# Optimization Problems - Homework

1. Find two numbers whose sum is 10 for which the sum of their squares is a minimum.

$$S = x^2 + (10 - x)^2$$
$$S = 2x^2 - 20x + 100$$
$$4x - 20 = 0$$
$$x = 5, y = 5$$

2. Find nonnegative numbers $x$ and $y$ whose sum is 75 and for which the value of $xy^2$ is as large as possible.

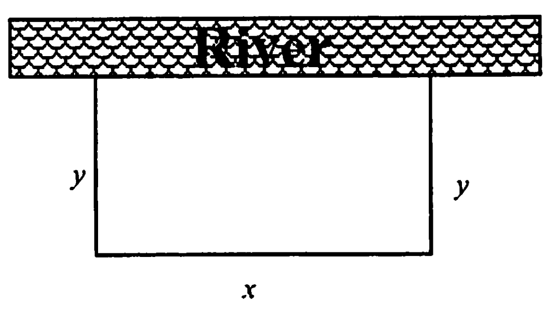
$$P = (75 - y)y^2$$
$$P = 75y^2 - y^3$$
$$0 = 150y - 3y^2$$

$$3y(50 - y) = 0$$
$$y = 50, x = 25$$

3. A ball is thrown straight up in the air from ground level. Its height after $t$ seconds is given by $s(t) = -16t^2 + 50t$. When does the ball reach it maximum height? What is its maximum height?

$$0 = -32t + 50$$
$$32t = 50$$
$$t = \frac{25}{16} \text{ sec}, \; s(t) = 39.063 \text{ ft}$$

4. A farmer has 2,000 feet of fencing to enclose a pasture area. The field will be in the shape of a rectangle and will be placed against a river where there is no fencing needed. What is the largest area field that can be created and what are its dimensions?




$$P = x + 2y = 2000$$
$$A = xy$$
$$A = (2000 - 2y)y$$
$$A = 2000y - 2y^2$$
$$0 = 2000 - 4y$$
$$4y = 2000$$
$$y = 500 \text{ ft}, x = 1000 \text{ ft}, \text{Area} = 500{,}000 \text{ ft}^2$$

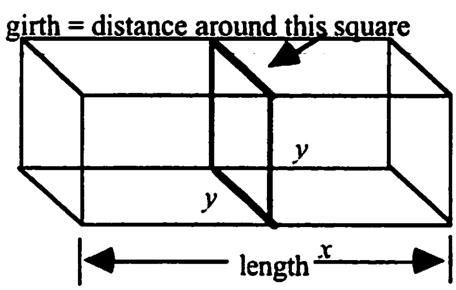
5. A fisheries biologist is stocking fish in a lake. She knows that when there are $n$ fish per unit of water, the average weight of each fish will be $W(n) = 500 - 2n$, measured in grams. What is the value of $n$ that will maximize the total fish weight after one season. *Hint: Total Weight = number of fish • average weight of a fish.*

$$W = n(500 - 2n)$$
$$W = 500n - 2n^2$$
$$0 = 500 - 4n$$
$$n = 125 \text{ fish}$$

6. The size of a population of bacteria introduced to a food grows according to the formula $P(t) = \dfrac{6000t}{60 + t^2}$ where $t$ is measured in weeks. Determine when the bacteria will reach its maximum size. What is the maximum size of the population?

$$0 = \frac{(60 + t^2)(6000) - 6000t(2t)}{(60 + t^2)^2}$$
$$0 = 360000 - 6000t^2$$
$$t^2 = 60$$
$$t = \sqrt{60} \text{ - Week 8}$$
$$\text{Size} = 387 \text{ bacteria}$$

7. The U.S. Postal Service will accept a box for domestic shipping only if the sum of the length and the girth (distance around) does not exceed 108 inches. Find the dimensions of the largest volume box with a square end that can be sent.

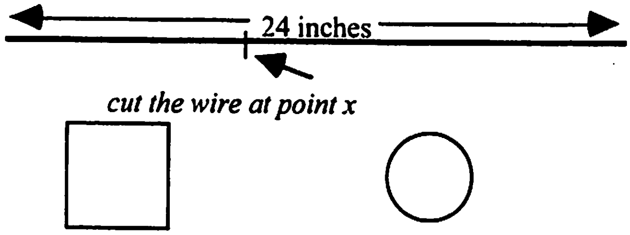



$$V = xy^2$$
$$V = (108 - 4y)y^2$$
$$V = 108y^2 - 4y^3$$
$$0 = 216y - 12y^2$$
$$0 = 12y(18 - y)$$
$$y = 18 \text{ in}, x = 36 \text{ in}$$

8. Blood pressure in a patient will drop by an amount $D(x)$ where $D(x) = 0.025x^2(30 - x)$ where $x$ is the amount of drug injected in cm³. Find the dosage that provides the greatest drop in blood pressure. What is the drop in blood pressure?

$$D(x) = 0.025x^2(30 - x)$$
$$D = .75x^2 - .025x^3$$
$$0 = 1.5x - .075x^2$$
$$0 = .075(20 - x)$$
$$x = 20 \text{ cm}^3, \text{ Drop } = 100 \text{ pts}$$

9. A wire 24 inches long is cut into two pieces. One piece is to be shaped into a square and the other piece into a circle. Where should the wire be cut to maximize the total area enclosed by the square and circle?




$$S = \frac{x^2}{16} + \frac{x^2 - 48x + 576}{4\pi}$$

$$0 = \frac{x}{8} + \frac{2x - 48}{4\pi} \Rightarrow 0 = \pi x + 4x - 96$$

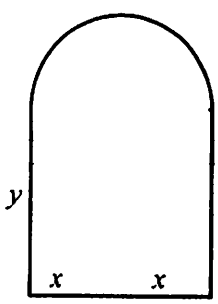
$$\pi x + 4x = 96 \Rightarrow x = \frac{96}{4+\pi} = 13.442$$

This is a minimum - max occurs when it is all circle

Let $x$ be the point where the cut is made. Assume the square is on the left and the circle on the right. Complete the chart.

| $x$ | 4 | 8 | 12 | 20 | $x$ |
|---|---|---|---|---|---|
| Area square | 1 | 4 | 9 | 16 | $\left(\frac{x}{4}\right)^2$ |
| Area circle | $\frac{400}{4\pi}$ | $\frac{256}{4\pi}$ | $\frac{144}{4\pi}$ | $\frac{16}{4\pi}$ | $\frac{(24-x)^2}{4\pi}$ |
| Total area | 32.1 | 24.3 | 20.5 | 20.0 | |

10. A designer of custom windows wishes to build a Norman Window with a total outside perimeter of 60 feet. How should the window be designed to maximize the area of the window. A Norman Window contains a rectangle bordered above by a semicircle.




$$A = 2xy + \frac{1}{2}\pi x^2$$

$$A = 2x\left(\frac{60 - 2x - \pi x}{2}\right) + \frac{1}{2}\pi x^2$$

$$0 = 60 - 4x - \pi x$$

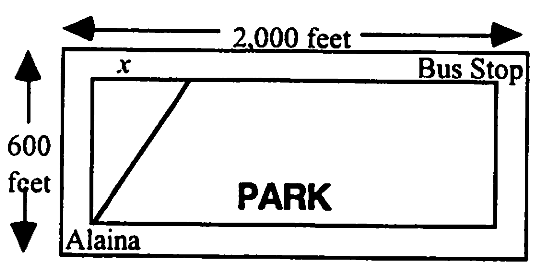
$$x = \frac{60}{\pi + 4} = 8.401$$

$$2x + 2y + \pi x = 60$$

$$y = \frac{60 - 2x - \pi x}{2}$$

Dimensions are 16.402 in wide by 8.401 in tall

11. Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2,000 feet east and 600 feet north of her starting position. Alaina can walk west along the edge of the park on the sidewalk at a speed of 6 feet/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec (dogs are walked here, so she must move with care). What path will get her to the bus stop the fastest.




$$T = \frac{\sqrt{x^2 + 360000}}{4} + \frac{2000 - x}{6}$$

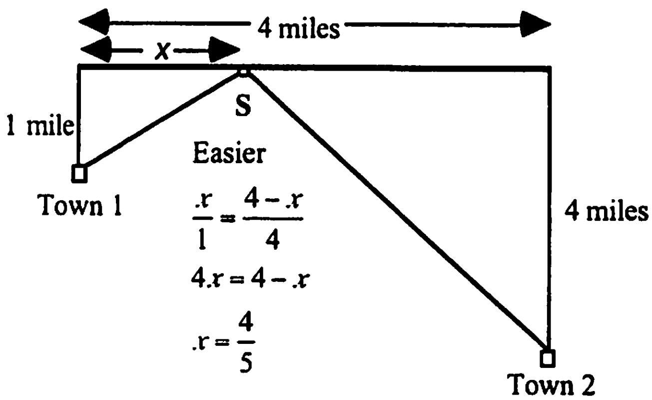
$$0 = \frac{x}{4\sqrt{x^2 + 360000}} - \frac{1}{6}$$

$$6x = 4\sqrt{x^2 + 360000}$$

$$9x^2 = 4x^2 + 1440000$$

$x = 536.656$ or $1463.344$ ft from bus stop

12. On the same side of a straight river are two towns, and the townspeople want to build a pumping station, S, that supplies water to them. The pumping station is to be at the river's edge with pipes extending straight to the two towns. The distances are shown in the figure below. Where should the pumping station be located to minimize the total length of pipe?

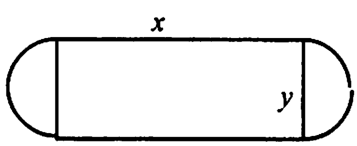



Easier

$$\frac{x}{1}=\frac{4-x}{4}$$
$$4x=4-x$$
$$x=\frac{4}{5}$$

$$D=\sqrt{x^2+1}+\sqrt{16-8x+x^2+16}$$
$$0=\frac{x}{\sqrt{x^2+1}}+\frac{x-4}{\sqrt{x^2-8x+32}}$$
$$-\frac{x}{\sqrt{x^2+1}}=\frac{x-4}{\sqrt{x^2-8x+32}}$$
$$\frac{x^2}{x^2+1}=\frac{x^2-8x+16}{x^2-8x+32}$$
$$x^4-8x^3+17x^2-8x+16=x^4-8x^3+32x^2$$
$$15x^2+8x-16=0$$
$$(5x-4)(3x+4)=0\Rightarrow x=\frac{4}{5}\text{ miles}$$

13. A physical fitness room consists of a rectangular region with a semicircle on each end. If the perimeter of the room is to be a 200-meter running track, find the dimensions that will make the area of the rectangular region as large as possible.



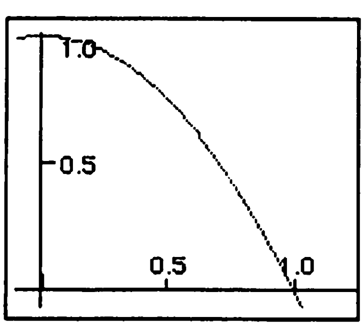

Total distance around track = 200 feet

$$A=xy$$
$$A=\left(100-\frac{\pi y^2}{8}\right)y$$
$$A=100y-\frac{\pi y^3}{8}$$
length = 50 f

$$0=100-\frac{3\pi y^2}{8}$$
$$y=\sqrt{\frac{800}{3\pi}}$$
width $=\dfrac{100}{\pi}$ f

$$2x+\pi\left(\frac{y}{2}\right)^2=200$$
$$2x+\frac{\pi y^2}{4}=200$$
$$x=100-\frac{\pi y^2}{8}$$

14. Below is the graph of $y=1-x^2$. Find the point on this curve which is closest to the origin. *(Remember, you need a primary equation. What is it that you wish to minimize?)*




$$D=\sqrt{x^2+y^2}$$
$$D=\sqrt{x^2+\left(1-x^2\right)^2}$$
$$D=\sqrt{x^2+x^2-2x+1}$$
$$0=\frac{x\left(2x^2-1\right)}{\sqrt{x^4-x^2+1}}$$
$$2x^2=1$$
$$x=\sqrt{\frac{1}{2}},y=\frac{3}{4}\qquad\left(\sqrt{\frac{1}{2}},\frac{3}{4}\right)$$