54. E. $n = 2k, \lim_{n\to\infty}\left((-1)^n \frac{3n-1}{n+2}\right) = 1$

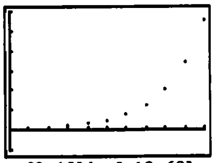
$n = 2k-1, \lim_{n\to\infty}\left((-1)^n \frac{3n-1}{n+2}\right) = -1$

55. (b) $\lim_{n\to\infty}\left(2n \sin\left(\frac{\pi}{n}\right)\right)$

$= \lim_{n\to\infty}\left(2n\left(\frac{\pi}{n}\right)\right) = 2\pi$

56. (a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b)



[0, 10] by [–10, 60]

57. $a_n = ar^{n-1}$ implies that $\log a_n = \log a + (n-1)\log r$. Thus $\{\log a_n\}$ is an arithmetic sequence with first term $\log a$ and common ratio $\log r$.

58. $a_n = a + (n-1)d$ implies that $10^{a_n} = 10^{a+(n-1)d} = 10^a (10^d)^{n-1}$. Thus $\{10^{a_n}\}$ is a geometric sequence with first term 10^a and common ratio 10^d.

59. Given $\varepsilon > 0$ choose $M = \frac{1}{\varepsilon}$. Then $\left|\frac{1}{n} - 0\right| < \varepsilon$ if $n > M$.

Section 8.2 L'Hôpital's Rule (pp. 444–452)

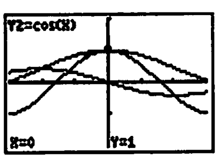
Exploration 1 Exploring L'Hôpital's Rule Graphically

1. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

2. The two graphs suggest that $\lim_{x\to 0}\frac{y_1}{y_2} = \lim_{x\to 0}\frac{y_1'}{y_2'}$.

3. $y_5 = \frac{x\cos x - \sin x}{x^2}$. The graphs of y_3 and y_5 clearly show that l'Hôpital's Rule does not say that $\lim_{x\to 0}\frac{y_1}{y_2}$ is equal to

$\lim_{x\to 0}\left(\frac{y_1}{y_2}\right)'$.



[–3, 3] by [–2, 2]

Quick Review 8.2

1.

x	$\left(1+\frac{0.1}{x}\right)^x$
1	1.1000
10	1.1046
100	1.1051
1000	1.1052
10,000	1.1052
1,000,000	1.1052

As $x \to \infty$, $\left(1+\frac{0.1}{x}\right)^x$ approach 1.1052.

2.

x	$x^{1/(\ln x)}$
0.1	2.7183
0.01	2.7183
0.001	2.7183
0.0001	2.7183
0.00001	2.7183

As $x \to 0^+$, $x^{1/(\ln x)}$ approaches 2.7183.

3.

x	$\left(1-\frac{1}{x}\right)^x$
−1	0.5
−0.1	0.78679
−0.01	0.95490
−0.001	0.99312
−0.0001	0.99908
−0.00001	0.99988
−0.000001	0.99999

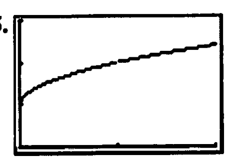
As $x \to 0^-$, $\left(1-\frac{1}{x}\right)^x$ approaches 1.

4.

x	$\left(1+\frac{1}{x}\right)^x$
−1.1	13.981
−1.01	105.77
−1.001	1007.9
−1.0001	10010

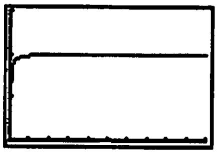
As $x \to -1^-$, $\left(1+\frac{1}{x}\right)^x$ goes to ∞.

5.



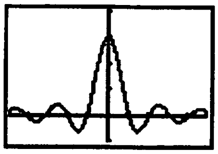
[0, 2] by [0, 3]

As $t \to 1$, $\frac{t-1}{\sqrt{t}-1}$ approaches 2.

6. 

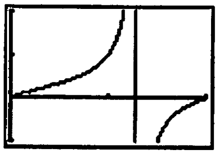
[0, 500] by [0, 3]

As $x \to \infty$, $\frac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

7. 

[−5, 5] by [−1, 4]

As $x \to 0$, $\frac{\sin 3x}{x}$ approaches 3.

8. 

[0, π] by [−1, 2]

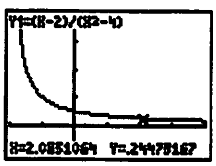
As $\theta \to \frac{\pi}{2}$, $\frac{\tan\theta}{2+\tan\theta}$ approaches 1.

9. $y = \frac{1}{h}\sin h$

10. $y = (1+h)^{1/h}$

Section 8.2 Exercises

1. $\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)$ appears to be about $\frac{1}{4}$;

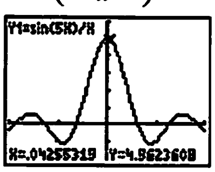


[−2, 4] by [−1, 4]

By L'Hôpital's Rule:

$$\lim_{x\to 2} = \left(\frac{x-2}{x^2-4}\right) = \lim_{x\to 2}\frac{1}{2(2)} = \frac{1}{4}$$

2. $\lim_{x\to 0} = \left(\frac{\sin(5x)}{x}\right)$ appears to be about 5;

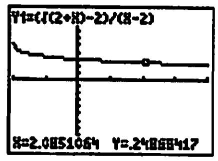


[−2, 2] by [−2, 6]

By L'Hôpital's Rule:

$$\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right) = \lim_{x\to 0}\left(\frac{5\cos(5(0))}{1}\right) = 5$$

3. $\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right)$ appears to be about $\frac{1}{4}$;

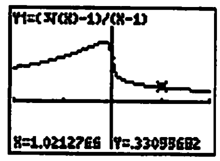


[−2, 4] by [−1, 1]

By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right) = \lim_{x\to 2}\left(\frac{\frac{1}{2}(2+2)^{-1/2}}{1}\right) = \frac{1}{4}$$

4. $\lim_{x\to 1}\left(\frac{\sqrt[3]{x}-1}{x-1}\right)$ appears to be about $\frac{1}{3}$;



[−2, 2] by [−1, 2]

By L'Hôpital's Rule:

$$\lim_{x\to 1}\left(\frac{\sqrt[3]{x}-1}{x-2}\right) = \lim_{x\to 1}\left(\frac{\frac{1}{3}(1)^{-2/3}}{1}\right) = \frac{1}{3}$$

5. $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \lim_{x\to 0}\left(\frac{\sin x}{2x}\right) = \lim_{x\to 0}\left(\frac{\cos(0)}{2}\right) = \frac{1}{2}$

6. $\lim_{\theta\to\pi/2}\left(\frac{1-\sin\theta}{1+\cos(2\theta)}\right) = \lim_{\theta\to\pi/2}\left(\frac{-\cos\theta}{-2\sin(2\theta)}\right)$

$$= \lim_{\theta\to\pi/2}\left(\frac{-\sin\frac{\pi}{2}}{-4\cos\left(2\frac{\pi}{2}\right)}\right) = \frac{1}{4}$$

7. $\lim_{t\to 0}\left(\frac{\cos t-1}{e^t-t-1}\right) = \lim_{t\to 0}\left(\frac{-\sin t}{e^t-1}\right) = \lim_{t\to 0}\left(\frac{-\cos 0}{e^0}\right) = -1$

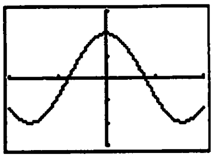
8. $\lim_{x\to 2}\left(\frac{x^2-4x+4}{x^3-12x+16}\right) = \lim_{x\to 2}\left(\frac{2x-4}{3x^2-12}\right) = \lim_{x\to 2}\left(\frac{2}{6(2)}\right) = \frac{1}{6}$

9. (a) $\lim_{x\to 0^-}\left(\frac{\sin 4x}{\sin 2x}\right) = \lim_{x\to 0^-}\left(\frac{4\cos(4(0))}{2\cos(2(0))}\right) = 2$

(b) $\lim_{x\to 0^+}\left(\frac{\sin 4x}{\sin 2x}\right) = \lim_{x\to 0^+}\left(\frac{4\cos(4(0))}{2\cos(2(0))}\right) = 2$

9. Continued

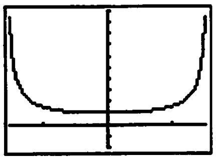
(b)



[–2, 2] by [–3, 3]

10. (a) $\lim_{x\to 0^-}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{1}\right)=1$

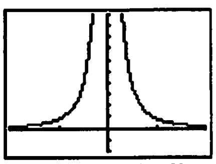
(b) $\lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{1}\right)=1$



[–1.5, 1.5] by [–2, 10]

11. (a) $\lim_{x\to 0^-}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^-}\left(\frac{\cos(0)}{3(0)^2}\right)=\infty$

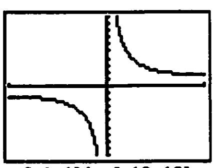
(b) $\lim_{x\to 0^+}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^+}\left(\frac{\cos(0)}{3(0)^2}\right)=\infty$



[–2, 2] by [–2, 10]

12. (a) $\lim_{x\to 0^-}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2(0)}{2(-0)}\right)=-\infty$

(b) $\lim_{x\to 0^+}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2(0)}{2(+0)}\right)=\infty$



[–1, 1] by [–10, 10]

13. Left:

$\lim_{x\to \pi^-}\left(\frac{\csc x}{1+\cot x}\right)=\frac{\infty}{-\infty}$

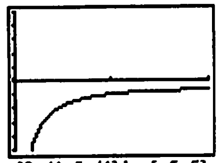
$\lim_{x\to \pi^-}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=-1$

Right:

$\lim_{x\to \pi^+}\left(\frac{\csc x}{1+\cot x}\right)=\frac{-\infty}{\infty}$

$\lim_{x\to \pi^+}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=-1$

limit = –1



[3π/4, 5π/4] by [–5, 5]

14. Left:

$\lim_{x\to \pi/2^-}\left(\frac{1+\sec x}{\tan x}\right)=\frac{\infty}{\infty}$

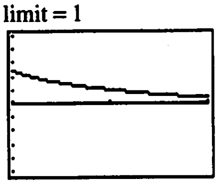
$\lim_{x\to \pi/2^-}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=1$

Right:

$\lim_{x\to \pi/2^+}\left(\frac{1+\sec x}{\tan x}\right)=\frac{-\infty}{-\infty}$

$\lim_{x\to \pi/2^+}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=1$

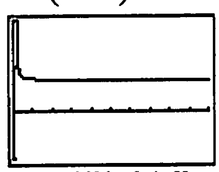
limit = 1



[π/4, 3π/4] by [–5, 5]

15. $\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\frac{\infty}{\infty}$

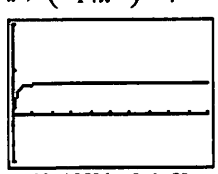
$\lim_{x\to\infty}\left(\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}\right)=\ln 2$



[0, 100] by [–1, 2]

16. $\lim_{x\to\infty}\left(\frac{5x^2-3x}{7x^2+1}\right)=\frac{\infty}{\infty}$

$\lim_{x\to\infty}\left(\frac{10x-3}{14x}\right)=\frac{5}{7}$



[0, 100] by [–1, 2]

17. $\lim_{x\to 0^+}(x\ln x)=0\bullet\infty$

$\lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right)=\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}(-x)=0$

18. $\lim_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right)=\infty\cdot 0$

$\lim_{x\to\infty}\left(\frac{1}{h}\tan(h)\right)=\lim_{h\to\infty}\left(\frac{1}{h}h\right)=1$

19. $\lim_{x\to 0^+}(\csc x-\cot x+\cos x)=\infty-\infty$

$\lim_{x\to 0^+}\left(\frac{1-\cos x+\cos x\sin x}{\sin x}\right)=\lim_{x\to 0^+}\left(\frac{\sin x-\sin^2 x+\cos^2 x}{\cos x}\right)=1$

20. $\lim_{x\to\infty}(\ln(2x)-\ln(x+1))=\infty-\infty$

$\lim_{x\to\infty}\left(\frac{\ln(2x)}{\ln(x+1)}\right)=\ln(2)$

21. $\lim_{x\to 0}(e^x+x)^{1/x}=(1+0)^\infty=1^\infty$

$\lim_{x\to 0}\left(\frac{\ln(e^x+x)}{x}\right)=\lim_{x\to 0}\left(\frac{x+\ln x}{x}\right)=\lim_{x\to 0}\left(\frac{1+\frac{1}{x}}{1}\right)=2$

$\lim_{x\to 0}e^{\ln f(x)}=e^2$

22. $\lim_{x\to 1}(x^{1/(x-1)})=1^\infty$

$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{1/x}{1}=1$

$\lim_{x\to 1}e^{\ln f(x)}=e^1=e$

23. $\lim_{x\to 1}(x^2-2x+1)^{x-1}=(1^2-2(1)+1)^{1-1}=0^0$

$\lim_{x\to 1}\frac{\ln(x^2-2x+1)}{1/x-1}=\lim_{x\to 1}\frac{\frac{2}{x-1}}{-1/(x-1)^2}=\lim_{x\to 1}\left(\frac{-2(x-1)^2}{x-1}\right)=0$

$\lim_{x\to 1}e^{\ln f(x)}=e^0=1$

24. $\lim_{x\to 0^+}(\sin x)^x=0^0$

$\lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x}=\lim_{x\to 0^+}\left(\frac{\frac{1}{\tan x}}{-1/x^2}\right)=\lim_{x\to 0^+}\left(-\frac{x^2}{\tan x}\right)$

$=\lim_{x\to 0^+}\left(-\frac{2x}{\sec^2 x}\right)=-\frac{0}{1}=0$

$\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$

25. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=(1+\infty)^0=\infty^0$

$\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0^+}\frac{\ln(1+\frac{1}{x})}{1/x}=\lim_{x\to 0^+}\left(\frac{-\frac{1}{x(x+1)}}{-1/x^2}\right)$

$=\lim_{x\to 0^+}\frac{x^2}{x(x+1)}$

$=\lim_{x\to 0^+}\frac{x}{x+1}=0$

$\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$

26. $\lim_{x\to\infty}(\ln x)^{1/x}=\infty^\infty$

$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0$

$\lim_{x\to\infty}e^{\ln f(x)}=e^0=1$

27. (a)

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.1513	0.2303	0.0345	0.00461	0.00058

Estimate the limit to be 0.

(b) $\lim_{x\to\infty}\frac{\ln x^5}{x}=\lim_{x\to\infty}\frac{5\ln x}{x}=\lim_{x\to\infty}\frac{5/x}{1}=\frac{0}{1}=0$

28. (a)

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$	0.1585	0.1666	0.1667	0.1667	0.1667

Estimate the limit to be $\frac{1}{6}$.

(b) $\lim_{x\to 0^+}\frac{x-\sin x}{x^3}=\lim_{x\to 0^+}\frac{1-\cos x}{3x^2}$

$=\lim_{x\to 0^+}\frac{\sin x}{6x}$

$=\lim_{x\to 0^+}\frac{\cos x}{6}$

$=\frac{1}{6}$

29. Let $f(\theta)=\frac{\sin 3\theta}{\sin 4\theta}$.

θ	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$	$\pm 10^{-4}$
$f(\theta)$	−0.1865	0.7589	0.7501	0.7500	0.7500

Estimate the limit to be $=\frac{3}{4}$.

$\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin 4\theta}=\lim_{\theta\to 0}\frac{3\cos 3\theta}{4\cos 4\theta}=\frac{3}{4}$

30. Let $f(t)=\frac{1}{\sin t}-\frac{1}{t}=\frac{t-\sin t}{t\sin t}$.

t	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$
$f(t)$	± 0.1884	± 0.0167	± 0.0017	± 0.00017

Estimate the limit to be 0.

$\lim_{t\to 0}\left(\frac{1}{\sin t}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{t-\sin t}{t\sin t}$

$=\lim_{t\to 0}\frac{1-\cos t}{t\cos t+\sin t}$

$=\lim_{t\to 0}\frac{\sin t}{-t\sin t+\cos t+\cos t}=0$

31. Let $f(x) = (1+x)^{1/x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.2710	1.0472	1.0069	1.0009	1.0001

Estimate the limit to be 1.

$\ln f(x) = \dfrac{\ln(1+x)}{x}$

$$\lim_{x\to\infty} \frac{\ln(1+x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = \frac{0}{1} = 0$$

$$\lim_{x\to\infty} (1+x)^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

32. Let $f(x) = \dfrac{x-2x^2}{3x^2+5x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	-0.5429	-0.6525	-0.6652	-0.6665	-0.6667

Estimate the limit to be $-\dfrac{2}{3}$.

$$\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x} = \lim_{x\to\infty} \frac{1-4x}{6x+5} = \lim_{x\to\infty} -\frac{4}{6} = -\frac{2}{3}.$$

33. $$\lim_{\theta\to 0} \frac{\sin\theta^2}{\theta} = \lim_{\theta\to 0} \frac{2\theta\cos\theta^2}{1} = (2)(0)\cos(0)^2 = 0$$

34. $$\lim_{t\to 1} \frac{t-1}{\ln t - \sin \pi t} = \lim_{t\to 1} \frac{1}{\frac{1}{t} - \pi\cos \pi t} = \frac{1}{1-\pi(-1)} = \frac{1}{\pi+1}$$

35. $$\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x\to\infty} \frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}} = \lim_{x\to\infty} \frac{(x+3)\ln 3}{x\ln 2} = \lim_{x\to\infty} \frac{x\ln 3 + 3\ln 3}{x\ln 2} = \lim_{x\to\infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}$$

36. $$\lim_{y\to 0^+} \frac{\ln(y^2+2y)}{\ln y} = \lim_{y\to 0^+} \frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}} = \lim_{y\to 0^+} \frac{y(2y+2)}{y^2+2y} = \lim_{y\to 0^+} \frac{(2y^2+2y)}{y^2+2y} = \lim_{y\to 0^+} \frac{4y+2}{2y+2} = \frac{4(0)+2}{2(0)+2} = \frac{2}{2} = 1$$

37. $$\lim_{y\to\pi/2} \left(\frac{\pi}{2} - y\right)\tan y = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y} = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2}-y\right)\cos y + (-1)\sin y}{-\sin y} = \frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2} + (-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = \frac{(-1)(1)}{-(1)} = 1$$

38. $$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \ln\frac{x}{\sin x}$$

Let $f(x) = \dfrac{x}{\sin x}$.

$\displaystyle\lim_{x\to 0^+} \frac{x}{\sin x} = \lim_{x\to 0^+} \frac{1}{\cos x} = 1$ Therefore,

$$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \ln f(x) = \ln 1 = 0$$

39. $$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim_{x\to 0^+} \frac{1-\sqrt{x}}{x} = \infty$$

40. The limit leads to the indeterminate form ∞^0.

Let $f(x) = \left(\dfrac{1}{x^2}\right)^x$.

$$\ln\left(\frac{1}{x^2}\right)^x = x\ln\left(\frac{1}{x^2}\right) = \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{-2/x^3}{1/x^2}}{-1/x^2} = \lim_{x\to 0} 2x = 0$$

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)^x = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$$

41. $$\lim_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2} = \lim_{x\to\pm\infty} \frac{3}{4x-1} = 0$$

42. $\lim_{x\to 0}\frac{\sin 7x}{\tan 11x}=\lim_{x\to 0}\frac{7\cos 7x}{11\sec^2 11x}=\frac{7}{11}$

43. The limit leads to the indeterminate form ∞^0.

Let $f(x)=(1+2x)^{1/(2\ln x)}$.

$\ln(1+2x)^{1/(2\ln x)}=\frac{\ln(1+2x)}{2\ln x}$

$\lim_{x\to\infty}\frac{\ln(1+2x)}{2\ln x}=\lim_{x\to\infty}\frac{\frac{2}{1+2x}}{\frac{2}{x}}=\lim_{x\to\infty}\frac{x}{1+2x}=\lim_{x\to\infty}\frac{1}{2}=\frac{1}{2}$

$\lim_{x\to\infty}(1+2x)^{1/(2\ln x)}=\lim_{x\to\infty}e^{\ln f(x)}=e^{1/2}=\sqrt{e}$

44. The limit leads to the indeterminate form 0^0.

Let $f(x)=(\cos x)^{\cos x}$.

$\ln(\cos x)^{\cos x}=(\cos x)\ln(\cos x)=\frac{\ln(\cos x)}{\sec x}$

$\lim_{x\to\pi/2^-}\frac{\ln(\cos x)}{\sec x}=\lim_{x\to\pi/2^-}\frac{\frac{-\sin x}{\cos x}}{\sec x\tan x}$

$=\lim_{x\to\pi/2^-}\frac{-\tan x}{\sec x\tan x}$

$=\lim_{x\to\pi/2^-}-\cos x=0$

$\lim_{x\to\frac{\pi}{2}^-}=(\cos x)^{\cos x}=\lim_{x=\frac{\pi}{2}^-}e^{\ln f(x)}=e^0=1$

45. The limit leads to the indeterminate form 1^∞.

Let $f(x)=(1+x)^{1/x}$.

$\ln(1+x)^{1/x}=\frac{\ln(1+x)}{x}$

$\lim_{x\to 0^+}\frac{\ln(1+x)}{x}=\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}=1$

$\lim_{x\to 0^+}(1+x)^{1/x}=\lim_{x\to 0^+}e^{\ln f(x)}=e^1=e$

46. The limit leads to the indeterminate form

Let $f(x)=(\sin x)^{\tan x}$

$\ln(\sin x)^{\tan x}=\tan x\ln(\sin x)=\frac{\ln(\sin x)}{\cot x}$

$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x}=\lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x}=\lim_{x\to 0^+}(-\sin x\cos x)=0$

$\lim_{x\to 0^+}(\sin x)^{\tan x}=\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$

47. The limit leads to the indeterminate form $1^{-\infty}$.

Let $f(x)=x^{1/(1-x)}$.

$\ln x^{1/(1-x)}=\frac{\ln x}{1-x}$

$\lim_{x\to 1^+}\frac{\ln x}{1-x}=\lim_{x\to 1^+}\frac{\frac{1}{x}}{-1}=-1$

$\lim_{x\to 1^+}x^{1/(1-x)}=\lim_{x\to 1^+}e^{\ln f(x)}=e^{-1}=\frac{1}{e}$

48. $\int_x^{2x}\frac{dt}{t}=\left[\ln|t|\right]_x^{2x}=\ln|2x|-\ln|x|=\ln\left|\frac{2x}{x}\right|$

$\lim_{x\to\infty}\int_x^{2x}\frac{dt}{t}=\lim_{x\to\infty}\ln\left|\frac{2x}{x}\right|=\lim_{x\to\infty}\ln 2=\ln 2$

49. $\lim_{x\to 1}\frac{x^3-1}{4x^3-x-3}=\lim_{x\to 1}\frac{3x^2}{12x^2-1}=3/11$

50. $\lim_{x\to\infty}\frac{2x^2+3x}{x^3+x+1}=\lim_{x\to\infty}\frac{4x+3}{3x^2+1}=\lim_{x\to\infty}\frac{4}{6x}=0$

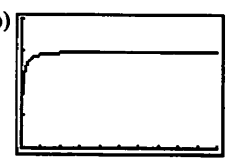
51. $\lim_{x\to 1}\frac{\int_1^x\cos t\,dt}{x^2-1}=\lim_{x\to 1}\frac{\sin x-\sin 1}{x^2-1}=\lim_{x\to 1}\frac{\cos x}{2x}=\frac{\cos 1}{2}$

52. $\lim_{x\to 1}\frac{\int_1^x\frac{dt}{t}}{x^3-1}=\lim_{x\to 1}\frac{\ln x-\ln 1}{x^3-1}=\lim_{x\to 1}\frac{1/x}{3x^2}=1/3$

53. (a) L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}=\lim_{x\to\infty}\frac{(9/2)(9x+1)^{-1/2}}{(1/2)(x+1)^{-1/2}}=\lim_{x\to\infty}\frac{9\sqrt{x+1}}{\sqrt{9x+1}}$

(b)



[0, 100] by [0, 4]

The limit appears to be 3.

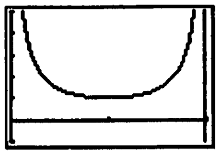
(c) $\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}=\lim_{x\to\infty}\frac{\sqrt{9+\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}}=\frac{\sqrt{9}}{\sqrt{1}}=3$

54. (a) L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to\pi/2}\frac{\tan x}{\sec x}$

54. Continued

(b)



[0, π] by [–1, 5]

The limit appears to be 1.

(c) $\lim_{x\to\pi/2} \frac{\sec x}{\tan x} = \lim_{x\to\pi/2} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim_{x\to\pi/2} \frac{1}{\sin x} = 1$

55. Find c such that $\lim_{x\to 0} f(x) = c$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{9x - 3\sin 3x}{5x^3}$$
$$= \lim_{x\to 0} \frac{9 - 9\cos 3x}{15x^2}$$
$$= \lim_{x\to 0} \frac{27\sin 3x}{30x}$$
$$= \lim_{x\to 0} \frac{81\cos 3x}{30} = \frac{81}{30} = \frac{27}{10}$$

Thus, $c = \frac{27}{10}$. This works since $\lim_{x\to 0} f(x) = c = f(0)$, so f is continuous.

56. $f(x)$ is defined at $x \neq 0$. $\lim_{x\to 0} f(x)$ leads to the indeterminate form 0^0.

$$\ln|x|^x = x\ln|x| = \frac{\ln|x|}{\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{\ln|x|}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0$$

$$\lim_{x\to 0} |x|^x = \lim_{x\to 0} e^{x\ln|x|} = e^0 = 1$$

Thus, f has a removable discontinuity at $x = 0$. Extend the definition of f by letting $f(0) = 1$.

57. (a) The limit leads to the indeterminate form 1^∞.

Let $f(k) = \left(1 + \frac{r}{k}\right)^{kt}$.

$$\ln f(k) = kt\ln\left(1\frac{r}{k}\right) = \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}$$

$$\lim_{k\to\infty} \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{t\left(-\frac{r}{k^2}\right)\left(1+\frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}}$$

$$= \lim_{k\to\infty} \frac{rt}{1+\frac{r}{k}} = \frac{rt}{1} = rt$$

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} = A_0 \lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{kt}$$
$$= A_0 \lim_{k\to\infty} e^{\ln f(k)}$$
$$= A_0 e^{rt}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

58. (a) For $x \neq 0$, $\frac{f'(x)}{g'(x)} = \frac{1}{1} = 1$.

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{2}{1} = 2$$

(b) This does not contradict L'Hôpital's Rule since $\lim_{x\to 0} f(x) = 2$ and $\lim_{x\to 0} g(x) = 1$.

59. (a) $A(t) = \int_0^t e^{-x}dx = \left[-e^{-x}\right]_0^t = e^{-t} + 1$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}(-e^{-1} + 1) = \lim_{t\to\infty}\left(-\frac{1}{e^t} + 1\right) = 1$$

(b) $V(t) = \pi\int_0^t (e^{-x})^2\,dx$

$$= \pi\int_0^t e^{-2x}\,dx$$
$$= \pi\left[-\frac{1}{2}e^{-2x}\right]_0^t$$
$$= \pi\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right)$$
$$= \frac{\pi}{2}\left(-e^{-2t} + 1\right)$$

$$\lim_{t\to\infty} \frac{V(t)}{A(t)} = \lim_{t\to\infty} \frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1} = \frac{\frac{\pi}{2}(1)}{1} = \frac{\pi}{2}$$

(c) $\lim_{t\to 0^+} \frac{V(t)}{A(t)} = \lim_{t\to 0^+} \frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{t}+1}$

$$= \lim_{t\to 0^+} \frac{\frac{\pi}{2}(2e^{-2t})}{e^{-t}}$$

$$= \frac{\frac{\pi}{2}(2)}{1} = \pi$$

60. (a)

x	$f(x)$
0.1	0.04542
0.01	0.00495
0.00	0.00050
0.0001	0.00005

The limit appears to be 0.

(b) $\lim_{x\to 0}\frac{\sin x}{1+2x}=\frac{0}{1}=0$

L'Hôpital's Rule is not applied here because the limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, since the denominator has limit 1.

61. (a) $f(x)=e^{x\ln(1+1/x)}$

$1+\frac{1}{x}>0$ when $x<-1$ or $x>0$

Domain: $(-\infty,-1)\cup(0,\infty)$

(b) The form is 0^{-1}, so $\lim_{x\to -1^-} f(x)=\infty$

(c) $\lim_{x\to-\infty} x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to-\infty}\frac{\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$=\lim_{x\to-\infty}\frac{\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{-1}}{\frac{-1}{x^2}}$

$=\lim_{x\to-\infty}\frac{1}{1+\frac{1}{x}}=1$

$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty} e^{x\ln(1+1/x)}=e$

62. False. Need $g'(a)\neq 0$. Consider $f(x)=\sin^2 x$ and $g(x)=x^2$ with $a=0$. Here $\lim_{x\to 0} f'(x)=\lim_{x\to 0} g'(x)=0$.

63. False. The limit is 1.

64. C. $\lim_{x\to 0}\frac{x}{\tan x}=\frac{1}{\sec^2 x}=\frac{1}{1}=1$

65. D. $\lim_{x\to 1}\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}}=\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{2}{x^3}}=\lim_{x\to 1}\frac{x^3}{2x^2}=1/2$

66. B. $\lim_{x\to\infty}\frac{\log_2 x}{\log_3 x}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}}=\frac{\ln 3}{\ln 2}$

67. E. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{1/3x}\right)=\lim_{x\to\infty}-\frac{\frac{1}{x(x+1)}}{-1/3x^2}$

$=\lim_{x\to\infty}\frac{3x^2}{x(x+1)}=\lim_{x\to\infty}\frac{3}{1}=3$

$\lim_{x\to\infty} e^{\ln f(x)}=e^3$

68. Possible answers:

(a) $f(x)=7(x-3); g(x)=x-3$

$\lim_{x\to 3}\frac{f(x)}{g(x)}=\lim_{x\to 3}\frac{7(x-3)}{x-3}\lim_{x\to 3}\frac{7}{1}=7$

(b) $f(x)=(x-3)^2; g(x)=x-3$

$\lim_{x\to 3}\frac{f(x)}{g(x)}=\lim_{x\to 3}\frac{(x-3)^2}{x-3}\lim_{x\to 3}\frac{2(2x-3)}{1}=0$

(c) $f(x)=x-3; g(x)=(x-3)^3$

$\lim_{x\to 3}\frac{f(x)}{g(x)}=\lim_{x\to 3}\frac{x-3}{(x-3)^3}=\lim_{x\to 3}\frac{1}{3(x-3)^2}=\infty$

69. Answers may vary.

(a) $f(x)=3x+1; g(x)=x$

$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{3x+1}{x}=\lim_{x\to\infty}\frac{3}{1}=3$

(b) $f(x)=x+1;\ g(x)=x^2$

$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x+1}{x^2}=\lim_{x\to\infty}\frac{1}{2x}=0$

(c) $f(x)=x^2;\ g(x)=x+1$

$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^2}{x+1}=\lim_{x\to\infty}\frac{2x}{1}=\infty$

70. (a) Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1-\cos x^6$ is 0 for even moderately small values of x. For example, at $x=0.1$, $\cos x^6\approx 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6=1$ and $1-\cos x^6=0$.

(b) Same reason as in part (a) applies.

(c) $\lim_{x\to 0}\frac{1-\cos x^6}{x^{12}}=\lim_{x\to 0}\frac{6x^5\sin x^6}{12x^{11}}$

$=\lim_{x\to 0}\frac{\sin x^6}{2x^6}$

$=\lim_{x\to 0}\frac{6x^5\cos x^6}{12x^5}$

$=\lim_{x\to 0}\frac{\cos x^6}{2}=\frac{1}{2}$

70. Continued

(d) The graph and/or table on a grapher show the value of the function to be 0 for x-values moderately close to 0, but the limit is 1/2. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

71. (a) $f'(x)=3x^2,\ g'(x)=2x-1$

$$f(1)-f(-1)=2,\ g(1)-g(-1)=-2$$
$$\frac{3c^2}{2c-1}=\frac{2}{-2}$$
$$3c^2=-2c+1$$
$$3c^2+2c-1=0$$
$$(3c-1)(c+1)=0$$
$$c=\frac{1}{3}\text{ or } c=-1$$

The value of c that satisfies the property is $c=\frac{1}{3}$.

(b) $f'(x)=-\sin x,\ g'(x)=\cos x$

$$f\left(\frac{\pi}{2}\right)-f(0)=-1,\ g\left(\frac{\pi}{2}\right)-g(0)=1$$
$$\frac{-\sin c}{\cos c}=\frac{-1}{1}$$
$$\tan c=1$$
$$c=\tan^{-1}1=\frac{\pi}{4}\text{ on }\left(0,\frac{\pi}{2}\right)$$

72. (a) $\ln f(x)^{g(x)}=g(x)\ln f(x)$

$$\lim_{x\to c}(g(x)\ln f(x))=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)$$
$$=\infty(-\infty)=-\infty$$
$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{-\infty}=0$$

(b) $\lim_{x\to c}\left(g(x)\ln f(x)\right)=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)$

$$=(-\infty)(-\infty)=\infty$$
$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{\infty}=\infty$$

Quick Quiz Sections 8.1 and 8.2

1. C. $\lim_{x\to 0}\frac{(x+1)^{4/3}-(4/3)x-1}{x^2}$

$$=\lim_{x\to 0}\frac{4/3(x+1)^{1/3}-(4/3)}{2x}$$
$$=\lim_{x\to 0}\frac{4/9(x+1)^{-2/3}}{2}$$
$$=\frac{2}{9}$$

2. D. $\lim_{x\to 0^+}(3x^{2x})$

$$=\lim_{x\to 0^+}\frac{\ln x}{1/2x}$$
$$=\lim_{x\to 0^+}\frac{1/x}{1/2x^2}=0$$
$$\lim_{x\to 0}3e^{\ln f(x)}=3e^0=3$$

3. B. $\lim_{x\to 2}\frac{\int_2^x \sin t\,dt}{x^2-4}$

$$=\lim_{x\to 2}\frac{\cos x-\cos 2}{x^2-4}$$
$$=\lim_{x\to 2}\frac{\sin x}{2x}=\frac{\sin 2}{4}$$

4. (a) $\left(\frac{1/2}{-4}\right)^{1/3}=-\frac{1}{2}$

$$\frac{-4}{-1/2}=8$$

(b) $-\frac{1}{2}$

(c) $a_n=8\left(-\frac{1}{2}\right)^n=(-1)^{n-1}(2^{4-n})$

(d) $a_n=\left(-\frac{1}{2}\right)a_{n-1}$

Section 8.3 Relative Rates of Growth (pp. 453–458)

Exploration 1 Comparing Rates of Growth as $x\to\infty$

1. $\lim_{x\to\infty}\frac{a^x}{x^2}=\lim_{x\to\infty}\frac{(\ln a)(a^x)}{2x}=\lim_{x\to\infty}\frac{(\ln a)^2a^x}{2}=\infty$, so a^x grows faster than x^2 as $x\to\infty$.

2. $\lim_{x\to\infty}\frac{3^x}{2^x}=\lim_{x\to\infty}1.5^x=\infty$

3. $\lim_{x\to\infty}\frac{a^x}{b^x}=\lim_{x\to\infty}\left(\frac{a}{b}\right)^x=\infty$ because $\frac{a}{b}>1$.

Quick Review 8.3

1. $\lim_{x\to\infty}\frac{\ln x}{e^x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{e^x}=0$

2. $\lim_{x\to\infty}\frac{e^x}{x^3}=\lim_{x\to\infty}\frac{e^x}{3x^2}=\lim_{x\to\infty}\frac{e^x}{6x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty$

3. $\lim_{x\to-\infty}\frac{x^2}{e^{2x}}=\infty$

4. $\lim_{x\to\infty}\frac{x^2}{e^{2x}}=\lim_{x\to\infty}\frac{2x}{2e^{2x}}=\lim_{x\to\infty}\frac{2}{4e^{2x}}=0$

5. $-3x^4$

6. $\frac{2x^3}{x}=2x^2$

7. $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x+\ln x}{x}=\lim_{x\to\infty}\frac{1+\frac{1}{x}}{1}=1$

8. $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sqrt{4x^2+5x}}{2x}=\lim_{x\to\infty}\sqrt{1+\frac{5}{4x}}=1$

9. (a) $f(x)=\frac{e^x+x^2}{e^x}=1+\frac{x^2}{e^x}$

$f'(x)=\frac{2xe^x-x^2e^x}{e^{2x}}=\frac{2x-x^2}{e^x}$

$\frac{2x-x^2}{e^x}=0$

$x(2-x)=0$

$x=0$ or $x=2$

$f'(x)<0$ for $x<0$ or $x>2$

The graph decreases, increases, and then decreases.

$f(0)=1; f(2)=1+\frac{4}{e^2}\approx 1.541$

f has a local maximum at $\approx(2, 1.541)$ and has a local minimum at $(0, 1)$.

(b) f is increasing on $[0, 2]$

(c) f is decreasing on $(-\infty, 0]$ and $[2, \infty)$.

10. $f(x)=\frac{x+\sin x}{x}=1+\frac{\sin x}{x}, x\neq 0$

Observe that $\left|\frac{\sin x}{x}\right|<1$ since $|\sin x|<|x|$ for $x\neq 0$.

$\lim_{x\to 0}f(x)=1+\lim_{x\to 0}\frac{\sin x}{x}=1+1=2$

Thus the values of f get close to 2 as x gets close to 0, so f doesn't have an absolute maximum value. f is not defined at 0.

Section 8.3 Exercises

1. $\lim_{x\to\infty}\frac{e^x}{x^3-3x+1}=\lim_{x\to\infty}\frac{e^x}{3x^2-3}=\lim_{x\to\infty}\frac{e^x}{6^x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty$

2. $\lim_{x\to\infty}\frac{e^x}{x^{20}}=\lim_{x\to\infty}\frac{e^x}{20!}=\infty$

3. $\lim_{x\to\infty}\frac{e^x}{e^{\cos x}}=\lim_{x\to\infty}\frac{e^x}{-\sin x\, e^{\cos x}}, -1\leq\cos x\leq 1,$

$\lim_{x\to\infty}\frac{e^x}{-\sin x\, e^{\cos y}}=\infty$

4. $\lim_{x\to\infty}\frac{e^x}{(5/2)^x}=\frac{e^x}{x!(5/2)}=\infty$

5. $\lim_{x\to\infty}\frac{\ln x}{x-\ln x}=\lim_{x\to\infty}\frac{1/x}{x-1/x}=\lim_{x\to\infty}\frac{1}{x^2-1}=0$

6. $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/2(x)^{-1/2}}=\frac{1}{1/2(x)^{-1/2}x}=0$

7. $\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}=\lim_{x\to\infty}\frac{1/x}{1/3(x)^{-2/3}}=\lim_{x\to\infty}\frac{1}{1/3(x)^{-2/3}x}=0$

8. $\lim_{x\to\infty}\frac{\ln x}{x^3}=\lim_{x\to\infty}\frac{1/x}{3x^2}=\lim_{x\to\infty}\frac{1}{3x^3}=0$

9. $\lim_{x\to\infty}\frac{x^2+4x}{x^2}=\lim_{x\to\infty}\frac{2x+4}{2x}=\lim_{x\to\infty}\frac{2}{2}=1$

10. $\lim_{x\to\infty}\frac{\sqrt{x^4+5x}}{x^2}=\lim_{x\to\infty}\frac{x^4+5x}{x^4}=\lim_{x\to\infty}\frac{12x^2}{12x^2}=1$

11. $\lim_{x\to\infty}\frac{(x^6+x^2)^{1/3}}{x^2}=\lim_{x\to\infty}\frac{x^6+x^2}{x^6}=\lim_{x\to\infty}\frac{120x^3}{120x^3}=1$

12. $\lim_{x\to\infty}\frac{x^2+\sin x}{x^2}=\lim_{x\to\infty}\frac{2x+\cos x}{2x}, -1\leq\cos x\leq 1, \lim_{x\to\infty}\frac{2x}{2x}=1$

13. $\lim_{x\to\infty}\frac{\log\sqrt{x}}{\ln x}=\frac{\frac{1}{\frac{1}{2}x\ln 10}}{1/x}=\frac{1}{2\ln 10}$

14. $\lim_{x\to\infty}\frac{e^{x+1}}{e^x}=e$

15. First observe that $\sqrt{1+x^4}$ grows at the same rate as x^2.

$\lim_{x\to\infty}\frac{\sqrt{1+x^4}}{x^2}=\lim_{x\to\infty}=\frac{\sqrt{1+x^4}}{x^4}=\lim_{x\to\infty}\sqrt{\frac{1}{x^4}+1}=1$

Next compare x^2 with e^x.

$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$

x^2 grows slower than e^x as $x\to\infty$, so $\sqrt{1+x^4}$ grows slower than e^x as $x\to\infty$.

16. $\lim_{x\to\infty}\frac{4^x}{e^x}=\lim_{x\to\infty}\left(\frac{4}{e}\right)^x=\infty$ since $\frac{4}{e}>1$.

4^x grows faster than e^x as $x\to\infty$.