AP Calculus
4.1 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Earlier this year we had the Intermediate Value Theorem (IVT) and now we have the Extreme Value Theorem (EVT). The *hypothesis* of each theorem is what is needed to apply each theorem.

a) What is the hypothesis of the IVT (i.e. what is needed to apply the IVT) ?

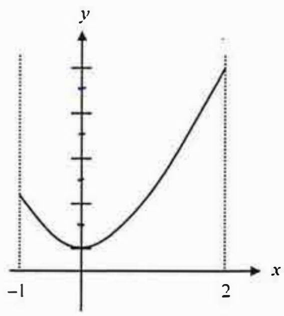
f continuous on a closed interval [a; b]

b) What is the hypothesis of the EVT?

same

2. Using the graphs provided, find the minimum and maximum values on the given interval. *If there is no maximum or minimum value*, explain which part of the hypothesis of the Extreme Value Theorem is not satisfied.

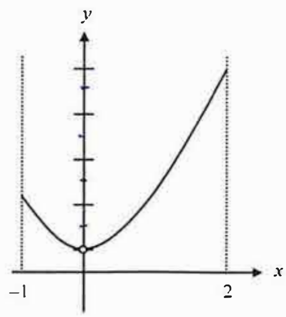
(a) $[-1, 2]$



Max: 9
Min: 1

OK

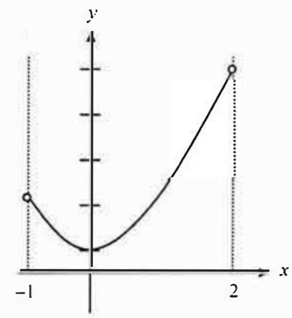
(b) $[-1, 2]$



Max: 9
Min: $\emptyset$

not continuous

(c) $(-1, 2)$



Max: $\emptyset$
Min: 1

open interval

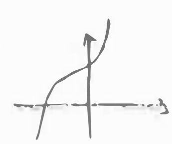
3. When looking for extrema, where do you find the candidates for the "candidates test"?

the critical points and the end points

Each of the following statements is NOT ALWAYS TRUE. Explain/Show why each statement is false.

4. If $f'(5) = 0$, then there is a maximum or a minimum at $x = 5$.

eg



eg : $f'(x) = (x-5)^2$

$f(x) = \frac{1}{3}x^3 - 5x^2 + 25x$

x		5	
$f'(x)$	+	0	+
$f(x)$	$-\infty$	↗	$+\infty$

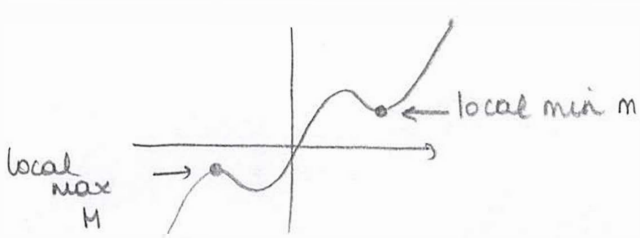
5. If $x = 2$ is a critical number, then $f'(2) = 0$.

no : f' could be undefined at 2.

6. An extrema occurs at every critical number.

no : cf q° 4. or end points

7. If m is a local minimum and M is a local maximum of a continuous function, then $m < M$.



M < m

8. If f is a continuous, decreasing function on [0, 10] with a critical point at (4, 2), which of the following statements MUST BE FALSE?

A $f(10)$ is an absolute minimum of f on [0, 10]
B f (4) is neither a relative maximum nor a relative minimum
C $f'(4)$ does not exist.
D $f'(4)=0$
E $f'(4)<0$

9. Find the extrema on each interval and where they occur … Use a "candidates test".

a) $f(x)=\frac{1}{x}+\ln x$ when $0.5\le x\le 4$

$f'(x) = -\frac{1}{x^2}+\frac{1}{x} = \frac{x-1}{x^2}$

$f'(x)=0 \Leftrightarrow x=1$

$f(0.5)\simeq 1.31$
$f(1) = 1$ (MIN)
$f(4)\simeq 1.64$ (MAX)

b) $g(x)=\ln(x+1)$ when $0\le x\le 3$

$g'(x) = \frac{1}{x+1}$ g' exists on [0;3] and is never 0.

$g(0) = 0$ (MIN)
$g(3) = \ln 4 \approx 1.39$ (MAX)

c) $k(x)=x^{2/5}$ when $-3\le x<1$

$k'(x) = \frac{2}{5}x^{-3/5}$ k' is never 0 but DNE when $x=0$

critical point

$k(-3)\simeq 1.55$ (MAX)
$k(0) = 0$ (MIN)
$k(1) = 1$

10. Find the extrema of $h(\theta)=2\sin\theta-\cos(2\theta)$ for $0\le\theta\le 2\pi$. ~~Use your graphing calculator to investigate first.~~

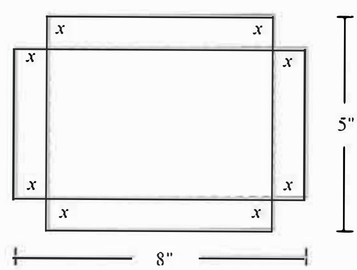
$h'(\theta) = 2\cos\theta + 2\sin(2\theta)$

critical points: $2\cos\theta + 2\sin 2\theta = 0$
$\cos\theta + 2\cos\theta.\sin\theta = 0$
$\cos\theta(1+2\sin\theta) = 0$
$\cos\theta = 0$ $\sin\theta = -1/2$
$\theta = \pi/2$ or $3\pi/2$ $\theta = 7\pi/6$ or $11\pi/6$

$h(0) = -1$
$h(\pi/2) = 3$ MAX
$h(\frac{7\pi}{6}) = -1.5$ MIN
$h(3\pi/2) = -1$
$h(11\pi/6) = -1.5$ MIN
$h(2\pi) = -1$

θ	0	$\pi/2$	$7\pi/6$	$3\pi/2$	$11\pi/6$	2π
$h(\theta)$	-1 ↗	3 ↘	-1.5 ↗	-1 ↘	-1.5 ↗	-1

11. [No Calculator] An open-top box is to be made by cutting congruent squares of side length x from the corners of a 5- by 8-inch sheet of tin and bending up the sides (see figure below).



a) Write an equation for the Volume of the box.

$V = x(8-2x)(5-2x)$

b) What is the domain of this function? [0;2.5]

c) How large should the squares be to maximize the volume?

$V(x) = x(40-16x-10x+4x^2)$
$V(x) = 4x^3-26x^2+40x$
$V'(x) = 12x^2-52x+40$

zeros of V': $\Delta = 784$

$x = \frac{52\pm 28}{24}$ ↗ $10/3 \notin D$ ↘ 1

$V(0) = 0$ $V(1) = 18$ $V(2.5) = 0$

response: $\boxed{x=1}$

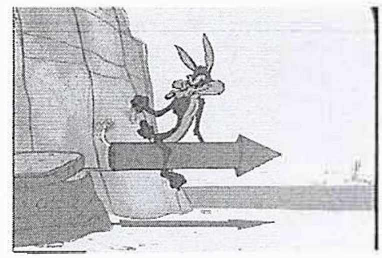
d) What are the dimensions of the box with maximum volume? What is the maximum volume?

$V_{max} = 18$ dimensions : 1 by 6 by 3

12. [Calculator Allowed] Wile E. is after Road Runner again! This time he's got it figured out. Sitting on his ACME rocket he hides behind a hill anxiously awaiting the arrival that "beeping" bird. In his excitement to light the rocket he tips the rocket up. Instead of thrusting himself parallel to the ground where he can catch the Road Runner, he sends himself widely into the air following a path given by function

$$h(t) = .1t^3 - 1.3t^2 + 4.2t + 2,$$

where h is the height of the rocket after t seconds. The rocket fuel lasts for 10 seconds. At that point, Wile E. Coyote stops suddenly and falls straight down to the ground.



a) What is the domain of this function? $D = [0;10]$

b) What is the highest point reached by Wile E. Coyote? [Use Calculus!]

$h'(t) = 0.3t^2 - 2.6t + 4.2$

critical points: $\Delta = 1.72$

$t = \frac{2.6 \pm \sqrt{1.72}}{0.6} \to 6.5$, $\to 2.1$ (approx)

$h(0) = 2$

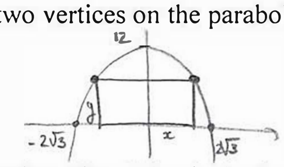
$h(6.5) \simeq 1.8$

$h(2.1) \simeq 6.0$

$h(10) = 14$ ← highest point

13. [No Calculator] A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$.

a) Draw a sketch the rectangle inscribed under the parabola.



b) Write a formula for the area of the inscribed rectangle as a function of x. What is the domain of this function?

$A(x) = 2x \cdot y = 2x(12 - x^2)$ $D = [0; 2\sqrt{3}]$

c) What is the largest area the rectangle can have, and what are its dimensions?

$A(x) = -2x^3 + 24x$

$A'(x) = -6x^2 + 24$

critical points: $-6x^2 + 24 = 0$

$x^2 = 4$

$x = \pm 2$

$A(0) = 0$

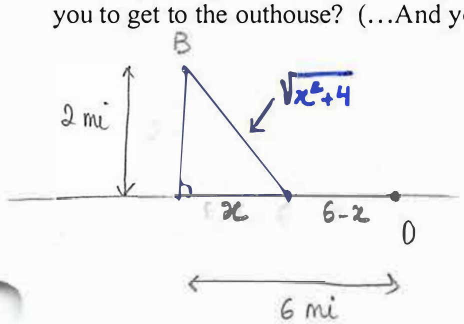
$A(2) = 32$

$A(2\sqrt{3}) = 0$

Largest Area: 32

dimensions: 4 by 8

14. [Challenge … Calculator Allowed] You are in a rowboat on Lake Perris, 2 miles from a straight shoreline taking your potential in-laws for a boat ride. Six miles down the shoreline from the nearest point on shore is an outhouse. You suddenly feel the need for its use. It is November, so the water is too cold to go in, and besides, your in-laws are already pretty unimpressed with your "yacht". It wouldn't help matters to jump over the side and relieve your distended bladder. Also, the shoreline is populated with lots of houses, all owned by people who already have restraining orders against you (apparently you've been out here before!). If you can row at 2 mph and run at 6 mph (you can run faster when you don't have to keep your knees together), for what point along the shoreline should you aim in order to minimize the amount of time it will take you to get to the outhouse? (…And you thought calculus wasn't useful!)



$s = \frac{d}{t}$

$t = \frac{d}{s}$

$D = [0;6]$

$$T = \frac{\sqrt{x^2+4}}{2} + \frac{6-x}{6}$$

$T'(x) = \frac{1}{2} \cdot \frac{1}{2\sqrt{x^2+4}} \times 2x - \frac{1}{6}$

$T'(x) = \frac{x}{2\sqrt{x^2+4}} - \frac{1}{6}$

$= \frac{3x - \sqrt{x^2+4}}{6\sqrt{x^2+4}}$

critical points: • T' is always defined on D

• $3x - \sqrt{x^2+4} = 0$

$3x = \sqrt{x^2+4}$

$9x^2 = x^2 + 4$

$8x^2 = 4$

$x^2 = \frac{1}{2}$

$x = \pm \frac{\sqrt{2}}{2}$

$T(0) = 2$

$T(\sqrt{2}/2) \simeq 1.9$ ← MIN

$T(6) \simeq 3.2$

Answer: where $x = \frac{\sqrt{2}}{2}$

AP Calculus
4.2 Worksheet

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1. State the hypothesis of each of the following theorems:

a) IVT
f continuous on a closed interval

b) EVT
f continuous on a closed interval

c) MVT
f continuous on a closed interval $[a;b]$
differentiable on $(a;b)$

2. State the MVT two different ways …

a) … in words
There is a point on $[a;b]$ where the instantaneous rate of change equals the average rate of change of the whole interval.

b) … algebraically
$$\exists c \in (a,b) \;/\; f'(c) = \frac{f(b)-f(a)}{b-a}$$

3. For each of the following, (a) state whether or not the function satisfies the hypotheses of the MVT on the given interval, and (b) if it does, find the value of c that the MVT guarantees.

a) $f(x) = -2x^2 + 14x - 12$ on the interval $[1, 6]$

- f is continuous on $[1,6]$
- f is differentiable on $(1,6)$

$\Rightarrow \exists c \in (1,6) \;/\; f'(c) = \frac{f(6)-f(1)}{6-1}$

$f'(x) = -4x + 14$ $\qquad f'(x) = 0 \Leftrightarrow \boxed{x = 7/2}$

$\frac{f(6)-f(1)}{6-1} = 0$

b) $h(x) = x^{1/3}$ on $[-1, 1]$

h is continuous on $[-1,1]$
h is not differentiable at 0.

$\Rightarrow$ MVT cannot be used!

4. When a trucker came to his second toll booth in a 169-mile stretch of road, he handed in a ticket stub that was stamped 2 hours earlier. The trucker was cited for speeding. Why?

average velocity/speed : $\frac{169}{2} = 84.5$ mi/h.

The displacement function is continuous on $[0;2]$ and differentiable on $(0;2)$.

$\Rightarrow$ There is a moment (between 0 and 2) when the velocity was 84.5 mi/h

5. Suppose $f(x)$ is a differentiable function on the interval $[-7, 1]$ such that $f(-7) = 4$ and $f(1) = -1$.

a) *Explain* why f must have at least one value in the interval $(-7, 1)$, where the function equals 2.

f is continuous on $[-7,1]$ because it is differentiable

$\Rightarrow$ (MVT) $\exists c \in (-7;1) \;/\; f'(c) = \frac{f(-7)-f(1)}{-7-1} = -\frac{5}{8}$

b) *Explain* why there must be at least one point in the interval $(-7, 1)$ whose derivative is $-\frac{5}{8}$.

f is continuous on $[-7,1]$ (because diff)

$\left.\begin{array}{l} f(-7)=4 \\ f(1)=-1 \end{array}\right\} \; 2 \in [-1;4]$

$\Rightarrow$ (IVT) $\exists k \in [-7,1] \;/\; f(k) = 2$

6. Make a sign chart for the following functions:

a) $f(x)=(x-3)^2(x+4)(7-x)$

x	$-\infty$		-4		3		7		$+\infty$
$f(x)$		$-$	0	$+$	0	$+$	0	$-$	

b) $g(x)=\dfrac{5(2x-7)}{(x+1)(3x-5)}$

$7/2$

x	$-\infty$		-1		$5/3$		$7/2$		$+\infty$
$g(x)$		$-$	$\Vert$	$+$	$\Vert$	$-$	0	$+$	

7. Summarize how we will use calculus to determine whether a function is increasing or decreasing.

Determining the sign of the derivative.

8. For each function, find the critical numbers of f and where the function is increasing or decreasing.

a) $h(x)=\dfrac{2}{x}$ $\quad D=\mathbb{R}\setminus\{0\}$

$h'(x) = -\dfrac{2}{x^2}$

$\lim\limits_{x\to-\infty} h(x) = 0$

$\lim\limits_{x\to0^-} h(x) = -\infty$

$\lim\limits_{x\to0^+} h(x) = +\infty$

$\lim\limits_{x\to+\infty} h(x) = 0$

x	$-\infty$		0		$+\infty$
$h'(x)$		$-$	$\Vert$	$-$	
$h(x)$	0	↘	$-\infty \Vert +\infty$	↘	0

b) $f(x)=x^3-6x^2+15$ $\quad D=\mathbb{R}$

$f'(x) = 3x^2-12x = 3x(x-4)$

x	$-\infty$		0		4		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	↗	15	↘	-17	↗	$+\infty$

c) $k(x)=\dfrac{-x}{x^2+4}$ $\quad D=\mathbb{R}$

$k'(x) = \dfrac{-(x^2+4)-2x(-x)}{(x^2+4)^2}$

$= \dfrac{x^2-4}{(x^2+4)^2}$

x	$-\infty$		-2		2		$+\infty$
$k'(x)$		$+$	0	$-$	0	$+$	
$k(x)$	0	↗	$\frac{1}{4}$	↘	$-\frac{1}{4}$	↗	0

9. [Calculator] The Profit P in dollars made by a fast food restaurant selling x hamburgers is given by

$$P = 2.44x - \frac{x^2}{20000} - 5000, \quad 0 \le x \le 35000.$$

a) Find the intervals on which P is increasing or decreasing. [Use Calculus!]

$P'(x) = 2.44 - \dfrac{x}{10000} = \dfrac{24400-x}{10000}$

x	0		24400		35000
$P'(x)$		$+$	0	$-$	
$P(x)$	-5000	↗	24768	↘	19150

b) Find the maximum profit.

P_{max} : 24768

10. Find *all* possible functions f with the given derivative.

a) $f'(x)=x$

$f(x)=\frac{1}{2}x^2+k$

$k\in\mathbb{R}$

b) $f'(x)=3x^2-2x+1$

$f(x)=x^3-x^2+x+k$

$k\in\mathbb{R}$

c) $f'(x)=e^x$

$f(x)=e^x+k$

$k\in\mathbb{R}$

11. If you know that the acceleration of gravity is $-32\,{}^{ft}/_{s^2}$, for an falling object, we could write the acceleration of the object at time t as $a(t)=-32$.

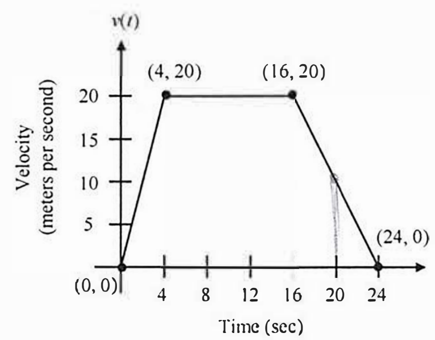
a) Find a function for the velocity of the object at time t. What does the constant equal (in words)?

$v(t)=-32t+k$ k is the original velocity (V_0)

b) Find a function for the position of the object at time t. What does the constant equal (in words)?

$p(t)=-16t^2+V_0t+k$ k is the original position at t=0

12. A car is traveling on a straight road. For $0\le t\le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph below.



a) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

$v'(4)$ DNE (corner)

$v'(20)=-2.5\ m/s^2$

b) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0<t<24$, write a piecewise-defined function for $a(t)$.

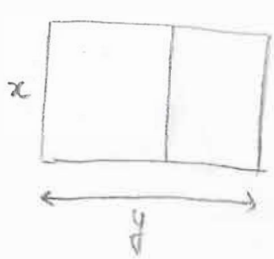
$$a(t)=v'(t)=\begin{cases}5 \text{ if } 0<x<4\\ 0 \text{ if } 4<x<16\\ -2.5 \text{ if } 16<x\le 24\end{cases}$$

c) Find the average rate of change of v over the interval $8\le t\le 20$. Does the Mean Value Theorem guarantee a value of c, for $8<c<20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

$$\frac{v(20)-v(8)}{20-8}=\frac{10-20}{12}=-\frac{5}{6}$$

v is not differentiable over $(8,20)$ so MVT doesn't guarantee it...

13. A 216-m^2 pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?



$216=xy$

$\hookrightarrow y=\frac{216}{x}$

$l=3x+2y$

$\Rightarrow l=3x+\frac{432}{x}$

$l'=3-\frac{432}{x^2}$

$l'=\frac{3(x^2-144)}{x^2}$

$l'=\frac{3(x+12)(x-12)}{x^2}$

x	0		12		$+\infty$
$l'(x)$		—	0	+	
$l(x)$	$+\infty$	$\searrow$	72	$\nearrow$	$+\infty$

min at 12 because l' changes from neg to positive at 12.

$\Rightarrow$ dim: 12m by 18m $\leftarrow\left(\frac{216}{12}\right)$

$\Rightarrow$ fence: $l=72\,m$

AP Calculus
4.3 Worksheet

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1. Complete each statement with the correct word.

   a) When f' is ___positive___, the graph of f is increasing,.

   b) When f' is ___negative___, the graph of f is decreasing,.

   c) When f'' is ___positive___, the graph of f is concave upward.

   d) When f'' is ___negative___, the graph of f is concave downward.

   e) When f' is ___increasing___, the graph of f is concave upward.

   f) When f' is ___decreasing___, the graph of f is concave downward.

2. Use the function $y = 3x - x^3 + 5$. [No calculator allowed]

   a) Where is the function increasing? Justify your response.

   $$\frac{dy}{dx} = -3x^2 + 3 = -3(x^2 - 1) = -3(x+1)(x-1)$$

x	$-\infty$		-1		1		$+\infty$
y'		$-$	0	$+$	0	$-$	
y		↘		↗		↘	

   increasing on $[-1, 1]$ because the derivative is positive on $(-1, 1)$

   b) Where is the function decreasing? Justify your response.

   decreasing on $(-\infty, -1]$ and on $[1; +\infty)$ because its derivative is negative on $(-\infty, -1)$ and on $(1, +\infty)$

   c) Where is the function concave up? Justify your response.

   $$y'' = -6x$$

x	$-\infty$		0		$+\infty$
y''		$+$	0	$-$	

   concave up on $(-\infty; 0)$ because the second derivative is positive

   concave down on $(0; +\infty)$ because the second derivative is negative

   d) Where is the function concave down? Justify your response.

   e) Where are the point(s) of inflection? Justify your response.

   $I(0; 5)$ that's where the second derivative changes signs.

   f) Find ALL extrema and justify your response.

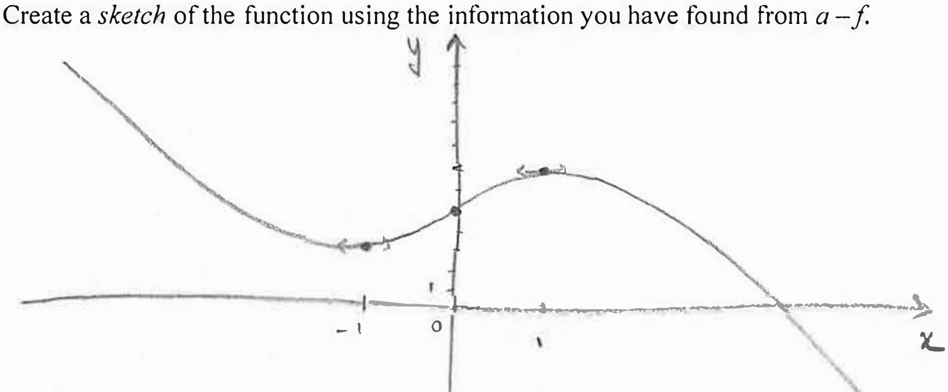
   $\lim_{x \to -\infty} y = +\infty$ $\quad f(-1) = 3$

   $\lim_{x \to +\infty} y = -\infty$ $\quad f(1) = 7$

   NO absolute MIN or absolute MAX

   local min : $(-1; 3)$ $\qquad$ local max : $(1; 7)$

   g) Create a *sketch* of the function using the information you have found from $a - f$.



3. Find all local extrema and justify your response for each function:

a) $y = -2x^3 + 6x^2 - 3$

$y' = -6x^2 + 12x$

$= -6x(x-2)$

x	$-\infty$		0		2		$+\infty$
y'		$-$	0	$+$	0	$-$	
y	$+\infty$	$\searrow$	-3	$\nearrow$	5	$\searrow$	$-\infty$

no abs MIN

no abs MAX

local min: $(0;-3)$

local max: $(2;5)$

because the derivative changes sign from neg. to positive

b) $y = xe^{1/x}$

$y' = e^{1/x} - \frac{1}{x^2} \cdot x \cdot e^{1/x}$

$y' = e^{1/x}\left(1 - \frac{1}{x}\right)$

$y' = e^{1/x} \cdot \frac{x-1}{x}$

x	$-\infty$		0		1		$+\infty$
y'		$+$	$\|$	$-$	0	$+$	
y	$-\infty$	$\nearrow$	$0 \,\|\, +\infty$	$\searrow$	e	$\nearrow$	$+\infty$

no abs MIN

no abs MAX

local min $(1;e)$

no local max

because the derivative changes sign from negative to positive.

4. Determine the intervals on which the graph of each function is concave up or concave down and determine all points of inflection. Justify your responses.

a) $y = \frac{1}{20}x^5 + \frac{1}{4}x^4 - \frac{3}{2}x^3 - \frac{27}{2}x^2 + x - 4$

$y' = \frac{1}{4}x^4 + x^3 - \frac{9}{2}x^2 - 27x + 1$

$y'' = x^3 + 3x^2 - 9x - 27$

$= (x-3)(x+3)^2$

x	$-\infty$		-3		3		$+\infty$
y''		$-$	0	$-$	0	$+$	

concave down on $(-\infty; 3)$

concave up on $(3;+\infty)$

because... pt of inflex°: $(3; -\frac{653}{5})$

b) $y = 2x^{1/5} + 3$

$y' = \frac{2}{5}x^{-4/5}$

$y'' = -\frac{8}{25}x^{-9/5}$

x	$-\infty$		0		$+\infty$
y''		$+$	$\|$	$-$	

concave up on $(-\infty;0)$

concave down on $(0;+\infty)$

point of inflection: $(0;3)$

because ...

5. If f is continuous on [0, 3] and satisfies the following:

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$	3
$f(x)$	0	+	2	+	0	–	–2
$f'(x)$	3	+	0	–	DNE	–	–3
$f''(x)$	0	–	–1	–	DNE	–	0

concave down

a) Find the absolute extrema of f and where they occur. Justify your response.

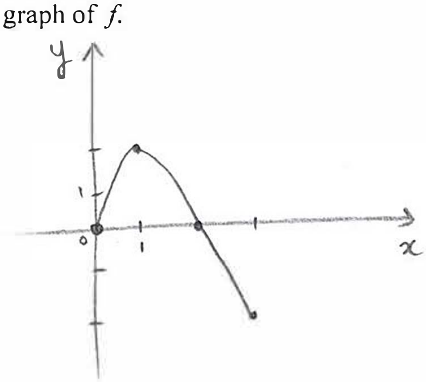
absolute MAX: $(1;2)$ local min: $(0;0)$

absolute MIN: $(3;-2)$

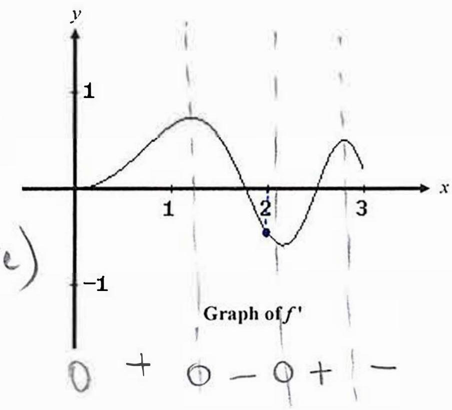
b) Find any points of inflection. Justify your response.

none: the second derivative never changes sign.

c) Sketch a possible graph of f.



7. [Calculator Required] Let f be a function defined for $x \geq 0$ with $f(0) = 5$ and f', the first derivative of f, given by $f'(x) = e^{(-x/4)} \sin(x^2)$. The graph of $y = f'(x)$ is shown below.



a) Use the graph of f' to determine whether the graph of f is concave up, concave down, or neither on the interval $1.7 < x < 1.9$. Explain your reasoning.

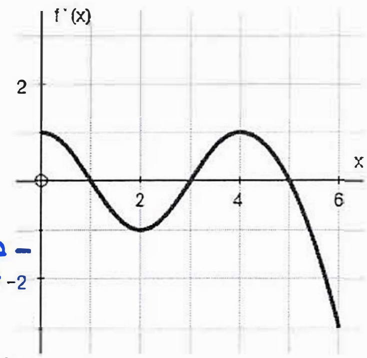
concave down because f' is decreasing (therefore, f'' is negative)

b) Write an equation for the line tangent to the graph of f at the point (2, 5.623).

$f'(2) = e^{-\frac{1}{2}} \cdot \sin 4$

$y - 5.623 = e^{-\frac{1}{2}} \cdot \sin 4 \, (x - 2)$

8. Use the graph of $f'(x)$ defined on [0, 6] provided below to estimate the following:



a) When is f increasing? When is f decreasing? Justify your response.

incr : on $[0;1]$ and on $[3;5]$ because f' is positive on $(0,1)$ and on $(3,5)$

decr : on $[1;3]$ and on $[5;6]$ because ...

b) Determine the x-coordinates of all local extrema. Justify your response.

- at 1 : local max because f' changes sign from + to −
- at 3 : — min — to +
- at 5 : local max

c) Determine when f is concave up and concave down. Justify your response.

x	0		2		4		6
f''	0	−	0	+	0	−	

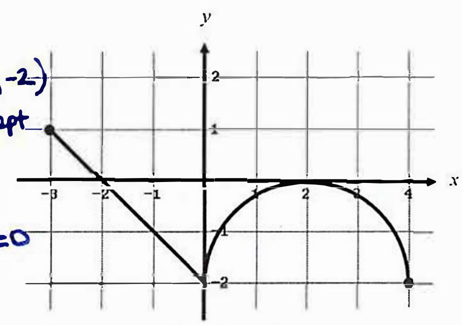
f is concave up on $[2;4]$

f is concave down on $[0;2]$ and $[4;6]$

d) Determine whether f has any points of inflection. Justify your response.

points of inflexion at 2 and at 4 because f'' changes sign.

9. [No Calculator Allowed] Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown below.



Graph of f'

a) On what intervals, if any, is f increasing? Decreasing? Justify your answer.

increasing on $[-3;-2]$ because $f'(x)>0$ on $(-3,-2)$

decreasing on $[-2;4]$ because $f'(x)<0$ on $(-2,4)$ except at 2 where it is 0.

b) Find all values of x for which f assumes a relative maximum. Justify your answer.

max when $x=-2$ because $f'(-2)=0$ and changes sign from positive to negative.

c) Where is the graph of f concave up? concave down? Justify your answers.

x	−3		0		2		4
f''		−	‖	+	0	−	

concave up on $[0;2]$ because f' is increasing on $[0,2]$

concave down on $[-3;0]$ and on $[2;4]$ because ... decreasing ...

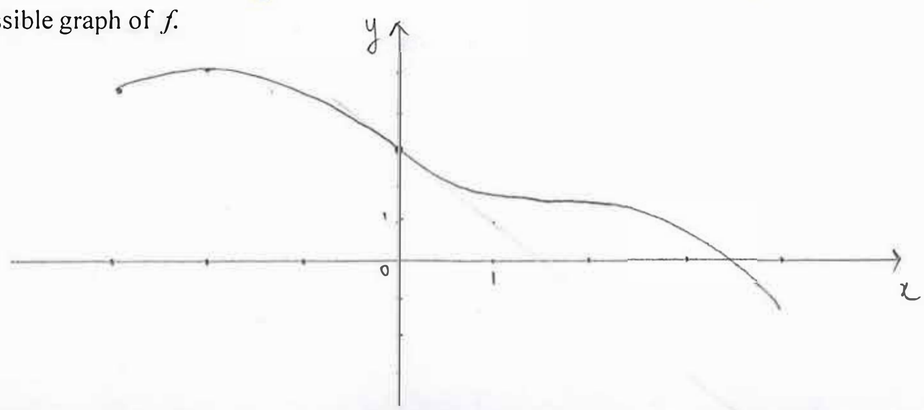
d) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

points of inflection for $x=0$ and $x=2$

e) Find an equation for the line tangent to the graph of f at the point (0, 3).

$f'(0)=-2$ $\qquad y-3=-2(x) \Rightarrow y=-2x+3$

f) Sketch a possible graph of f.



10. If g is a differentiable function such that $g(x) < 0$ for all real numbers x, and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true?

	$-\infty$		-3		3		$+\infty$
f'		$-$	0	$+$	0	$-$	
f		↘		↗		↘	

~~A)~~ f has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

B) f has a relative minimum at $x = -3$ and a relative maximum at $x = 3$.

~~C)~~ f has relative minima at $x = -3$ and at $x = 3$.

~~D)~~ f has relative maxima at $x = -3$ and at $x = 3$.

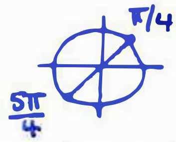
~~E)~~ It cannot be determined if f has any relative extrema.

11. Suppose that at any time t (sec) the current I (amp) in an alternating current circuit is $I = 2\cos t + 2\sin t$. What is the peak (largest magnitude) current for this circuit? Justify your response.

$I' = -2\sin t + 2\cos t$ period : 2π

critical points: $-2\sin t + 2\cos t = 0$

$\tan t = 1$



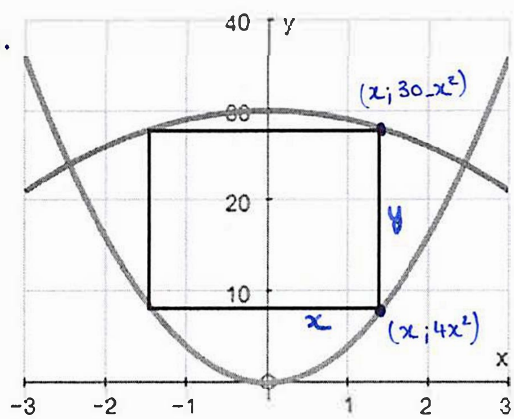
t	0		$\pi/4$		$5\pi/4$		2π
I'		$+$	0	$-$	0	$+$	
I	2	↗	$2\sqrt{2}$	↘	$-2\sqrt{2}$	↗	2

largest magnitude: $2\sqrt{2}$

4.4

12. A rectangle is inscribed between the parabolas $y = 4x^2$ and $y = 30 - x^2$ as shown in the picture. What is the maximum area of such a rectangle? Justify your response.

- There is a symmetry about the y-axis.



- $A = 2x \cdot y$

$A = 2x(30 - x^2 - 4x^2)$

$A = -10x^3 + 60x$

- Domain : intersection of 2 parabolas:

$4x^2 = 30 - x^2$

$5x^2 = 30$

$x^2 = 6$

$x = \pm\sqrt{6}$

$D = [0; \sqrt{6}]$

x	0		$\sqrt{2}$		$\sqrt{6}$
A'		$+$	0	$-$	
A		↗		↘	

- $A' = -30x^2 + 60$

critical points: $-30x^2 + 60 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

maximum area when $x = \sqrt{2}$

$A_{max} = -20\sqrt{2} + 60\sqrt{2}$

$\boxed{A_{max} = 40\sqrt{2}}$

AP Calculus
4.5 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Consider the function $y = \sin x$.

a) Find the equation of the tangent line when $x = 0$.

$y' = \cos x$ point: $(0,0)$

$y'|_{x=0} = 1$ $\Rightarrow \boxed{y = x}$

b) Graph both equations on your calculator in a standard viewing window.
When would the tangent line be a good approximation for the curve? (Try zooming in at the origin)

$[-0.5 ; 0.5]$?

c) Use the tangent line to approximate sin (0.2).
[Why is using the tangent line from part *a* a good approximation?]

$y \approx 0.2$

2. The approximate value of $y = \sqrt{4 + \sin x}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$, is
[Why does the question ask you to use the tangent line at $x = 0$?]

A 2.00
B 2.03
C 2.06
D 2.12
E 2.24

$y' = \dfrac{\cos x}{2\sqrt{4+\sin x}}$

$y'|_{x=0} = \frac{1}{4}$

point: $(0, 2)$

$\Rightarrow y - 2 = \frac{1}{4}x$

$y = \frac{1}{4}x + 2$

$\Rightarrow y \simeq \frac{1}{4}(0.12) + 2$ $y \simeq 2.03$

3. Use linearization to approximate $f(5.02)$ if $f(x) = -\dfrac{1}{\sqrt{4+x}}$. Find the error for your approximation.

$f'(x) = -\frac{1}{2}(4+x)^{-3/2}$

$f'(5) = -\frac{1}{2}(9)^{-3/2}$

$= -\dfrac{1}{2\sqrt{9}^3}$

$= -\frac{1}{54}$

point: $(5 ; \frac{1}{3})$

$\Rightarrow L(x) = -\frac{1}{54}(x-5) + \frac{1}{3}$

$f(5.02) \simeq L(5.02)$

$\simeq -0.332963$ $|\text{Error}| < 10^{-6}$

4. Suppose you were asked to determine the value of $(2.003)^4$.

a) Use linearization to approximate the value of $(2.003)^4$.

$f(x) = x^4$
$f'(x) = 4x^3$
$f'(2) = 32$
point: $(2;16)$

$L(x) = 32(x-2)+16$
$L(x) = 32x - 48$

$f(2.003) \simeq L(2.003) = 16.096$

$(2.003)^4 \simeq 16.096$

~~b) Use differentials to approximate the value of $(2.003)^4$~~

5. [With calculator] Let f be the function given by $f(x) = x^2 - 2x + 3$. The tangent line to the graph of f at $x = 2$ is used to approximate the values of $f(x)$. Which of the following is the greatest value for which the error resulting from this tangent line approximation is less than 0.5?

A 2.4

B 2.5

C 2.6

D 2.7

E 2.8

$f'(x) = 2x - 2$
$f'(2) = 2$
point: $(2;3)$

$L(x) = 2(x-2)+3$
$L(x) = 2x - 1$

$|Error| = |x^2 - 2x + 3 - 2x + 1|$
$= |x^2 - 4x + 4| = (x-2)^2$

$(x-2)^2 < 0.5$

$-\sqrt{0.5} < x - 2 < \sqrt{0.5}$
$2 - \sqrt{0.5} < x < 2 + \sqrt{0.5}$
≈ 2.71

~~6. Find the differential dy when $dx = -0.2$ and $x = 1$, if $y = x^2 e^x$. Explain what you've found.~~

~~7. If $y = \sin(x^2 - 3)$, find dy if $x = \sqrt{3}$ and $dx = \frac{1}{10}$. Explain what you've found.~~

AP Calculus
4.6 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

~~OK … I couldn't find a decent looney tunes picture for the next problem, so I thought I'd just throw in this cartoon (which by the way has nothing to do with related rates!) since I found it looking for any other good pictures. Besides, poor Wile E. Coyote has been working so much this year, it's about time he finally got a good meal. ☺~~

1. The radius r and area A of a circle are related by the equation: $A = \pi r^2$

Write an equation that relates $\frac{dA}{dt}$ and $\frac{dr}{dt}$.

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

2. A spherical container is deflated such that its volume is decreasing at a constant rate of 3141cm³/min. [The Surface area of a sphere is $S = 4\pi r^2$ The volume of a sphere is $V = \frac{4}{3}\pi r^3$]

$$\frac{dV}{dt} = -3141$$

a) At what rate is the radius changing when the radius is 5 cm? Indicate units of measure.

$$\left.\frac{dr}{dt}\right|_{r=5} ?$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$-3141 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = -\frac{3141}{4\pi r^2}$$

$$\boxed{\left.\frac{dr}{dt}\right|_{r=5} = -\frac{3141}{100\pi} \text{ cm/min}}$$

b) At that same moment, how fast is the Surface Area changing? Indicate units of measure.

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

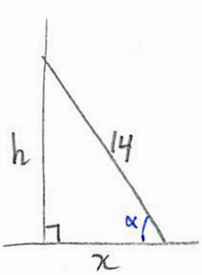
$$\left.\frac{dS}{dt}\right|_{r=5} = 8\cancel{\pi} \times 5 \times \left(-\frac{3141}{100\cancel{\pi}}\right)$$

$$= \boxed{-\frac{6282}{5} \text{ cm}^2/\text{min}}$$

The surface area is decreasing by 1256.4 cm² per min.

3. A 14 ft ladder is leaning against a wall. The top of the ladder is slipping down the wall at a rate of 2 ft/s.

$$\frac{dh}{dt} = -2$$



a) How fast will the end of the ladder be moving away from the wall when the top is 6 ft above the ground? Indicate units of measure.

$$\left.\frac{dx}{dt}\right|_{h=6} ?$$

$$h^2 + x^2 = 14^2$$

$$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{h}{x} \cdot \frac{dh}{dt}$$

when $h=6$: $x^2 + h^2 = 14^2$

$x^2 = 160$

$x = \pm\sqrt{160}$

$$\left.\frac{dx}{dt}\right|_{h=6} = -\frac{6}{\sqrt{160}} \cdot (-2)$$

$$= \frac{12}{4\sqrt{10}}$$

$$\boxed{\simeq 0.95 \text{ ft/s}}$$

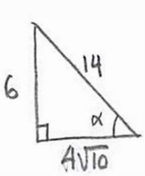
b) At the same moment, how fast is the angle between the ground and the ladder changing?

$$\sin\alpha = \frac{h}{14}$$

$$\to \cos\alpha \cdot \frac{d\alpha}{dt} = \frac{1}{14}\frac{dh}{dt}$$

when $h=6$: $\sin\alpha = \frac{6}{14}$

$$\cos\alpha = \frac{2\sqrt{10}}{7}$$



$$\left.\frac{d\alpha}{dt}\right|_{h=6} = \frac{1}{14} \cdot (-2) \cdot \frac{7}{2\sqrt{10}}$$

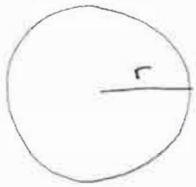
$$= \boxed{-\frac{1}{2\sqrt{10}} \text{ rad/s}}$$

$$\simeq -0.158 \text{ rad/s}$$

4. A pebble is dropped into a still pool and sends out a circular ripple whose radius increases at a constant rate of 4 ft/s. How fast is the area of the region enclosed by the ripple increasing at the end of 8 s? Indicate units of measure.

$\frac{dr}{dt} = 4$

$\left.\frac{dA}{dt}\right|_{t=8}$?



- $A = \pi r^2$

$\rightarrow \frac{dA}{dt} = 2\pi r . \frac{dr}{dt}$

- when $t = 8$:

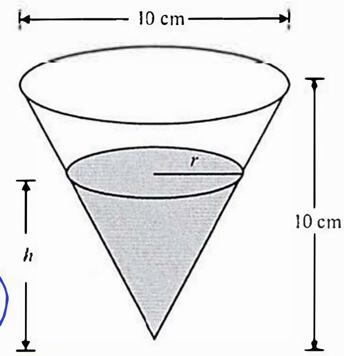
$r = 32$ ft

$\rightarrow \left.\frac{dA}{dt}\right|_{t=8} = 2\pi \times 32 \times 4$

$= \boxed{256\pi \text{ ft}^2/s}$

5. A container has the shape of an open right circular cone, as shown in the figure to the right. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr. $\frac{dh}{dt} = -\frac{3}{10}$

(The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)



a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

$V = \frac{1}{3}\pi r^2 h$

- when $h = 5$: $\underline{r = 2.5}$ $\left(\frac{h}{10} = \frac{r}{5} : \text{similar tr}\right)$

- $V_{|h=5} = \frac{1}{3}\pi (2.5)^2 \times 5$ $\boxed{V_{|h=5} = \frac{125\pi}{12} \text{ cm}^3}$

b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

$\frac{dV}{dt} = \frac{1}{3}\pi \left(2r . h \frac{dr}{dt} + r^2 \frac{dh}{dt}\right)$

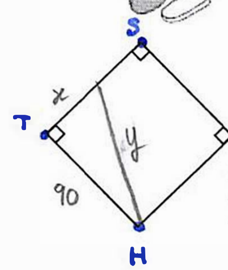
$r = \frac{h}{2}$

$\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt} = -\frac{3}{20}$

$\boxed{\left.\frac{dV}{dt}\right|_{h=5} = -\frac{15\pi}{8} \text{ cm}^3/h}$

$\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{3}\pi \left(2 \times 2.5 \times 5 \times \left(-\frac{3}{20}\right) + 2.5^2 \times \left(-\frac{3}{10}\right)\right)$

6. A baseball diamond has the shape of a square with sides 90 feet long. Tweety is just flying around the bases, running from 2nd base (top of the diamond) to third base (left side of diamond) at a speed of 28 feet per second. When Tweety is 30 feet from third base, at what rate is Tweety's distance from home plate (bottom of diamond) changing? Indicate units of measure.



$\frac{dx}{dt} = -28$

$\left.\frac{dy}{dt}\right|_{x=30}$?

$y^2 = x^2 + 90^2$

$\rightarrow 2y\frac{dy}{dt} = 2x . \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{x}{y} . \frac{dx}{dt}$

- when $x = 30$:

$y^2 = 30^2 + 90^2$

$y^2 = 9000$

$y = 30\sqrt{10}$

$\Rightarrow \left.\frac{dy}{dt}\right|_{x=30} = \frac{30}{30\sqrt{10}} . (-28)$

$\left.\frac{dy}{dt}\right|_{x=30} = \boxed{-\frac{28}{\sqrt{10}} \text{ ft/s}}$

$\simeq -8.85$ ft/s.

7. The radius r, height h, and volume V of a right circular cylinder are related by the equation $V = \pi r^2 h$.

a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant?

$$\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt}$$

b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if h is constant?

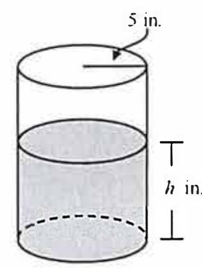
$$\frac{dV}{dt} = \pi h \cdot 2r \cdot \frac{dr}{dt}$$

c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither r nor h is constant?

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

8. A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure to the right. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.



$$\frac{dV}{dt} = -5\pi\sqrt{h}$$

$$V = 25\pi h$$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt}$$

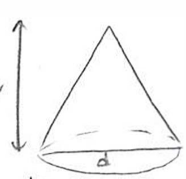
$$-5\pi\sqrt{h} = 25\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{5\pi\sqrt{h}}{25\pi}$$

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{5} \checkmark$$

9. Sand pours out of a chute into a conical pile whose height is always one half its diameter. If the height increases at a constant rate of 4 ft/min, at what rate is sand pouring from the chute when the pile is 15 ft high? Indicate units of measure.

$\frac{1}{2}d = r = h$



$\frac{dh}{dt} = 4 \Rightarrow \frac{dr}{dt} = 4$

$\left.\frac{dV}{dt}\right|_{h=15}$?

when $h=15$, $r=15$

$$V = \frac{1}{3}\pi r^2 h$$

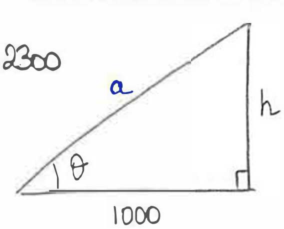
$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

$$\left.\frac{dV}{dt}\right|_{h=15} = \frac{1}{3}\pi\left(2 \times 15^2 \times 4 + 15^2 \times 4\right)$$

$$\boxed{\left.\frac{dV}{dt}\right|_{h=15} = 900\pi \text{ ft}^3/\text{min}}$$

10. A camera man is standing 1000 feet from the launch of a rocket. As the rocket launches, the camera man must change the angle of elevation of his camera to keep the rocket in the camera's view. How fast is the angle of elevation changing when the rocket is 1 mile (5280 feet) in the air if the rocket is moving at 2300 feet per second?

$\frac{dh}{dt} = 2300$



$\left.\frac{d\theta}{dt}\right|_{h=5280}$

$$\tan\theta = \frac{h}{1000}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{1000} \cdot \frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{1000} \cdot \frac{dh}{dt}$$

when $h = 5280$, $a = \sqrt{28878400}$

$$\cos\theta = \frac{1000}{\sqrt{28878400}}$$

$$\left.\frac{d\theta}{dt}\right|_{h=5280} = \frac{1000}{28878400} \cdot 2300$$

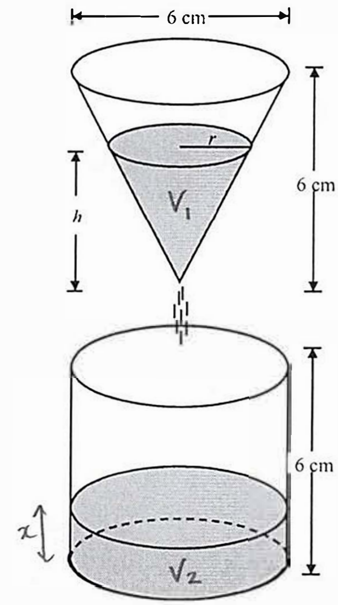
$$\boxed{\simeq 0.08 \text{ rad/s}}$$

11. Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.

$\frac{dV_1}{dt} = -10$

a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

$\left.\frac{dx}{dt}\right|_{h=5}$?



$V_1 = \frac{1}{3}\pi r^2 h \quad V_2 = 9\pi x \quad \frac{dV_2}{dt} = 10 \quad \left.\frac{dx}{dt}\right|_{h=5}$?

$\frac{dV_2}{dt} = 9\pi \frac{dx}{dt}$

$\boxed{\frac{dx}{dt} = \frac{10}{9\pi} \text{ in/min}}$

b) How fast is the level in the cone falling at that moment?

$\frac{dV_1}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) \qquad \frac{r}{3} = \frac{h}{6}$

$\left.\frac{dV_1}{dt}\right|_{h=5} = \frac{1}{3}\pi\left(2 \times \frac{5}{2} \times 5 \times \frac{1}{2}\frac{dh}{dt} + \frac{25}{4} \times \frac{dh}{dt}\right)_{h=5} \qquad r = \frac{1}{2}h$

$-10 = \frac{1}{3}\pi \times \frac{75}{4} \times \left.\frac{dh}{dt}\right|_{h=5}$

$\left.\frac{dh}{dt}\right|_{h=5} = \frac{-40}{25\pi}$

$\boxed{\left.\frac{dh}{dt}\right|_{h=5} = -\frac{8}{5\pi} \text{ in/min}}$