81. Continued

(e) $v(t)=3t^2-2>0$

$3t^2>2$

$t>\frac{\sqrt{6}}{2}$

82. (a) $\frac{d}{dx}e^u=e^u\frac{du}{dx}$ where $u=x$

$\frac{d}{dx}\frac{e^x+e^{-x}}{2}=\frac{e^x-e^{-x}}{2}$

(b) $\frac{d}{dx}\frac{e^x-e^{-x}}{2}=\frac{e^x+e^{-x}}{2}$

(c) $y(1)=\frac{e^1+e^{-1}}{2}=1.543$

$y'(1)=\frac{e^1-e^{-1}}{2}=1.175$

$y=1.175(x-1)+1.543$

$y=1.175x+0.368$

(d) $m_2=-\frac{1}{m_1}=-\frac{1}{1.175}=-0.851$

$y=-0.851(x-1)+1.543$

$y=-0.851x+2.394$

(e) $y'=0=\frac{e^x-e^{-x}}{2}$

$0=e^x-e^{-x}$

$e^x=e^{-x}$

$x=-x$ or $x=0$

83. (a) $1-x^2>0$

$x^2>1, -1<x<1$

(b) $f'(x)=\frac{d}{dx}\ln(1-x^2)$ $\quad u=1-x^2$

$\frac{d}{dx}\ln(u)=\frac{1}{u}\frac{du}{dx}$ $\quad \frac{du}{dx}=-2x$

$=\frac{-2x}{(1-x^2)}$

(c) $1-x^2>0, -1<x<1$

(d) $y'\left(\frac{1}{2}\right)=\frac{-2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}=-\frac{1}{3/4}=-4/3$

Chapter 4
Applications of Derivatives

Section 4.1 Extreme Values of Functions (pp. 187–195)

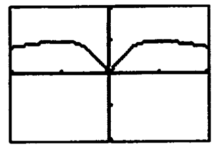
Exploration 1 Finding Extreme Values

1. From the graph we can see that there are three critical points: $x=-1, 0, 1$.

Critical point values: $f(-1)=0.5$, $f(0)=0$, $f(1)=0.5$

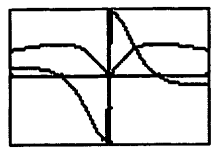
Endpoint values: $f(-2)=0.4$, $f(2)=0.4$

Thus f has absolute maximum value of 0.5 at $x=-1$ and $x=1$, absolute minimum value of 0 at $x=0$, and local minimum value of 0.4 at $x=-2$ and $x=2$.



[−2, 2] by [−1, 1]

2. The graph of f' has zeros at $x=-1$ and $x=1$ where the graph of f has local extreme values. The graph of f' is not defined at $x=0$, another extreme value of the graph of f.



[−2, 2] by [−1, 1]

3. Using the chain rule and $\frac{d}{dx}(|x|)=\frac{|x|}{x}$, we find

$\frac{df}{dx}=\frac{|x|}{x}\cdot\frac{1-x^2}{(x^2+1)^2}$.

Quick Review 4.1

1. $f'(x)=\frac{1}{2\sqrt{4-x}}\cdot\frac{d}{dx}(4-x)=\frac{-1}{2\sqrt{4-x}}$

2. $f'(x)=\frac{d}{dx}2(9-x^2)^{-1/2}=-(9-x^2)^{-3/2}\cdot\frac{d}{dx}(9-x^2)$

$=-(9-x^2)^{-3/2}(-2x)=\frac{2x}{(9-x^2)^{3/2}}$

3. $g'(x)=-\sin(\ln x)\cdot\frac{d}{dx}\ln x=-\frac{\sin(\ln x)}{x}$

4. $h'(x)=e^{2x}\cdot\frac{d}{dx}2x=2e^{2x}$

5. Graph (c), since this is the only graph that has positive slope at *c*.

6. Graph (b), since this is the only graph that represents a differentiable function at *a* and *b* and has negative slope at c.

7. Graph (d), since this is the only graph representing a function that is differentiable at *b* but not at *a*.

8. Graph (a), since this is the only graph that represents a function that is not differentiable at *a* or *b*.

9. As $x\to3^-$, $\sqrt{9-x^2}\to0^+$. Therefore, $\lim_{x\to3^-}f(x)=\infty$.

10. As $x\to3^+$, $\sqrt{9-x^2}\to0^+$. Therefore, $\lim_{x\to-3^+}f(x)=\infty$.

11. (a) $\frac{d}{dx}(x^3-2x)=3x^2-2$

$f'(1)=3(1)^2-2=1$

(b) $\frac{d}{dx}(x+2)=1$

$f'(3)=1$

(c) Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[(2+h)^3-2(2+h)]-4}{h}$$
$$=\lim_{h\to 0^-}\frac{h^3+6h^2+10h}{h}$$
$$=\lim_{h\to 0^-}(h^2+6h+10)$$
$$=10$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[(2+h)+2]-4}{h}$$
$$=\lim_{h\to 0^+}\frac{h}{h}$$
$$=\lim_{h\to 0^+}1$$
$$=1$$

Since the left-and right-hand derivatives are not equal, $f'(2)$ is underfined.

12. (a) The domain is $x\neq 2$. (See the solution for 11.(c)).

(b) $f'(x)=\begin{cases}3x^2-2, & x<2\\ 1, & x>2\end{cases}$

Section 4.1 Exercises

1. Minima at (–2, 0) and (2, 0), maximum at (0, 2)
2. Local minimum at (–1, 0), local maximum at (1, 0)
3. Maximum at (0, 5) Note that there is no minimum since the endpoint (2, 0) is excluded from the graph.
4. Local maximum at (–3, 0), local minimum at (2, 0), maximum at (1, 2), minimum at (0, –1)
5. Maximum at $x=b$, minimum at $x=c_2$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.
6. Maximum at $x=c$, minimum at $x=b$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.
7. Maximum at $x=c$, no minimum;
The Extreme Value Theorem does not apply, because the function is not defined on a closed interval.
8. No maximum, no minimum;
The Extreme Value Theorem does not apply, because the function is not continuous or defined on a closed interval.
9. Maximum at $x=c$, minimum at $x=a$;
The Extreme Value Theorem does not apply, because the function is not continuous.
10. Maximum at $x=a$, minimum at $x=c$;
The Extreme Value Theorem does not apply since the function is not continuous.
11. The first derivative $f'(x)=-\frac{1}{x^2}+\frac{1}{x}$ has a zero at $x=1$.
Critical point value: $f(1)=1+\ln 1=1$
Endpoint values: $f(0.5)=2+\ln 0.5\approx 1.307$
$f(4)=\frac{1}{4}+\ln 4\approx 1.636$
Maximum value is $\frac{1}{4}+\ln 4$ at $x=4$;
minimum value is 1 at $x=1$;
local maximum at $\left(\frac{1}{2}, 2-\ln 2\right)$
12. The first derivative $g'(x)=-e^{-x}$ has no zeros, so we need only consider the endpoints.
$g(-1)=e^{-(-1)}=e \quad g(1)=e^{-1}=\frac{1}{e}$
Maximum value is e at $x=-1$;
minimum value is $\frac{1}{e}$ at $x=1$.
13. The first derivative $h'(x)=\frac{1}{x+1}$ has no zeros, so we need only consider the endpoints.
$h(0)=\ln 1=0 \qquad h(3)=\ln 4$
Maximum value is ln 4 at $x=3$;
minimum value is 0 at $x=0$.
14. The first derivative $k'(x)=-2xe^{-x^2}$ has a zero at $x=0$.
Since the domain has no endpoints, any extreme value must occur at $x=0$. Since $k(0)=e^{-0^2}=1$ and $\lim_{x\to\pm\infty}k(x)=0$, the maximum value is 1 at $x=0$.
15. The first derivative $f'(x)=\cos\left(x+\frac{\pi}{4}\right)$, has zeros at $x=\frac{\pi}{4}$, $x=\frac{5\pi}{4}$.
Critical point values: $x=\frac{\pi}{4}$ $\quad f(x)=1$
$x=\frac{5\pi}{4}$ $\quad f(x)=-1$
Endpoint values: $x=0$ $\quad f(x)=\frac{1}{\sqrt{2}}$
$x=\frac{7\pi}{4}$ $\quad f(x)=0$
Maximum value is 1 at $x=\frac{\pi}{4}$;
minimum value is –1 at $x=\frac{5\pi}{4}$;

15. Continued

local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$;

local maximum at $\left(\frac{7\pi}{4}, 0\right)$

16. The first derivative $g'(x) = \sec x \tan x$ has zeros at $x = 0$ and $x = \pi$ and is undefined at $x = \frac{\pi}{2}$.

Since $g(x) = \sec x$ is also undefined at $x = \frac{\pi}{2}$, the critical points occur only at $x = 0$ and $x = \pi$.

Critical point values: $x = 0$ $\quad g(x) = 1$

$x = \pi$ $\quad g(x) = -1$

Since the range of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$, these values must be a local minimum and local maximum, respectively.

Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$

17. The first derivative $f'(x) = \frac{2}{5}x^{-3/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $\quad f(x) = 0$

Endpoint value: $x = -3$ $\quad f(x) = (-3)^{2/5}$

$= 3^{2/5} \approx 1.552$

Since $f(x) > 0$ for $x \neq 0$, the critical point at $x = 0$ is a local minimum, and since $f(x) \leq (-3)^{2/5}$ for $-3 \leq x < 1$, the endpoint value at $x = -3$ is a global maximum.

Maximum value is $3^{2/5}$ at $x = -3$;

minimum value is 0 at $x = 0$.

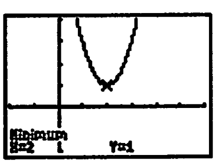
18. The first derivative $f'(x) = \frac{3}{5}x^{-2/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $\quad f(x) = 0$

Endpoint value: $x = 3$ $\quad f(x) = 3^{3/5} \approx 1.933$

Since $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$, the critical point is not a local minimum or maximum. The maximum value is $3^{3/5}$ at $x = 3$.

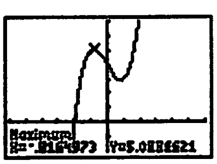
19.



$[-2, 6]$ by $[-2, 4]$

Minimum value is 1 at $x = 2$.

20.



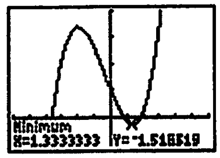
$[-6, 6]$ by $[-2, 7]$

To find the exact values, note that $y' = 3x^2 - 2$, which is zero when $x = \pm\sqrt{\frac{2}{3}}$. Local maximum at

$\left(-\sqrt{\frac{2}{3}}, 4 + \frac{4\sqrt{6}}{9}\right) \approx (-0.816, 5.089)$; local minimum at

$\left(\frac{\sqrt{2}}{3}, 4 - \frac{4\sqrt{6}}{9}\right) \approx (0.816, 2.911)$

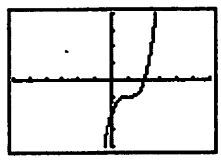
21.



$[-6, 6]$ by $[-5, 20]$

To find the exact values, note that $y' = 3x^2 + 2x - 8 = (3x - 4)(x + 2)$, which is zero when $x = -2$ or $x = \frac{4}{3}$. Local maximum at $(-2, 17)$; local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$

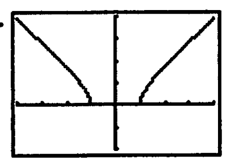
22.



$[-6, 6]$ by $[-4, 4]$

Note that $y' = 3x^2 - 6x + 3 = 3(x - 1)^2$, which is zero at $x = 1$. The graph shows that the function assumes lower values to the left and higher values to the right of this point, so the function has no local or global extreme values.

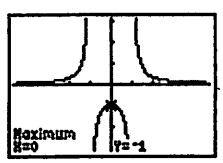
23.



$[-4, 4]$ by $[-2, 4]$

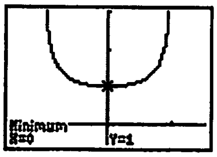
Minimum value is 0 at $x = -1$ and at $x = 1$.

24.



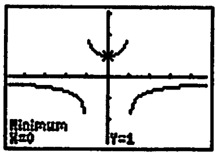
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

To confirm that there are no "hidden" extrema, note that $y' = -(x^2 - 1)^{-2}(2x) = \frac{-2x}{(x^2 - 1)^2}$ which is zero only at $x = 0$ and is undefined only where y is undefined. There is a local maximum at $(0, -1)$.

25.


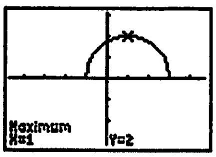
[−1.5, 1.5] by [−0.5, 3]

The minimum value is 1 at $x = 0$.

26.


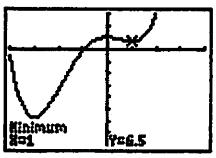
[−4.7, 4.7] by [−3.1, 3.1]

The actual graph of the function has asymptotes at $x = \pm 1$, so there are no extrema near these values. (This is an example of *grapher failure.*) There is a local minimum at (0, 1).

27.


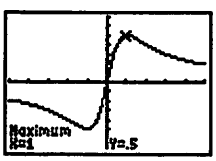
[−4.7, 4.7] by [−3.1, 3.1]

Maximum value is 2 at $x = 1$;
minimum value is 0 at $x = -1$ and at $x = 3$.

28.


[−4, 4] by [−80, 30]

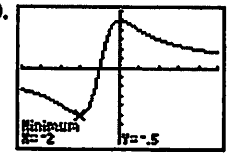
Minimum value is $-\frac{115}{2}$ at $x = -3$;
local maximum at (0, 10);
local minimum at $\left(1, \frac{13}{2}\right)$

29.


[−5, 5] by [−0.7, 0.7]

Maximum value is $\frac{1}{2}$ at $x = 1$;

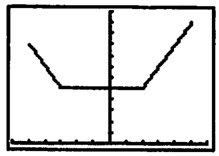
minimum value is $-\frac{1}{2}$ at $x = -1$.

30.


[−5, 5] by [−0.8, 0.6]

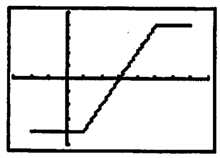
Maximum value is $\frac{1}{2}$ at $x = 0$;

minimum value is $-\frac{1}{2}$ at $x = -2$.

31.


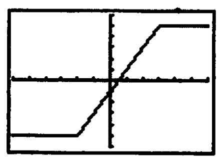
[−6, 6] by [0, 12]

Maximum value is 11 at $x = 5$;
minimum value is 5 on the interval [−3, 2];
local maximum at (−5, 9)

32.


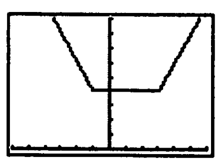
[−3, 8] by [−5, 5]

Maximum value is 4 on the interval [5, 7];
minimum value is −4 on the interval [−2, 1].

33.


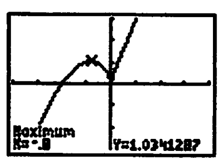
[−6, 6] by [−6, 6]

Maximum value is 5 on the interval [3, ∞);
minimum value is −5 on the interval (−∞, −2].

34.


[−6, 6] by [0, 9]

Minimum value is 4 on the interval [−1, 3]

35.


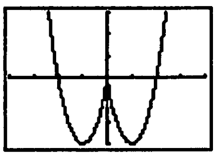
[−4, 4] by [−3, 3]

$$y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$$

35. Continued

crit. pt.	derivative	extremum	value
$x=-\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} \approx 1.034$
$x=0$	undefined	local min	0

36.

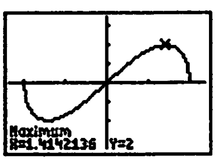


[−4, 4] by [−3, 3]

$$y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2-4) = \frac{8x^2-8}{3\sqrt[3]{x}}$$

crit. pt.	derivative	extremum	value
$x=-1$	0	minimum	−3
$x=0$	undefined	local max	0
$x=1$	0	minumum	−3

37.

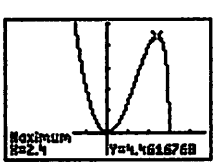


[−2.35, 2.35] by [−3.5, 3.5]

$$y' = x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) + (1)\sqrt{4-x^2}$$
$$= \frac{-x^2+(4-x^2)}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$$

crit. pt.	derivative	extremum	value
$x=-2$	undefined	local max	0
$x=-\sqrt{2}$	0	minimum	−2
$x=\sqrt{2}$	0	maximum	2
$x=2$	undefined	local min	0

38.

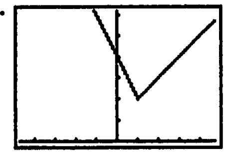


[−4.7, 4.7] by [−1, 5]

$$y = x^2 \cdot \frac{1}{2\sqrt{3-x}}(-1) + 2x\sqrt{3-x}$$
$$= \frac{-x^2+4x(3-x)}{2\sqrt{3-x}} = \frac{-5x^2+12x}{2\sqrt{3-x}}$$

crit. pt.	derivative	extremum	value
$x=0$	0	minimum	0
$x=\frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2} \approx 4.462$
$x=3$	undefined	minimum	0

39.

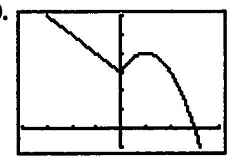


[−4.7, 4.7] by [0, 6.2]

$$y' = \begin{cases} -2, & x<1 \\ 1, & x>1 \end{cases}$$

crit. pt.	derivative	extremum	value
$x=1$	undefined	minimum	2

40.

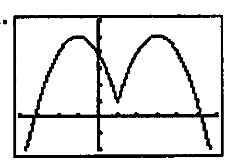


[−4, 4] by [−1, 6]

$$y' = \begin{cases} -1, & x<0 \\ 2-2x, & x>0 \end{cases}$$

crit. pt.	derivative	extremum	value
$x=0$	undefined	local min	3
$x=1$	0	local max	4

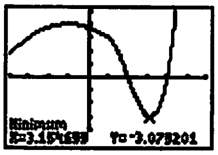
41.



[−4, 6] by [−2, 6]

$$y' = \begin{cases} -2x-2, & x<1 \\ -2x+6, & x>1 \end{cases}$$

crit. pt.	derivative	extremum	value
$x=-1$	0	maximum	5
$x=1$	undefined	local min	1
$x=3$	0	maximum	5

42. 

[–4, 6] by [–5, 5]

We begin by determining whether $f'(x)$ is defined at $x = 1$, where

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$

Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{-\frac{1}{4}(1+h)^2 - \frac{1}{2}(1+h) + \frac{15}{4} - 3}{h}$$
$$= \lim_{h\to 0^-} \frac{-h^2-4h}{4h}$$
$$= \lim_{h\to 0^-} \frac{1}{4}(-h-4)$$
$$= -1$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0^+} \frac{(1+h)^3 - 6(1+h)^2 + 8(1+h) - 3}{h}$$
$$= \lim_{h\to 0^+} \frac{h^3 - 3h^2 - h}{h}$$
$$= \lim_{h\to 0^+} (h^2 - 3h - 1)$$
$$= -1$$

Thus $f'(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2}, & x \le 1 \\ 3x^2 - 12x + 8, & x > 1 \end{cases}$

Note that $-\frac{1}{2}x - \frac{1}{2} = 0$ when $x = -1$, and

$3x^2 - 12x + 8 = 0$ when $x = \frac{12 \pm \sqrt{12^2 - 4(3)(8)}}{2(3)}$

$= \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$

But $2 - \frac{2\sqrt{3}}{3} \approx 0.845 < 1$, so the only critical points occur at

$x = -1$ and $x = 2 + \frac{2\sqrt{3}}{3} \approx 3.155.$

crit. pt.	derivative	extremum	value
$x = -1$	0	local max	4
$x \approx 3.155$	0	local max	≈ -3.079

43. (a) $V(x) = 160x - 52x^2 + 4x^3$

$V'(x) = 160 - 104x + 12x^2 = 4(x-2)(3x-20)$

The only critical point in the interval (0, 5) is at $x = 2$. The maximum value of $V(x)$ is 144 at $x = 2$.

(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x = 2$.

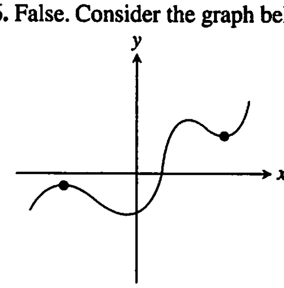
44. (a) $P'(x) = 2 - 200x^{-2}$

The only critical point in the interval $(0, \infty)$ is at $x = 10$. The minimum value of $P(x)$ is 40 at $x = 10$.

(b) The smallest possible perimeter of the rectangle is 40 units and it occurs at $x = 10$, which makes the rectangle a 10 by 10 square.

45. False. For example, the maximum could occur at a corner, where $f'(c)$ would not exist.

46. False. Consider the graph below.



47. E. $\frac{d}{dx}(4x - x^2 + 6) = 4 - 2x$

$4 - 2x = 0$

$x = 2$

$f(2) = 4(2) - (2)^2 + 6 = 10$

48. E. See Theorem 2.

49. B. $\frac{d}{dx}(x^3 - 6x + 5) = 3x^2 - 6$

$3x^2 - 6 = 0$

$x = \pm\sqrt{2}$

50. B.

51. (a) No, since $f'(x) = \frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x = 2$.

(b) The derivative is defined and nonzero for all $x \ne 2$. Also, $f(2) = 0$ and $f(x) > 0$ for all $x \ne 2$.

(c) No, $f(x)$ need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form $[a, b]$ would have both a maximum value and a minimum value on the interval.

(d) The answers are the same as (a) and (b) with 2 replaced by a.

52. Note that $f(x) = \begin{cases} -x^3 + 9x, & x \le -3 \text{ or } 0 \le x < 3 \\ x^3 - 9x, & -3 < x < 0 \text{ or } x \ge 3. \end{cases}$

Therefore, $f'(x) = \begin{cases} -3x^2 + 9, & x < -3 \text{ or } 0 < x < 3 \\ 3x^2 - 9, & -3 < x < 0 \text{ or } x > 3. \end{cases}$

(a) No, since the left- and right-hand derivatives at $x = 0$ are –9 and 9, respectively.

52. Continued

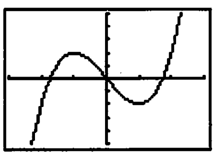
(b) No, since the left- and right-hand derivatives at $x = 3$ are -18 and 18, respectively.

(c) No, since the left- and right-hand derivatives at $x = -3$ are -18 and 18, respectively.

(d) The critical points occur when $f'(x) = 0$ (at $x = \pm\sqrt{3}$) and when $f'(x)$ is undefined (at $x = 0$ or $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $(-\sqrt{3}, 6\sqrt{3})$ and $(\sqrt{3}, 6\sqrt{3})$.

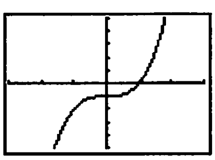
53. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of f.

Examples:



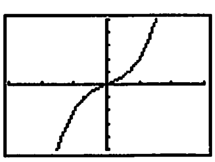
[–3, 3] by [–5, 5]

The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$.



[–3, 3] by [–5, 5]

The function $f(x) = x^3 - 1$ has one critical point at $x = 0$.



[–3, 3] by [–5, 5]

The function $f(x) = x^3 + x$ has no critical points.

(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

54. (a) By the definition of local maximum value, there is an open interval containing c where $f(x) \le f(c)$, so $f(x) - f(c) \le 0$.

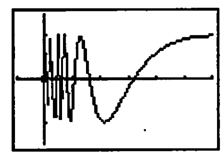
(b) Because $x \to c^+$, we have $(x - c) > 0$, and the sign of the quotient must be negative (or zero). This means the limit is nonpositive.

(c) Because $x \to c^-$, we have $(x - c) < 0$, and the sign of the quotient must be positive (or zero). This means the limit is nonnegative.

(d) Assuming that $f'(c)$ exists, the one-sided limits in (b) and (c) above must exist and be equal. Since one is nonpositive and one is nonnegative, the only possible common value is 0.

(e) There will be an open interval containing c where $f(x) - f(c) \ge 0$. The difference quotient for the left-hand derivative will have to be negative (or zero), and the difference quotient for the right-hand derivative will have to be positive (or zero). Taking the limit, the left-hand derivative will be nonpositive, and the right-hand derivative will be nonnegative. Therefore, the only possible value for $f'(c)$ is 0.

55. (a)



[–0.1, 0.6] by [–1.5, 1.5]

$f(0) = 0$ is not a local extreme value because in any open interval containing $x = 0$, there are infinitely many points where $f(x) = 1$ and where $f(x) = -1$.

(b) One possible answer, on the interval [0, 1]:

$$f(x) = \begin{cases} (1-x)\cos\dfrac{1}{1-x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

This function has no local extreme value at $x = 1$. Note that it is continuous on [0, 1].

Section 4.2 Mean Value Theorem (pp. 196–204)

Quick Review 4.2

1. $2x^2 - 6 < 0$

$2x^2 < 6$

$x^2 < 3$

$-\sqrt{3} < x < \sqrt{3}$

Interval: $(-\sqrt{3}, \sqrt{3})$

2. $3x^2 - 6 > 0$

$3x^2 > 6$

$x^2 > 2$

$x < -\sqrt{2}$ or $x > \sqrt{2}$

Intervals: $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

3. Domain: $8 - 2x^2 \ge 0$

$8 \ge 2x^2$

$4 \ge x^2$

$-2 \le x \le 2$

The domain is [–2, 2].

4. f is continuous for all x in the domain, or, in the interval [–2, 2].

5. f is differentiable for all x in the interior of its domain, or, in the interval $(-2, 2)$.

6. We require $x^2 - 1 \neq 0$, so the domain is $x \neq \pm 1$.

7. f is continuous for all x in the domain, or, for all $x \neq \pm 1$.

8. f is differentiable for all x in the domain, or, for all $x \neq \pm 1$.

9. $7 = -2(-2) + C$
$7 = 4 + C$
$C = 3$

10. $-1 = (1)^2 + 2(1) + C$
$-1 = 3 + C$
$C = -4$

Section 4.2 Exercises

1. (a) Yes.

(b) $f'(x) = \frac{d}{dx} x^2 + 2x - 1 = 2x + 2$

$2c + 2 = \frac{2-(-1)}{1-0} = 3$

$c = \frac{1}{2}.$

2. (a) Yes.

(b) $f'(x) = \frac{d}{dx} x^{2/3} = \frac{2}{3} x^{-1/3}$

$\frac{2}{3} c^{-1/3} = \frac{1-0}{1-0} = 1$

$c = \frac{8}{27}.$

3. (a) No. There is a verticle tangent at $x = 0$.

4. (a) No. There is a corner at $x = 1$.

5. (a) Yes.

(b) $f'(x) = \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\frac{1}{\sqrt{1-c^2}} = \frac{(\pi/2) - (-\pi/2)}{1-(-1)} = \frac{\pi}{2}$

$\sqrt{1-c^2} = \frac{2}{\pi}$

$c = \sqrt{1 - 4/\pi^2} \approx 0.771.$

6. (a) Yes.

(b) $f'(x) = \frac{d}{dx} \ln(x-1) = \frac{1}{x-1}$

$\frac{1}{c-1} = \frac{\ln 3 - \ln 1}{4-2}$

$c = \frac{4-2}{\ln 3 - \ln 1} + 1 \approx 2.820$

7. (a) No. The function is discontinuous at $x = \frac{\pi}{2}$

8. (a) No. The split function is discontinuous at $x = 1$

9. (a) The secant line passes through $(0.5, f(0.5)) = (0.5, 2.5)$ and $(2, f(2)) = (2, 2.5)$, so its equation is $y = 2.5$.

(b) The slope of the secant line is 0, so we need to find c such that $f'(c) = 0$.

$1 - c^{-2} = 0$
$c^{-2} = 1$
$c = 1$
$f(c) = f(1) = 2$

The tangent line has slope 0 and passes through $(1, 2)$, so its equation is $y = 2$.

10. (a) The secant line passes through $(1, f(1)) = (1, 0)$ and $(3, f(3)) = (3, \sqrt{2})$, so its slope is

$\frac{\sqrt{2}-0}{3-1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$

The equation is $y = \frac{1}{\sqrt{2}}(x-1) + 0$

or $y = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}$, or $y \approx 0.707x - 0.707$.

(b) We need to find c such that $f'(c) = \frac{1}{\sqrt{2}}$.

$\frac{1}{2\sqrt{c-1}} = \frac{1}{\sqrt{2}}$
$2\sqrt{c-1} = \sqrt{2}$
$c - 1 = \frac{1}{2}$
$c = \frac{3}{2}$
$f(c) = f\left(\frac{3}{2}\right) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

The tangent line has slope $\frac{1}{\sqrt{2}}$ and passes through $\left(\frac{3}{2}, \frac{1}{\sqrt{2}}\right)$. Its equation is $y = \frac{1}{\sqrt{2}}\left(x - \frac{3}{2}\right) + \frac{1}{\sqrt{2}}$ or

$y = \frac{1}{\sqrt{2}}x - \frac{1}{2\sqrt{2}}$, or $y \approx 0.707x - 0.354$.

11. Because the trucker's average speed was 79.5 mph, and by then Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

12. Let $f(t)$ denote the temperature indicated after t seconds. We assume that $f'(t)$ is defined and continuous for $0 \leq t \leq 20$. The average rate of change is $10.6°$ F/sec. Therefore, by the Mean Value Theorem, $f'(c) = 10.6°$ F/sec for some value of c in $[0, 20]$. Since the temperature was constant before $t = 0$, we also know that $f'(0) = 0°$ F/min. But f' is continuous, so by the Intermediate Value Theorem, the rate of change $f'(t)$ must have been $10.1°$F/sec at some moment during the interval.

13. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must been going that speed at least once during the trip.

14. The runner's average speed for the marathon was approximately 11.909 mph. Therefore, by the Mean Value Theroem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 11 mph at least twice.

15. (a) $f'(x)=5-2x$

Since $f'(x)>0$ on $\left(-\infty, \frac{5}{2}\right)$, $f'(x)=0$ at $x=\frac{5}{2}$, and $f'(x)<0$ on $\left(\frac{5}{2}, \infty\right)$, we know that $f(x)$ has a local maximum at $x=\frac{5}{2}$. Since $f\left(\frac{5}{2}\right)=\frac{25}{4}$, the local maximum occurs at the point $\left(\frac{5}{2}, \frac{25}{4}\right)$. (This is also a global maximum.)

(b) Since $f'(x)>0$ on $\left(-\infty, \frac{5}{2}\right)$, $f(x)$ is increasing on $\left(-\infty, \frac{5}{2}\right]$.

(c) Since $f'(x)<0$ on $\left(\frac{5}{2}, \infty\right)$, $f(x)$ is decreasing on $\left[\frac{5}{2}, \infty\right)$.

16. (a) $g'(x)=2x-1$

Since $g'(x)<0$ on $\left(-\infty, \frac{1}{2}\right)$, $g'(x)=0$ at $x=\frac{1}{2}$, and $g'(x)>0$ on $\left(\frac{1}{2}, \infty\right)$, we know that $g(x)$ has a local minimum at $x=\frac{1}{2}$.

Since $g\left(\frac{1}{2}\right)=-\frac{49}{4}$, the local minimum occurs at the point $\left(\frac{1}{2}, -\frac{49}{4}\right)$. (This is also a global minimum.)

(b) Since $g'(x)>0$ on $\left(\frac{1}{2}, \infty\right)$, $g(x)$ is increasing on $\left[\frac{1}{2}, \infty\right)$.

(c) Since $g'(x)<0$ on $\left(-\infty, \frac{1}{2}\right)$, $g(x)$ is decreasing on $\left(-\infty, \frac{1}{2}\right]$.

17. (a) $h'(x)=-\frac{2}{x^2}$

Since $h'(x)$ is never zero is undefined only where $h(x)$ is undefined, there are no critical points. Also, the domain $(-\infty, 0)\cup(0, \infty)$ has no endpoints. Therefore, $h(x)$ has no local extrema.

(b) Since $h'(x)$ is never positive, $h(x)$ is not increasing on any interval.

(c) Since $h'(x)<0$ on $(-\infty, 0)\cup(0, \infty)$, $h(x)$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

18. (a) $k'(x)=-\frac{2}{x^3}$

Since $k'(x)$ is never zero and is undefined only where $k(x)$ is undefined, there are no critical points. Also, the domain $(-\infty, 0)\cup(0, \infty)$ has no endpoints. Therefore, $k(x)$ has no local extrema.

(b) Since $k'(x)>0$ on $(-\infty, 0)$, $k(x)$ is increasing on $(-\infty, 0)$.

(c) Since $k'(x)<0$ on $(0, \infty)$, $k(x)$ is decreasing on $(0, \infty)$.

19. (a) $f'(x)=2e^{2x}$

Since $f'(x)$ is never zero or undefined, and the domain of $f(x)$ has no endpoints, $f(x)$ has no extrema.

(b) Since $f'(x)$ is always positive, $f(x)$ is increasing on $(-\infty, \infty)$.

(c) Since $f'(x)$ is never negative, $f(x)$ is not decreasing on any interval.

20. (a) $f'(x)=-0.5e^{-0.5x}$

Since $f'(x)$ is never zero or undefined, and the domain of $f(x)$ has no endpoints, $f(x)$ has no extrema.

(b) Since $f'(x)$ is never positive, $f(x)$ is not increasing on any interval.

(c) Since $f'(x)$ is always negative, $f(x)$ is decreasing on $(-\infty, \infty)$.

21. (a) $y'=-\frac{1}{2\sqrt{x+2}}$

In the domain $[-2, \infty)$, y' is never zero and is undefined only at the endpoint $x=-2$. The function y has a local maximum at $(-2, 4)$. (This is also a global maximum.)

(b) Since y' is never positive, y is not increasing on any interval.

(c) Since y' is negative on $(-2, \infty)$, y is decreasing on $[-2, \infty)$.

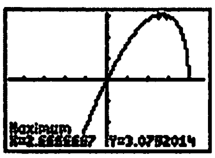
22. (a) $y' = 4x^3 - 20x = 4x(x+\sqrt{5})(x-\sqrt{5})$

The function has critical points at $x = -\sqrt{5}$, $x = 0$, and $x = \sqrt{5}$. Since $y' < 0$ on $(-\infty, -\sqrt{5})$ and $(0, \sqrt{5})$ and $y' > 0$ on $(-\sqrt{5}, 0)$ and $(\sqrt{5}, \infty)$, the points at $x = \pm\sqrt{5}$ are local minima and the point at $x = 0$ is a local maximum. Thus, the function has a local maximum at $(0, 9)$ and local minima at $(-\sqrt{5}, -16)$ and $(\sqrt{5}, -16)$. (These are also global minima.)

(b) Since $y' > 0$ on $(-\sqrt{5}, 0)$ and $(\sqrt{5}, \infty)$, y is increasing on $[-\sqrt{5}, 0]$ and $[\sqrt{5}, \infty)$.

(c) Since $y' > 0$ on $(-\infty, -\sqrt{5})$ and $(0, \sqrt{5})$, y is decreasing on $(-\infty, -\sqrt{5}]$ and $[0, \sqrt{5}]$.

23.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(a) $f'(x) = x \cdot \dfrac{1}{2\sqrt{4-x}}(-1) + \sqrt{4-x}$

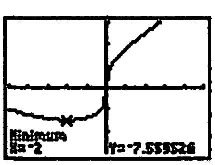
$= \dfrac{-3x+8}{2\sqrt{4-x}}$

The local extrema occur at the critical point $x = \dfrac{8}{3}$ and at the endpoint $x = 4$. There is a local (and absolute) maximum at $\left(\dfrac{8}{3}, \dfrac{16}{3\sqrt{3}}\right)$ or approximately $(2.67, 3.08)$, and a local minimum at $(4, 0)$.

(b) Since $f'(x) > 0$ on $\left(-\infty, \dfrac{8}{3}\right)$, $f(x)$ is decreasing on $\left(-\infty, \dfrac{8}{3}\right]$.

(c) Since $f'(x) < 0$ on $\left(\dfrac{8}{3}, 4\right)$, $f(x)$ is decreasing on $\left[\dfrac{8}{3}, 4\right]$.

24.



$[-5, 5]$ by $[-15, 15]$

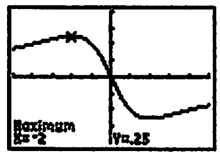
(a) $g'(x) = x^{1/3}(1) + \dfrac{1}{3}x^{-2/3}(x+8) = \dfrac{4x+8}{3x^{2/3}}$

The local extrema can occur at the critical points $x = -2$ and $x = 0$, but the graph shows that no extrema occurs at $x = 0$. There is a local (and absolute) minimum at $(-2, -6\sqrt[3]{2})$ or approximately $(-2, -7.56)$.

(b) Since $g'(x) > 0$ on the intervals $(-2, 0)$ and $(0, \infty)$, and $g(x)$ is continuous at $x = 0$, $g(x)$ is increasing on $[-2, \infty)$.

(c) Since $g'(x) < 0$ on the interval $(-\infty, -2)$, $g(x)$ is decreasing on $(-\infty, -2]$.

25.



$[-5, 5]$ by $[-0.4, 0.4]$

(a) $h'(x) = \dfrac{(x^2+4)(-1)-(-x)(2x)}{(x^2+4)^2} = \dfrac{x^2-4}{(x^2+4)^2}$

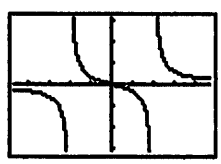
$= \dfrac{(x+2)(x-2)}{(x^2+4)^2}$

The local extrema occur at the critical points, $x = \pm 2$. There is a local (and absolute) maximum at $\left(-2, \dfrac{1}{4}\right)$ and a local (and absolute) minimum at $\left(2, -\dfrac{1}{4}\right)$.

(b) Since $h'(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $h(x)$ is increasing on $(-\infty, -2]$ and $[2, \infty)$.

(c) Since $h'(x) < 0$ on $(-2, 2)$, $h(x)$ is decreasing on $[-2, 2]$.

26.



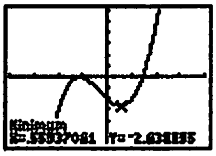
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(a) $k'(x) = \dfrac{(x^2-4)(1)-x(2x)}{(x^2-4)^2} = -\dfrac{x^2+4}{(x^2-4)^2}$

Since $k'(x)$ is never zero and is undefined only where $k(x)$ is undefined, there are no critical points. Since there are no critical points and the domain includes no endpoints, $k(x)$ has no local extrema.

(b) Since $k'(x)$ is never positive, $k(x)$ is not increasing on any interval.

(c) Since $k'(x)$ is negative wherever it is defined, $k(x)$ is decreasing on each interval of its domain; on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

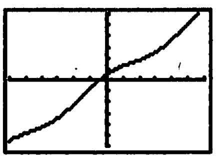
27. 

[−4, 4] by [−6, 6]

(a) $f'(x) = 3x^2 - 2 + 2\sin x$
Note that $3x^2 - 2 > 2$ for $|x| \geq 1.2$ and $|2 \sin x| \leq 2$ for all x, so $f'(x) > 0$ for $|x| \geq 1.2$. Therefore, all critical points occur in the interval (−1.2, 1.2), as suggested by the graph. Using grapher techniques, there is a local maximum at approximately (−1.126, −0.036), and a local minimum at approximately (0.559, −2.639).

(b) $f(x)$ is increasing on the intervals $(-\infty, -1.126]$ and $[0.559, \infty)$, where the interval endpoints are approximate.

(c) $f(x)$ is decreasing on the interval [−1.126, 0.559], where the interval endpoints are approximate.

28. 

[−6, 6] by [−12, 12]

(a) $g'(x) = 2 - \sin x$
Since $1 \leq g'(x) \leq 3$ for all x, there are no critical points. Since there are no critical points and the domain has no endpoints, there are no local extrema.

(b) Since $g'(x) > 0$ for all x, $g(x)$ is increasing on $(-\infty, \infty)$.

(c) Since $g'(x)$ is never negative, $g(x)$ is not decreasing on any interval.

29. $f(x) = \frac{x^2}{2} + C$

30. $f(x) = 2x + C$

31. $f(x) = x^3 - x^2 + x + C$

32. $f(x) = -\cos x + C$

33. $f(x) = e^x + C$

34. $f(x) = \ln(x-1) + C$

35. $f(x) = \frac{1}{x} + C,\ x > 0$

$f(2) = 1$

$\frac{1}{2} + C = 1$

$C = \frac{1}{2}$

$f(x) = \frac{1}{x} + \frac{1}{2},\ x > 0$

36. $f(x) = x^{1/4} + C$

$f(1) = -2$

$1^{1/4} + C = -2$

$1 + C = -2$

$C = -3$

$f(x) = x^{1/4} - 3$

37. $f(x) = \ln(x+2) + C$

$f(-1) = 3$

$\ln(-1+2) + C = 3$

$0 + C = 3$

$C = 3$

$f(x) = \ln(x+2) + 3$

38. $f(x) = x^2 + x - \sin x + C$

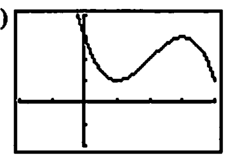
$f(0) = 3$

$0 + C = 3$

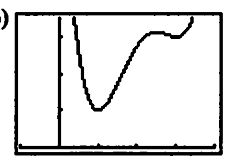
$C = 3$

$f(x) = x^2 + x - \sin x + 3$

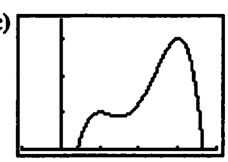
39. Possible answers:

(a) 

[−2, 4] by [−2, 4]

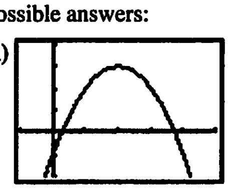
(b) 

[−1, 4] by [0, 3.5]

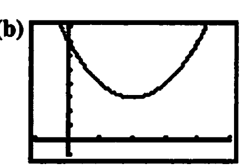
(c) 

[−1, 4] by [0, 3.5]

40. Possible answers:

(a) 

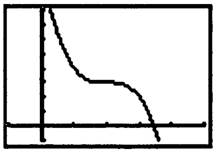
[−1, 5] by [−2, 4]

(b) 

[−1, 5] by [−1, 8]

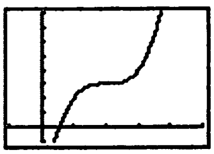
40. Continued

(c)



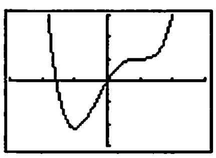
$[-1, 5]$ by $[-1, 8]$

(d)



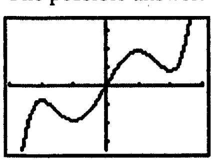
$[-1, 5]$ by $[-1, 8]$

41. One possible answer:



$[-3, 3]$ by $[-15, 15]$

42. One possible answer:



$[-3, 3]$ by $[-70, 70]$

43. **(a)** Since $v'(t) = 1.6$, $v(t) = 1.6t + C$. But $v(0) = 0$, so $C = 0$ and $v(t) = 1.6t$. Therefore, $v(30) = 1.6(30) = 48$. The rock will be going 48 m/sec.

(b) Let $s(t)$ represent position. Since $s'(t) = v(t) = 1.6t$, $s(t) = 0.8t^2 + D$. But $s(0) = 0$, so $D = 0$ and $s(t) = 0.8t^2$. Therefore, $s(30) = 0.8(30)^2 = 720$. The rock travels 720 meters in the 30 seconds it takes to hit bottom, so the bottom of the crevasse is 720 meters below the point of release.

(c) The velocity is now given by $v(t) = 1.6t + C$, where $v(0) = 4$. (Note that the sign of the initial velocity is the same as the sign used for the acceleration, since both act in a downward direction.) Therefore, $v(t) = 1.6t + 4$, and $s(t) = 0.8t^2 + 4t + D$, where $s(0) = 0$ and so $D = 0$. Using $s(t) = 0.8t^2 + 4t$ and the known crevasse depth of 720 meters, we solve $s(t) = 720$ to obtain the positive solution $t \approx 27.604$, and so $v(t) = v(27.604) = 1.6(27.604) + 4 \approx 48.166$. The rock will hit bottom after about 27.604 seconds, and it will be going about 48.166 m/sec.

44. **(a)** We assume the diving board is located at $s = 0$ and the water at $s = 0$, so that downward velocities are positive. The acceleration due to gravity is 9.8 m/sec^2, so $v'(t) = 9.8$ and $v(t) = 9.8t + C$. Since $v(0) = 0$, we have $v(t) = 9.8t$. Then the position is given by $s(t)$ where $s'(t) = v(t) = 9.8t$, so $s(t) = 4.9t^2 + D$. Since $s(0) = 0$, we have $s(t) = 4.9t^2$. Sloving $s(t) = 10$ gives $t^2 = \frac{10}{4.9} = \frac{100}{49}$, so the positive solution is $t = \frac{10}{7}$. The velocity at this time is $v\left(\frac{10}{7}\right) = 9.8\left(\frac{10}{7}\right) = 14$ m/sec.

(b) Again $v(t) = 9.8t + C$, but this time $v(0) = -2$ and so $v(t) = 9.8t - 2$. The $s'(t) = 9.8t - 2$, so $s(t) = 4.9t^2 - 2t + D$. Since $s(0) = 0$, we have $s(t) = 4.9t^2 - 2t$. Sloving $s(t) = 10$ gives the positive solution $t = \frac{2 + 10\sqrt{2}}{9.8} \approx 1.647$ sec.
The velocity at this time is
$v\left(\frac{2+10\sqrt{2}}{9.8}\right) = 9.8\left(\frac{2+10\sqrt{2}}{9.8}\right) - 2 = 10\sqrt{2}$ m/sec or about 14.142 m/sec.

45. Because the function is not continuous on [0, 1]. The function does not satisfy the hypotheses of the Mean Value Theorem, and so it need not satisfy the conclusion of the Mean Value Theorem.

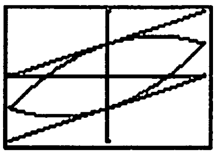
46. Because the Mean Value Theorem applies to the function $y = \sin x$ on any interval, and $y = \cos x$ is the derivative of $\sin x$. So, between any two zeros of $\sin x$, its derivative, $\cos x$, must be zero at least once.

47. $f(x)$ must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that $f(x)$ is zero twice between a and b. Then by the Mean Value Theorem, $f'(x)$ would have to be zero at least once between the two zeros of $f(x)$, but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, $f(x)$ is zero once and only once between a and b.

48. Let $f(x) = x^4 + 3x + 1$. Then $f(x)$ is continuous and differentiable everywhere. $f'(x) = 4x^3 + 3$, which is never zero between $x = -2$ and $x = -1$. Since $f(-2) = 11$ and $f(-1) = -1$, exercise 47 applies, and $f(x)$ has exactly one zero between $x = -2$ and $x = -1$.

49. Let $f(x) = x + \ln(x + 1)$. Then $f(x)$ is continuous and differentiable everywhere on [0, 3]. $f'(x) = 1 + \frac{1}{x+1}$, which is never zero on [0, 3]. Now $f(0) = 0$, so $x = 0$ is one solution of the equation. If there were a second solution, $f(x)$ would be zero twice in [0, 3], and by the Mean Value Theorem, $f'(x)$ would have to be zero somewhere between the two zeros of $f(x)$ .But this can't happen, since $f'(x)$ is never zero on [0, 3]. Therefore, $f(x) = 0$ has exactly one solution in the interval [0, 3].

50. Consider the function $k(x) = f(x) - g(x)$. $k(x)$ is continuous and differentiable on $[a, b]$, and since $k(a) = f(a) - g(a) = 0$ and $k(b) = f(b) - g(b) = 0$, by the Mean Value Theorem, there must be a point c in (a, b) where $k'(c) = 0$. But since $k'(c) = f'(c) - g'(c)$, this means that $f'(c) - g'(c)$, and c is a point where the graphs of f and g have parallel or identical tangent lines.



(–1, 1) by [–2, 2]

51. False. For example, the function x^3 is increasing on (–1, 1), but $f'(0) = 0$.

52. True. In fact, f is the increasing on $[a, b]$ by Corollary to the Mean Value Theorem.

53. A. $f'(x) = \dfrac{\frac{1}{2} - 1}{\frac{\pi}{3}} = -\dfrac{3}{2\pi}$.

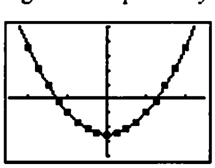
54. B. $f'(x) = \dfrac{f(4) - f(0)}{4 - 0}$
$= \dfrac{3.78 - 2980.96}{4 - 0}$
$= -744.30$, negative slope.

55. E. $\dfrac{d}{dx}(2\sqrt{x} - 10)$
$= \dfrac{2}{2\sqrt{x}} = \dfrac{1}{\sqrt{x}}$.

56. D. $x^{3/5}$ is not differentiable at $x = 0$.

57. (a) Increasing: [–2, –1.3] and [1.3, 2];
decreasing: [–1.3, 1.3];
local max: $x \approx -1.3$
local min: $x \approx 1.3$

(b) Regression equation: $y = 3x^2 - 5$



[–2.5, 2.5] by [–8, 10]

(c) Since $f'(x) = 3x^2 - 5$, we have $f(x) = x^3 - 5x + C$.
But $f(0) = 0$, so $C = 0$. Then $f(x) = x^3 - 5x$.

58. (a) Toward: $0 < t < 2$ and $5 < t < 8$; away: $2 < t < 5$

(b) A local extremum in this problem is a time/place where Priya changes the direction of her motion.

(c) Regression equation:
$y = -0.0820x^3 + 0.9163x^2 - 2.5126x + 3.3779$

[–0.5, 8.5] by [–0.5, 5]

(d) Using the unrounded values from the regression equation, we obtain $f'(t) = -0.2459t^2 + 1.8324t - 2.5126$. According to the regression equation, Priya is moving toward the motion detector when $f'(t) < 0$ ($0 < t < 1.81$ and $5.64 < t < 8$), and away from the detector when $f'(t) > 0$ ($1.81 < t < 5.64$).

59. $\dfrac{f(b) - f(a)}{b - a} = \dfrac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\dfrac{1}{ab}$

$f'(c) = -\dfrac{1}{c^2}$, so $-\dfrac{1}{c^2} = -\dfrac{1}{ab}$ and $c^2 = ab$.

Thus, $c = \sqrt{ab}$.

60. $\dfrac{f(b) - f(a)}{b - a} = \dfrac{b^2 - a^2}{b - a} = b + a$

$f'(c) = 2c$, so $2c = b + a$ and $c = \dfrac{a + b}{2}$.

61. By the Mean Value Theorem, $\sin b - \sin a = (\cos c)(b - a)$ for some c between a and b. Taking the absolute value of both sides and using $|\cos c| \le 1$ gives the result.

62. Apply the Mean Value Theorem to f on $[a, b]$.
Since $f(b) < f(a)$, $\dfrac{f(b) - f(a)}{b - a}$ is negative, and hence $f'(x)$ must be negative at some point between a and b.

63. Let $f(x)$ be a monotonic function defined on an interval D. For any two values in D, we may let x be the smaller value and let x be the larger value, so $x_1 < x_2$. Then either $f(x_1) < f(x_2)$ (if f is increasing), or $f(x_1) > f(x_2)$ (if f is decreasing), which means $f(x_1) \ne f(x_2)$. Therefore, f is one-to-one.

Section 4.3 Connecting f' and f'' with the Graph of f (pp. 205–218)

Exploration 1 Finding f from f''

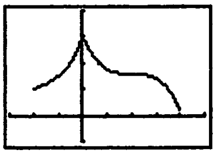
1. Any function $f(x) = x^4 - 4x^3 + C$ where C is a real number. For example, let $C = 0, 1, 2$. Their graphs are all vertical shifts of each other.

2. Their behavior is the same as the behavior of the function f of Example 8.

Exploration 2 Finding f from f' and f''

1. f has an absolute maximum at $x = 0$ and an absolute minimum of 1 at $x = 4$. We are not given enough information to determine $f(0)$.

2. f has a point of inflection at $x = 2$.

3. 

[−3, 5] by [−5, 20]

Quick Review 4.3

1. $x^2 - 9 < 0$
$(x+3)(x-3) < 0$

Intervals	$x < -3$	$-3 < x < 3$	$3 < x$
Sign of $(x+3)(x-3)$	+	−	+

Solution set: (−3, 3)

2. $x^3 - 4x > 0$
$x(x+2)(x-2) > 0$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x$
Sign of $x(x+2)(x-2)$	−	+	−	+

Solution set: $(-2, 0) \cup (2, \infty)$

3. f: all reals
f': all reals, since $f'(x) = xe^x + e^x$

4. f: all reals
f': $x \neq 0$, since $f'(x) = \frac{3}{5}x^{-2/5}$

5. f: $x \neq 2$
f': $x \neq 2$, since $f'(x) = \frac{(x-2)(1)-(x)(1)}{(x-2)^2} = \frac{-2}{(x-2)^2}$

6. f: all reals
f': $x \neq 0$, since $f'(x) = \frac{2}{5}x^{-3/5}$

7. Left end behavior model: 0
Right end behavior model: $-x^2e^x$
Horizontal asymptote: $y = 0$

8. Left end behavior model: x^2e^{-x}
Right end behavior model: 0
Horizontal asymptote: $y = 0$

9. Left end behavior model: 0
Right end behavior model: 200
Horizontal asymptote: $y - 0$, $y = 200$

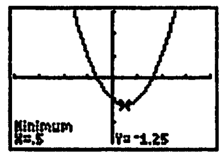
10. Left end behavior model: 0
Right end behavior model: 375
Horizontal asymptotes: $y = 0$, $y = 375$

Section 4.3 Exercises

1. $y' = 2x - 1$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y'	−	+
Behavior of y	Decreasing	Increasing

Graphical support:



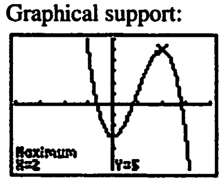
[−4, 4] by [−3, 3]

Local (and absolute) minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$.

2. $y' = -6x^2 + 12x = -6x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

Graphical support:



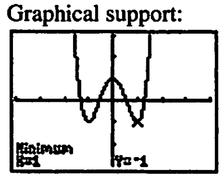
[−4, 4] by [−6, 6]

Local maximum: (2, 5);
local minimum: (0, −3)

3. $y' = 8x^3 - 8x = 8x(x-1)(x+1)$

Intervals	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	−	+	−	+
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

Graphical support:



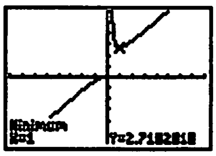
[−4, 4] by [−3, 3]

Local maximum: (0, 1);
local (and absolute) minima: (−1, −1) and (1, −1)

4. $y' = xe^{1/x}(-x^{-2}) + e^{1/x} = e^{1/x}\left(1 - \frac{1}{x}\right)$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	+	−	+
Behavior of y	Increasing	Decreasing	Increasing

Graphical support:



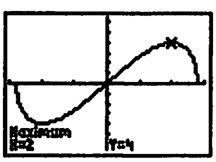
$[-8, 8]$ by $[-6, 6]$

Local minimum: $(1, e)$

5. $y' = x\frac{1}{2\sqrt{8 - x^2}}(-2x) + (\sqrt{8 - x^2})(1) = \frac{8 - 2x^2}{\sqrt{8 - x^2}}$

Intervals	$-\sqrt{8} < x < -2$	$-2 < x < 2$	$2 < x < \sqrt{8}$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

Graphical support:



$[-3.02, 3.02]$ by $[-6.5, 6.5]$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$;

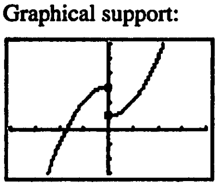
local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$

Note that the local extrema at $x = \pm 2$ are also absolute extrema.

6. $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

Graphical support:



$[-4, 4]$ by $[-3, 6]$

Local minimum: $(0, 1)$

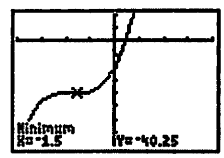
7. $y' = 12x^2 + 42x + 36 = 6(x + 2)(2x + 3)$

Intervals	$x < -2$	$-2 < x < -\frac{3}{2}$	$-\frac{3}{2} < x$
Sign of y'	+	−	+
Behavior of y	Increasing	Decreasing	Increasing

$y'' = 24x + 42 = 6(4x + 7)$

Intervals	$x < -\frac{7}{4}$	$-\frac{7}{4} < x$
Sign of y'	−	+
Behavior of y	Concave down	Concave up

Graphical support:



$[-4, 4]$ by $[-80, 20]$

(a) $\left(-\frac{7}{4}, \infty\right)$

(b) $\left(-\infty, -\frac{7}{4}\right)$

8. $y' = -4x^3 + 12x^2 - 4$

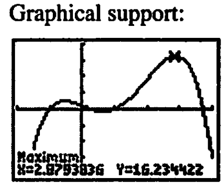
Using grapher techniques, the zeros of y' are $x \approx -0.53$, $x \approx 0.65$, and $x \approx 2.88$.

Intervals	$x < -0.53$	$-0.53 < x < 0.65$	$0.65 < x < 2.88$	$2.88 < x$
Sign of y'	+	−	+	−
Behavior of y	Increasing	Decreasing	Increasing	Decreasing

$y'' = -12x^2 + 24x = -12x(x - 2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	−	+	−
Behavior of y	Concave down	Concave up	Concave down

Graphical support:



$[-2, 4]$ by $[-20, 20]$

(a) $(-\infty, -0.53]$ and $[0.65, 2.88]$

(b) $[-0.53, 0.65]$ and $[2.88, \infty)$

(c) $(0, 2)$

(d) $(-\infty, 0)$ and $(2, \infty)$

8. Continued

(e) Local maxima: (–0.53, 2.45) and (2.88, 16.23); local minimum: (0.65, –0.68)
Note that the local maximum at $x \approx 2.88$ is also an absolute maximum.

(f) (0, 1) and (2, 9)

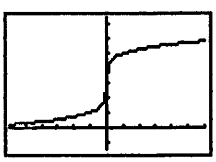
9. $y' = \frac{2}{5}x^{-4/5}$

Intervals	$x<0$	$0<x$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

$y'' = -\frac{8}{25}x^{-9/5}$

Intervals	$x<0$	$0<x$
Sign of y''	+	–
Behavior of y	Concave up	Concave down

Graphical support:



[–6, 6] by [–1.5, 7.5]

(a) $(-\infty, \infty)$

(b) None

(c) $(-\infty, 0)$

(d) $(0, \infty)$

(e) None

(f) (0, 3)

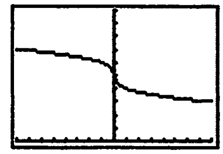
10. $y' = -\frac{1}{3}x^{-2/3}$

Intervals	$x<0$	$0<x$
Sign of y'	–	–
Behavior of y	Decreasing	Decreasing

$y'' = \frac{2}{9}x^{-5/3}$

Intervals	$x<0$	$0<x$
Sing of y''	–	+
Behavior of y	Concave down	Concave up

Graphical support:



[–8, 8] by [0, 10]

(a) $(0,\infty)$

(b) $(-\infty,0)$

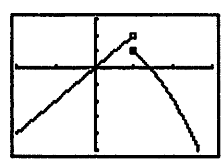
11. $y' = \begin{cases} 2, & x<1 \\ -2x, & x>1 \end{cases}$

Intervals	$x<1$	$1<x$
Sign of y'	+	–
Behavior of y	Increasing	Decreasing

$y'' = \begin{cases} 0, & x<1 \\ -2, & x>1 \end{cases}$

Intervals	$x<1$	$1<x$
Sing of y''	0	–
Behavior of y	Linear	Concave down

Graphical support:



[–2, 3] by [–5, 3]

(a) None

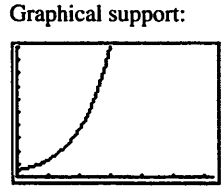
(b) $(1, \infty)$

12. $y' = e^x$

$y'' = e^x$

Since y' and y'' are both positive on the entire domain, y is increasing and concave up on the entire domain.

Graphical support:



[0, 2π] by [0, 20]

(a) $(0, 2\pi)$

(b) None

13. $y = xe^x$

$y' = e^x + xe^x$

Intervals	$x < -1$	$x > -1$
Sign of y'	$-$	$+$
Behavior of y	Decreasing	Increasing

$y'' = 2e^x + xe^x$

Intervals	$x < -2$	$x > -2$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

$\left(-2, -\frac{2}{e^2}\right)$

14. $y = x\sqrt{9 - x^2}$

$y' = \sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}} = 0$

$x = \pm\frac{3\sqrt{2}}{2}$

Intervals	$-3 < x < -\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2} < x < \frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2} < x < 3$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$y'' = -\frac{3x}{(9 - x^2)^{1/2}} + \frac{x^3}{(9 - x^2)^{3/2}} = 0$

$y'' = 0$ at $x = 0$

Intervals	$-3 < x < 0$	$0 < x < 3$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

15. $y' = \frac{1}{1 + x^2}$

since $y' > 0$ for all x, y is always increasing:

$y'' = \frac{d}{dx}(1 + x^2)^{-1} = -(1 + x^2)^{-2}(2x) = \frac{-2x}{(1 + x^2)^2}$

Intervals	$x < 0$	$0 < x$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

(0, 0)

16. $y = x^3(4 - x)$

$y' = 12x^2 - 4x^3$

Intervals	$x < 0$	$0 < x < 3$	$x > 3$
Sign of y'	$+$	$+$	$-$
Behavior of y	Increasing	Increasing	Decreasing

$y'' = 24x - 12x^2$

Intervals	$x < 0$	$0 < x < 2$	$x > 2$
Sign of y''	$-$	$+$	$-$
Behavior of y	Concave down	Concave up	Concave down

(0, 0) and (2, 16)

17. $y = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$

$y' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x - 4}{3x^{2/3}}$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	$-$	$-$	$+$
Behavior of y	Decreasing	Decreasing	Increasing

$y'' = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4x + 8}{9x^{5/3}}$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

$(-2, 6\sqrt[3]{2}) \approx (-2, 7.56)$ and (0, 0)

18. $y = x^{1/2}(x + 3)$

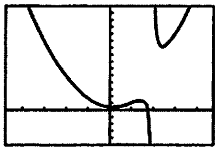
$y' = \frac{1}{2}x^{-1/2}(x + 3) + x^{1/2}$ y is always increasing, so there are no critical points for y'.

$y'' = \frac{1}{(x)^{1/2}} - \frac{x - 3}{4(x)^{3/2}} = 0$

Intervals	$0 < x < 1$	$x > 1$
Sing of y''	$+$	$-$
Behavior of y	Concave up	Concave down

(1, 4)

19. We use a combination of analytic and grapher techniques to solve this problem. Depending on the viewing window chosen, graphs obtained using NDER may exhibit strange behavior near $x = 2$ because, for example, NDER $(y, 2) \approx 1{,}000{,}000$ while y' is actually undefined at $x = 2$. The graph of $y = \frac{x^3 - 2x^2 + x - 1}{x - 2}$ is shown below.

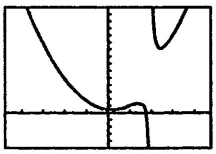


[−4.7, 4.7] by [−5, 15]

19. Continued

$$y' = \frac{(x-2)(3x^2-4x+1)-(x^3-2x^2+x-1)(1)}{(x-2)^2}$$
$$= \frac{2x^3-8x^2+8x-1}{(x-2)^2}$$

The graph of y' is shown below.



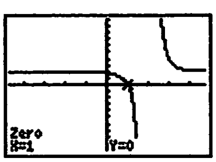
[−4,7, 4.7] by [−10, 10]

The zeros of y' are $x \approx 0.15$, $x \approx 1.40$, and $x \approx 2.45$.

Intervals	$x < 0.15$	$0.15 < x < 1.40$	$1.40 < x < 2$	$2 < x < 2.45$	$2.45 < x$
Sign of y'	−	+	−	−	+
Behavior of y	Decreasing	Increasing	Decreasing	Decreasing	Increasing

$$y'' = \frac{(x-2)^2(6x^2-16x+8)-(2x^3-8x^2+8x-1)(2)(x-2)}{(x-2)^4}$$
$$= \frac{(x-2)(6x^2-16x+8)-2(2x^3-8x^2+8x-1)}{(x-2)^3}$$
$$= \frac{2x^3-12x^2+24x-14}{(x-2)^3}$$
$$= \frac{2(x-1)(x^2-5x+7)}{(x-2)^3}$$

The graph of y'' is shown below.



[−4,7, 4.7] by [−10, 10]

Note that the discriminant of x^2-5x+7 is $(-5)^2-4(1)(7)=-3$, so the only solution of $y''=0$ is $x=1$.

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

(1, 1)

20. $y' = \frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

Intervals	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{(x^2+1)^2(-2x)-(-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4}$$
$$= \frac{(x^2+1)(-2x)-4x(-x^2+1)}{(x^2+1)^3}$$
$$= \frac{2x^3-6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of y''	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

$(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

21. **(a)** Zero: $x = \pm 1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$

(b) Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

22. **(a)** Zero: $x \approx 0, \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$

(b) Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

23. **(a)** $(-\infty, -2]$ and $[0, 2]$

(b) $[-2, 0]$ and $[2, \infty)$

(c) Local maxima: $x = -2$ and $x = 2$;
local minimum: $x = 0$

24. **(a)** $[-2, 2]$

(b) $(-\infty, -2]$ and $[2, \infty)$

(c) Local maximum: $x = 2$;
local minimum: $x = -2$

25. **(a)** $v(t) = x'(t) = 2t - 4$

(b) $a(t) = v'(t) = 2$

(c) It begins at position 3 moving in a negative direction. It moves to position −1 when $t = 2$, and then changes direction, moving in a positive direction thereafter.

26. **(a)** $v(t) = x'(t) = -2 - 2t$

(b) $a(t) = v'(t) = -2$

(c) In begins at position 6 and moves in the negative direction thereafter.

27. **(a)** $v(t) = x'(t) = 3t^2 - 3$

(b) $a(t) = v'(t) = 6t$

27. Continued

(c) It begins at position 3 moving in a negative direction. It moves to position 1 when $t = 1$, and then changes direction, moving in a positive direction thereafter.

28. (a) $v(t) = x'(t) = 6t - 6t^2$

(b) $a(t) = v'(t) = 6 - 12t$

(c) It begins at position 0. It starts moving in the positive direction until it reaches position 1 when $t = 1$, and then it changes direction. It moves in the negative direction thereafter.

29. (a) The velocity is zero when the tangent line is horizontal, at approximately $t = 2.2, t = 6$ and $t = 9.8$.

(b) The acceleration is zero at the inflection points, approximately $t = 4,\ t = 8$ and $t = 11$.

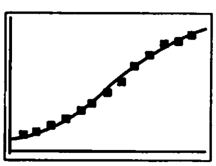
30. (a) The velocity is zero when the tangent line is horizontal, at approximately $t = -0.2,\ t = 4$, and $t = 12$.

(b) The acceleration is zero at the inflection points, approximately $t = 1.5,\ t = 5.2,\ t = 8,\ t = 11$, and $t = 13$.

31. Some calculators use different logistic regression equations, so answers may vary.

(a) $y = \dfrac{12655.179}{1 + 12.871e^{-0.0326t}}$

(b)



[0, 140] by [−200, 12000]

(c) $y = \dfrac{12655.179}{1 + 12.871e^{-0.0326(180)}} = 12{,}209{,}870$. (This is remarkably close to the 2000 census number of 12,281,054.)

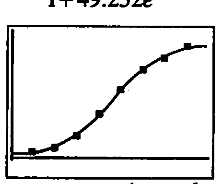
(d) The second derivative has a zero at about 78, indicating that the population was growing fastest in 1898. This corresponds to the inflection point on the regression curve.

(e) The regression equation predicts a population limit of about 12,655,179.

32. Some calculators use different logistic regression equations, so answers may vary.

(a) $y = \dfrac{28984386.288}{1 + 49.252e^{-0.851t}}$

(b)



[0, 9] by [-3.1×10^6, 3.2×10^7]

(c) The zero of the second derivative is about 4.6, which puts the fastest growth during 1981. This corresponds to the inflection point on the regression curve.

(d) The regression curve predicts that cable subscribers will approach a limit of 28,984,386 + 12,168,450 subscribers (about 41 million).

33. $y = 3x - x^3 + 5$

$y' = 3 - 3x^2$
$y'' = -6x$
$y' = 0$ at ± 1.

$y''(-1) > 0$ and $y''(1) < 0$, so there is a local minimum at $(-1, 3)$ and a local maximum at $(1, 7)$.

34. $y = x^5 - 80x + 100$

$y' = 5x^4 - 80$
$y'' = 20x^3$
$y' = 0$ at ± 2

$y''(-2) < 0$ and $y''(2) > 0$, so there is a local maximum at $(-2, 228)$ and a local minimum at $(2, -28)$.

35. $y = x^3 + 3x^2 - 2$
$y' = 3x^2 + 6x$
$y'' = 6x + 6$
$y' = 0$ at -2 and 0.
$y''(-2) < 0,\ y''(0) > 0,$

so there is a local maximum at $(-2, 2)$ and a local minimum at $(0, -2)$.

36. $y = 3x^5 - 25x^3 + 60x + 20$

$y' = 15x^4 - 75x^2 + 60$
$y'' = 60x^3 - 150x$
$y' = 0$ at ± 1 and ± 2.
$y''(-2) < 0,\ y''(-1) > 0$
$y''(1) < 0$, and $y''(2) > 0$;

so there are local maxima at $(-2, 4)$ and $(1, 58)$, and there are local minima at $(-1, -18)$ and $(2, 36)$.

37. $y = xe^x$

$y' = (x + 1)e^x$
$y'' = (x + 2)e^x$
$y' = 0$ at -1.
$y''(-1) > 0$, so there is a local minimum at $(-1, -1/e)$.

38. $y = xe^{-x}$
$y' = (1 - x)e^{-x}$
$y'' = (x - 2)e^{-x}$
$y' = 0$ at 1
$y''(1) < 0$, so there is a local maximum at $(1, 1/e)$.

39. $y' = (x - 1)^2(x - 2)$

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y'	−	−	+
Behavior of y	Decreasing	Decreasing	Increasing

39. Continued

$$y'' = (x-1)^2(1) + (x-2)(2)(x-1)$$
$$= (x-1)[(x-1)+2(x-2)]$$
$$= (x-1)(3x-5)$$

Intervals	$x < 1$	$1 < x < \frac{5}{3}$	$\frac{5}{3} < x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x = 2$.

(c) There are points of inflection at $x = 1$ and at $x = \frac{5}{3}$.

40. $y' = (x-1) \quad (x-2)(x-4)$

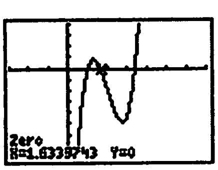
Intervals	$x < 1$	$1 < x < 2$	$2 < x < 4$	$4 < x$
Sign of y'	+	+	−	+
Behavior of y	Increasing	Increasing	Decreasing	Increasing

$$y'' = \frac{d}{dx}[(x-1)^2(x^2-6x+8)]$$
$$= (x-1)^2(2x-6) + (x^2-6x+8)(2)(x-1)$$
$$= (x-1)[(x-1)(2x-6) + 2(x^2-6x+8)]$$
$$= (x-1)(4x^2-20x+22)$$
$$= 2(x-1)(2x^2-10x+11)$$

Note that the zeros of y'' are $x = 1$ and

$$x = \frac{10 \pm \sqrt{10^2 - 4(2)(11)}}{4} = \frac{10 \pm \sqrt{12}}{4}$$
$$= \frac{5 \pm \sqrt{3}}{2} \approx 1.63 \text{or } 3.37.$$

The zeros of y'' can also be found graphically, as shown.



[−3, 7] by [−8, 4]

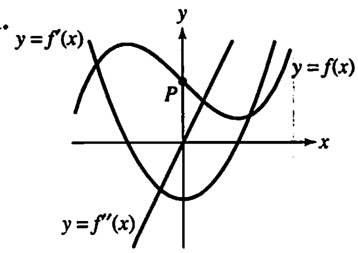
Intervals	$x < 1$	$1 < x < 1.63$	$1.63 < x < 3.37$	$3.37 < x$
Sign of y''	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

(a) Local maximum at $x = 2$

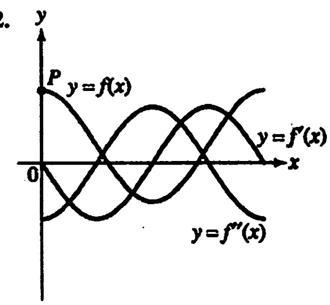
(b) Local minimum at $x = 4$

(c) Points of inflection at $x = 1$, at $x \approx 1.63$, and at $x \approx 3.37$.

41.



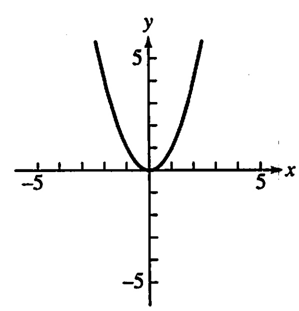
42.



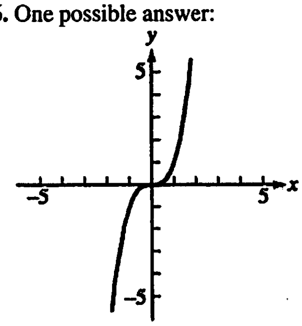
43. No f must have a horizontal tangent at that point, but f could be increasing (or decreasing), and there would be no local extremum. For example, if $f(x) = x^3$, $f'(0) = 0$ but there is no local extremum at $x = 0$.

44. No. $f''(x)$ could still be positive (or negative) on both sides of $x = c$, in which case the concavity of the function would not change at $x = c$. For example, if $f(x) = x^4$, then $f''(0) = 0$, but f has no inflection point at $x = 0$.

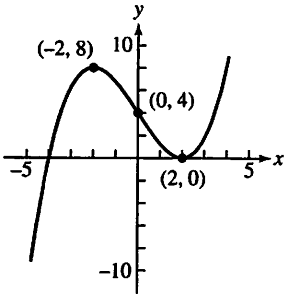
45. One possible answer:



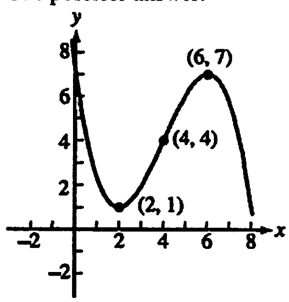
46. One possible answer:



47. One possible answer:



48. One possible answer:



49. **(a)** [0, 1], [3, 4], and [5.5, 6]

(b) [1, 3] and [4, 5.5]

(c) Local maxima: $x = 1, x = 4$ (if f is continuous at $x = 4$), and $x = 6$; local minima: $x = 0, x = 3$, and $x = 5.5$

50. If f is continuous on the interval [0, 3]:

(a) [0, 3]

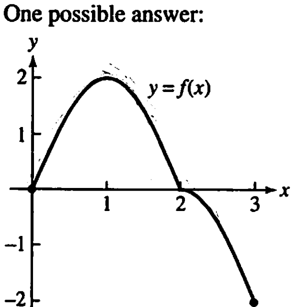
(b) Nowhere

(c) Local maximum: $x = 3$; local minimum: $x = 0$

51. **(a)** Absolute maximum at (1, 2); absolute minimum at (3, –2)

(b) None

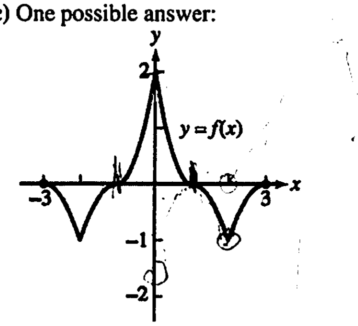
(c) One possible answer:



52. **(a)** Absolute maximum at (0, 2); absolute minimum at (2, –1) and (–2, –1)

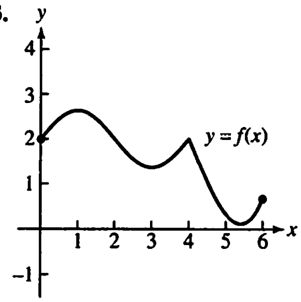
(b) At (1, 0) and (–1, 0)

(c) One possible answer:

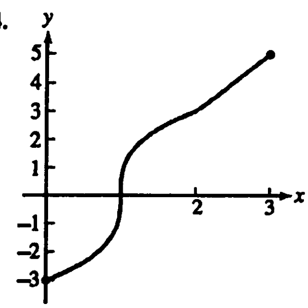


(d) Since f is even, we know $f(3) = f(-3)$. By the continuity of f, since $f(x) < 0$ when $2 < x < x < 3$, we know that $f(3) \le 0$, and since $f(2) = -1$ and $f'(x) > 0$ when $2 < x < 3$, we know that $f(3) > -1$. In summary, we know that $f(3) = f(-3), -1 < f(3) \le 0$, and $-1 < f(-3) \le 0$.

53.



54.



55. False. For example, consider $f(x) = x^4$ at $c = 0$.

56. True. This is the Second Derivative Test for a local maximum.

57. A. $y = ax^3 + 3x^2 = 4x + 5$ say $a = -2$

$y' = -6x^2 + 6x + 4$

$y'' = -12x + 6$

$y'' = 0$ at $\frac{1}{2}$

Interval	$x < 1/2$	$x > 1/2$
Sign of y''	+	–
Behavior of y	Concave up	Concave down

58. E.

59. C. $y=x^5-5x^4+3x+7$

$y'=5x^4-20x^3+3$

$y''=20x^3-60x^2$

$y''=0$ at 3

Interval	$x<3$	$x>3$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

3 is an inflection point.

60. A.

61. **(a)** In exercise 13, $a=4$ and $b=21$, so $-\frac{b}{3a}=-\frac{7}{4}$, which is the x-value where the point of inflection occurs. The local extrema are at $x=-2$ and $x=-\frac{3}{2}$, which are symmetric about $x=-\frac{7}{4}$.

(b) In exercise 8, $a=-2$ and $b=6$, so $-\frac{b}{3a}=1$, which is the x-value where the point of inflection occurs. The local extrema are at $x=0$ abd $x=2$, which are symmetric about $x=1$.

(c) $f'(x)=3ax^2+2bx+c$ and

$f''(x)=6ax+2b$.

The point of inflection will occur where

$f''(x)=0$, which is at $x=-\frac{b}{3a}$.

If there are local extrema, they will occur at the zeros of $f'(x)$. Since $f'(x)$ is quadratic, its graph is a parabola and any zeros will be symmetric about the vertex which will also be where $f''(x)=0$.

62. **(a)** $f'(x)=\frac{(1+ae^{-bx})(0)-(c)(-abe^{-bx})}{(1+ae^{-bx})^2}$

$=\frac{abce^{-bx}}{(1+ae^{-bx})^2}$

$=\frac{abce^{-bx}}{(e^{bx}+a)^2}$,

so the sign of $f'(x)$ is the same as the sign of abc.

(b) $f''(x)=\frac{(e^{bx}+a)^2(ab^2ce^{bx})-(abce^{bx})2(e^{bx}+a)(be^{bx})}{(e^{bx}+a)^4}$

$=\frac{(e^{bx}+a)(ab^2ce^{bx})-(abce^{bx})(2be^{bx})}{(e^{bx}+a)^3}$

$=-\frac{ab^2ce^{bx}(e^{bx}-a)}{(e^{bx}+a)^3}$

Since $a>0$, this changes sign when $x=\frac{\ln a}{b}$ due to the $e^{bx}-a$ factor in the numerator, and $f(x)$ has a point of inflection at the location.

63. **(a)** $f'(x)=4ax^3+3bx^2+2cx+d$

$f''(x)=12ax^2+6bx+2c$

Since is $f''(x)$ quadratic, it must have 0, 1, or 2 zeros. If $f(x)$ has 0 or 1 zeros, it will not change sign and the concavity of $f(x)$ will not change, so there is no point of inflection. If $f''(x)$ has 2 zeros, it will change sign twice, and $f(x)$ will have 2 points of inflection.

(b) If f has no points of inflection, then $f''(x)$ has 0 or 1 zeros, so the discriminant of $f''(x)$ is ≤ 0. This gives $(6b)^2-4(12a)(2c)\le 0$, or $3b^2\le 8ac$. If f has 2 points of inflection, then $f''(x)$ has 2 zeros and the inequality is reversed, so $3b^2>8ac$. In summary, f has 2 points of inflection if and only if $3b^2>8ac$.

Quick Quiz Sections 4.1–4.3

1. (C) $f'(x)=5(x-2)^4(x+3)^4+4(x-2)^5(x+3)^3=0$

$x=-3,-\frac{7}{9},2$

2. (D) $f'(x)=(x-3)^2+2(x-2)(x-3)=0$

$f'(x)=(x-3)(3x-7)=0$

$x=\frac{7}{3},3$

3. (B) $x^2-9=0$

$x=\pm 3$

4. **(a)** $\frac{d}{dx}3\ln(x^2+2)-2x$

$=3\frac{2x}{x^2+2}-2=0$

$x=1,2$

Intervals	$-2<x<1$	$1<x<2$	$2<x<4$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

f has relative minima at $x=1$ and $x=4$ f has relative maxima at $x=\pm 2$

(b) $f''(x)=\frac{d}{dx}\left(\frac{6x}{x^2+2}-2\right)$

$f''(x)=\frac{6}{x^2+2}-\frac{12x^2}{(x^2+2)^2}=0$

$x=\pm\sqrt{2}$

f has points of inflection at $x=\pm\sqrt{2}$

(c) The absolute maximum is at $x=-2$ and $f(x)=3\ln 6+4$.

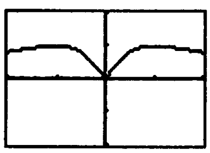
Section 4.4 Modeling and Optimization (pp. 219–232)

Exploration 1 Constructing Cones

1. The circumference of the base of the cone is the circumference of the circle of radius 4 minus x, or $8\pi - x$. Thus, $r = \frac{8\pi - x}{2\pi}$. Use the Pythagorean Theorem to find h, and the fomula for the volume of a cone to find V.

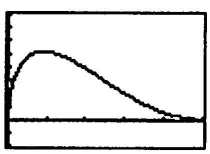
2. The expression under the radical must be nonnegative, that is, $16 - \left(\frac{8\pi - x}{2\pi}\right)^2 \geq 0$.

   Solving this inequality for x gives: $0 \leq x \leq 16\pi$.



[0, 16π] by [−10, 40]

3. The circumference of the original circle of radius 4 is 8π. Thus, $0 \leq x \leq 8\pi$.



[0, 8π] by [−10, 40]

4. The maximum occurs at about $x = 4.61$. The maximum volume is about $V = 25.80$.

5. Start with $\frac{dV}{dx} = \frac{2\pi}{3} rh \frac{dr}{dx} + \frac{\pi}{3} r^2 \frac{dh}{dx}$.

   Compute $\frac{dr}{dx}$ and $\frac{dh}{dx}$, substitute these values in $\frac{dV}{dx}$, set $\frac{dV}{dx} = 0$, and solve for x to obtain

   $x = \frac{8(3 - \sqrt{6})\pi}{3} \approx 4.61$.

   Then $V = \frac{128\pi\sqrt{3}}{27} \approx 25.80$.

Quick Review 4.4

1. $y' = 3x^2 - 12x + 12 = 3(x-2)^2$

   Since $y' \geq 0$ for all x (and $y' > 0$ for $x \neq 2$), y is increasing on $(-\infty, \infty)$ and there are no local extrema.

2. $y' = 6x^2 + 6x - 12 = 6(x+2)(x-1)$

   $y'' = 12x + 6$

   The critical points occur at $x = -2$ or $x = 1$, since $y' = 0$ at these points. Since $y''(-2) = -18 < 0$, the graph has a local maximum at $x = -2$. Since $y''(1) = 18 > 0$, the graph has a local minimum at $x = 1$. In summary, there is a local maximum at $(-2, 17)$ and a local minimum at $(1, -10)$.

3. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5)^2(8) = \frac{200\pi}{3}\text{ cm}^3$

4. $V = \pi r^2 h = 1000$

   $SA = 2\pi rh + 2\pi r^2 = 600$

   Solving the volume equation for h gives $= \frac{1000}{\pi r^2}$.

   Substituting into the surface area equation gives

   $\frac{2000}{r} + 2\pi r^2 = 600$. Solving graphically, we have

   $r \approx -11.14, r \approx 4.01$, or $r \approx 7.13$. Discarding the negative value and using $h = \frac{1000}{\pi r^2}$ to find the corresponding values of h, the two possibilities for the dimensions of the cylinder are:

   $r \approx 4.01$ cm and $h \approx 19.82$ cm, or,
   $r \approx 7.13$ cm and $h \approx 6.26$ cm.

5. Since $y = \sin x$ is an odd function, $\sin(-\alpha) = -\sin\alpha$.

6. Since $y = \cos x$ is an even function, $\cos(-\alpha) = \cos\alpha$.

7. $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha$
   $= 0\cos\alpha - (-1)\sin\alpha$
   $= \sin\alpha$

8. $\cos(\pi - \alpha) = \cos\pi\cos\alpha - \sin\pi\sin\alpha$
   $= (-1)\cos\alpha + 0\sin\alpha$
   $= -\cos\alpha$

9. $x^2 + y^2 = 4$ and $y = \sqrt{3}x$

   $x^2 + (\sqrt{3}x)^2 = 4$
   $x^2 + 3x^2 = 4$
   $4x^2 = 4$
   $x = \pm 1$

   Since $y = \sqrt{3}x$, the solution are:

   $x = 1$ and $y = \sqrt{3}$, or, $x = -1$ and $y = -\sqrt{3}$.

   In ordered pair notation, the solutions are $(1, \sqrt{3})$ and $(-1, -\sqrt{3})$.

10. $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and $y = x + 3$

    $\frac{x^2}{4} + \frac{(x+3)^2}{9} = 1$
    $9x^2 + 4(x+3)^2 = 36$
    $9x^2 + 4x^2 + 24x + 36 = 36$
    $13x^2 + 24x = 0$
    $x(13x + 24) = 0$
    $x = 0$ or $x = -\frac{24}{13}$

10. Continued

Since $y = x + 3$, the solutions are:

$x = 0$ and $y = 3$, or, $x = -\frac{24}{13}$ and $y\frac{15}{13}$.

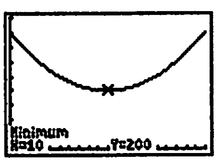
In ordered pair notation, the solution are (0, 3) and $\left(-\frac{24}{13}, \frac{15}{13}\right)$.

Section 4.4 Exercises

1. Represent the numbers by x and $20 - x$, where $0 \le x \le 20$.

(a) The sum of the squares is given by $f(x) = x^2 + (20-x)^2 = 2x^2 - 40x + 400$. Then $f'(x) = 4x - 40$. The critical point and endpoints occur at $x = 0, x = 10$, and $x = 20$. Then $f(0) = 400, f(10) = 200$, and $f(20) = 400$. The sum of the squares is as large as possible for the numbers 0 and 20, and is as small as possible for the numbers 10 and 10.

Graphical support:



[0, 20] by [0, 450]

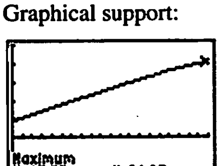
(b) The sum of one number plus the square root of the other is given by $g(x) = x + \sqrt{20 - x}$. Then

$g'(x) = 1 - \frac{1}{2\sqrt{20-x}}$. The critical point occurs when

$2\sqrt{20-x} = 1$, so $20 - x = \frac{1}{4}$ and $x = \frac{79}{4}$. Testing the endpoints and critical point, we find $g(0) = \sqrt{20} \approx 4.47$, $g\left(\frac{79}{4}\right) = \frac{81}{4} = 20.25$, and $g(20) = 20$. The sum is as large as possible when the numbers are

$\frac{79}{4}$ and $\frac{1}{4}$ $\left(\text{summing} \frac{79}{4} + \sqrt{\frac{1}{4}}\right)$, and is as small as possible when the numbers are 0 and 20 (summing $0 + \sqrt{20}$).

Graphical support:



[0, 20] by [−10, 25]

2. Let x and y represent the legs of the triangle, and note that $0 < x < 5$. Then $x^2 + y^2 = 25$, so $y = \sqrt{25 - x^2}$ (since $y > 0$). The area is $A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{25-x^2}$,

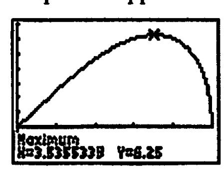
$$\text{so } \frac{dA}{dx} = \frac{1}{2}x\frac{1}{2\sqrt{25-x^2}}(-2x) + \frac{1}{2}\sqrt{25-x^2} = \frac{25-2x^2}{2\sqrt{25-x2}}.$$

The critical point occurs when $25 - 2x^2 = 0$, which means $x = \frac{5}{\sqrt{2}}$, (since $x > 0$). This value corresponds to the largest possible area, since $\frac{dA}{dx} > 0$ for $0 < x < \frac{5}{\sqrt{2}}$ and $< \frac{dA}{dx} < 0$ for $\frac{5}{\sqrt{2}} < x < 5$. When $x = \frac{5}{\sqrt{2}}$, we have

$$y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} = \frac{5}{\sqrt{2}} \text{ and } A = \frac{1}{2}xy = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{4}.$$

Thus, the largest possible area is $\frac{25}{4}\text{cm}^2$, and the dimensions (legs) are $\frac{5}{\sqrt{2}}$ cm by $\frac{5}{\sqrt{2}}$ cm.

Graphical support:



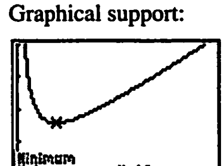
[0, 5] by [−2, 7]

3. Let x represent the length of the rectangle in inches ($x > 0$). Then the width is $\frac{16}{x}$ and the perimeter is

$$P(x) = 2\left(x + \frac{16}{x}\right) = 2x + \frac{32}{x}.$$

Since $P'(x) = 2 - 32x^{-2} = \frac{2(x^2 - 16)}{x^2}$ this critical point occurs at $x = 4$. Since $P'(x) < 0$ for $0 < x < 4$ and $P'(x) > 0$ for $x > 4$, this critical point corresponds to the minimum perimeter. The smallest possible perimeter is $P(4) = 16$ in., and the rectangle's dimensions are 4 in. by 4 in.

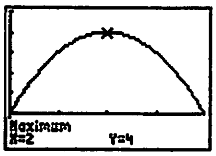
Graphical support:



[0, 20] by [0, 40]

4. Let x represent the length of the rectangle in meters $(0 < x < 4)$. Then the width is $4 - x$ and the area is $A(x) = x(4-x) = 4x - x^2$. Since $A'(x) = 4 - 2x$, the critical point occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < 4$, this critical point corresponds to the maximum area. The rectangle with the largest area measures 2 m by 4 – 2 = 2m, so it is a square.

Graphical support:



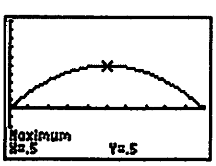
[0, 4] by [−1.5, 5]

5. (a) The equation of line AB is $y = -x + 1$, so the y-coordinate of P is $-x + 1$.

(b) $A(x) = 2x(1 - x)$

(c) Since $A'(x) = \frac{d}{dx}(2x - 2x^2) = 2 - 4x$, the critical point occurs at $x = \frac{1}{2}$. Since $A'(x) > 0$ for $0 < x < \frac{1}{2}$ and $A'(x) < 0$ for $- < x < 1$, this critical point corresponds to the maximum area. The largest possible area is $A\left(\frac{1}{2}\right) = \frac{1}{2}$ square unit, and the dimensions of the rectangle are $\frac{1}{2}$ unit by 1 unit.

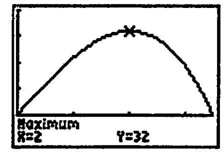
Graphical support:



[0, 1] by [−0.5, 1]

6. If the upper right vertex of the rectangle is located at $(x, 12 - x^2)$ for $0 < x < \sqrt{12}$, then the rectangle's dimensions are $2x$ by $12 - x^2$ and the area is $A(x) = 2x(12 - x^2) = 24x - 2x^3$. Then $A'(x) = 24 - 6x^2 = 6(4 - x^2)$, so the critical point (for $0 < x < \sqrt{12}$) occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < \sqrt{12}$, this critical point corresponds to the maximum area. The largest possible area is $A(2) = 32$, and the dimensions are 4 by 8.

Graphical support:



[0, $\sqrt{12}$] by [−10, 40]

7. Let x be the side length of the cut-out square $(0 < x < 4)$. Then the base measures $8 - 2x$ in. by $15 - 2x$ in., and the volume is

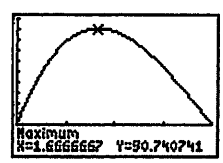
$V(x) = x(8 - 2x)(15 - 2x) = 4x^3 - 46x^2 + 120x$. Then

$V'(x) = 12x^2 - 92x + 120 = 4(3x - 5)(x - 6)$.

Then the critical point (in $0 < x < 4$) occurs at $x = \frac{5}{3}$. Since $V'(x) > 0$ for $0 < x < \frac{5}{3}$ and $V'(x) < 0$ for $\frac{5}{3} < x < 4$, the critical point corresponds to the maximum volume.

The maximum volume is $V\left(\frac{5}{3}\right) = \frac{2450}{27} \approx 90.74\,\text{in}^3$, and the dimensions are $\frac{5}{3}$ in. by $\frac{14}{3}$ in. by $\frac{35}{3}$ in.

Graphical support:



[0, 4] by [−25, 100]

8. Note that the values a and b must satisfy $a^2 + b^2 = 20^2$ and so $b = \sqrt{400 - a^2}$. Then the area is given by

$A = \frac{1}{2}ab = \frac{1}{2}a\sqrt{400 - a^2}$ for $0 < a < 20$, and

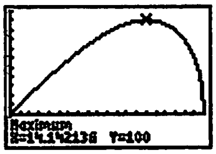
$$\frac{dA}{da} = \frac{1}{2}a\left(\frac{1}{2\sqrt{400 - a^2}}\right)(-2a) + \frac{1}{2}\sqrt{400 - a^2}$$

$$= \frac{-a^2 + (400 - a^2)}{2\sqrt{400 - a^2}} = \frac{200 - a^2}{\sqrt{400 - a^2}}.$$

The critical point occurs when $a^2 = 200$. Since $\frac{dA}{da} > 0$ for $0 < a < \sqrt{200}$ and $\frac{dA}{da} < 0$ for $\sqrt{200} < a < 20$, this critical point corresponds to the maximum area. Furthermore, if $a = \sqrt{200}$ then $b = \sqrt{400 - a^2} = \sqrt{200}$, so the maximum area occurs when $a = b$.

8. Continued

Graphical support:



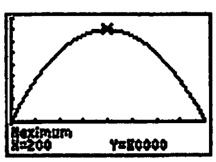
[0, 20] by [−30, 110]

9. Let x be the length in meters of each side that adjoins the river. Then the side parallel to the river measures $800 - 2x$ meters and the area is

$A(x) = x(800 - 2x) = 800x - 2x^2$ for $0 < x < 400$.

Therefore, $A'(x) = 800 - 4x$ and the critical point occurs at $x = 200$. Since $A'(x) > 0$ for $0 < x < 200$ and $A'(x) < 0$ for $200 < x < 400$, the critical point corresponds to the maximum area. The largest possible area is $A(200) = 80{,}000\ \text{m}^2$ and the dimensions are 200 m (perpendicular to the river) by 400 m (parallel to the river).

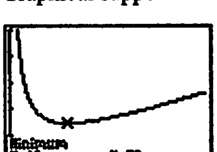
Graphical support:



[0, 400] by [−25,000, 90,000]

10. If the subdividing fence measures x meters, then the pea patch measures x m by $\frac{216}{x}$ m and the amount of fence needed is $f(x) = 3x + 2\frac{216}{x} = 3x + 432x^{-1}$. Then $f'(x) = 3 - 432x^{-2}$ and the critical point (for $x > 0$) occurs at $x = 12$. Since $f'(x) < 0$ for $0 < x < 12$ and $f'(x) > 0$ for $x > 12$, the critical point corresponds to the minimum total length of fence. The pea patch will measure 12 m by 18 m (with a 12-m divider), and the total amount of fence needed is $f(12) = 72$ m.

Graphical support:



[0, 40] by [0, 250]

11. (a) Let x be the length in feet of each side of the square base. Then the height is $\frac{500}{x^2}$ ft and the surface area (not including the open top) is

$S(x) = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + 2000x^{-1}$. Therefore,

$S'(x) = 2x - 2000x^{-2} = \frac{2(x^3 - 1000)}{x^2}$ and the critical point occurs at $x = 10$. Since $S'(x) < 0$ for $0 < x < 10$ and $S'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of steel used. The dimensions should be 10 ft by 10 ft by 5 ft, where the height is 5 ft.

(b) Assume that the weight is minimized when the total area of the bottom and the four sides is minimized.

12. (a) Note that $x^2y = 1125$, so $y = \frac{1125}{x^2}$. Then

$$\begin{aligned} c &= 5(x^2 + 4xy) + 10xy \\ &= 5x^2 + 30xy \\ &= 5x^2 + 30x\left(\frac{1125}{x^2}\right) \\ &= 5x^2 + 33{,}750x^{-1} \end{aligned}$$

$$\frac{dc}{dx} = 10x - 33{,}750x^{-2} = \frac{10(x^3 - 3375)}{x^2}$$

The critical point occurs at $x = 15$. Since $\frac{dc}{dx} < 0$ for $0 < x < 15$ and $\frac{dc}{dx} > 0$ for $x > 15$, the critical point corresponds to the minimum cost. The values of x and y are $x = 15$ ft and $y = 5$ ft.

(b) The material for the tank costs 5 dollars/sq ft and the excavation charge is 10 dollars for each square foot of the cross-sectional area of one wall of the hole.

13. Let x be the height in inches of the printed area. Then the width of the printed area is $\frac{50}{x}$ in. and the overall dimensions are $x + 8$ in. by $\frac{50}{x} + 4$ in. The amount of paper used is $A(x) = (x + 8)\left(\frac{50}{x} + 4\right) = 4x + 82 + \frac{400}{x}$ in^2. Then

$A'(x) = 4 - 400x^{-2} = \frac{4(x^2 - 100)}{x^2}$ and the critical point (for $x > 0$) occurs at $x = 10$. Since $A'(x) < 0$ for $0 < x < 10$ and $A'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of paper. Using $x + 8$ and $\frac{50}{x} + 4$ for $x = 10$, the overall dimensions are 18 in. high by 9 in. wide.

14. (a) $s(t) = -16t^2 + 96t + 112$

$v(t) = s'(t) = -32t + 96$

At $t = 0$, the velocity is $v(0) = 96$ ft/sec.

(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 256$ ft and it occurs at $t = 3$ sec.

14. Continued

(c) Note that $s(t) = -16t^2 + 96t + 112 = -16(t+1)(t-7)$, so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value, of t, the velocity when $s = 0$ is $v(7) = -128$ ft/sec.

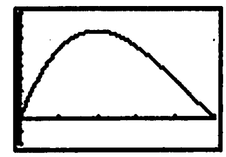
15. We assume that a and b are held constant. Then $A(\theta) = \frac{1}{2}ab\sin\theta$ and $A'(\theta) = \frac{1}{2}ab\cos\theta$. The critical point (for $0 < \theta < \pi$) occurs at $\theta = \frac{\pi}{2}$. Since $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$ and $A'(\theta)$ 0 for $\frac{\pi}{2} < \theta < \pi$, the critical point corresponds to the maximum area. The angle that maximizes the triangle's area is $\theta = \frac{\pi}{2}$ (or 90°).

16. Let the can have radius r cm and height h cm. Then $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$. The area of material used is $A = \pi r^2 + 2\pi rh = \pi r^2 + \frac{2000}{r}$, so $\frac{dA}{dr} = 2\pi r - 2000r^{-2}$ $= \frac{2\pi r^3 - 2000}{r^2}$. The critical point occurs at $r = \sqrt[3]{\frac{1000}{\pi}} = 10\pi^{-1/3}$ cm. Since $\frac{dA}{dr} < 0$ for $0 < r < 10\pi^{-1/3}$ and $\frac{dA}{dr} > 0$ for $r > 10\pi^{1/3}$, the critical point corresponds to the least amount of material used and hence the lightest possible can. The dimensions are $r = 10\pi^{-1/3} \approx 6.83$cm and $h = 10\pi^{-1/3} \approx 6.83$cm. In Example 2, because of the top of the can, the "best" design is less big around and taller.

17. Note that $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$. Then $A = 8r^2 + 2\pi rh = 8r^2 + \frac{2000}{r}$, so $\frac{dA}{dr} = 16r - 2000r^{-2} = \frac{16(r^3 - 125)}{r^2}$. The critical point occurs at $r = \sqrt[3]{125} = 5$ cm. Since $\frac{dA}{dr} < 0$ for $0 < r < 5$ and $\frac{dA}{dr} > 0$ for $r > 5$, the critical point corresponds to the least amount of aluminium used or wasted and hence the most economical can. The dimensions are $r = 5$ cm and $h = \frac{40}{\pi}$, so the ratio of h to r is $\frac{8}{\pi}$ to 1.

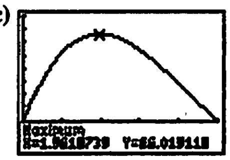
18. (a) The base measures $10 - 2x$ in. by $\frac{15-2x}{2}$ in, so the volume formula is
$V(x) = \frac{x(10-2x)(15-2x)}{2} = 2x^3 - 25x^2 + 75x$.

(b) We require $x > 0$, $2x < 10$, and $2x < 15$. Combining these requirements, the domain is the interval (0, 5).



[0, 5] by [−20, 80]

(c)



[0, 5] by [−20, 80]

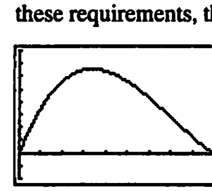
The maximum volume is approximately 66.02 when $x \approx 1.96$ in.

(d) $V'(x) = 6x^2 - 50x + 75$

The critical point occurs when $V'(x) = 0$, at
$x = \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{700}}{12} = \frac{25 \pm 5\sqrt{7}}{6}$,
that is, $x \approx 1.96$ or $x \approx 6.37$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x - 50$, which is negative when $x \approx 1.96$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \frac{25 - 5\sqrt{7}}{6} \approx 1.96$, which confirms the result in (c).

19. (a) The "sides" of the suitcase will measure $24 - 2x$ in. by $18 - 2x$ in. and will be $2x$ in. apart, so the volume formula is
$V(x) = 2x(24-2x)(18-2x) = 8x^3 - 168x^2 + 864x$.

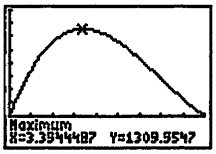
(b) We require $x > 0$, $2x < 18$, and $2x < 24$. Combining these requirements, the domain is the interval (0, 9).



[0, 9] by [−400, 1600]

19. Continued

(c)



[0, 9] by [−400, 1600]

The maximum volume is approximately 1309.95 when $x \approx 3.39$ in.

(d) $V'(x) = 24x^2 - 336x + 864 = 24(x^2 - 14x + 36)$

The critical point is at

$$x = \frac{4 \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)} = \frac{14 \pm \sqrt{52}}{2} = 7 \pm \sqrt{13},$$ that

is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$, which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$, which confirms the results in (c).

(e) $8x^3 - 168x^2 + 864x = 1120$

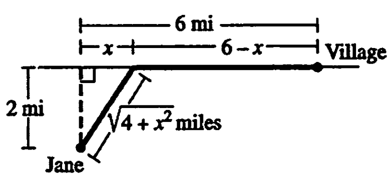
$8(x^3 - 21x^2 + 108x - 140) = 0$

$8(x - 2)(x - 5)(x - 14) = 0$

Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

(f) The dimensions of the resulting box are $2x$ in., $(24 - 2x)$ in., and $(18 - 2x)$ in. Each of these measurements must be positive, so that gives the domain of (0, 9)

20.



Let x be the distance from the point on the shoreline nearest Jane's boat to the point where she lands her boat. Then she needs to row $\sqrt{4 + x^2}$ mi at 2 mph and walk $6 - x$ mi at 5 mph. The total amount of time to reach the village is

$$f(x) = \frac{\sqrt{4 + x^2}}{2} + \frac{6 - x}{5} \text{ hours } (0 \le x \le 6).$$ Then

$$f'(x) = \frac{1}{2}\frac{1}{2\sqrt{4 + x^2}}(2x) - \frac{1}{5} = \frac{x}{2\sqrt{4 + x^2}} - \frac{1}{5}.$$

Solving $f'(x) = 0$, we have:

$$\frac{x}{2\sqrt{4 + x^2}} = \frac{1}{5}$$

$$5x = 2\sqrt{4 + x^2}$$

$$25x^2 = 4(4 + x^2)$$

$$21x^2 = 16$$

$$x = \pm\frac{4}{\sqrt{21}}$$

We discard the negative value of x because it is not in the domain. Checking the endpoints and critical point, we have $f(0) = 2.2, f\left(\frac{4}{\sqrt{21}}\right) \approx 2.12$, and $f(6) \approx 3.16$. Jane should land her boat $\frac{4}{\sqrt{21}} \approx 0.87$ miles down the shoreline from the point nearest her boat.

21. If the upper right vertex of the rectangle is located at $(x, 4 \cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4 \cos 0.5x$, so the area is $A(x) = 8x \cos 0.5x$.

Then $A'(x) = 8x(-0.5 \sin 0.5x) + 8(\cos 0.5x)(1)$

$= -4x \sin 0.5x + 8 \cos 0.5x$.

Solving $A'(x)$ graphically for $0 < x < \pi$, we find that $x \approx 1.72$. Evaluating $2x$ and $4 \cos 0.5x$ for $x \approx 1.72$, the dimensions of the rectangle are approximately 3.44 (width) by 2.61 (height), and the maximum area is approximately 8.98.

22. Let the radius of the cylinder be r cm, $0 < r < 10$. Then the height is $2\sqrt{100 - r^2}$ and the volume is

$V(r) = 2\pi r^2 \sqrt{100 - r^2}\text{ cm}^3$. Then

$$V'(r) = 2\pi r^2\left(\frac{1}{2\sqrt{100 - r^2}}\right)(-2r) + (2\pi\sqrt{100 - r^2})(2r)$$

$$= \frac{-2\pi r^3 + 4\pi r(100 - r^2)}{\sqrt{100 - r^2}}$$

$$= \frac{2\pi r(200 - 3r^2)}{\sqrt{100 - r^2}}$$

The critical point for $0 < r < 10$ occurs at

$r = \sqrt{\frac{200}{3}} = 10\sqrt{\frac{2}{3}}$. Since $V'(r) > 0$ for $0 < r < 10\sqrt{\frac{2}{3}}$ and $V'(r) > 0$ for $10\sqrt{\frac{2}{3}} < r < 10$, the critical point corresponds to the maximum volume. The dimensions are

$r = 10\sqrt{\frac{2}{3}} \approx 8.16$ cm and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{3\sqrt{3}} \approx 2418.40\text{ cm}^3$.

23. Set $r'(x) = c'(x): 4x^{-1/2} = 4x$. The only positive critical value is $x = 1$, so profit is maximized at a production level of 1000 units. Note that $(r - c)''(x) = -2(x)^{-3/2} - 4 < 0$ for all positive x, so the Second Derivative Test confirms the Maximum.

24. Set $r'(x) = c'(x): 2x / (x^2 + 1)^2 = (x - 1)^2$. We solve this equation grpahically to find that $x \approx 0.294$. The graph of $y = r(x) - c(x)$ shows a minimum at $x \approx 0.294$ and a maximum at $x \approx 1.525$, so profit is maximized at a production level of about 1,525 units.

25. Set $c'(x) = \frac{c(x)}{x}$: $3x^2 - 20x + 30 = x^2 - 10x + 30$. The only positive solution is $x = 5$, so average cost is minimized at a production level of 5000 units. Note that $\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$ for all positive x, so the Second Derivative Test Confirms the minimum.

26. Set $c'(x) = c(x)/x : xe^x + e^x - 4x = e^x - 2x$. The only positive solution is $x = \ln 2$, so average cost is minimized at a production level of 1000 ln 2, which is about 693 units. Note that $\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = e^x > 0$ for all positive x, so the Second Derivative Test confirms the minimum.

27. Revenue: $r(x) = [200 - 2(x - 50)]x = -2x^2 + 300x$

Cost: $c(x) = 6000 + 32x$

Profit: $p(x) = r(x) - c(x)$

$= -2x^2 + 268x - 6000, 50 \le x \le 80$

Since $p'(x) = -4x + 268 = -4(x - 67)$, the critical point occurs at $x = 67$. This value represents the maximum because $p''(x) = -4$, which is negative for all x in the domain. The maximum profit occurs if 67 people go on the tour.

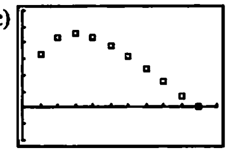
28. **(a)** $f'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1 - x)$

The critical point occurs at $x = 1$. Since $f'(x) > 0$ for $0 \le x < 1$ and $f'(x) < 0$ for $x > 1$, the critical point corresponds to the maximum value of f. The absolute maximum of f occurs at $x = 1$.

(b) To find the values of b, use grapher techniques to solve $xe^{-x} = 0.1e^{-0.1}$, $xe^{-x} = 0.2e^{-0.2}$, and so on. To find the values of A, calculate $(b - a)\,ae^{-2}$, using the unrounded values of b. (Use the *list* features of the grapher in order to keep track of the unrounded values for part (d).)

a	b	A
0.1	3.71	0.33
0.2	2.86	0.44
0.3	2.36	0.46
0.4	2.02	0.43
0.5	1.76	0.38
0.6	1.55	0.31
0.7	1.38	0.23
0.8	1.23	0.15
0.9	1.11	0.08
1.0	1.00	0.00

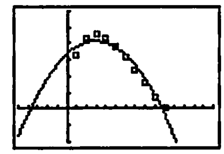
(c)



[0, 1.1] by [–0.2, 0.6]

(d) Quadratic:

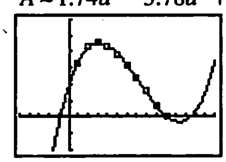
$A \approx -0.91a^2 + 0.54a + 0.34$



[–0.5, 1.5] by [–0.2, 0.6]

Cubic:

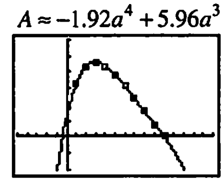
$A \approx 1.74a^3 - 3.78a^2 + 1.86a + 0.19$



[–0.5, 1.5] by [–0.2, 0.6]

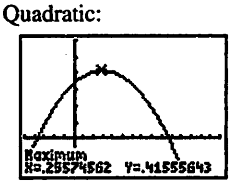
Quartic:

$A \approx -1.92a^4 + 5.96a^3 - 6.87a^2 + 2.71a + 0.12$



[–0.5, 1.5] by [–0.2, 0.6]

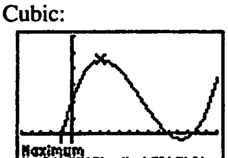
(e) Quadratic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quadratic regression equation, the maximum area occurs at $a \approx 0.30$ and is approximately 0.42.

Cubic:

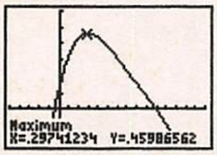


[−0.5, 1.5] by [−0.2, 0.6]

According to the cubic regression equation, the maxiumu area occurs at $a \approx 0.31$ and is approximately 0.45.

28. Continued

Quartic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quartic regression equation the maximum area occurs at $a \approx 0.30$ and is approximately 0.46.

29. (a) $f'(x)$ is a quadratic polynominal, and as such it can have 0, 1, or 2 zeros. If it has 0 or 1 zeros, then its sign never changes, so $f(x)$ has no local extrema.
If $f'(x)$ has 2 zeros, then its sign changes twice, and $f(x)$ has 2 local extrema at those points.

(b) Possible answers:
No local extrema: $y = x^3$;
2 local extrema: $y = x^3 - 3x$

30. Let x be the length in inches of each edge of the square end, and let y be the length of the box. Then we require $4x + y \le 108$. Since our goal is to maximize volume, we assume $4x + y = 108$ and so $y = 108 - 4x$. The volume is $V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$, where $0 < x < 27$. Then $V' = 216x - 12x^2 = -12x(x - 18)$, so the critical point occurs at $x = 18$ in. Since $V'(x) > 0$ for $0 < x < 18$ and $V'(x) < 0$ for $18 < x < 27$, the critical point corresponds to the maximum volume. The dimensions of the box with the largest possible volume are 18 in. by 18 in. by 36 in.

31. Since $2x + 2y = 36$, we know that $y = 18 - x$. In part (a), the radius is $\frac{x}{2\pi}$ and the height is $18 - x$, and so the volume is given by

$$\pi r^2 h = \pi\left(\frac{x}{2\pi}\right)^2(18 - x) = \frac{1}{4\pi}x^2(18 - x).$$

In part (b), the radius is and the height is $18 - x$, and so the volume is given by $\pi r^2 h = \pi x^2(18 - x)$. Thus, each problem requires us to find the value of x that maximizes $f(x) = x^2(18 - x)$ in the interval $0 < x < 18$, so the two problems have the same answer.
To solve either problem, note that $f(x) = 18^2 - x^3$ and so $f'(x) = 36x - 3x^2 = -3x(x - 12)$. The critical point occursat $x = 12$. Since $f'(x) > 0$ for $0 < x < 12$ and $f'(x) < 0$ for $12 < x < 18$, the critical point corresponds to the maximum value of $f(x)$. To maximize the volume in either part (a) or (b), let $x = 12$ cm and $y = 6$ cm.

32. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by $V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}(3 - h^2)h = \pi h - \frac{\pi}{3}h^3$ for $0 < h < \sqrt{3}$, and so $\frac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2)$. The critical point (for $h > 0$) occurs at $h = 1$. Since $\frac{dV}{dh} > 0$ for $0 < h < 1$ and $\frac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\frac{2\pi}{3}\text{m}^3$.

33. (a) We require $f(x)$ to have a critical point at $x = 2$. Since $f'(x) = 2x - ax^{-2}$, we have $f'(2) = 4 - \frac{a}{4}$ and so our requirement is that $4 - \frac{a}{4} = 0$. Therefore, $a = 16$. To verify that the critical point corresponds to a local minimum, note that we now have $f'(x) = 2x - 16x^{-2}$ and so $f''(x) = 2 + 32x^{-3}$, so $f''(2) = 6$, which is positive as expected. So, use $a = -16$.

(b) We require $f''(1) = 0$. Since $f'' = 2 + 2ax^{-3}$, we have $f''(1) = 2 + 2a$, so our requirement is that $2 + 2a = 0$. Therefore, $a = -1$. To verify that $x = 1$ is in fact an inflection point, note that we now have $f''(x) = 2 - 2x^{-3}$, which is negative for $0 < x < 1$ and positive for $x > 1$. Therefore, the graph of f is concave down in the interval (0, 1) and concave up in the interval $(1, \infty)$, So, use $a = -1$.

34. $f'(x) = 2x - ax^{-2} = \frac{2x^3 - a}{x^2}$, so the only sign change in $f'(x)$ occurs at $x = \left(\frac{a}{2}\right)^{1/3}$, where the sign changes from negative to positive. This means there is a local minimum at that point, and there are local maxima.

35. (a) Note that $f'(x) = 3x^2 + 2ax + b$. We require $f'(-1) = 0$ and $f'(3) = 0$, which give $3 - 2a + b = 0$ and $27 + 6a + b = 0$. Substracting the first equation from the second, we have $24 + 8a = 0$ and so $a = -3$. Substituting into the first equation, we have $9 + b = 0$, so $b = -9$. Therefore, our equation for $f(x)$ is $f(x) = x^3 - 3x^2 - 9x$. To verify that we have a local maximum at $x = -1$ and a local minimum at $x = 3$, note that $f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$, which is positive for $x < -1$, negative for $-1 < x < 3$, and positive for $x >$ 3. So, use $a = -3$ and $b = -9$.

(b) Note that $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$. We require $f'(4) = 0$ and $f''(1) = 0$, which give $48 + 8a + b = 0$ and $6 + 2a = 0$. By the second equation, $a = -3$, and so the first equation becomes $48 - 24 + b = 0$. Thus $b = -24$. To verify that we have a local minimum at $x = 4$, and an inflection point *at* $x = 1$, note that we now have $f''(x) = 6x - 6$. Since f'' changes sign at $x = 1$ and is positive at $x = 4$, the desired conditions are satisfied. So, use $a = -3$ and $b = -24$.

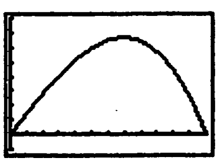
36. Refer to the illustration in the problem statement. Since $x^2+y^2=9$, we have $x=\sqrt{9-y^2}$. Then the volume of the cone is given by

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi x^2(y+3)$$
$$=\frac{1}{3}\pi(9-y^2)(y+3)$$
$$=\frac{\pi}{3}(-y^3-3y^2+9y+27),$$

for $-3<y<3$.

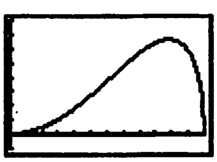
Thus $\frac{dV}{dy}=\frac{\pi}{3}(-3y^2-6y+9)=-\pi(y^2+2y-3)$
$=-\pi(y+3)(y-1)$, so the critical point in the interval $(-3, 3)$ is $y=1$. Since $\frac{dV}{dy}>0$ for $-3<y<1$ and $\frac{dV}{dy}<0$ for $1<y<3$, the critical point does correspond to the maximum value, which is $V(1)=\frac{32\pi}{3}$ cubic units.

37. (a) Note that $w^2+d^2=12^2$, so $d=\sqrt{144-w^2}$. Then we may write $S=kwd^2=kw(144-w^2)=144kw-kw^3$ for $0<w<12$, so $\frac{dS}{dw}=144k-3kw^2=-3k(w^2-48)$.
The critical point (for $0<w<12$) occurs at $w=\sqrt{48}=4\sqrt{3}$. Since $\frac{dS}{dw}>0$ for $0<w<4\sqrt{3}$ and $\frac{dS}{dw}<0$ for $4\sqrt{3}<w<12$, the critical point corresponds to the maximum strength. The dimensions are $4\sqrt{3}$ in. wide by $4\sqrt{6}$ in. deep.

(b)


[0, 12] by [−100, 800]

The graph of $S=144w-w^3$ is shown. The maximum strength shown in the graph occurs at $w=4\sqrt{3}\approx 6.9$, which agrees with the answer to part (a).

(c)


[0, 12] by [−100, 800]

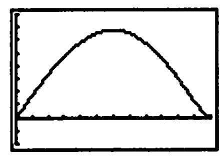
The graph of $S=d^2\sqrt{144-d^2}$ is shown. The maximum strength shown in the graph occurs at $d=4\sqrt{6}\approx 9.8$, which agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.

Changing the value of k changes the maximum strength, but not the dimensions of the strongest beam. The graphs for different values of k look the same except that the vertical scale is different.

38. (a) Note that $w^2+d^2=12^2$, so $d=\sqrt{144-w^2}$. Then we may write $S=kwd^3=kw(144-w^2)^{3/2}$, so

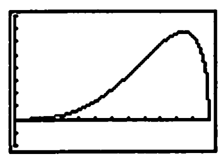
$$\frac{dS}{dw}=kw\cdot\frac{3}{2}(144-w^2)^{1/2}(-2w)+k(144-w^2)^{3/2}(1)$$
$$=(k\sqrt{144-w^2})(-3w^2+144-w^2)$$
$$=(-4k\sqrt{144-2^2})(w^2-36)$$

The critical point (for $0<w<12$) occurs at $w=6$. Since $\frac{dS}{dw}>0$ for $0<w<6$ and $\frac{dS}{dw}<0$ for $6<w<12$, the critical point corresponds to the maximum stiffness. The dimensions are 6 in. wide by $6\sqrt{3}$ in. deep.

(b)


[0, 12] by [−2000, 8000]

The graph of $S=w(144-w^2)^{3/2}$ is shown. The maximum stiffness shown in the graph occurs at $w=6$, which agrees with the answer to part (a).

(c)


[0, 12] by [−2000, 8000]

The graph of $S=d^3\sqrt{144-d^2}$ is shown. The maximum stiffness shown in the graph occurs at $d=6\sqrt{3}\approx 10.4$ agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.
Changing the value of k changes the maximum stiffness, but not the dimensions of the stiffest beam. The graphs for different values of k look the same except that the vertical scale is different.

39. (a) $v(t)=s'(t)=-10\pi\sin\pi t$
The speed at time t is $10\pi|\sin\pi t|$. The maximum speed is 10π cm/sec and it occurs at $t=\frac{1}{2}$, $t=\frac{3}{2}$, $t=\frac{5}{2}$, and $t=\frac{7}{2}$ sec. The position at these times is $s=0$ cm (rest position), and the acceleration $a(t)=v'(t)=-10\pi^2\cos\pi t$ is $0\ \text{cm}/\text{sec}^2$ at these times.

39. Continued

(b) Since $a(t) = -10\pi^2 \cos \pi t$, the greatest magnitude of the acceleration occurs at $t = 0$, $t = 1$, $t = 2$, $t = 3$, and $t = 4$. At these times, the position of the cart is either $s = -10$ cm or $s = 10$ cm, and the speed of the cart is 0 cm/sec.

40. Since $\frac{di}{dt} = -2\sin t + 2\cos t$, the largest magnitude of the current occurs when $-2 \sin t + 2 \cos t = 0$, or $\sin t = \cos t$. Squaring both sides gives $\sin^2 t = \cos^2 t$, and we know that $\sin^2 t + \cos^2 t = 1$, so $\sin^2 t = \cos^2 t = \frac{1}{2}$. Thus the possible values of t are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and so on. Eliminating extraneous solutions, the solutions of $\sin t = \cos t$ are $t = \frac{\pi}{4} + k\pi$ for integers k, and at these times $|i| = |2\cos t + 2\sin t| = 2\sqrt{2}$. The peak current is $2\sqrt{2}$ amps.

41. The square of the distance is

$$D(x) = \left(x - \frac{3}{2}\right)^2 + (\sqrt{x} + 0)^2 = x^2 - 2x + \frac{9}{4},$$

so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

42. Calculus method:

The square of the distance from the point $(1, \sqrt{3})$ to $(x, \sqrt{16 - x^2})$ is given by

$$\begin{aligned} D(x) &= (x-1)^2 + (\sqrt{16 - x^2} - \sqrt{3})^2 \\ &= x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 \\ &= -2x + 20 - 2\sqrt{48 - 3x^2}. \text{ Then} \end{aligned}$$

$$D'(x) = -2 - \frac{2}{2\sqrt{48 - 3x^2}}(-6x) = -2 + \frac{6x}{\sqrt{48 - 3x^2}}.$$

Solving $D'(x) = 0$, we have:

$$\begin{aligned} 6x &= 2\sqrt{48 - 3x^2} \\ 36x^2 &= 4(48 - 3x^2) \\ 9x^2 &= 48 - 3x^2 \\ 12x^2 &= 48 \\ x &= \pm 2 \end{aligned}$$

We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry method:

The semicircle is centered at the origin and has radius 4. The distance from the origin to $(1, \sqrt{3})$ is $\sqrt{1^2 + (\sqrt{3})^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $(1, \sqrt{3})$. That distance is $4 - 2 = 2$.

43. No. Since $f(x)$ is a quadratic function and the coefficient of x^2 is positive, it has an absolute minimum at the point where $f'(x) = 2x - 1 = 0$, and the point is $\left(\frac{1}{2}, \frac{3}{4}\right)$.

44. (a) Because $f(x)$ is periodic with period 2π.

(b) No. Since $f(x)$ is continuous on $[0, 2\pi]$, its absolute minimum occurs at a critical point or endpoint.
Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = -4\sin x - 2\sin 2x &\doteq 0 \\ -4\sin x - 4\sin x \cos x &= 0 \\ -4(\sin x)(1 + \cos x) &= 0 \\ \sin x &= 0 \text{ or } \cos x = -1 \\ x &= 0, \pi, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 8)$, $(\pi, 0)$, and $(2\pi, 8)$. Thus, $f(x)$ has an absolute minimum at $(\pi, 0)$ and it is never negative.

45. (a)

$$\begin{aligned} 2\sin t &= \sin 2t \\ 2\sin t &= 2\sin t \cos t \\ 2(\sin t)(1 - \cos t) &= 0 \\ \sin t &= 0 \text{ or } \cos t = 1 \end{aligned}$$

$t = k\pi$, where k is an integer

The masses pass each other whenever t is an integer multiple of π seconds.

(b) The vertical distance between the objects is the absolute value of $f(x) = \sin 2t - 2\sin t$.
Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = 2\cos 2t - 2\cos t &= 0 \\ 2(2\cos^2 t - 1) - 2\cos t &= 0 \\ 2(2\cos^2 t - \cos t - 1) &= 0 \\ 2(2\cos t + 1)(\cos t - 1) &= 0 \end{aligned}$$

$$\cos t = -\frac{1}{2} \text{ or } \cos t = 1$$

$$t = \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi$$

The critical points (and endpoints) are $(0, 0)$, $\left(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$, $\left(\frac{4\pi}{3}, \frac{3\sqrt{3}}{2}\right)$, and $(2\pi, 0)$

The distance is greatest when $t = \frac{2\pi}{3}$ sec and when $t = \frac{4\pi}{3}$ sec. The distance at those times is $\frac{3\sqrt{3}}{2}$ meters.

46. (a) $\sin t = \sin\left(t + \frac{\pi}{3}\right)$

$$\sin t = \sin t \cos\frac{\pi}{3} + \cos t \sin\frac{\pi}{3}$$
$$\sin t = \frac{1}{2}\sin t + \frac{\sqrt{3}}{2}\cos t$$
$$\frac{1}{2}\sin t = \frac{\sqrt{3}}{2}\cos t$$
$$\tan t = \sqrt{3}$$

Solving for *t*, the particles meet at $t = \frac{\pi}{3}$ sec and at $t = \frac{4\pi}{3}$ sec.

(b) The distance between the particles is the absolute value of $f(t) = \sin\left(t + \frac{\pi}{3}\right) - \sin t = \frac{\sqrt{3}}{2}\cos t - \frac{1}{2}\sin t$. Find the critical points in $[0, 2\pi]$:

$$f'(t) = -\frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t = 0$$
$$-\frac{\sqrt{3}}{2}\sin t = \frac{1}{2}\cos t$$
$$\tan t = -\frac{1}{\sqrt{3}}$$

The solutions are $t = \frac{5\pi}{6}$ and $t = \frac{11\pi}{6}$, so the critical points are at $\left(\frac{5\pi}{6}, -1\right)$ and $\left(\frac{11\pi}{6}, 1\right)$, and the interval endpoints are at $\left(0, \frac{\sqrt{3}}{2}\right)$, and $\left(2\pi, \frac{\sqrt{3}}{2}\right)$. The particles are farthest apart at $t = \frac{5\pi}{6}$ sec and at $t = \frac{11\pi}{6}$ sec, and the maximum distance between the particles is 1 m.

(c) We need to maximize $f'(t)$, so we solve $f''(t) = 0$.

$$f''(t) = -\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t = 0$$
$$\frac{1}{2}\sin t = \frac{\sqrt{3}}{2}\cos t$$

This is the same equation we solved in part (a), so the solutions are $t = \frac{\pi}{3}$ sec and $t = \frac{4\pi}{3}$ sec.

For the function $y = f'(t)$, the critical points occur at $\left(\frac{\pi}{3}, -1\right)$ and $\left(\frac{4\pi}{3}, 1\right)$, and the interval endpoints are at $\left(0, -\frac{1}{2}\right)$ and $\left(2\pi, -\frac{1}{2}\right)$.

Thus, $|f'(t)|$ is maximized at $t = \frac{\pi}{3}$ and $t = \frac{4\pi}{3}$. But these are the instants when the particles pass each other, so the graph of $y = |f(t)|$ has corners at these points and $\frac{d}{dt}|f(t)|$ is undefined at these instants. We cannot say that the distance is changing the fastest at any particular instant, but we can say that near $t = \frac{\pi}{3}$ or $t = \frac{4\pi}{3}$ the distance is changing faster than at any other time in the interval.

47. The trapezoid has height $(\cos\theta)$ft and the trapezoid bases measure 1 ft and $(1 + 2\sin\theta)$ft, so the volume is given by

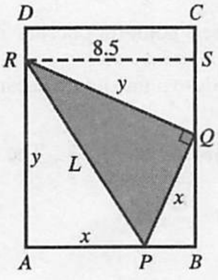
$$V(\theta) = \frac{1}{2}(\cos\theta)(1 + 1 + 2\sin\theta)(20)$$
$$= 20(\cos\theta)(1 + \sin\theta).$$

Find the critical points for $0 \le \theta < \frac{\pi}{2}$:

$$V'(\theta) = 20(\cos\theta)(\cos\theta) + 20(1 + \sin\theta)(-\sin\theta) = 0$$
$$20\cos^2\theta - 20\sin\theta - 20\sin^2\theta = 0$$
$$20(1 - \sin^2\theta) - 20\sin\theta - 20\sin^2\theta = 0$$
$$-20(2\sin^2\theta + \sin\theta - 1 = 0$$
$$-20(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2} \text{ or } \sin\theta = -1$$
$$\theta = \frac{\pi}{6}$$

The critical point is at $\left(\frac{\pi}{6}, 15\sqrt{3}\right)$. Since $V'(\theta) > 0$ for $0 \le \theta < \frac{\pi}{6}$ and $V'(\theta) < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$, the critical point corresponds to the maximum possible trough volume. The volume is maximized when $\theta = \frac{\pi}{6}$.

48. (a)



Sketch segment *RS* as shown, and let *y* be the length of segment *QR*. Note that $PB = 8.5 - x$, and so

$$QB = \sqrt{x^2 - (8.5 - x)^2} = \sqrt{8.5(2x - 8.5)}.$$

Also note that triangles *QRS* and *PQB* are similar.

$$\frac{QR}{RS} = \frac{PQ}{QB}$$
$$\frac{y}{8.5} = \frac{x}{\sqrt{8.5(2x - 8.5)}}$$

48. Continued

(a) $\frac{y^2}{8.5^2}=\frac{x^2}{8.5(2x-8.5)}$

$y^2=\frac{8.5x^2}{2x-8.5}$

$L^2=x^2+y^2$

$L^2=x^2+\frac{8.5x^2}{2x-8.5}$

$L^2=\frac{x^2(2x-8.5)+8.5x^2}{2x-8.5}$

$L^2=\frac{2x^3}{2x-8.5}$

(b) Note that $x>4.25$, and let $f(x)=L^2=\frac{2x^2}{2x-8.5}$. Since $y\le 11$, the approximate domain of f is $5.20\le x\le 8.5$. Then

$f'(x)=\frac{(2x-8.5)(6x^2)-(2x^3)(2)}{(2x-8.5)^2}=\frac{x^2(8x-51)}{(2x-8.5)^2}$

For $x>5.20$, the critical point occurs at $x=\frac{51}{8}=6.375$ in., and this corresponds to a minimum value of $f(x)$ because $f'(x)<0$ for $5.20<x<6.375$ and $f'(x)>0$ for $x>6.375$. Therefore, the value of x that minimizes L^2 is $x=6.375$ in.

(c) The minimum value of L is

$\sqrt{\frac{2(6.375)^3}{2(6.375)-8.5}}\approx 11.04$ in.

49. Since $R=M^2\left(\frac{C}{2}-\frac{M}{3}\right)=\frac{C}{2}M^2-\frac{1}{3}M^3$, we have

$\frac{dR}{dM}=CM-M^2$. Let $f(M)=CM-M^2$. Then $f'(M)=C-2M$, and the critical point for f occurs at $M=\frac{C}{2}$. This value corresponds to a maximum because $f'(M)>0$ for $M<\frac{C}{2}$ and $f'(M)<0$ for $M>\frac{C}{2}$. The value of M that maximizes $\frac{dR}{dM}$ is $M=\frac{C}{2}$.

50. The profit is given by

$P(x)=(n)(x-c)\ =\ a+b(100-x)(x-c)$

$=\ bx^2+(100+c)bx+(a-100bc)$.

Then $P'(x)=-2bx+(100+c)b$

$=b(100+c-2x)$.

The critical point occurs at $x=\frac{100+c}{2}=50+\frac{c}{2}$, and this value corresponds to the maximum profit because

$P'(x)>0$ for $x<50+\frac{c}{2}$ and $P'(x)<0$ for $x>50+\frac{c}{2}$.

A selling price of $50+\frac{c}{2}$ will bring the maximum profit.

51. True. This is guaranteed by the Extreme Value Theorem (Section 4.1).

52. False. For example, consider $f(x)=x^3$ at $c=0$.

53. D. $f(x)=x^2(60-x)$

$f'(x)=x^2(-1)+(60-x)(2x)$

$=-x^2+120x-2x^2$

$=-3x^2+120x$

$=-3x(x-40)$

$x=0$ or $x=40$

$60-x=60$ $\quad$ $60-x=20$

$x^2(60-x)=0$

$(40)^2(20)=(1600)(20)$

$=32{,}000$

54. B. Since $f'(x)$ is negative, $f(x)$ is always decreasing, so $f(25)=3$.

55. B. $A=\frac{1}{2}bh$

$b^2+h^2=100$

$b=\sqrt{100-h^2}$

$A=\frac{h}{2}\sqrt{100-h^2}$

$A'=\frac{\sqrt{100-h^2}}{2}-\frac{h^2}{2\sqrt{100-h^2}}$

$A'=0$ when $h=\sqrt{50}$

$b=\sqrt{100-\sqrt{50}^2}=\sqrt{50}$

$A_{max}=\frac{1}{2}\sqrt{50}\sqrt{50}=25$

56. E. length $=2x$

height $=30-x^2-4x^2=30-5x^2$

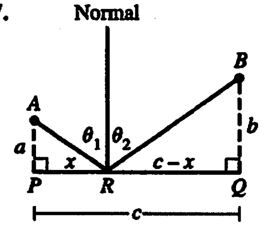
$2x(30-5x^2)=60x-10x^3$

$\frac{dA}{dx}(60x-10x^3)=60-30x^2$

$x=\sqrt{2}$

$2\sqrt{2}(30-\sqrt{2}^2-4(\sqrt{2})^2)=40\sqrt{2}$.

57.



Let P be the foot of the perpendicular from A to the mirror, and Q be the foot of the perpendicular from B to the mirror.

57. Continued

Suppose the light strikes the mirror at point R on the way from A to B. Let:

a = distance from A to P
b = distance from B to Q
c = distance from P to Q
x = distance from P to R

To minimize the time is to minimize the total distance the light travels going from A to B. The total distance is

$$D(x)=\sqrt{x^2+a^2}+\sqrt{(c-x)^2+b^2}$$

Then

$$D'(x)=\frac{1}{2\sqrt{x^2+a^2}}(2x)+\frac{1}{2\sqrt{(c-x)^2+b^2}}[-2(c-x)]$$
$$=\frac{x}{\sqrt{x^2+a^2}}-\frac{c-x}{\sqrt{(c-x)^2+b^2}}$$

Solving $D'(x)=0$ gives the equation

$\frac{x}{\sqrt{x^2+a^2}}=\frac{c-x}{\sqrt{(c-x)^2+b^2}}$ which we will refer to as Equation 1. Squaring both sides, we have:

$$\frac{x^2}{x^2+a^2}=\frac{(c-x)^2}{(c-x)^2+b^2}$$
$$x^2\left[(c-x)^2+b^2\right]=(c-x)^2(x^2+a^2)$$
$$x^2(c-x)^2+x^2b^2=(c-x)^2x^2+(c-x)^2a^2$$
$$x^2b^2=(c-x)^2a^2$$
$$x^2b^2=\left[c^2-2xc+x^2\right]a^2$$
$$0=(a^2-b^2)x^2-2a^2cx+a^2c^2$$
$$0=[(a+b)x-ac][(a-b)x-ac]$$
$$x=\frac{ac}{a+b} \text{ or } x=\frac{ac}{a-b}$$

Note that the value $x=\frac{ac}{a-b}$ is an extraneous solution because x and $c-x$ have opposite signs for this value. The only critical point occurs at $x=\frac{ac}{a+b}$.

To verify that critical point represents the minimum distance, note that

$$D''(x)=\frac{(\sqrt{x^2+a^2})(1)-(x)\left(\frac{x}{\sqrt{x^2+a^2}}\right)}{x^2+a^2}-\frac{(\sqrt{(c-x)^2+b^2})(-1)-(c-x)\left(\frac{-(c-x)}{\sqrt{(c-x)^2+b^2}}\right)}{(c-x)^2+b^2}$$
$$=\frac{(x^2+a^2)-x^2}{(x^2+a^2)^{3/2}}-\frac{-[(c-x)^2+b^2]+(c-x)^2}{[(c-x)^2+b^2]^{3/2}}$$
$$=\frac{a^2}{(x^2+a^2)^{3/2}}+\frac{b^2}{[(c-x)^2+b^2]^{3/2}},$$

which is always positive.

We now know that $D(x)$ is minimized when Equation 1 is true, or, equivalently, $\frac{PR}{AR}=\frac{QR}{BR}$. This means that the two right triangles APR and BQR are similar, which in turn implies that the two angles must be equal.

58. $\frac{dv}{dx}=ka-2kx$

The critical point occurs at $x=\frac{ka}{2k}=\frac{a}{2}$, which represents a maximum value because $\frac{d^2v}{dx^2}=-2k$, which is negative for all x. The maximum value of v is

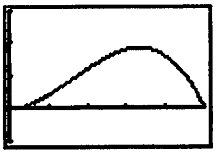
$$kax-kx^2=ka\left(\frac{a}{2}\right)-k\left(\frac{a}{2}\right)^2=\frac{ka^2}{4}.$$

59. (a) $v=cr_0r^2-cr^3$

$$\frac{dv}{dr}=2cr_0r-3cr^2=cr(2r_0-3r)$$

The critical point occurs at $r=\frac{2r_0}{3}$. (Note that $r=0$ is not in the domain of v.) The critical point represents a maximum because $\frac{d^2v}{dr^2}=2cr_0-6cr=2c(r_0-3r)$, which is negative in the domain $\frac{r_0}{2}\le r\le r_0$.

(b) We graph $v=(0.5-r)r^2$, and observe that the maximum indeed occurs at $v=\left(\frac{2}{3}\right)0.5=\frac{1}{3}$.



[0, 0.5] by [−0.01, 0.03]

60. (a) Since $A''(q)=-kmq^{-2}+\frac{h}{2}$, the critical point occurs when $\frac{km}{q^2}=\frac{h}{2}$, or $q=\frac{\sqrt{2km}}{h}$. This corresponds to the minimum value of $A(q)$ because $A''(q)=2kmq^{-3}$, which is positive for $q>0$.

(b) The new formula for average weekly cost is

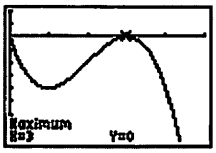
$$B(q)=\frac{(k+bq)m}{q}+cm+\frac{hq}{2}$$
$$=\frac{km}{q}+bm+cm+\frac{hq}{2}$$
$$=A(q)+bm$$

Since $B(q)$ differs from $A(q)$ by a constant, the minimum value of $B(q)$ will occur at the same q-value as the minimum value of $A(q)$. The most economical quantity is again $\frac{\sqrt{2km}}{h}$.

61. The profit is given by

$$p(x) = r(x) - c(x)$$
$$= 6x - (x^3 - 6x^2 + 15x)$$
$$= -x^3 + 6x^2 - 9x, \text{ for } x \geq 0.$$

Then $p'(x) = -3x^2 + 12x - 9 = -3(x-1)(x-3)$, so the critical points occur at $x = 1$ and $x = 3$. Since $p'(x) < 0$ for $0 \leq x < 1$, $p'(x) > 0$ for $1 < x < 3$, and $p'(x) < 0$ for $x > 3$, the relative maxima occur at the endpoint $x = 0$ and at the critical point $x = 3$. Since $p(0) = p(3) = 0$, this means that for $x \geq 0$, the function $p(x)$ has its absolute maximum value at the points (0, 0) and (3, 0). This result can also be obtained graphically, as shown.



[0, 5] by [−8, 2]

62. The average cost is given by

$a(x) = \dfrac{c(x)}{x} = x^2 - 20x + 20{,}000$. Therefore,

$a'(x) = 2x - 20$ and the critical value is $x = 10$, which represents the minimum because $a''(x) = 2$, which is positive for all x. The average cost is minimized at a production level of 10 items.

63. **(a)** According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$.

Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0)$ implies that $c = 0$. Therefore, $y(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. Then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 - 2bL = 0$, so we have two linear equations in the two unknowns a and b. The second equation gives $b = \dfrac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \dfrac{3aL^3}{2} = H$, or

$\dfrac{aL^3}{2} = H$, so $a = 2\dfrac{H}{L^3}$. Therefore, $b = 3\dfrac{H}{L^2}$ and the equation for y

is $y(x) = 2\dfrac{H}{L^3}x^3 + 3\dfrac{H}{L^2}x^2$, or

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$

64. **(a)** The base radius of the cone is $r = \dfrac{2\pi a - x}{2\pi}$ and so the height is $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \left(\dfrac{2\pi a - x}{2\pi}\right)^2}$. Therefore,

$$V(x) = \frac{\pi}{3}r^2 h = \frac{\pi}{3}\left(\frac{2\pi a - x}{2\pi}\right)^2 \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}.$$

(b) To simplify the calculations, we shall consider the volume as a function of r:

volume $= f(r) = \dfrac{\pi}{3}r^2\sqrt{a^2 - r^2}$, where $0 < r < a$.

$$f'(r) = \frac{\pi}{3}\frac{d}{dr}(r^2\sqrt{a^2 - r^2})$$
$$= \frac{\pi}{3}\left[r^2 \cdot \frac{1}{2\sqrt{a^2 - r^2}} \cdot (-2r) + (\sqrt{a^2 - r^2})(2r)\right]$$
$$= \frac{\pi}{3}\left[\frac{-r^3 + 2r(a^2 - r^2)}{\sqrt{a^2 - r^2}}\right]$$
$$= \frac{\pi}{3}\left[\frac{(2a^2 r - 3r^2)}{\sqrt{a^2 - r^2}}\right]$$
$$= \frac{\pi r(2a^2 - 2r^2)}{3\sqrt{a^2 - r^2}}$$

The critical point occurs when $r^2 = \dfrac{2a^2}{3}$, which gives $r = a\sqrt{\dfrac{2}{3}} = \dfrac{a\sqrt{6}}{3}$. Then

$h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \dfrac{2a^2}{3}} = \sqrt{\dfrac{a^2}{3}} = \dfrac{a\sqrt{3}}{3}$. Using

$r = \dfrac{a\sqrt{6}}{3}$ and $h = \dfrac{a\sqrt{3}}{3}$,

we may now find the values of r and h for the given values of a

when $a = 4: r = \dfrac{4\sqrt{6}}{3}, h = \dfrac{4\sqrt{3}}{3}$;

when $a = 5: r = \dfrac{5\sqrt{6}}{3}, h = \dfrac{5\sqrt{3}}{3}$;

when $a = 6: r = 2\sqrt{6}, h = 2\sqrt{3}$;

when $a = 8: r = \dfrac{8\sqrt{6}}{3}, h = \dfrac{8\sqrt{3}}{3}$

(c) Since $r = \dfrac{a\sqrt{6}}{3}$ and $h = \dfrac{a\sqrt{3}}{3}$, the relationship is $\dfrac{r}{h} = \sqrt{2}$.

65. **(a)** Let x_0 represent the fixed value of x at point P, so that P has coordinates (x_0, a) and let $m = f'(x_0)$ be the slope of line RT. Then the equation of line RT is $y = m(x - x_0) + a$. The y-intercept of this line is $m(0 - x_0) + a = a - mx_0$, and the x-intercept is the solution of $m(x - x_0) + a = 0$, or $x = \dfrac{mx_0 - a}{m}$. Let O designate the origin. Then (Area of triangle RST)

65. Continued

(a) $= 2$ (Area of triangle ORT)

$= 2 \cdot \frac{1}{2}$ (x-intercept of line RT) (y-intercept of line RT)

$$= 2 \cdot \frac{1}{2}\left(\frac{mx_0 - a}{m}\right)(a - mx_0)$$

$$= -m\left(\frac{mx_0 - a}{m}\right)\left(\frac{mx_0 - a}{m}\right)$$

$$= -\left(\frac{mx_0 - a}{m}\right)^2$$

$$= -m\left(x_0 - \frac{a}{m}\right)^2$$

Substituting x for x_0, $f'(x)$ for m, and $f(x)$ for a, we have $A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$.

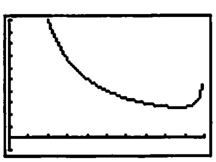
(b) The domain is the open interval (0, 10).

To graph, let $y_1 = f(x) = 5 + 5\sqrt{1 - \frac{x^2}{100}}$,

$y_2 = f'(x) = \text{NDER}(y_1)$, and

$y_3 = A(x) = -y_2\left(x - \frac{y_1}{y^2}\right)^2$.

The graph of the area function $y_3 = A(x)$ is shown below.



[0, 10] by [−100, 1000]

The vertical asymptotes at $x = 0$ and $x = 10$ correspond to horizontal or vertical tangent lines, which do not form triangles.

(c) Using our expression for the y-intercept of the tangent line, the height of the triangle is

$$a - mx = f(x) - f'(x) \cdot x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} - \frac{-x}{2\sqrt{100 - x^2}}x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} + \frac{x^2}{2\sqrt{100 - x^2}}$$

We may use graphing methods or the analytic method in part (d) to find that the minimum value of $A(x)$ occurs at $x \approx 8.66$. Substituting this value into the expression above, the height of the triangle is 15. This is 3 times the y-coordinate of the center of the ellipse.

(d) Part (a) remains unchanged. The domain is $(0, C)$. To graph, note that

$f(x) = B + B\sqrt{1 - \frac{x^2}{C^2}} = B + \frac{B}{C}\sqrt{C^2 - x^2}$ and

$$f'(x) = \frac{B}{C}\frac{1}{2\sqrt{C^2 - x^2}}(-2x) = \frac{-Bx}{C\sqrt{C^2 - x^2}}.$$

Therefore, we have

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{B + \frac{B}{C}\sqrt{C^2 - x^2}}{\frac{-Bx}{C\sqrt{C^2 - x^2}}}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{(BC + B\sqrt{C^2 - x^2})\sqrt{C^2 - x^2}}{-Bx}\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + (BC + B\sqrt{C^2 - x^2})(\sqrt{C^2 - x^2})\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + BC\sqrt{C^2 - x^2} + B(C^2 - x^2\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[BC(C + \sqrt{C^2 - x^2})\right]^2$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})^2}{x\sqrt{C^2 - x^2}}$$

$$A'(x) = BC \cdot \frac{(x\sqrt{C^2 - x^2})(2)(C + \sqrt{C^2 - x^2})\left(\frac{-x}{\sqrt{C^2 - x^2}}\right) - (C + \sqrt{C^2 - x^2})^2\left(x\frac{-x}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}(1)\right)}{x^2(C^2 - x^2)}$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)}\left[-2x^2 - (C + \sqrt{C^2 - x^2})\left(\frac{-x^2}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}\right)\right]$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2\sqrt{C^2 - x^2}}\left[-2x^2 + \frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} + x^2 - (C^2 - x^2)\right]$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)}\left(\frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} - C^2\right)$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)^{3/2}}\left[Cx^2 - C(C^2 - x^2) - C^2\sqrt{C^2 - x^2}\right]$$

$$= \frac{BC^2(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)^{3/2}}(2x^2 - C^2 - C\sqrt{C^2 - x^2})$$

To find the critical points for $0 < x < C$, we solve:

$$2x^2 - C^2 = C\sqrt{C^2 - x^2}$$

$$4x^4 - 4C^2x^2 + C^4 = C^4 = C^2x^2$$

$$4x^4 - 3C^2x^2 = 0$$

$$x^2(4x^2 - 3C^2) = 0$$

The minimum value of $A(x)$ for $0 < x < C$ occurs at the critical point $x = \frac{C\sqrt{3}}{2}$, or $x^2 = \frac{3C^2}{4}$. The corresponding triangle height is

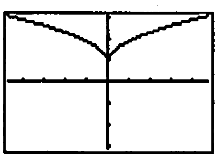
65. Continued

$$a-mx=f(x)-f'(x)\cdot x$$
$$=B+\frac{B}{C}\sqrt{C^2-x^2}+\frac{Bx^2}{C\sqrt{C^2-x^2}}$$
$$=B+\frac{B}{C}\sqrt{C^2-\frac{3C^2}{4}}+\frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2-\frac{3C^2}{4}}}$$
$$=B+\frac{B}{C}\left(\frac{C}{2}\right)+\frac{\frac{3BC^2}{4}}{\frac{C^2}{2}}$$
$$=B+\frac{B}{2}+\frac{3B}{2}$$
$$=3B$$

This shows that the triangle has minimum area when its height is $3B$.

Section 4.5 Linerization and Newton's Method (pp. 233–245)

Exploration 1 Appreciating Local Linearity

1. 

$y=(x^2+0.0001)^{1/4}+0.9$

The function appears to come to a point.

2. $f^1(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$$=\lim_{x\to a}\frac{(x^2+0.0001)^{1/4}+0.9-((0+0.0001)^{1/4}+0.9)}{x-0}$$
$$=\lim_{x\to a}\frac{(x^2+0.0001)^{1/4}-0.1}{x}=0$$

$f(x)$ is differentiable at $x=0$, and the equation of the tangent line is $y=1$.

3. The graph of the function at that point seems to become the graph of a straight line with repeated zooming.

4. The graph will eventually look like the tangent line.

Exploration 2 Using Newton's Method on Your Calculator

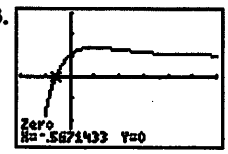
See text page 237.

Quick Review 4.5

1. $\frac{dy}{dx}=\cos(x^2+1)\cdot\frac{d}{dx}(x^2+1)=2x\cos(x^2+1)$

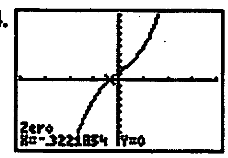
2. $\frac{dy}{dx}=\frac{(x+1)(1-\sin x)-(x+\cos x)(1)}{(x+1)^2}$

$$=\frac{x-x\sin x+1-\sin x-x-\cos x}{(x+1)^2}$$
$$=\frac{1-\cos x-(x+1)\sin x}{(x+1)^2}$$

3. 

$[-2, 6]$ by $[-3, 3]$

$x\approx-0.567$

4. 

$[-4, 4]$ by $[-10, 10]$

$x\approx-0.322$

5. $f'(x)=(x)(-e^{-x})+(e^{-x})(1)=e^{-x}-xe^{-x}$

$f'(0)=1$

The lines passes through (0, 1) and has slope 1. Its equation is $y=x+1$.

6. $f'(x)=(x)(-e^{-x})+(e^{-x})(1)=e^{-x}-xe^{-x}$

$f'(-1)=e^1-(-e^1)=2e$

The lines passes through $(-1, -e+1)$ and has slope $2e$. Its equation is $y=2e(x+1)+(-e+1)$, or $y=2ex+e+1$.

7. (a) $x+1=0$

$x=-1$

(b) $2ex+e+1=0$

$2ex=-(e+1)$

$x=-\frac{e+1}{2e}\approx-0.684$