore, g is increasing over $[4;6]$
As a consequence, $g(6)>g(4)$ i.e $g(6)-g(4)>0$

f) Find (if they exist) any value(s) of x, where $g'(x)=0$?

$g'(x)=0$ for $x=-3$ and $x=3$.

g) Is it possible to find $g(2)$ from this graph? Explain.

not yet …

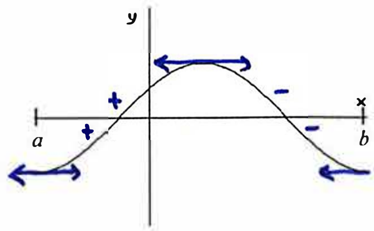
h) What interval is $g(x)$ increasing? What interval is $g(x)$ decreasing? How do you know?

$(-\infty;-3)$ and $(3;+\infty)$ $(-3;3)$ (sign of the derivative)

i) If you were told that $g(2)=1$, sketch a possible graph of $g(x)$?

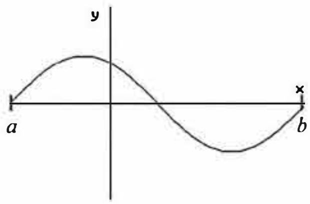
approx for now.

11. The graph of f is shown below.

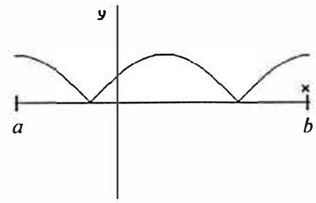


Which of the following could be the graph of the derivative of f?

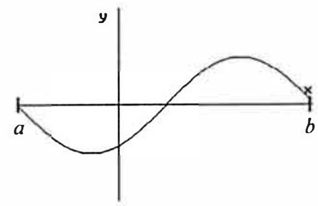
A.



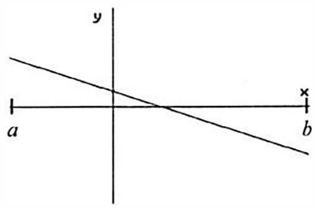
B.



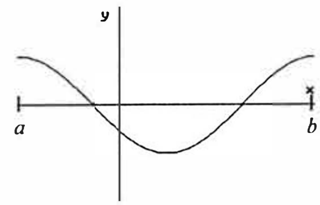
C.



D.

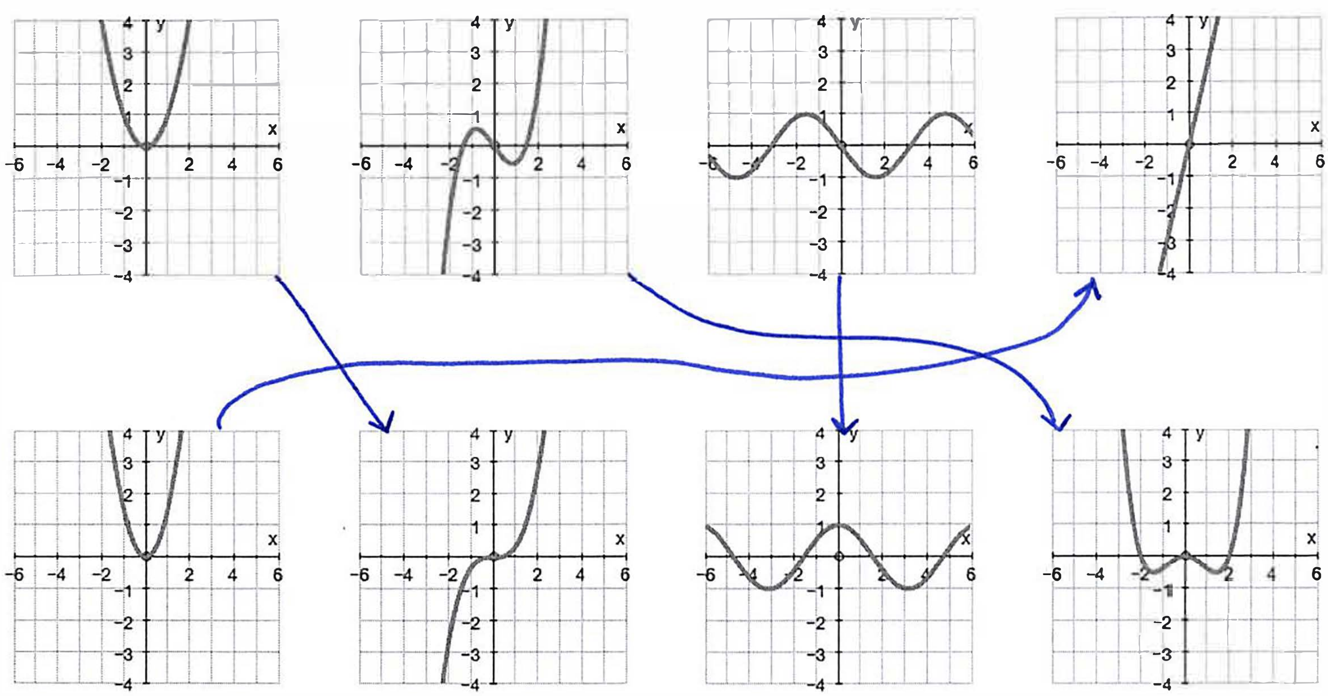


E.



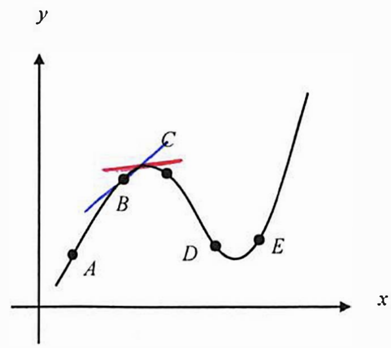
12. The graphs in the first row are the derivatives. Match them with the graph of their function shown in the second row.

(Graphs of Derivative)



(Graphs of Function)

13. Use the graph of f below to answer each question.



a) Between which two consecutive points is the average rate of change of the function greatest?

A and B

b) Is the average rate of change between A and B greater than or less than the instantaneous rate of change of B?

greater

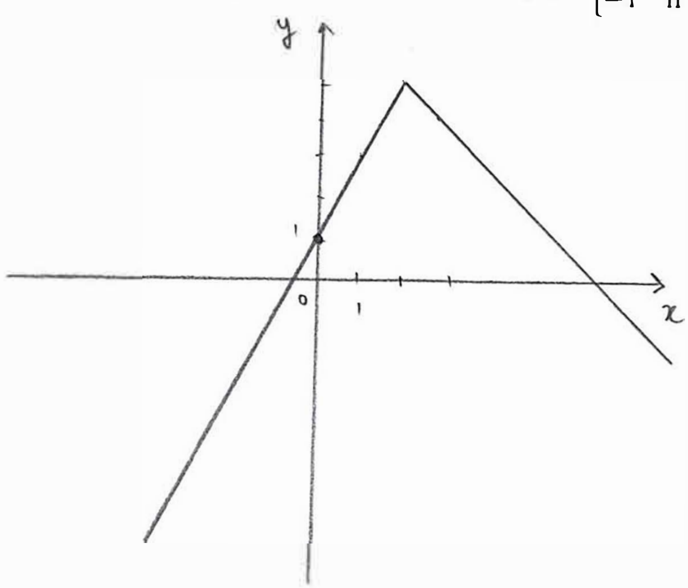
c) Give any sets of consecutive points for which the average rates of change of the function are approximately equal.

B and C vs. D and E

d) Sketch a tangent line to the graph somewhere between the points B and C such that the slope of the tangent line you draw is the same as the average rate of change of the function between B and C. (Do you think it would be possible to do this for ANY two points on a curve?)

(see red)

14. Sketch the graph of a continuous function f with $f(0) = 1$ and $f'(x) = \begin{cases} 2 & \text{if } x < 2 \\ -1 & \text{if } x > 2 \end{cases}$



AP Calculus
3.2 Worksheet

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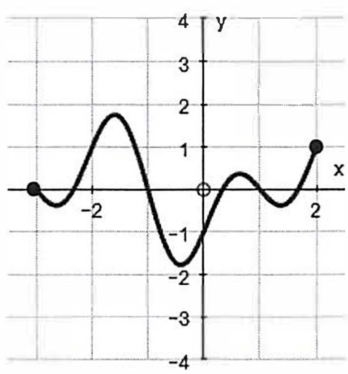
1. When does a derivative fail to exist? when there is a corner, a discontinuity or a vertical tangent line.

For questions 2 – 4, the graph of a function over a closed interval D is given. At what domain points does the function appear to be

a) differentiable? b) continuous but not differentiable c) neither continuous or differentiable?

2. $D: \; -3 \le x \le 2$

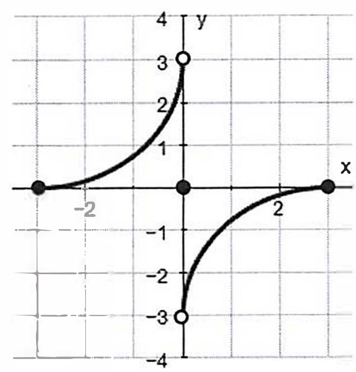


a) $\mathbb{R}$

b) $\emptyset$

c) $\emptyset$

3. $D: \; -3 \le x \le 3$

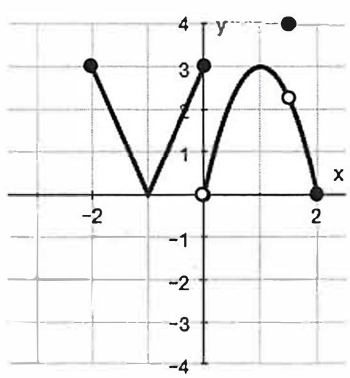


a) $[-3;0) \cup (0;3]$

b) none

c) 0

4. $D: \; -2 \le x \le 2$

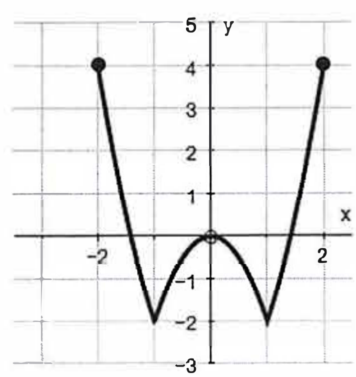


a) $[-2;-1)$
$(-1;0)$
$(0;1.5)$
$(1.5;2]$

b) -1

c) 0 and 1.5

5. $D: \; -2 \le x \le 2$



a) $[-2;-1)$
$(-1;1)$
$(1;2]$

b) -1 and 1

c) none

6. The graph of the function f shown in the figure below has a vertical tangent at the point (2, 0) and horizontal tangents at the points (1, –1) and (3, 1). For what values of x, $-2 < x < 4$, is f not differentiable?

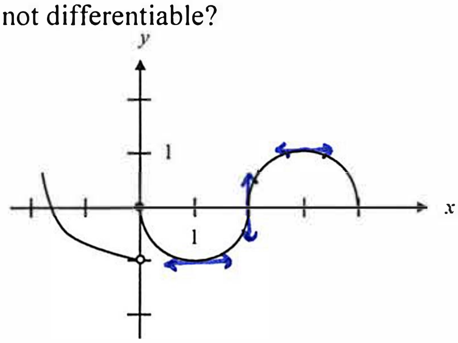
A) 0 only

B) 0 and 2 only

C) 1 and 3 only

D) 0, 1, and 3 only

E) 0, 1, 2, and 3



7. Suppose $f(x)=2+|x+3| = \begin{cases} x+5 & \text{if } x \geq -3 \\ -x-1 & \text{if } x < -3 \end{cases}$

a) What is the value of $f'(3)$? Explain your answer.

$f'(3) = 1$

b) What is the value of $f'(-3)$? Explain your answer.

$f'(-3)$ DNE (the function is not continuous at -3)

8. What are the three different derivative "formulas" ?... (don't forget to use a limit)

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ $\lim_{c \to x} \frac{f(x)-f(c)}{x-c}$

9. If f is a function such that $\lim_{x \to 2} \frac{f(x)-f(2)}{x-2} = 0$, which of the following must be true?

A) The limit of $f(x)$ as x approaches 2 does not exist.
B) f is not defined at $x = 2$.
C) The derivative of f at $x = 2$ is 0.
D) f is continuous at $x = 0$.
E) $f(2) = 0$

10. Let f be a function such that $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = 5$. Which of the following must be true?

I. f is continuous at $x = 2$.
II. f is differentiable at $x = 2$.
III. The derivative of f is continuous at $x = 2$.

A) I only B) II only C) I and II only D) I and III only E) II and III only

11. Let f be a function that is differentiable on the open interval (0, 10). If $f(2) = -5$, and $f(5) = 5$, and $f(9) = -5$, each of the following statements MUST be true. Explain why each statement must be true.

a) f has at least 2 zeros. f is differentiable Therefore, it is continuous.

$\left.\begin{array}{l} f(2)<0 \\ f(5)>0 \end{array}\right\} \Rightarrow$ there is a zero between 2 and 5. $\left.\begin{array}{l} f(5)>0 \\ f(9)<0 \end{array}\right\}$ same thing. (IVT)

b) The graph of f has at least one horizontal tangent line.

$f(2) = f(9)$ there has to be a change in direction...

c) For some c, $2 < c < 5$, $f(c) = 3$.

IVT

- f is continuous over $[2;5]$
- $\left.\begin{array}{l} f(2)=-5 \\ f(5)=5 \end{array}\right\}$ $3 \in [-5;5] \Rightarrow \dots$

12. The function f is defined on the closed interval [0, 8]. The graph of its derivative f' is shown below. The point (3, 5) is on the graph of $f(x)$. An equation of the tangent line to the graph of f at (3, 5) is

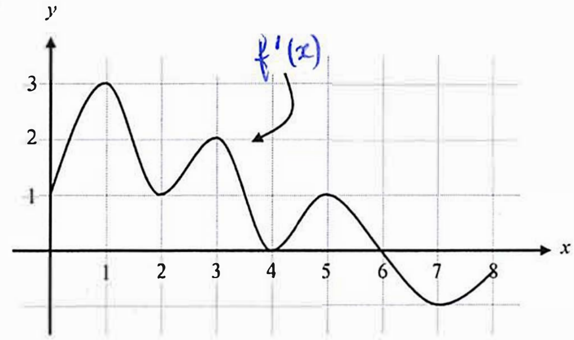
A) $y = 2$

B) $y = 5$

C) $y - 5 = 2(x-3)$

D) $y + 5 = 2(x-3)$

E) $y + 5 = 2(x+3)$



$y - 5 = \underbrace{f'(3)}_{=2}(x-3) \qquad y = 2x - 1$

13. Let $g(x) = \begin{cases} 3x-2 & \text{if } x \le 0 \\ x^2 - 1 & \text{if } x > 0 \end{cases}$. Which of the following is equal to the left-hand derivative of g at $x = 0$?

A) $2x$ B) 3 C) 0 D) ∞ E) $-\infty$

14. Suppose $f(x) = \begin{cases} 3-x & \text{if } x < 1 \\ mx^2 + nx & \text{if } x \ge 1 \end{cases}$

a) If the function is continuous, what is the relationship between m and n. (Use the definition of continuity!)

$\lim_{x \to 1^-} f(x) = 2 \qquad f(1) = \lim_{x \to 1^+} f(x) = m + n$

$\boxed{m + n = 2}$

b) What is the derivative of the portion of the graph where $x < 1$.

$f'(x) = -1$

c) Using whatever method you wish to show/explain, find the derivative of the portion of the graph where $x \ge 1$.

$f'(x) = 2mx + n$

$\frac{f(x) - f(c)}{x - c} = \frac{mx^2 + nx - mc^2 - nc}{x - c}$

$= \frac{m(x^2 - c^2) + n(x - c)}{x - c}$

$= \frac{\cancel{(x-c)}(m(x+c) + n)}{\cancel{x-c}} \xrightarrow[c \to x]{} 2mx + n$

d) In order for $f(x)$ to be differentiable at $x = 1$, what is the relationship between the answers in part b and c?

$2m(1) + n = -1$

e) Using your answers from part (d) and from part (a), solve for m and n.

$\begin{cases} 2m + n = -1 \\ m + n = 2 \end{cases}$

$\boxed{m = -3} \quad \boxed{n = 5}$

One of the four (4) required calculator skills on the AP exam is for you to take a derivative at a point. Use your calculator to answer the following questions. Be sure to use correct mathematical notation.

15. Using your calculator, find the equation of the tangent line to the graph of $f(x)=x^3+x^2$ when $x=2$. Show your work using correct notation.

point : (2; 12)

slope : $f'(2)$ nDeriv (x^3+x^2, x, 2) = 16

$\Rightarrow y-12=16(x-2)$

16. When an object falls its distance traveled (in meters) can be modeled by the equation $h(t)=4.9t^2$. The derivative of h with respect to t is the velocity of the object. Find the velocity of the object at $t=3$ seconds.

$h'(3)$ = nDeriv ($4.9x^2$, x, 3) = 29.4

17. Suppose $f(x)=\left|4-x^2\right|$.

a) Find the slope of the function when $x=3$.

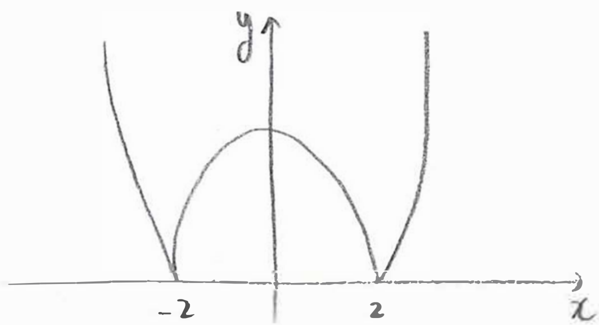
$$f(x)=\begin{cases} 4-x^2 & \text{if } -2\le x\le 2 \\ -4+x^2 & \text{if } x<-2 \text{ or } x>2 \end{cases}$$

$f'(3)=2$

b) Find the slope of the function when $x=2$.

DNE (corner)

c) Graph the function and explain any issues with your answer from part (b).



AP Calculus
3.3 Worksheet Day 1

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1. A derivative tells you the slope of a function.

2. What is the power rule for derivatives? (i.e. how do you take the derivative of $y = x^n$?)

$$\frac{dy}{dx} = nx^{n-1}$$

3. For each of the following functions, find $\frac{dy}{dx}$.

a) $y = -2x^3 + x$

$$\frac{dy}{dx} = -6x^2 + 1$$

b) $y = \frac{x^4}{3} - \frac{x^2}{7} + 5$

$$\frac{dy}{dx} = \frac{4}{3}x^3 - \frac{2}{7}x$$

c) $y = \frac{5}{x^2} + \frac{6}{x} - 8x^3$

$$\frac{dy}{dx} = -\frac{10}{x^3} - \frac{6}{x^2} - 24x^2$$

d) $y = \frac{x^{-3}}{2} + 5x^{-4} - 3x^{-6}$

$$\frac{dy}{dx} = -\frac{3}{2}x^{-4} - 20x^{-5} + 18x^{-7}$$

e) $y = 5x^4 + 2x^3 - 8x^2 - 7x + 11$

$$\frac{dy}{dx} = 20x^3 + 6x^2 - 16x - 7$$

f) $y = 7x - 8$

$$\frac{dy}{dx} = 7$$

g) $y = (x^2 - 3)(x + 4)$

$$= x^3 + 4x^2 - 3x - 12$$

$$\frac{dy}{dx} = 3x^2 + 8x - 3$$

h) $y = \frac{x^5 - 2x^4 + 3x^3}{x^5}$

$$y = 1 - 2x^{-1} + 3x^{-2}$$

$$\frac{dy}{dx} = 2x^{-2} - 6x^{-3}$$

$$= \frac{2x - 6}{x^3}$$

i) $y = \sqrt{x} + \frac{3}{\sqrt{x}} - 6x^{5/3} + \frac{7}{x^3}$

$$y = x^{1/2} + 3x^{-1/2} - 6x^{5/3} + 7x^{-3}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2} - 10x^{2/3} - 21x^{-4}$$

4. [Calculator Required] We want to find all points where the graph of $y = x^4 - 5x^3 - 3x^2 + 13x + 10$ has a horizontal tangent line.

a) First, find an equation for y'.

$$y' = 4x^3 - 15x^2 - 6x + 13$$

b) A horizontal tangent line will have a slope = 0. So set $y' =$ 0, and use your calculator to solve this equation.

$$4x^3 - 15x^2 - 6x + 13 = 0$$

$$(x+1)(4x^2 - 19x + 13) = 0$$

$$x = -1 \qquad x = \frac{19 \pm \sqrt{153}}{8}$$

5. Find the equation of the tangent line to the function $y = \dfrac{x^2 + x - 2}{2x}$ at the point where $x = 1$.

$y = \frac{1}{2}x + \frac{1}{2} - \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{2} + x^{-2}$ point: $(1;0)$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{2}$ $\Rightarrow$ $\boxed{y = \frac{3}{2}(x-1)}$

6. Find the equation of the normal line to the function $y = x^3 - 5x + 1$ at the point when $x = 2$.

point: $(2;-1)$

$\frac{dy}{dx} = 3x^2 - 5$

$\left.\frac{dy}{dx}\right|_{x=2} = 7$ $\Rightarrow$ slope of normal line: $-\frac{1}{7}$ $\Rightarrow$ $\boxed{y + 1 = -\frac{1}{7}(x-2)}$

7. Find the points on the curve $y = x^3 + 3x^2 - 9x + 7$ where the tangent line is parallel to the x-axis.

(horizontal: slope $= 0$)

$\frac{dy}{dx} = 3x^2 + 6x - 9$

$\frac{dy}{dx} = 0 \Leftrightarrow 3x^2 + 6x - 9 = 0$

$3(x^2 + 2x - 3) = 0$

$3(x+3)(x-1) = 0$

$x = -3$ $x = 1$

$(-3, 34)$ and $(1, 2)$

8. Consider the curve $y = x^3 + x$.

a) Find the tangents to the curve at all the points where the slope is 4. (be careful! … it doesn't say $x = 4$!)

$\frac{dy}{dx} = 3x^2 + 1$

$3x^2 + 1 = 4$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

Points $(1;2)$ and $(-1;-2)$

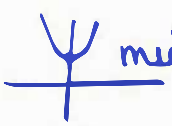
Tangents:

$y - 2 = 4(x-1)$ and $y + 2 = 4(x+1)$

b) What is the smallest slope of the curve? At what value of x does the curve have this value?

$\frac{dy}{dx} = 3x^2 + 1$

$3x^2 + 1 = 1$ when $\boxed{x = 0}$



min $= 1$

9. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

$\frac{dy}{dx} = 3x^2$ tangent line: $y + 8 = 12(x+2)$

$\left.\frac{dy}{dx}\right|_{x=-2} = 12$

x-intercept: $8 = 12x + 24$

$-16 = 12x$

$\boxed{x = -4/3}$

y-int: $y + 8 = 12(0+2)$

$y + 8 = 24$

$\boxed{y = 16}$

10. If the line normal to the graph of f at the point $(1, 2)$ passes through the point $(-1, 1)$, then which of the following gives the value of $f'(1)$?

A –2
B 2
C –1/2
D 1/2
E 3

normal line: $y - 2 = m_\perp(x-1)$

point $(-1;1)$ $1 - 2 = m_\perp(-1-1)$

$-1 = -2m_\perp$

$m_\perp = \frac{1}{2}$ $\Rightarrow$ $f'(1) = -2$

To be differentiable at a point, the left and right derivatives must be equal at that point (see last WS). Use this concept and the definition of continuity to solve for the parameters (those are those pesky little letters) in the next question.

11. Solve for a and b in order for $g(x)$ to be both continuous and differentiable at $x = 0$. (look back at the 3.2 WS if you need help)

$$g(x)=\begin{cases} ax+b & ;x>0 \\ 1-x+x^2 & ;x\le 0 \end{cases}$$

continuity:

$\lim_{x\to 0^-} g(x) = g(0) = 1$

$\lim_{x\to 0^+} g(x) = b$

$\boxed{b=1}$

differentiability — at 0

left derivative: $2x-1$ — -1

right derivative: a — a

$\boxed{a=-1}$

12. When $x = 8$, the rate at which $\sqrt[3]{x}$ is increasing is $\frac{1}{k}$ times the rate at which x is increasing. What is the value of k?

A) 3
B) 4
C) 6
D) 8
E) 12

$\left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{k}$

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x}^2}$

$\left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{12} \Rightarrow k=12$

13. Let $f(x)=\sqrt{x}$. If the rate of change of f at $x = c$ is twice its rate of change at $x = 1$, then $c =$

A) $\frac{1}{4}$
B) 1
C) 4
D) $\frac{1}{\sqrt{2}}$
E) $\frac{1}{2\sqrt{2}}$

$\left.\frac{df(x)}{dx}\right|_{x=c} = 2\left.\frac{df(x)}{dx}\right|_{x=1}$

$\frac{df(x)}{dx} = \frac{1}{2\sqrt{x}}$

$\frac{1}{2\sqrt{c}} = 2 \times \frac{1}{2}$

$\sqrt{c} = \frac{1}{2}$ so $c = \frac{1}{4}$

14. [Calculator] Which of the following is an equation of the tangent line to $f(x)=x^4+2x^2$ at the point where $f'(x)=1$?

A) $y = 8x - 5$
B) $y = x + 7$
C) $y = x + .763$
D) $y = x - .122$
E) $y = x - 2.146$

$f'(x) = 4x^3 + 4x$

$f'(x) = 1 \Leftrightarrow 4x^3+4x-1=0$

$x \simeq 0.237$ $y \simeq 0.115$

Tangent line: $y - 0.115 = 1(x - 0.237)$

$y = x - 0.122$

AP Calculus
3.3 Worksheet Day 2

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What is the product rule?

$$(u.v)' = u'.v + v'.u$$

2. What is the quotient rule?

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

3. Let $f(x)=\left(3x^3+4x^2\right)\left(2x^4-5x\right)$.

a) Find $f'(x)$ without using the product rule

$$f(x) = 6x^7 - 15x^4 + 8x^6 - 20x^3$$

$$f'(x) = 42x^6 - 60x^3 + 48x^5 - 60x^2$$

$$= 42x^6 + 48x^5 - 60x^3 - 60x^2$$

b) Find $f'(x)$ using the product rule.

$$f'(x) = (9x^2+8x)(2x^4-5x) + (8x^3-5)(3x^3+4x^2)$$

$$= 18x^6 - 45x^3 + 16x^5 - 40x^2 + 24x^6 + 32x^5 - 15x^3 - 20x^2$$

$$= 42x^6 + 48x^5 - 60x^3 - 60x^2$$

4. Let $f(x)=\dfrac{x^2+4}{x}$.

a) Find $f'(x)$ without using the quotient rule

$$f(x) = x + 4x^{-1}$$

$$f'(x) = 1 - \frac{4}{x^2}$$

$$= \frac{x^2-4}{x^2}$$

b) Find $f'(x)$ using the quotient rule.

$$f'(x) = \frac{2x.x - (x^2+4)}{x^2}$$

$$= \frac{2x^2 - x^2 - 4}{x^2}$$

$$= \frac{x^2-4}{x^2}$$

5. Find $\dfrac{dy}{dx}$ for each of the following functions.

a) $y=\dfrac{2x-5}{3x+2}$

$$\frac{dy}{dx} = \frac{2(3x+2) - 3(2x-5)}{(3x+2)^2}$$

$$= \frac{6x + 4 - 6x + 15}{(3x+2)^2}$$

$$= \frac{19}{(3x+2)^2}$$

b) $y=(3-x)\left(2+x^2\right)^{-1}$

$$y = \frac{3-x}{2+x^2}$$

$$\frac{dy}{dx} = \frac{-(2+x^2) - 2x(3-x)}{(2+x^2)^2}$$

$$= \frac{-2 - x^2 - 6x + 2x^2}{(2+x^2)^2}$$

$$= \frac{x^2 - 6x - 2}{(2+x^2)^2}$$

c) $y=\dfrac{x^3}{8-x^2}$

$$\frac{dy}{dx} = \frac{3x^2(8-x^2) + 2x(x^3)}{(8-x^2)^2}$$

$$= \frac{24x^2 - 3x^4 + 2x^4}{(8-x^2)^2}$$

$$= \frac{-x^2(x^2-24)}{(8-x^2)^2}$$

6. For $a-d$, write a expression for $f'(x)$ and then use it to find $f'(2)$ given the following information:

$$g(2)=3 \quad g'(2)=-2$$
$$h(2)=-1 \quad h'(2)=4$$

a) $f(x)=2g(x)+h(x)$

$f'(x)=2g'(x)+h'(x)$

$f'(2)=2(-2)+4=0$

b) $f(x)=4-h(x)$

$f'(x)=-h'(x)$

$f'(2)=-4$

c) $f(x)=g(x)h(x)$

$f'(x)=g'(x)h(x)+h'(x)g(x)$

$f'(2)=-2\cdot(-1)+4(3)$

$=14$

d) $f(x)=\dfrac{g(x)}{h(x)}$

$f'(x)=\dfrac{g'(x).h(x)-h'(x)g(x)}{(h(x))^2}$

$f'(2)=\dfrac{-2\cdot(-1)-4\cdot(3)}{(-1)^2}$

$=-10$

7. Suppose u and v are differentiable functions of $x=3$ and that $u(3)=4$, $\left.\dfrac{du}{dx}\right|_{x=3}=-3$, $v(3)=2$, and $\left.\dfrac{dv}{dx}\right|_{x=3}=3$. Find the values of the following derivatives at $x=3$.

a) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{u'.v-v'.u}{v^2}$

$\left.\dfrac{d}{dx}\left(\dfrac{u}{v}\right)\right|_{x=3}=\dfrac{-3\times2-3\times4}{2^2}$

$=-\dfrac{9}{2}$

b) $\dfrac{d}{dx}(uv)=u'.v+v'u$

$\left.\dfrac{d}{dx}(uv)\right|_{x=3}=-3\times2+3\times4$

$=6$

c) $\dfrac{d}{dx}(5u-2v+4uv)=5u'-2v'+4\dfrac{d}{dx}(uv)$

$\left.\dfrac{d}{dx}(5u-2v+4uv)\right|_{x=3}=5(-3)-2(3)+4\times6$ (4b)

$=-15-6+24=3$

d) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)=\dfrac{v'u-u'v}{u^2}$

$\left.\dfrac{d}{dx}\left(\dfrac{v}{u}\right)\right|_{x=3}=\dfrac{3\times4-(-3)\times2}{4^2}$

$=\dfrac{9}{8}$

8. Solve for a and b in order for $f(x)$ to be both continuous and differentiable at $x=1$. (be sure to use the definition of continuity)

$$f(x)=\begin{cases} x^2+2 & ;x\le 1 \\ a\left(x-\frac{1}{x}\right)+b & ;x>1 \end{cases}$$

continuity at 1:

$\lim_{x\to1^-} f(x)=f(1)=3$

$\lim_{x\to1^+} f(x)=b$

$\Rightarrow \boxed{b=3}$

differentiability at 1:

left side derivative: $2x \to 2$

right side derivative: $a\left(1+\frac{1}{x^2}\right)\to 2a$

$\Rightarrow \boxed{a=1}$

9. For each of the following, find the equation of the tangent line to the given function at the indicated point.

a) $f(x) = (x^3 - 3x + 1)(x + 2)$ at the point (1, –3).

$f'(x) = (3x^2 - 3)(x+2) + x^3 - 3x + 1$

$= 3x^3 + 6x^2 - 3x - 6 + x^3 - 3x + 1$

$= 4x^3 + 6x^2 - 6x - 5$

$f'(1) = -1$

$\Rightarrow \boxed{y + 3 = -(x - 1)}$

b) $y = \dfrac{8}{4 + x^2}$ at the point (–2, 1).

$\dfrac{dy}{dx} = \dfrac{-2x(8)}{(4+x^2)^2} = -\dfrac{16x}{(4+x^2)^2}$

$\left.\dfrac{dy}{dx}\right|_{x=-2} = \dfrac{1}{2}$

$\boxed{y - 1 = \frac{1}{2}(x + 2)}$

10. At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent line parallel to the line $2x - 4y = 3$?

A) $(\frac{1}{2}, \frac{1}{2})$
B) $(\frac{1}{2}, \frac{1}{8})$
C) $(1, -\frac{1}{4})$
D) $(1, \frac{1}{2})$
E) $(2, 2)$

$\dfrac{dy}{dx} = x$

$-4y = -2x + 3$

$y = \frac{1}{2}x - \frac{3}{4}$

$\uparrow$

parallel = same slope!

$x = \frac{1}{2}$ is when they are parallel.

$y = \frac{1}{2}(\frac{1}{2})^2 = \frac{1}{8}$

11. Let f be a differentiable function such that $f(3) = 2$ and $f'(3) = 5$. If the tangent line to the graph of f at $x = 3$ is used to find an approximation to a zero of f, that approximation is

A) 0.4
B) 0.5
C) 2.6
D) 3.4
E) 5.5

$\downarrow$

$y - 2 = 5(x - 3)$

$\rightarrow$ to approximate a zero:

$0 - 2 = 5(x - 3)$

$5x = 13$

$x = 13/5$

12. An equation of the line tangent to the graph of $y = \dfrac{2x+3}{3x-2}$ at the point (1, 5) is

A) $13x - y = 8$
B) $13x + y = 18$
C) $x - 13y = 64$
D) $x + 13y = 66$
E) $-2x + 3y = 13$

$\dfrac{dy}{dx} = \dfrac{2(3x-2) - 3(2x+3)}{(3x-2)^2}$

$= \dfrac{6x - 4 - 6x - 9}{(3x-2)^2}$

$= -\dfrac{13}{(3x-2)^2}$

$\left.\dfrac{dy}{dx}\right|_{x=1} = -13 \Rightarrow y - 5 = -13(x - 1)$

$y = -13x + 18$

13. What is the instantaneous rate of change at $x = 2$ of the function f given by $f(x) = \frac{x^2 - 2}{x - 1}$?

A) -2
B) $\frac{1}{6}$
C) $\frac{1}{2}$
D) 2
E) 6

$$f'(x) = \frac{2x(x-1) - (x^2-2)}{(x-1)^2}$$

$$= \frac{x^2 - 2x + 2}{(x-1)^2}$$

$$f'(2) = \frac{2}{1}$$

14. If u, v, and w are nonzero differentiable functions of x, then the $\frac{d}{dx}\left(\frac{uv}{w}\right)$ is

A) $\frac{uv' + u'v}{w'}$
B) $\frac{u'v'w - uvw'}{w^2}$
C) $\frac{uvw' - uv'w - u'vw}{w^2}$
D) $\frac{u'vw + uv'w + uvw'}{w^2}$
E) $\frac{uv'w + u'vw - uvw'}{w^2}$

$$\frac{d}{dx}\left(\frac{uv}{w}\right) = \frac{(uv)' . w - w'(uv)}{w^2}$$

$$= \frac{(u'v + v'u)w - w'uv}{w^2}$$

$$= \frac{u'vw + v'uw - w'uv}{w^2}$$

15. When an object is thrown off a 100 foot cliff with an initial velocity of 40 feet/second, the height h, in feet, of the object can be modeled as a function of time t, in seconds, using the function

$$h(t) = -16t^2 + 45t + 100.$$

a) Find $\frac{dh}{dt}$ ... What is the unit of measurement for this equation?

$$\frac{dh}{dt} = -32t + 45 \quad (ft/s)$$

b) Find $\frac{d^2h}{dt^2}$ ... What is the unit of measurement for this equation?

$$\frac{d^2h}{dt^2} = -32 \quad (ft/s^2)$$

16. Let $g(x) = x - \frac{1}{x}$. Find the following:

a) $g'(x) = 1 + \frac{1}{x^2}$

$$= \frac{x^2 + 1}{x^2}$$

b) $g''(x) = -\frac{2}{x^3}$

c) The tangent line equation when $x = 2$

$g'(2) = \frac{5}{4}$ point: $(2; \frac{3}{2})$

$$y - \frac{3}{2} = \frac{5}{4}(x - 2)$$

AP Calculus
3.4 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What is the relationship between position, velocity, and acceleration?

$p' = v \qquad v' = a$
$\qquad\qquad\quad p'' = a$

2. Once again trying to blow up earth because it interferes with his view of Venus, Marvin the Martian lands on the moon. Bugs Bunny, as always, interferes with his plan. Chasing Bugs, Marvin fires a warning shot straight up into the air with his Acme Disintegration Pistol. The height (in feet) after t seconds of the shot is given by

$$s(t) = -2.66t^2 + 135t + 3.$$

a) Find the velocity and acceleration as functions of time.
(What is the meaning of the acceleration function?)

velocity: $s'(t) = -5.32t + 135$ (ft/s)

acceleration: $s''(t) = -5.32$ (ft/s²) The shot slows down 5.32 ft/s each second...

b) What is the position of the shot when the velocity is 0?

$s'(t) = 0$ when $t_1 \simeq 25.4$ s.

$s(t_1) \simeq 1715.9$ ft.

3. Fill in the blanks.

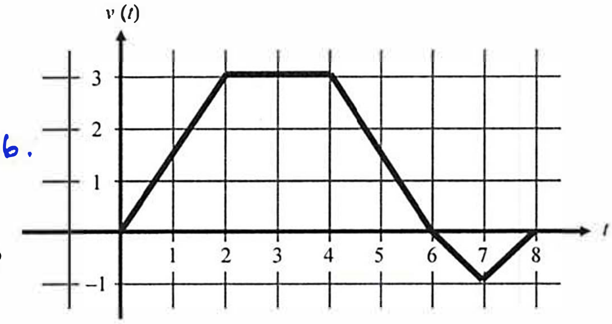
a) When the ___velocity___ is positive, the object is moving in a positive direction.

b) An object is ___slowing down___ when the velocity and acceleration have different signs.

c) An object is stopped when ___the velocity___ is zero.

d) Speed is always positive because it is the ___absolute value___ of velocity.

4. A bug begins to crawl up a vertical wire at time $t = 0$. The velocity, v, of the bug at time t, $0 \le t \le 8$ is given by the function whose graph is shown below.



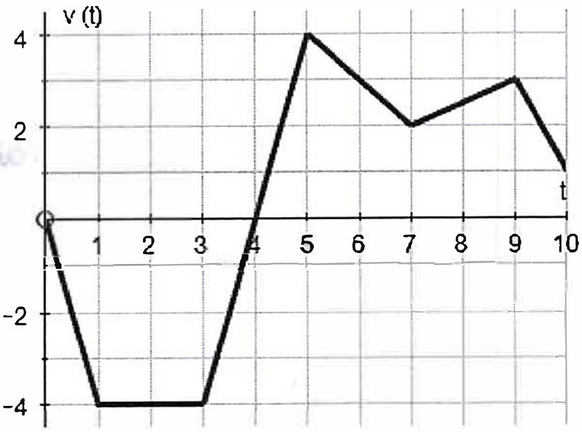
a) At what value of t does the bug change direction? Justify your response.

$t = 6$ because v changes sign at $t = 6$.

b) During which time intervals in the bug slowing down? Justify your response.

$(4;6)$ and $(7;8)$ because v and a $(a = v')$ have opposite signs on these intervals.

5. The figure graphed below shows the velocity of a particle moving along a coordinate line. Justify each response.



a) When is the particle moving right?

when $t \in (4, 10]$ because $v(t) > 0$

b) When is the particle moving left?

when $t \in (0, 4)$ because $v(t) < 0$

c) When is the particle stopped?

when $t = 0$ and $t = 4$ because $v(t) = 0$

d) When is the particle speeding up? when $t \in (0,1)$ $t \in (4,5]$ and $t \in (7,9)$ because v and a have the same sign ($a = v'$)

e) When does the particle change directions?

when $t = 4$ because v changes sign.

f) When is the particle slowing down? when $t \in (3,4)$, $t \in (5,7)$ and $t \in (9,10)$ because v and a have opposite signs

g) What is the particle moving at its greatest speed?

over $(1,3)$ and at $t = 5$ $s(t) = |v(t)| = 4$

h) When is the particle's acceleration positive?

on $(3,5)$ and $(7,9)$ because v is increasing

i) When is the particle's acceleration negative?

on $(0,1)$, $(5,7)$ and $(9,10)$ because v is decreasing.

6. Fill in the blanks with correct mathematical notation.

a) If you want the average velocity of a particle on the interval [2, 5], you must find $\frac{f(5) - f(2)}{5 - 2}$.

b) If you want the velocity of a particle at $t = 4$, you must find $\lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$ or $f'(4)$

7. Velocity is the rate of change of the position. If the position of a particle on the x – axis at time t is given by $-5t^2$, then what is the average velocity of the particle for $0 \le t \le 3$?

$$\frac{f(3) - f(0)}{3 - 0} = \frac{-45 - 0}{3} = -15$$

8. A particle moves along the x – axis so that its position at time t is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero?

$x'(t) = 2t - 6$

$x'(t) = 0 \iff t = 3$

9. Fill in the blanks with correct mathematical notation.

a) If you want the average acceleration of a particle on the interval [1, 3], you must find $\frac{f'(3) - f'(1)}{3 - 1}$.

b) If you want the acceleration of a particle at $t = 8$, you must find $f''(8)$.

10. Rocket A has a positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds as shown in the table below.

t (sec)	0	10	20	30	40	50	60	70	80
$v(t)$ (ft/sec)	5	14	22	29	35	40	44	47	49

a) Find the average acceleration of Rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

$$\frac{v(80)-v(0)}{80-0} = \frac{49-5}{80} = 0.55 \text{ ft/sec}^2$$

b) Using the data in the table, find an estimate for $v'(35)$. Indicate units of measure.

$v'(35)$ is close to $\frac{v(40)-v(30)}{40-30} = \frac{35-29}{10} \simeq 0.6 \text{ ft/sec}^2$

11. A particle moves along the x–axis so that its position at any time $t \ge 0$ is given by the function $x(t) = t^3 - 12t + 1$, where x is measured in feet and t is measured in seconds. Justify each response and indicate units of measure when appropriate.

a) Find the displacement during the first 3 seconds.

$x(3) - x(0) = -8 - 1 = -9$ (ft)

b) Find the average velocity during the first 3 seconds.

$$\frac{x(3)-x(0)}{3-0} = \frac{-9}{3} = -3 \text{ (ft/sec)}$$

c) Find the instantaneous velocity at $t = 3$ seconds.

$x'(t) = 3t^2 - 12$

$x'(3) = 15$ (ft/sec)

d) Find the acceleration when $t = 3$ seconds.

$x''(t) = 6t$

$x''(3) = 18$ (ft/sec^2)

e) When is the particle moving left?

$x'(t) < 0 \Leftrightarrow 3t^2 - 12 < 0$

$\Leftrightarrow t^2 - 4 < 0$

when $t \in [0; 2]$

f) At what value(s) t does the particle change direction?

$t = 2$ (sec)

g) When is the particle speeding up?

$x''(t) > 0 \Leftrightarrow 6t \ge 0$

$\Leftrightarrow t \ge 0$

t	0		2	
$x'(t)$		$-$	0	$+$
$x''(t)$	0	$+$		$+$

$\Rightarrow$ when $t > 2$: because a and v have the same sign.

13. [Calculator] The cost involved in maintaining annual inventory for a certain manufacturer is given by $C(x) = \frac{1{,}008{,}000}{x} + 6.3x$, where x is the number of items stored. Find the marginal cost of storing the 351st item.

$$C'(x) = -\frac{1008000}{x^2} + 6.3$$

$C'(350) \simeq \boxed{-1.9}$

14. [Calculator] Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

a) Find the marginal cost when 100 washing machines are produced.

$C'(x) = 100 - 0.2x$

$C'(100) = 80$ (dollars for the next machine produced)

b) Show that the marginal cost when 100 washing machines are produced (your answer from part b) is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

$C(101) - C(100) = 2000 + 100(101) - 0.1(101)^2 - (2000 + 100(100) - 0.1(100)^2)$

$= 79.9$ (≈ 80)

15. [Calculator] Suppose the weekly revenue (\$) from selling x custom-made office desks is $r(x) = 2000\left(1 - \frac{1}{x+1}\right)$.

Find the marginal revenue when a 6th desk is created.

$r'(x) = \frac{2000}{(x+1)^2}$ $\qquad r'(5) = \frac{2000}{(5+1)^2} \simeq 55.56$

The marginal revenue for the 6th desk is approx \$55.56.

16. a) Write the area A of a circle as a function of the circumference C.

$A = \pi r^2$ $\qquad C = 2\pi r$ $\qquad r = \frac{C}{2\pi}$ $\qquad$ (substitution) $\qquad A = \pi\left(\frac{C}{2\pi}\right)^2$

$\boxed{A(C) = \frac{C^2}{4\pi}}$

b) Evaluate the rate of change of A at $C = 4\pi$.

$\frac{dA}{dC} = \frac{C}{2\pi}$ $\qquad \left.\frac{dA}{dC}\right|_{C=4\pi} = 2$

c) If C is measured in miles and A is measured in square miles, what units are used for $\frac{dA}{dC}$?

$\frac{dA}{dC}$ will be in miles. $\quad$ (mi^2/mi)

17. a) Write the area A of an equilateral triangle as a function of the side length s.

$\cdot\ A = \frac{1}{2}(h \times s)$ $\qquad$ Pythagorean Theorem: $h^2 = s^2 - \left(\frac{s}{2}\right)^2 = \frac{3}{4}s^2$

$h = \sqrt{\frac{3s^2}{4}} = \frac{\sqrt{3}}{2}s$

Substitution: $\quad A = \frac{1}{2} \cdot \frac{\sqrt{3}}{2}s^2$ $\qquad \boxed{A(s) = \frac{\sqrt{3}}{4}s^2}$

b) Find $\left.\frac{dA}{ds}\right|_{s=12}$

$\frac{dA}{ds} = \frac{\sqrt{3}}{2}s$ $\qquad \left.\frac{dA}{ds}\right|_{s=12} = \boxed{6\sqrt{3}}$

18. The number of gallons of water in a tank t minutes after the tank has started to drain is given by the equation

$$G(t) = 300(20 - t^2).$$

a) How fast is the water draining at the end of 5 minutes? $G'(5)$?

$G'(t) = -600t$

$G'(5) = -3000$ gallons per minute

b) What is the average rate at which the water drains out of the tank during the first 5 minutes?

$\frac{G(5) - G(0)}{5} = -1500$ gallons/min

AP Calculus
3.5 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Find the following derivatives ... AND MEMORIZE THEM ASAP!

a) $\frac{d}{dx}[\sin x] = \cos x$ b) $\frac{d}{dx}[\cos x] = -\sin x$ c) $\frac{d}{dx}[\tan x] = \sec^2 x$

d) $\frac{d}{dx}[\sec x] = \sec x \tan x$ e) $\frac{d}{dx}[\csc x] = -\csc x . \cot x$ f) $\frac{d}{dx}[\cot x] = -\csc^2 x$

2. Find $\frac{dy}{dx}$ for each of the following:

a) $y = 3 - x - \tan x$

$$\frac{dy}{dx} = -1 - \sec^2 x$$

b) $y = x \csc x$

$$\frac{dy}{dx} = \csc x - x . \csc x . \cot x$$

c) $y = \frac{1}{x} + 7x^2 \sin x$

$$\frac{dy}{dx} = -\frac{1}{x^2} + 7(2x)\sin x + 7x^2 \cos x$$

$$= -\frac{1}{x^2} + 14x \sin x + 7x^2 \cos x$$

d) $y = \frac{\cot x}{5 - \cos x}$

$$\frac{dy}{dx} = \frac{-\csc^2 x(5 - \cos x) - \sin x . \cot x}{(5 - \cos x)^2}$$

$$= \frac{-\csc^2 x(5 - \cos x) - \cos x}{(5 - \cos x)^2}$$

3. If $y = \tan x - \cot x$, then $\frac{dy}{dx} =$

$$\frac{dy}{dx} = \sec^2 x + \csc^2 x$$

4. If $f(x) = \frac{x}{\tan x}$, then $f'\left(\frac{\pi}{4}\right) =$

$$f'(x) = \frac{\tan x - x \sec^2 x}{\tan^2 x}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{1 - \frac{\pi}{4} \cdot \left(\frac{2}{\sqrt{2}}\right)^2}{1}$$

$$= 1 - \frac{\pi}{2}$$

5. If $y = \sec x$, find $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \sec x . \tan x$$

$$\frac{d^2y}{dx^2} = (\sec x \tan x)\tan x + \sec^3 x$$

$$= \sec x . \tan^2 x + \sec^3 x$$

6. If $y = \theta \tan\theta$, find y''.

$$y' = \tan\theta + \theta . \sec^2\theta$$

$$y'' = \sec^2\theta + \sec^2\theta + \theta\,(\sec\theta.\tan\theta.\sec\theta + \sec\theta.\tan\theta.\sec\theta)$$

$$y'' = 2\sec^2\theta + 2\sec^2\theta.\tan\theta.\theta$$

$$= 2\sec^2\theta(1+\theta\tan\theta)$$

7. If $f(x) = \sin x$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. What do you think the function $f^{(100)}(x)$ is?

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f^{(100)}(x) = f^{(4\times 25)}(x) = \sin x$

every 4th derivative, you go back to $\sin x$.

8. Find an equation of the tangent line and the normal line to the graph of $y = x + \cos x$ at the point (0, 1).

tangent: $y-1 = 1(x-0) \rightarrow y = x+1$

normal: $y-1 = -1(x-0) \rightarrow y = -x+1$

$\left(\begin{array}{l} y' = 1-\sin x \\ y'|_{x=0} = 1 \end{array} \right)$

9. [Calculator] Find equations for the lines that are tangent and normal to the curve $y = x^3 \sin x$ when $x = 2$.

point: $(2; 8\sin 2)$ ≈ 7.27

$y' = 3x^2.\sin x + x^3\cos x$

$y'|_{x=2} = 12\sin 2 + 8\cos 2 \approx 7.58$

tangent: $y - 7.27 = 7.58(x-2)$

normal: $y - 7.27 = -\frac{1}{7.58}(x-2)$

10. Find the points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -2x$.

$y' = -\csc^2 x$

$-\csc^2 x = -2$

$\csc^2 x = 2$

$\sin^2 x = \frac{1}{2}$

$\sin x = \pm \frac{\sqrt{2}}{2}$ $\rightarrow x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ on this domain

↳ same slope: -2

Points: $(\frac{\pi}{4}; 1)$ & $(\frac{3\pi}{4}; -1)$

11. $\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} =$

A) 0
B) 1
C) $\sin x$
D) $\cos x$
E) nonexistent

$$\frac{d}{dx}(\sin x) = \cos x$$

12. [Calculator] A particle moves along a line so that at time t, $0 \le t \le \pi$, its position is given by $s(t) = -4\cos t - \frac{t^2}{2} + 10$. What is the velocity of the particle when its acceleration is zero?

$s'(t) = 4\sin t - t$

$s''(t) = 4\cos t - 1$

$s''(t) = 0 \iff \cos t = \frac{1}{4}$

$\iff t \doteq 1.32$ over the domain

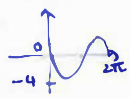
$s'(1.32) \simeq 2.55$

13. [No Calculator] A spring is bobbing up and down. Its position at any time $t \ge 0$ is given by $s(t) = -4\sin t$.

a) What is the initial position of the spring? when $t = 0$, $s = 0$

b) Which way is the particle moving to start? Justify your response.

graph of $y = -4\sin t$



It's going down at first

or $s'(t) = -4\cos t$ negative at the start...

c) At $t = \frac{5\pi}{4}$, is the spring moving up or down? Justify your response.

$s'\left(\frac{5\pi}{4}\right) = -4 \cdot \cos\left(\frac{5\pi}{4}\right) > 0$ moving up.

d) Is the spring speeding up or slowing down at $t = \frac{5\pi}{4}$? Justify your response.

$s''(t) = 4\sin t$

$s''\left(\frac{5\pi}{4}\right) = 4\sin\left(\frac{5\pi}{4}\right) < 0$ slowing down

14. [Calculator Required] A body is moving in simple harmonic motion (up/down) with position $s(t) = 3 + \cos t$, where $0 \le t < 2\pi$.

a) Find $v(t)$, the velocity function.

$v(t) = -\sin t$

b) Find the zeros of $v(t)$.

$t = 0$ or π or 2π

c) Find $a(t)$, the acceleration function.

$a(t) = -\cos t$

d) Find the zeros of $a(t)$.

$t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

e) When is the object stopped? Justify your response.

when $t = \pi$ and $t = 2\pi$ (and at the start)

f) When does the object change direction? Justify your response.

when $t = \pi$

g) When does the object speed up? Justify your response.

when $t \in \left(0, \frac{\pi}{2}\right)$ and $t \in \left(\pi, \frac{3\pi}{2}\right)$

because a and v have the same signs.

	0		$\pi/2$		π		$3\pi/2$		2π
$v(t)$	0	$-$		$-$	0	$+$		$+$	0
$a(t)$		$-$	0	$+$		$+$	0	$-$	

15. Suppose $h(x)=\begin{cases}\cos x & x<0\\ x+p & x\geq 0\end{cases}$.

a) Is there a value of p so that $h(x)$ is continuous at $x = 0$? Explain.

$$\left.\begin{array}{l}\lim_{x\to 0^-} h(x) = 1 \\ \lim_{x\to 0^+} h(x) = h(0) = p\end{array}\right\} \text{ continuous at } x=0 \text{ if } p=1$$

b) Is there a value of p so that $h(x)$ is differentiable at $x = 0$? Explain.

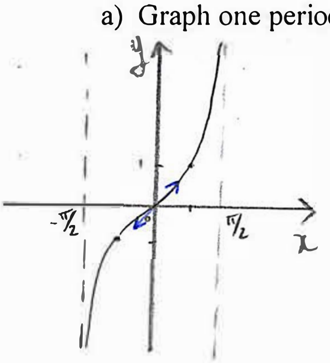
left side derivative: $-\sin x$ $\quad \lim_{x\to 0}(-\sin x) = 0 \neq 1$

right side derivative: 1

$\Rightarrow$ impossible.

16. Suppose $y = \tan x$.

a) Graph one period of the function on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



b) Find $\left.\frac{dy}{dx}\right|_{x=0}$. Explain what you have found.

$\frac{dy}{dx} = \sec^2 x \qquad \left.\frac{dy}{dx}\right|_{x=0} = \sec^2 0 = 1 \qquad$ the slope of the graph at 0

c) Does your answer in part (b) match the graph you made in part (a)? If not, fix it.

yes.

AP Calculus
3.6 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

For questions 1 - 6, find $\frac{dy}{dx}$.

1. $y=\cos\left(\frac{\pi}{2}-3x\right)$

$\frac{dy}{dx} = 3\sin\left(\frac{\pi}{2}-3x\right)$

2. $y=\left(x-\sqrt{x}\right)^{-2}$

$\frac{dy}{dx} = -2(x-\sqrt{x})^{-3}\cdot\left(1-\frac{1}{2\sqrt{x}}\right)$

3. $y=\sin^{-5}x-\cos^{3}x$

$\frac{dy}{dx} = -5\sin^{-6}x.\cos x + 3\cos^{2}x.\sin x$

4. $y=\frac{3}{\sqrt{2x+1}} = 3(2x+1)^{-1/2}$

$\frac{dy}{dx} = -\frac{3}{2}(2x+1)^{-3/2}\cdot(2)$

$= -3(2x+1)^{-3/2}$

5. $y=\sin^{2}(3x-2)$

$\frac{dy}{dx} = 2\sin(3x-2).\cos(3x-2).3$

$= 6\sin(3x-2).\cos(3x-2)$

6. $y=\sqrt{\tan(5x)}$

$\frac{dy}{dx} = \frac{1}{2}.(\tan 5x)^{-1/2}.\sec^{2}(5x).5$

$= \frac{5}{2}\frac{\sec^{2}5x}{\sqrt{\tan 5x}}$

For questions 7 and 8, find the indicated derivative.

7. Find $\frac{dr}{d\theta}$ if $r=\sqrt{\theta\sin\theta}$

$\frac{dr}{d\theta} = \frac{1}{2}(\theta.\sin\theta)^{-1/2}.(\sin\theta+\theta\cos\theta)$

8. Find $\frac{ds}{d\theta}$, if $s=20\sqrt{\sec\theta}$

$\frac{ds}{d\theta} = 2\left(\sqrt{\sec\theta} + \theta.\frac{1}{2}(\sec\theta)^{-1/2}.\sec\theta.\tan\theta\right)$

$= 2\left(\sqrt{\sec\theta} + \frac{\theta}{2}(\sec\theta)^{1/2}.\tan\theta\right)$

$= 2\sqrt{\sec\theta}\left(1+\theta\frac{\tan\theta}{2}\right)$

9. Find $\frac{d^2y}{dx^2}$ if $y=\tan(3x-1)$

$\frac{dy}{dx} = \sec^{2}(3x-1).3$

$= 3\sec^{2}(3x-1)$

$\frac{d^2y}{dx^2} = 3.2\sec(3x-1).\sec(3x-1).\tan(3x-1).3$

$= 18\sec^{2}(3x-1).\tan(3x-1)$

10. What is the largest possible value for the slope of the curve $g(x)=\sin\left(\frac{x}{3}\right)$?

$g'(x)=\frac{1}{3}\cos\left(\frac{x}{3}\right)$ ampl: $\frac{1}{3}$ centre line: $y=0$

$g'_{max}=\frac{1}{3}$

11. Find the equation of the tangent line when $x=4$ on the function $f(x)=\sqrt{25-x^2}$. point: $(4;3)$

$f'(x)=\frac{1}{2}(25-x^2)^{-1/2}\cdot(-2x)$

$=-x(25-x^2)^{-1/2}$

$f'(4)=-4\cdot 9^{-1/2}=-\frac{4}{3}$ $\Rightarrow$ $\boxed{y-3=-\frac{4}{3}(x-4)}$

12. The position of a particle moving along the x-axis is given by the equation $x(t)=\sqrt{1+4t}$.

a) Find $v(t)$.

$v(t)=x'(t)$

$=\frac{1}{2}\cdot(1+4t)^{-1/2}\cdot 4$

$\boxed{v(t)=\frac{2}{\sqrt{1+4t}}}$

b) Find $a(t)$

$a(t)=v'(t)$ or $x''(t)$

$=-(1+4t)^{-3/2}\cdot 4$

$\boxed{a(t)=-4\cdot(1+4t)^{-3/2}}$

c) When is the particle stopped?

$v(t)=0$ impossible.

d) Is the particle speeding up or slowing down at $t=6$? Justify your response.

$a(6)=-4(1+24)^{-3/2}$

$=-\frac{4}{125}<0$ slowing down.

13. Suppose that x is a function of t. Find $\frac{dy}{dt}$, if $y=\tan x$.

$\frac{dy}{dt}=\sec^2x\cdot\frac{dx}{dt}$

14. Suppose f and g are differentiable functions with the values given in the table below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	5	5	e	$\sqrt{2}$
5	2	8	π	7

a) If $h(x)=f(g(x))$, write an expression for $h'(x)$ and use it to find $h'(2)$.

$h'(x) = f'(g(x))\cdot g'(x)$ $\quad$ $h'(2) = f'(g(2))\cdot g'(2)$

$= f'(5)\cdot\sqrt{2}$ $\quad$ $\boxed{h'(2) = \pi\sqrt{2}}$

b) If $h(x)=g(f(x))$, write an expression for $h'(x)$ and use it to find find $h'(2)$.

$h'(x) = g'(f(x))\cdot f'(x)$ $\quad$ $h'(2) = g'(f(2))\cdot f'(2)$

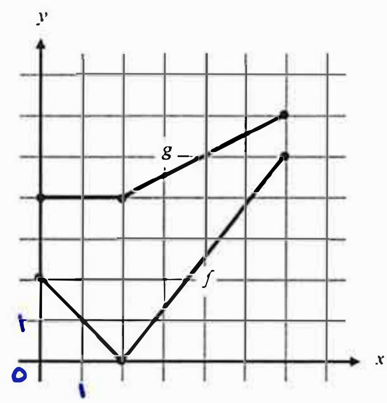
$= g'(5)\cdot e$ $\quad$ $\boxed{h'(2) = 7e}$

c) If $h(x)=f(f(x))$, write an expression for $h'(x)$ and use it to find find $h'(2)$.

$h'(x) = f'(f(x))\cdot f'(x)$ $\quad$ $h'(2) = f'(f(2))\cdot f'(2)$

$= f'(5)\cdot e$ $\quad$ $\boxed{h'(2) = e\pi}$

15. Let $r(x)=f(g(x))$ and $s(x)=g(f(x))$ where f and g are shown in the figure below.



a) Find $r'(1)$.

$r'(x) = f'(g(x))\cdot g'(x)$

$r'(1) = f'(g(1))\cdot g'(1)$

$= f'(4)\cdot 0$ $\quad$ $\boxed{r'(1)=0}$

b) Find $s'(4)$.

$s'(x) = g'(f(x))\cdot f'(x)$

$s'(4) = g'(f(4))\cdot f'(4)$

$= g'(2.5)\cdot \frac{5}{4} = \frac{1}{2}\cdot\frac{5}{4}$ $\quad$ $\boxed{s'(4) = \frac{5}{8}}$

16. Four functions (f, g, h, and j) are continuous and differentiable for all real numbers , and some of their values (and the values of their derivatives) are given by the table below. If you know that $h(x)=f(x)\cdot g(x)$ and $j(x)=g(f(x))$, fill in the correct numbers for each blank value in the table.

$h'(x) = f'(x)\cdot g(x) + f(x)\,g'(x)$

$j'(x) = g'(f(x))\cdot f'(x)$

x	0	1
$f(x)$	1	0
$g(x)$	-1	3
$h(x)$	-1	0
$j(x)$	3	-1
$f'(x)$	4	2
$g'(x)$	1	2
$h'(x)$	-3	6
$j'(x)$	8	2

- $h(1)=0 \Rightarrow f(1)\cdot\underbrace{g(1)}_{3} = 0 \Rightarrow f(1)=0$
- $j'(0)=8 \Rightarrow g'(f(0))\cdot f'(0) = 8$

  $\Rightarrow g'(1)\cdot\underbrace{f'(0)}_{4} = 8 \Rightarrow g'(1)=2$
- $h'(1)=6 \Rightarrow f'(1)\cdot g(1) + f(1)\cdot g'(1) = 6$

  $\Rightarrow f'(1)\cdot 3 + 0\cdot 1 = 6$

  $\Rightarrow f'(1)=2$

- $j'(1) = g'(f(1))\cdot f'(1) = g'(0)\cdot 2 = 2$
- $j(0) = g(f(0)) = g(1) = 3$
- $j(1) = -1 \Rightarrow g(f(1)) = -1$

  $\Rightarrow g(0) = -1$
- $h(0) = f(0)\cdot g(0) = -1$
- $h'(0) = f'(0)\cdot g(0) + g'(0)\cdot f(0) = -4+1$

AP Calculus
3.7 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Use implicit differentiation to find $\frac{dy}{dx}$.

a) $x^2y + xy^2 = 6$

$$2x \cdot y + x^2 \cdot \frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2 + 2xy) = -y^2 - 2xy$$

$$\frac{dx}{dy} = -\frac{y^2 + 2xy}{x^2 + 2xy}$$

b) $x + \tan(xy) = 0$

$$1 + \sec^2(xy) \cdot \left(y + x\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} \cdot x\sec^2 xy = -1 - y\sec^2(x \cdot y)$$

$$\frac{dy}{dx} = -\frac{1 + y\sec^2(xy)}{x\sec^2(xy)}$$

2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

a) $y^2 = x^2 + 2x$

- $$2y \cdot \frac{dy}{dx} = 2x + 2$$

$$\frac{dy}{dx} = \frac{x+1}{y}$$

- $$\frac{d^2y}{dx^2} = \frac{y - \frac{dy}{dx}(x+1)}{y^2}$$

$$= \frac{y - \frac{(x+1)^2}{y}}{y^2} = \frac{1}{y} - \frac{(x+1)^2}{y^3}$$

b) $x^{2/3} + y^{2/3} = 1$

- $$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

- $$\frac{d^2y}{dx^2} = -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3} \cdot \left(\frac{\frac{dy}{dx} \cdot x - y}{x^2}\right)$$

$$= -\frac{1}{3}\left(\frac{x}{y}\right)^{2/3} \frac{-\left(\frac{y}{x}\right)^{1/3} \cdot x - y}{x^2}$$

3. Find the equations of the tangent line and the normal line to the curves.

a) $y = \sin(\pi x - y)$ at the point $(1, 0)$

$$\frac{dy}{dx} = \cos(\pi x - y) \cdot \left(\pi - \frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \pi\cos(\pi x - y) - \frac{dy}{dx}\cos(\pi x - y)$$

$$\frac{dy}{dx}\left(1 + \cos(\pi x - y)\right) = \pi\cos(\pi x - y)$$

$$\frac{dy}{dx} = \frac{\pi\cos(\pi x - y)}{1 + \cos(\pi x - y)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=1 \\ y=0}} = \frac{\pi \cdot \cos\pi}{1 + \cos\pi} \leftarrow \text{impossible!}$$

vertical tangent line!

Tangent: $\boxed{x = 1}$

Normal: $\boxed{y = 0}$

b) $y^2(2 - x) = x^3$ at the point $(1,1)$

$$2y\frac{dy}{dx}(2-x) - y^2 = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2-x)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=1 \\ y=1}} = \frac{4}{2} = 2$$

Tangent

$$\Rightarrow \boxed{y - 1 = 2(x - 1)}$$

Normal

$$\boxed{y - 1 = -\frac{1}{2}(x - 1)}$$

4. Find the equations of the tangent line and the normal line to the curve $x^2 + xy - y^2 = 1$ when $x = 2$.

points: $2^2 + 2y - y^2 = 1$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y = 3 \quad y = -1$

points: $(2;-1)$ et $(2;3)$

$$2x + y + x\frac{dy}{dx} - 2y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x-2y) = -2x - y$$

$$\frac{dy}{dx} = -\frac{2x+y}{x-2y}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{3}{4} \qquad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=3}} = +\frac{7}{4}$$

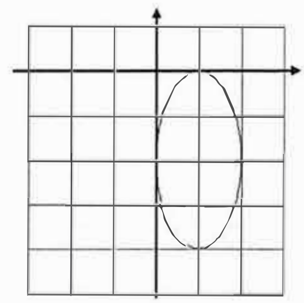
Tangentes: $y+1 = -\frac{3}{4}(x-2)$

$y-3 = \frac{7}{4}(x-2)$

Normal: $y+1 = \frac{4}{3}(x-2)$

$y-3 = -4/7\,(x-2)$

5. Find the points at which the graph of $4x^2 + y^2 - 8x + 4y + 4 = 0$ has a vertical tangent line. (Pretend the picture isn't there until *after* you have found the points!)



$$8x + 2y\cdot\frac{dy}{dx} - 8 + 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{8-8x}{2y+4} = \frac{4-4x}{y+2}$$

v. tangent: when $y = -2$

$4x^2 + 4 - 8x - 8 + 4 = 0$

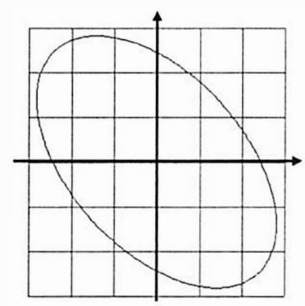
$4x^2 - 8x = 0$

$4x(x-2) = 0$

$\boxed{x=0}$ or $\boxed{x=2}$

points: $(0;-2)$ and $(2;-2)$

6. Find the point(s) (if any) of horizontal tangent lines: $x^2 + xy + y^2 = 6$
(Does your answer make sense given the picture to the right?)



$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y) = -2x - y$$

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

$$\frac{dy}{dx} = 0 \Leftrightarrow y = -2x$$

$x^2 + x(-2x) + (-2x)^2 = 6$

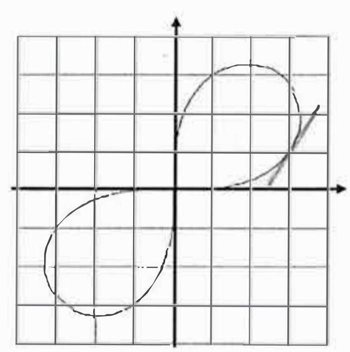
$x^2 - 2x^2 + 4x^2 = 6$

$3x^2 = 6$

$x^2 = 2$

$x = \pm\sqrt{2}$ points: $(-\sqrt{2};\,2\sqrt{2})$ and $(\sqrt{2};\,-2\sqrt{2})$

7. Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point (3, 1).
(Does your answer make sense given the picture to the right?)



$$3.2\,(x^2+y^2).\left(2x + 2y\cdot\frac{dy}{dx}\right) = 100\left(y + x\frac{dy}{dx}\right)$$

$$12(x^2+y^2)x + 12(x^2+y^2)y\cdot\frac{dy}{dx} = 100y + 100x\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(12y(x^2+y^2) - 100x\right) = 100y - 12x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{100y - 12x(x^2+y^2)}{12y(x^2+y^2) - 100x}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=3\\y=1}} = \frac{100 - 36(10)}{12(10) - 300} = \frac{-260}{-180} = \frac{13}{9}$$

8. Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$.

a) Show that $\frac{dy}{dx} = \frac{4x - 2xy}{x^2 + y^2 + 1}$

$$6y^2 \cdot \frac{dy}{dx} + 6\left(2xy + x^2\frac{dy}{dx}\right) - 24x + 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(6y^2 + 6x^2 + 6) = 24x - 12xy$$

$$\frac{dy}{dx} = \frac{6(4x - 2xy)}{6(y^2 + x^2 + 1)} \quad \therefore \quad \frac{dy}{dx} = \frac{4x - 2xy}{x^2 + y^2 + 1}$$

CALC b) Write an equation of each horizontal tangent line to the curve.

$$\frac{dy}{dx} = 0 \Rightarrow 4x - 2xy = 0$$

$$2x - xy = 0$$

$$x(2 - y) = 0$$

$$x = 0 \text{ or } y = 2$$

$x = 0$: $2y^3 + 6y = 1$

$2y^3 + 6y - 1 = 0$

$y \approx 0.165165$

$\boxed{y = 0.165165}$

$y = 2$: $16 + 12x^2 - 12x^2 + 12 = 1$ impossible!

CALC c) The line through the origin with slope -1 is tangent to the curve at point P. Find the x– and y– coordinates of point P.

$$y - 0 = -1(x - 0) \Rightarrow y = -x$$

$$\hookrightarrow 2(-x)^3 + 6x^2(-x) - 12x^2 + 6(-x) = 1$$

$$-2x^3 - 6x^3 - 12x^2 - 6x - 1 = 0$$

$$-8x^3 - 12x^2 - 6x - 1 = 0$$

$$(x + \tfrac{1}{2})(\underbrace{-8x^2 - 8x - 2}_{\Delta < 0}) = 0$$

$\Rightarrow x = -\frac{1}{2} \quad y = \frac{1}{2}$

$P(-\frac{1}{2}; \frac{1}{2})$

when $x = -\frac{1}{2}$ and $y = \frac{1}{2}$: $\frac{dy}{dx} = \frac{-2 + \frac{1}{2}}{3/2} = -1$ ✓

9. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at the point $(1,1)$ intersects the curve at what other point?

• $2x + 2\left(y + x\frac{dy}{dx}\right) - 6y\frac{dy}{dx} = 0$

$$x + y + x\frac{dy}{dx} - 3y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x - 3y) = -x - y$$

$$\frac{dy}{dx} = -\frac{x + y}{x - 3y}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{2}{2} = 1$$

$\Rightarrow$ normal line:

$$y - 1 = -1(x - 1)$$

$$\boxed{y = -x + 2}$$

Intersection:

$$x^2 + 2x(-x + 2) - 3(-x + 2)^2 = 0$$

$$x^2 - 2x^2 + 4x - 3(x^2 - 4x + 4) = 0$$

$$-4x^2 + 16x - 12 = 0$$

$$-4(x^2 - 4x + 3) = 0$$

$$(x - 1)(x - 3) = 0$$

$\begin{cases} x = 1 \\ y = 1 \end{cases} \qquad \begin{cases} x = 3 \\ y = -1 \end{cases}$

Point: $(3, -1)$

AP Calculus
3.8 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What are the following derivatives … these MUST be memorized forwards (now) and backwards (later).

a) $\frac{d}{dx}\left[\sin^{-1}x\right]= \frac{1}{\sqrt{1-x^2}}$ b) $\frac{d}{dx}\left[\tan^{-1}x\right]= \frac{1}{1+x^2}$ c) $\frac{d}{dx}\left[\sec^{-1}x\right]= \frac{1}{x\sqrt{x^2-1}}$

2. Find the derivative of the following functions:

a) $y=t\sin^{-1}\left(t\sqrt{2}\right)$

$$\frac{dy}{dt} = \sin^{-1}(t\sqrt{2}) + \frac{t}{\sqrt{1-2t^2}}\cdot\sqrt{2}$$

b) $f(x)=x\sqrt{1-x^2}+\cos^{-1}\left(x^3\right)$

$$f'(x) = \sqrt{1-x^2} + \frac{x}{2\sqrt{1-x^2}}\cdot(-2x) + \frac{-1}{\sqrt{1-x^6}}\cdot 3x^2$$

$$= \frac{1-x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} - \frac{3x^2}{\sqrt{1-x^6}}$$

$$= \frac{1-2x^2}{\sqrt{1-x^2}} - \frac{3x^2}{\sqrt{1-x^6}}$$

3. A particle is moving along the x-axis so that its position at any time $t \geq 0$ is given by $x(t)$. Find the velocity at the indicated value of t. Do without a calculator … then check it with your calculator.

a) $x(t)=\sin^{-1}\left(\frac{\sqrt{t}}{4}\right)$ when $t = 4$.

$$v(t) = x'(t)$$

$$= \frac{1}{\sqrt{1-\frac{t}{16}}}\cdot\frac{1}{4}\cdot\frac{1}{2\sqrt{t}}$$

$$v(4) = \frac{2}{\sqrt{3}}\cdot\frac{1}{4}\cdot\frac{1}{4}$$

$$= \frac{1}{8\sqrt{3}}$$

$$= \frac{\sqrt{3}}{24}$$

b) $x(t)=\tan^{-1}\left(t^2\right)$ when $t = 1$.

$$v(t) = x'(t)$$

$$= \frac{1}{1+t^4}\cdot 2t$$

$$v(1) = \frac{1}{2}\cdot 2 = 1$$

4. Find an equation for the line tangent to the graph of $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

$\frac{dy}{dx} = \sec^2 x$ $\quad \boxed{y - 1 = 2\left(x - \frac{\pi}{4}\right)}$

$m = 2$

5. Find an equation for the line tangent to the graph of $y = \tan^{-1} x$ at the point $\left(1, \frac{\pi}{4}\right)$.

$\frac{dy}{dx} = \frac{1}{1+x^2}$ $\quad \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2}$ $\quad \boxed{y - \frac{\pi}{4} = \frac{1}{2}(x-1)}$

6. What is the relationship between the slopes of the tangent lines in questions 4 and 5? How does this help you remember the rule for finding the derivative of an inverse function? They are reciprocals.

7. [illegible] Let $f(x) = x^3 + 2x - 1$.

a) Find $f(1)$ and $f'(1)$. $\quad f(1) = 0$ $\quad f'(x) = 3x^2 + 2$

$f'(1) = 5$

b) How can you find $f^{-1}(-4)$? Find it.

let $x = f^{-1}(-4)$ then $f(x) = -4$

$x^3 + 2x - 1 = -4$

$x^3 + 2x + 3 = 0$

$(x+1)(x^2 - x + 3) = 0$

$\boxed{x = -1}$ $\quad \Delta = 1 - 12 < 0$

c) What is $\left(f^{-1}\right)'(-4)$?

$(f^{-1})'(-4) = \frac{1}{f'(-1)} = \frac{1}{5}$

d) What is $\left(f^{-1}\right)'(2)$

$x^3 + 2x - 1 = 2$

$x^3 + 2x - 3 = 0$

$(x-1)(x^2 + x + 3) = 0$

$\boxed{x = 1}$ $\quad \Delta = 1 - 12 < 0$

$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{5}$

8. [Calculator Allowed] Suppose $f(x) = x^3 - \frac{4}{x}$, $x > 0$

a) Find $f(4)$ and $f'(4)$ $\quad f(4) = 64 - 1 = 63$ $\quad f'(x) = 3x^2 + \frac{4}{x^2}$ $\quad f'(4) = 48\frac{1}{4}$

b) What is $\left(f^{-1}\right)'(6)$? $\quad x^3 - \frac{4}{x} = 6 \iff x^4 - 6x - 4 = 0$, $x \neq 0$

$\iff (x-2)(x^3 + 2x^2 + 4x + 2) = 0$

$x = 2$ or $x \simeq \cancel{-0.639}$ $(x>0)$

$(f^{-1})'(6) = \frac{1}{f'(2)} = \frac{1}{13}$

c) What is $\left(f^{-1}\right)'(-3)$?

$x^3 - \frac{4}{x} = -3$

$x^4 + 3x - 4 = 0$, $x \neq 0$

$(x-1)(x^3 + x^2 + x + 4) = 0$

$x = 1$ $\quad x = ?\ \cancel{-1.7}$ $(x>0)$ $\quad (f^{-1})'(-3) = \frac{1}{f'(1)} = \frac{1}{7}$

9. The rule for inverse functions is that $f\left(f^{-1}(x)\right)=x$.

a) Take the derivative of both sides of the above expression.

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

b) Solve your equation from part *a* for the derivative of $f^{-1}(x)$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

10. The functions f and g are differentiable for all real numbers and g is strictly increasing. The table below gives values of the functions and their derivatives at selected values of x.

x	$g(x)$	$g'(x)$
1	2	5
2	3	1
3	4	2
4	6	7

If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y=g^{-1}(x)$ at $x=2$.

Point : $(2; g^{-1}(2))$ $\quad g^{-1}(2) = 1$ $\quad$ Point : $(2; 1)$

$$\frac{dg^{-1}(x)}{dx} = \frac{1}{g'(g^{-1}(x))}$$

$$\left.\frac{dg^{-1}(x)}{dx}\right|_{x=2} = \frac{1}{g'(g^{-1}(2))}$$

$$= \frac{1}{g'(1)}$$

$$= \frac{1}{5} \quad \Rightarrow \quad \boxed{y - 1 = \frac{1}{5}(x-2)}$$

AP Calculus
3.9 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Find the derivative of each of the following functions:

a) $f(x)=7^{3x+1}$

$f(x)=e^{(3x+1)\ln 7}$

$f'(x)=3\ln 7 \cdot 7^{3x+1}$

b) $g(x)=9^{x^2}$

$g(x)=e^{x^2\ln 9}$

$g'(x)=2x\ln 9 \cdot 9^{x^2}$

c) $h(x)=e^{2x^3-5}$

$h'(x)=6x^2 e^{2x^3-5}$

d) $k(x)=\log_2(x^2-9)$

$k(x)=\frac{\ln(x^2-9)}{\ln 2}$

$k'(x)=\frac{2x}{\ln 2(x^2-9)}$

e) $j(x)=\log(3x+7)$

$j(x)=\frac{\ln(3x+7)}{\ln 10}$

$j'(x)=\frac{3}{\ln 10(3x+7)}$

f) $m(x)=\ln(3x^5+8)$

$m'(x)=\frac{15x^4}{3x^5+8}$

2. Find $\frac{dy}{dx}$ for the following functions:

a) $y=e^{-5x}$

$\frac{dy}{dx}=-5e^{-5x}$

b) $y=xe^2-e^{-x}$

$\frac{dy}{dx}=e^2+e^{-x}$

c) $y=x^2e^x-xe^x$

$\frac{dy}{dx}=2xe^x+x^2e^x-e^x-xe^x$

$=e^x(x^2+x-1)$

d) $y=3^{\csc x}=e^{\csc x \cdot \ln 3}$

$\frac{dy}{dx}=-\ln 3\csc x.\cot x \cdot 3^{\csc x}$

e) $y=(\ln x)^2$

$\frac{dy}{dx}=2\ln x \times \frac{1}{x}$

$=\frac{2\ln x}{x}$

f) $y=\log_5\sqrt{x}=\frac{\ln\sqrt{x}}{\ln 5}$

$\frac{dy}{dx}=\frac{1}{2\sqrt{x}.\ln 5.\sqrt{x}}$

$=\frac{1}{2x\ln 5}$

g) $y=\log_3(1+x\ln x)=\frac{\ln(1+x\ln x)}{\ln 3}$

$\frac{dy}{dx}=\frac{1}{\ln 3}\cdot\frac{1}{1+x\ln x}\cdot(\ln x+1)$

$=\frac{1+\ln x}{\ln 3(1+x\ln x)}$

h) $y=\ln 2\cdot\log_2 x=\ln x$

$\frac{dy}{dx}=\frac{1}{x}$

i) $y=5^{\sin^{-1}(3x^2)}=e^{\sin^{-1}(3x^2).\ln 5}$

$\frac{dy}{dx}=\ln 5.\frac{6x}{\sqrt{1-9x^4}}.5^{\sin^{-1}(3x^2)}$

j) $y=x^e e^x$

$\frac{dy}{dx}=ex^{e-1}.e^x+x^e.e^x$

$=x^e.e^x\left(\frac{e}{x}+1\right)$

k) $y=\ln(\ln x)$

$\frac{dy}{dx}=\frac{1}{\ln x}\cdot\frac{1}{x}$

$=\frac{1}{x\ln x}$

l) $y=\log_2\left(\frac{1}{x}\right)=\frac{\ln\left(\frac{1}{x}\right)}{\ln 2}$

$\frac{dy}{dx}=\frac{1}{\ln 2}\cdot\frac{1}{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)$

$=\frac{1}{\ln 2}\cdot x\cdot\left(-\frac{1}{x^2}\right)$

$=-\frac{1}{x\ln 2}$

3. Suppose $10 = e^{xy} + x^2 + y^2$, find $\frac{dy}{dx}$.

$$0 = e^{x \cdot y}\left(y + x\frac{dy}{dx}\right) + 2x + 2y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(xe^{xy} + 2y\right) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} + 2y}$$

4. [Calculator Required] At what point on the graph of $y = 3^x + 1$ is the tangent line parallel to $5x - y = 1$?

$y = 5x - 1$ slope $\underline{\underline{5}}$

$$\frac{dy}{dx} = \ln 3 . 3^x$$

$$\frac{dy}{dx} = 5 \Leftrightarrow 3^x = \frac{5}{\ln 3} \Leftrightarrow x = \frac{\ln 5 - \ln(\ln 3)}{\ln 3}$$

$$P\left(\frac{\ln 5 - \ln(\ln 3)}{\ln 3} ; \frac{5}{\ln 3} + 1\right)$$

5. [redacted] At what point on $y = 2e^x - 1$ is the tangent line perpendicular to the line $3x + y = 2$?

$y = -3x + 2$

$\Rightarrow$ slope : $\frac{1}{3}$

$$\frac{dy}{dx} = 2e^x$$

$$2e^x = \frac{1}{3} \Leftrightarrow e^x = \frac{1}{6} \Leftrightarrow x = \ln\left(\frac{1}{6}\right)$$

$$P\left(\ln\left(\frac{1}{6}\right) ; -\frac{2}{3}\right)$$

6. If $y = \tan u$, $u = v - \frac{1}{v}$, and $v = \ln x$, what is the value of $\frac{dy}{dx}$ at $x = e$?

A 0
B $1/e$
C 1
D $2/e$
E $\sec^2(e)$

$$y = \tan\left(\ln x - \frac{1}{\ln x}\right)$$

$$\frac{dy}{dx} = \sec^2\left(\ln x - \frac{1}{\ln x}\right) \cdot \left(\frac{1}{x} + \frac{1}{(\ln x)^2} \cdot \frac{1}{x}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=e} = \sec^2(0) \cdot \frac{2}{e} = \frac{2}{e}$$

7. Use logarithmic differentiation to find the derivative of the following functions:

a) $y = x^{\ln x}$

$$\ln y = \ln x . \ln x$$

$$\ln y = (\ln x)^2$$

$$\rightarrow \frac{1}{y}\frac{dy}{dx} = 2\ln x \cdot \frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{2\ln x}{x} \cdot x^{\ln x}}$$

b) $y = \sqrt[5]{\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}}$

$$\ln y = \frac{1}{5}\left(4\ln(x-3) + \ln(x^2+1) - 3\ln(2x+5)\right)$$

$$\rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{5}\left(\frac{4}{x-3} + \frac{2x}{x^2+1} - \frac{6}{2x+5}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{5}\left(\frac{4}{x-3} + \frac{2x}{x^2+1} - \frac{6}{2x+5}\right) \cdot \sqrt[5]{\frac{(x-3)^4(x^2+1)}{(2x+5)^3}}}$$

8. Use logarithmic differentiation: If $f(x)=(x^2+1)^{(2-3x)}$, then $f'(1)=$

$$\ln f(x) = (2-3x)\ln(x^2+1)$$

$$\rightarrow \frac{1}{f(x)}\cdot f'(x) = -3\ln(x^2+1) + (2-3x)\cdot\frac{2x}{x^2+1}$$

$$f'(x) = \left(-3\ln(x^2+1) + (2-3x)\cdot\frac{2x}{x^2+1}\right)(x^2+1)^{2-3x}$$

$$f'(1) = (-3\ln 2 - 1)2^{-1}$$

$$= -\frac{3\ln 2 + 1}{2}$$

10. If $y=x^2e^x$, find where $\frac{dy}{dx}=0$.

$$\frac{dy}{dx} = 2x\cdot e^x + x^2e^x$$

$$= e^x(x^2+2x)$$

$$\frac{dy}{dx} = 0 \Leftrightarrow e^x\cdot x(x+2) = 0$$

$$\Leftrightarrow \boxed{x=0} \text{ or } \boxed{x=-2}$$

11. If $y=\frac{e^x}{x^3}$, find where $\frac{dy}{dx}$ is equal to zero and undefined.

$D = \mathbb{R}\setminus\{0\}$

$$\frac{dy}{dx} = \frac{e^x\cdot x^3 - 3x^2\cdot e^x}{x^6}$$

$$= \frac{e^x(x-3)}{x^4}$$

$$\frac{dy}{dx} = 0 \Leftrightarrow x = 3$$

$\frac{dy}{dx}$ is undefined when $x=0$.

12. Differentiate the following functions:

a) $f(x) = \sqrt{\log x}$

$$f'(x) = \frac{1}{2\sqrt{\log x}}\cdot\frac{1}{x\ln 10}$$

$$\boxed{f'(x) = \frac{1}{2\ln 10\cdot x\sqrt{\log x}}}$$

b) $g(x) = (\ln x)^x$

$$\ln g(x) = x\ln(\ln x)$$

$$\frac{g'(x)}{g(x)} = \ln(\ln x) + \frac{1}{\ln x}\cdot\frac{1}{x}\cdot x$$

$$\boxed{g'(x) = (\ln x)^x\left(\ln(\ln x) + \frac{1}{\ln x}\right)}$$

c) $h(x) = (x^2+e^{3x})^{4x}$

$$\ln h(x) = 4x\ln(x^2+e^{3x})$$

$$\frac{h'(x)}{h(x)} = 4\left[\ln(x^2+e^{3x}) + \frac{x}{x^2+e^{3x}}\cdot(2x+3e^{3x})\right]$$

$$\boxed{h'(x) = 4(x^2+e^{3x})^{4x}\left[\ln(x^2+e^{3x}) + \frac{x(2x+3e^{3x})}{x^2+e^{3x}}\right]}$$

d) $k(x) = x^{\cos x}$

$$\ln k(x) = \cos x\ln x$$

$$\frac{k'(x)}{k(x)} = -\sin x\ln x + \frac{\cos x}{x}$$

$$\boxed{k'(x) = x^{\cos x}\left(-\sin x\ln x + \frac{\cos x}{x}\right)}$$