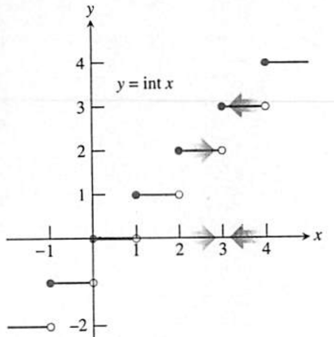


Figure 2.5 At each integer, the greatest integer function $y = \text{int } x$ has different right-hand and left-hand limits. (Example 7)

right the **right-hand limit** of f at c and the limit as x approaches c from the left the **left-hand limit** of f at c. Here is the notation we use:

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

EXAMPLE 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int } x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 2.5. For example,

$$\lim_{x \to 3^+} \text{int } x = 3 \quad \text{and} \quad \lim_{x \to 3^-} \text{int } x = 2.$$

The limit of int x as x approaches an integer n from the right is n, while the limit as x approaches n from the left is $n - 1$.

Now try Exercises 31 and 32.

We sometimes call $\lim_{x \to c} f(x)$ the **two-sided limit** of f at c to distinguish it from the *one-sided* right-hand and left-hand limits of f at c. Theorem 3 shows how these limits are related.

On the Far Side

If f is not defined to the left of $x = c$, then f does not have a left-hand limit at c. Similarly, if f is not defined to the right of $x = c$, then f does not have a right-hand limit at c.

> **THEOREM 3 One-sided and Two-sided Limits**
>
> A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,
>
> $$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Thus, the greatest integer function $f(x) = \text{int } x$ of Example 7 does not have a limit as $x \to 3$ even though each one-sided limit exists.

EXAMPLE 8 Exploring Right- and Left-Hand Limits

All the following statements about the function $y = f(x)$ graphed in Figure 2.6 are true.

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

f has no limit as $x \to 1$. (The right- and left-hand limits at 1 are not equal, so $\lim_{x \to 1} f(x)$ does not exist.)

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2 = f(3) = \lim_{x \to 3} f(x)$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$.

At noninteger values of c between 0 and 4, f has a limit as $x \to c$.

Now try Exercise 37.

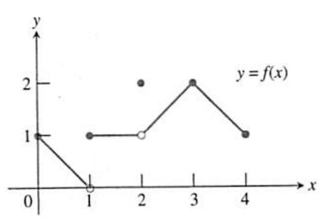


Figure 2.6 The graph of the function

$$f(x) = \begin{cases} -x + 1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x - 1, & 2 < x \le 3 \\ -x + 5, & 3 \le x \le 4. \end{cases}$$

(Example 8)

Sandwich Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Sandwich Theorem. The theorem refers to a function f whose values are sandwiched between the values of two other functions, g and h. If g and h have the same limit as $x \to c$, then f has that limit too, as suggested by Figure 2.7.

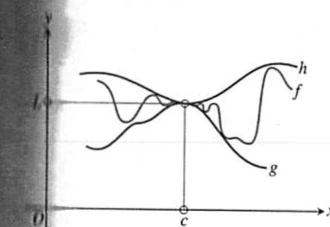


Figure 2.7 Sandwiching f between g and h forces the limiting value of f to be between the limiting values of g and h.

> **THEOREM 4 The Sandwich Theorem**
>
> If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and
>
> $$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$
>
> then
>
> $$\lim_{x \to c} f(x) = L.$$

EXAMPLE 9 Using the Sandwich Theorem

Show that $\lim_{x \to 0} [x^2 \sin (1/x)] = 0$.

SOLUTION

We know that the values of the sine function lie between –1 and 1. So, it follows that

$$\left| x^2 \sin \frac{1}{x} \right| = |x^2| \cdot \left| \sin \frac{1}{x} \right| \le |x^2| \cdot 1 = x^2$$

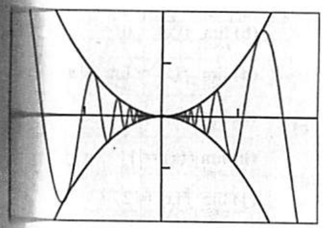
and

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2.$$

Because $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, the Sandwich Theorem gives

$$\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0.$$

The graphs in Figure 2.8 support this result.



[–0.2, 0.2] by [–0.02, 0.02]

Figure 2.8 The graphs of $y_1 = x^2$, $y_2 = x^2 \sin (1/x)$, and $y_3 = -x^2$. Notice that $y_3 \le y_2 \le y_1$. (Example 9)

Quick Review 2.1 *(For help, go to Section 1.2.)*

In Exercises 1–4, find $f(2)$.

1. $f(x) = 2x^3 - 5x^2 + 4$
2. $f(x) = \dfrac{4x^2 - 5}{x^3 + 4}$
3. $f(x) = \sin \left(\pi \dfrac{x}{2} \right)$
4. $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \dfrac{1}{x^2 - 1}, & x \ge 2 \end{cases}$

In Exercises 5–8, write the inequality in the form $a < x < b$.

5. $|x| < 4$
6. $|x| < c^2$
7. $|x - 2| < 3$
8. $|x - c| < d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\dfrac{x^2 - 3x - 18}{x + 3}$
10. $\dfrac{2x^2 - x}{2x^2 + x - 1}$

Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall.
2. Find the average speed during the first 4 seconds of fall.
3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.
4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically.

In Exercises 5 and 6, use $\lim_{x\to c} k = k$, $\lim_{x\to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x\to c} (2x^3 - 3x^2 + x - 1)$
6. $\lim_{x\to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$

In Exercises 7–14, determine the limit by substitution. Support graphically.

7. $\lim_{x\to -1/2} 3x^2(2x - 1)$
8. $\lim_{x\to -4} (x + 3)^{1998}$
9. $\lim_{x\to 1} (x^3 + 3x^2 - 2x - 17)$
10. $\lim_{y\to 2} \dfrac{y^2 + 5y + 6}{y + 2}$
11. $\lim_{y\to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$
12. $\lim_{x\to 1/2} \text{int } x$
13. $\lim_{x\to -2} (x - 6)^{2/3}$
14. $\lim_{x\to 2} \sqrt{x + 3}$

In Exercises 15–18, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

15. $\lim_{x\to -2} \sqrt{x - 2}$
16. $\lim_{x\to 0} \dfrac{1}{x^2}$
17. $\lim_{x\to 0} \dfrac{|x|}{x}$
18. $\lim_{x\to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 19–28, determine the limit graphically. Confirm algebraically.

19. $\lim_{x\to 1} \dfrac{x - 1}{x^2 - 1}$
20. $\lim_{t\to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$
21. $\lim_{x\to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$
22. $\lim_{x\to 0} \dfrac{\frac{1}{2 + x} - \frac{1}{2}}{x}$
23. $\lim_{x\to 0} \dfrac{(2 + x)^3 - 8}{x}$
24. $\lim_{x\to 0} \dfrac{\sin 2x}{x}$
25. $\lim_{x\to 0} \dfrac{\sin x}{2x^2 - x}$
26. $\lim_{x\to 0} \dfrac{x + \sin x}{x}$
27. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$
28. $\lim_{x\to 0} \dfrac{3 \sin 4x}{\sin 3x}$

In Exercises 29 and 30, use a graph to show that the limit does not exist.

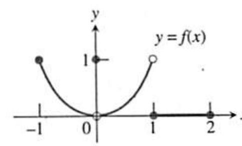
29. $\lim_{x\to 1} \dfrac{x^2 - 4}{x - 1}$
30. $\lim_{x\to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 31–36, determine the limit.

31. $\lim_{x\to 0^+} \text{int } x$
32. $\lim_{x\to 0^-} \text{int } x$
33. $\lim_{x\to 0.01} \text{int } x$
34. $\lim_{x\to 2^-} \text{int } x$
35. $\lim_{x\to 0^+} \dfrac{x}{|x|}$
36. $\lim_{x\to 0^-} \dfrac{x}{|x|}$

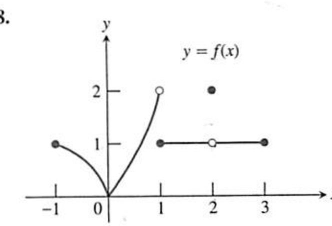
In Exercises 37 and 38, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

37.



(a) $\lim_{x\to -1^+} f(x) = 1$
(b) $\lim_{x\to 0^-} f(x) = 0$
(c) $\lim_{x\to 0^-} f(x) = 1$
(d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$
(e) $\lim_{x\to 0} f(x)$ exists
(f) $\lim_{x\to 0} f(x) = 0$
(g) $\lim_{x\to 0} f(x) = 1$
(h) $\lim_{x\to 1} f(x) = 1$
(i) $\lim_{x\to 1} f(x) = 0$
(j) $\lim_{x\to 2^-} f(x) = 2$

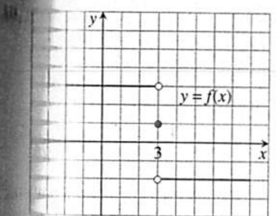
38.



(a) $\lim_{x\to -1^+} f(x) = 1$
(b) $\lim_{x\to 2} f(x)$ does not exist.
(c) $\lim_{x\to 2} f(x) = 2$
(d) $\lim_{x\to 1^-} f(x) = 2$
(e) $\lim_{x\to 1^+} f(x) = 1$
(f) $\lim_{x\to 1} f(x)$ does not exist.
(g) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$
(h) $\lim_{x\to c} f(x)$ exists at every c in $(-1, 1)$.
(i) $\lim_{x\to c} f(x)$ exists at every c in $(1, 3)$.

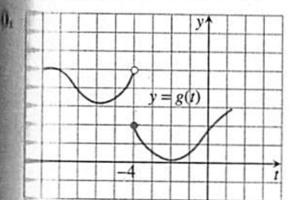
In Exercises 39–44, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

39.



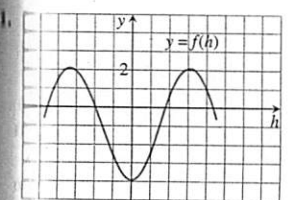
(a) $\lim_{x\to 3^-} f(x)$
(b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$
(d) $f(3)$

40.



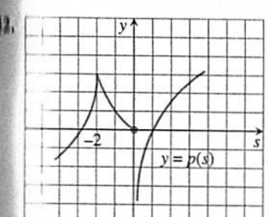
(a) $\lim_{t\to -4^-} g(t)$
(b) $\lim_{t\to -4^+} g(t)$
(c) $\lim_{t\to -4} g(t)$
(d) $g(-4)$

41.



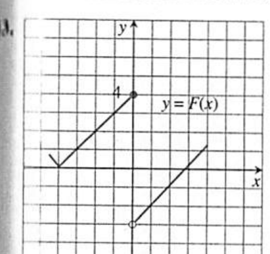
(a) $\lim_{h\to 0^-} f(h)$
(b) $\lim_{h\to 0^+} f(h)$
(c) $\lim_{h\to 0} f(h)$
(d) $f(0)$

42.



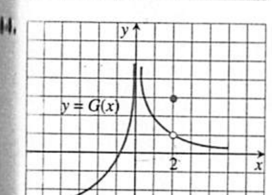
(a) $\lim_{s\to -2^-} p(s)$
(b) $\lim_{s\to -2^+} p(s)$
(c) $\lim_{s\to -2} p(s)$
(d) $p(-2)$

43.



(a) $\lim_{x\to 0^-} F(x)$
(b) $\lim_{x\to 0^+} F(x)$
(c) $\lim_{x\to 0} F(x)$
(d) $F(0)$

44.



(a) $\lim_{x\to 2^-} G(x)$
(b) $\lim_{x\to 2^+} G(x)$
(c) $\lim_{x\to 2} G(x)$
(d) $G(2)$

In Exercises 45–48, match the function with the table.

45. $y_1 = \dfrac{x^2 + x - 2}{x - 1}$
46. $y_1 = \dfrac{x^2 - x - 2}{x - 1}$
47. $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$
48. $y_1 = \dfrac{x^2 + x - 2}{x + 1}$

X	Y_1
.7	–.4765
.8	–.3111
.9	–.1526
1	0
1.1	.14762
1.2	.29091
1.3	.43043
X = .7	

(a)

X	Y_1
.7	7.3667
.8	10.8
.9	20.9
1	ERROR
1.1	–18.9
1.2	–8.8
1.3	–5.367
X = .7	

(b)

X	Y_1
.7	2.7
.8	2.8
.9	2.9
1	ERROR
1.1	3.1
1.2	3.2
1.3	3.3
X = .7	

(c)

X	Y_1
.7	–.3
.8	–.2
.9	–.1
1	ERROR
1.1	.1
1.2	.2
1.3	.3
X = .7	

(d)

In Exercises 49 and 50, determine the limit.

49. Assume that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$.
(a) $\lim_{x\to 4} (g(x) + 3)$
(b) $\lim_{x\to 4} x f(x)$
(c) $\lim_{x\to 4} g^2(x)$
(d) $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

50. Assume that $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$.
(a) $\lim_{x\to b} (f(x) + g(x))$
(b) $\lim_{x\to b} (f(x) \cdot g(x))$
(c) $\lim_{x\to b} 4\, g(x)$
(d) $\lim_{x\to b} \dfrac{f(x)}{g(x)}$

In Exercises 51–54, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of f.
(b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.
(c) **Writing to Learn** Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If not, explain.

51. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

52. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

53. $c = 1, f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

54. $c = -1, f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

In Exercises 55–58, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of f.

(b) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?

(c) At what points c does only the left-hand limit exist?

(d) At what points c does only the right-hand limit exist?

55. $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$

56. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$

57. $f(x) = \begin{cases} \sqrt{1-x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

58. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

In Exercises 59–62, find the limit graphically. Use the Sandwich Theorem to confirm your answer.

59. $\lim_{x\to 0} x \sin x$ **60.** $\lim_{x\to 0} x^2 \sin x$

61. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x^2}$ **62.** $\lim_{x\to 0} x^2 \cos \dfrac{1}{x^2}$

63. ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in t sec. Find the balloon's

(a) average speed during the first 3 sec of fall.

(b) speed at the instant $t = 3$.

64. ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in t sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g.

(b) Find the average speed for the fall.

(c) With what speed did the rock hit the bottom?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

65. True or False If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer.

66. True or False $\lim_{x\to 0} \dfrac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 67–70, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \dfrac{x}{2} + 1, & x > 1 \end{cases}$$

67. Multiple Choice What is the value of $\lim_{x\to 1^-} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

68. Multiple Choice What is the value of $\lim_{x\to 1^+} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

69. Multiple Choice What is the value of $\lim_{x\to 1} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

70. Multiple Choice What is the value of $f(1)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

Explorations

In Exercises 71–74, complete the following tables and state what you believe $\lim_{x\to 0} f(x)$ to be.

(a)

x	−0.1	−0.01	−0.001	−0.0001	...
$f(x)$	?	?	?	?	

(b)

x	0.1	0.01	0.001	0.0001	...
$f(x)$	?	?	?	?	

71. $f(x) = x \sin \dfrac{1}{x}$ **72.** $f(x) = \sin \dfrac{1}{x}$

73. $f(x) = \dfrac{10^x - 1}{x}$ **74.** $f(x) = x \sin(\ln |x|)$

75. Group Activity To prove that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ when θ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

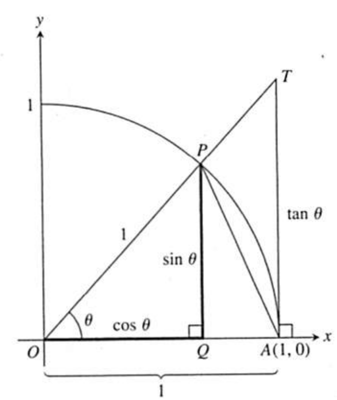
(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$.

(b) Use the figure to show that

$$\text{area of } \triangle OAP = \frac{1}{2}\sin\theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \triangle OAT = \frac{1}{2}\tan\theta.$$



(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

(f) Use the Sandwich Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

(g) Show that $(\sin\theta)/\theta$ is an even function.

(h) Use part (g) to show that

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

(i) Finally, show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Extending the Ideas

76. ***Controlling Outputs*** Let $f(x) = \sqrt{3x - 2}$.

(a) Show that $\lim_{x\to 2} f(x) = 2 = f(2)$.

(b) Use a graph to estimate values for a and b so that $1.8 < f(x) < 2.2$ provided $a < x < b$.

(c) Use a graph to estimate values for a and b so that $1.99 < f(x) < 2.01$ provided $a < x < b$.

77. ***Controlling Outputs*** Let $f(x) = \sin x$.

(a) Find $f(\pi/6)$.

(b) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.3 < f(x) < 0.7$ provided $a < x < b$.

(c) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.49 < f(x) < 0.51$ provided $a < x < b$.

78. ***Limits and Geometry*** Let $P(a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let O be the origin and $(0, b)$ the y-intercept of the perpendicular bisector of line segment OP. Find $\lim_{P\to O} b$.

DEFINITION End Behavior Model

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$

(b) a **left end behavior model** for f if and only if $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1.$

If one function provides both a left and right end behavior model, it is simply called an **end behavior model.** Thus, $g(x) = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ (Example 6).

In general, $g(x) = a_n x^n$ is an end behavior model for the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, a_n \neq 0$. Overall, the end behavior of all polynomials behave like the end behavior of monomials. This is the key to the end behavior of rational functions, as illustrated in Example 7.

EXAMPLE 7 Finding End Behavior Models

Find an end behavior model for

(a) $f(x) = \dfrac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ (b) $g(x) = \dfrac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

SOLUTION

(a) Notice that $2x^5$ is an end behavior model for the numerator of f, and $3x^2$ is one for the denominator. This makes

$$\frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

an end behavior model for f.

(b) Similarly, $2x^3$ is an end behavior model for the numerator of g, and $5x^3$ is one for the denominator of g. This makes

$$\frac{2x^3}{5x^3} = \frac{2}{5}$$

an end behavior model for g. ***Now try Exercise 41.***

Notice in Example 7b that the end behavior model for g, $y = 2/5$, is also a horizontal asymptote of the graph of g, while in 7a, the graph of f does not have a horizontal asymptote. We can use the end behavior model of a rational function to identify any horizontal asymptote.

We can see from Example 7 that a rational function always has a simple power function as an end behavior model.

A function's right and left end behavior models need not be the same function.

EXAMPLE 8 Finding End Behavior Models

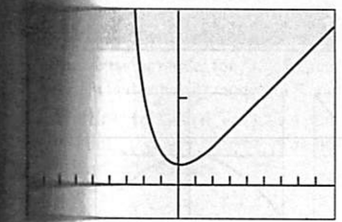
Let $f(x) = x + e^{-x}$. Show that $g(x) = x$ is a right end behavior model for f while $h(x) = e^{-x}$ is a left end behavior model for f.

SOLUTION

On the right,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + e^{-x}}{x} = \lim_{x\to\infty} \left(1 + \frac{e^{-x}}{x}\right) = 1 \text{ because } \lim_{x\to\infty} \frac{e^{-x}}{x} = 0.$$

continued



[–9, 9] by [–2, 10]

Figure 2.14 The graph of $f(x) = x + e^{-x}$ looks like the graph of $g(x) = x$ to the right of the y-axis, and like the graph of $h(x) = e^{-x}$ to the left of the y-axis. (Example 8)

On the left,

$$\lim_{x\to-\infty} \frac{f(x)}{h(x)} = \lim_{x\to-\infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x\to-\infty} \left(\frac{x}{e^{-x}} + 1\right) = 1 \text{ because } \lim_{x\to-\infty} \frac{x}{e^{-x}} = 0.$$

The graph of f in Figure 2.14 supports these end behavior conclusions.

Now try Exercise 45.

"Seeing" Limits as $x \to \pm\infty$

We can investigate the graph of $y = f(x)$ as $x \to \pm\infty$ by investigating the graph of $y = f(1/x)$ as $x \to 0$.

EXAMPLE 9 Using Substitution

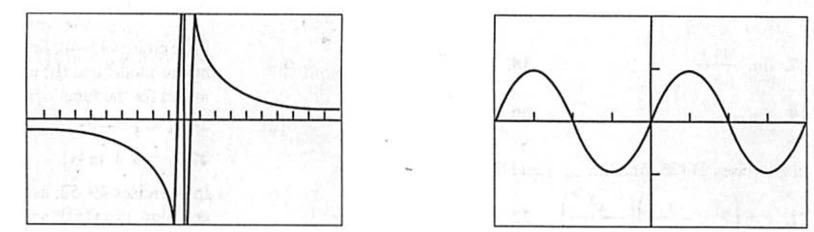
Find $\lim_{x\to\infty} \sin (1/x)$.

SOLUTION

Figure 2.15a suggests that the limit is 0. Indeed, replacing $\lim_{x\to\infty} \sin(1/x)$ by the equivalent $\lim_{x\to 0^+} \sin x = 0$ (Figure 2.15b), we find

$$\lim_{x\to\infty} \sin 1/x = \lim_{x\to 0^+} \sin x = 0.$$

Now try Exercise 49.



[–10, 10] by [–1, 1]
(a)

[–2π, 2π] by [–2, 2]
(b)

Figure 2.15 The graphs of (a) $f(x) = \sin(1/x)$ and (b) $g(x) = f(1/x) = \sin x$. (Example 9)

Quick Review 2.2 *(For help, go to Section 1.2 and 1.5.)*

In Exercises 1–4, find f^{-1} and graph f, f^{-1}, and $y = x$ in the same square viewing window.

1. $f(x) = 2x - 3$
2. $f(x) = e^x$
3. $f(x) = \tan^{-1} x$
4. $f(x) = \cot^{-1} x$

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

5. $f(x) = 2x^3 - 3x^2 + x - 1, \quad g(x) = 3x^3 + 4x - 5$
6. $f(x) = 2x^5 - x^3 + x - 1, \quad g(x) = x^3 - x^2 + 1$

In Exercises 7–10, write a formula for **(a)** $f(-x)$ and **(b)** $f(1/x)$. Simplify where possible.

7. $f(x) = \cos x$
8. $f(x) = e^{-x}$
9. $f(x) = \dfrac{\ln x}{x}$
10. $f(x) = \left(x + \dfrac{1}{x}\right)\sin x$

Section 2.2 Exercises

In Exercises 1–8, use graphs and tables to find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$ **(c)** Identify all horizontal asymptotes.

1. $f(x) = \cos\left(\frac{1}{x}\right)$ **2.** $f(x) = \frac{\sin 2x}{x}$

3. $f(x) = \frac{e^{-x}}{x}$ **4.** $f(x) = \frac{3x^3 - x + 1}{x + 3}$

5. $f(x) = \frac{3x + 1}{|x| + 2}$ **6.** $f(x) = \frac{2x - 1}{|x| - 3}$

7. $f(x) = \frac{x}{|x|}$ **8.** $f(x) = \frac{|x|}{|x| + 1}$

In Exercises 9–12, find the limit and confirm your answer using the Sandwich Theorem.

9. $\lim_{x\to\infty} \frac{1 - \cos x}{x^2}$ **10.** $\lim_{x\to-\infty} \frac{1 - \cos x}{x^2}$

11. $\lim_{x\to-\infty} \frac{\sin x}{x}$ **12.** $\lim_{x\to\infty} \frac{\sin(x^2)}{x}$

In Exercises 13–20, use graphs and tables to find the limits.

13. $\lim_{x\to 2^+} \frac{1}{x - 2}$ **14.** $\lim_{x\to 2^-} \frac{x}{x - 2}$

15. $\lim_{x\to -3^-} \frac{1}{x + 3}$ **16.** $\lim_{x\to -3^+} \frac{x}{x + 3}$

17. $\lim_{x\to 0^+} \frac{\text{int } x}{x}$ **18.** $\lim_{x\to 0^-} \frac{\text{int } x}{x}$

19. $\lim_{x\to 0^+} \csc x$ **20.** $\lim_{x\to(\pi/2)^+} \sec x$

In Exercises 21–26, find $\lim_{x\to\infty} y$ and $\lim_{x\to-\infty} y$.

21. $y = \left(2 - \frac{x}{x + 1}\right)\left(\frac{x^2}{5 + x^2}\right)$ **22.** $y = \left(\frac{2}{x} + 1\right)\left(\frac{5x^2 - 1}{x^2}\right)$

23. $y = \frac{\cos(1/x)}{1 + (1/x)}$ **24.** $y = \frac{2x + \sin x}{x}$

25. $y = \frac{\sin x}{2x^2 + x}$ **26.** $y = \frac{x \sin x + 2 \sin x}{2x^2}$

In Exercises 27–34, **(a)** find the vertical asymptotes of the graph of $f(x)$. **(b)** Describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

27. $f(x) = \frac{1}{x^2 - 4}$ **28.** $f(x) = \frac{x^2 - 1}{2x + 4}$

29. $f(x) = \frac{x^2 - 2x}{x + 1}$ **30.** $f(x) = \frac{1 - x}{2x^2 - 5x - 3}$

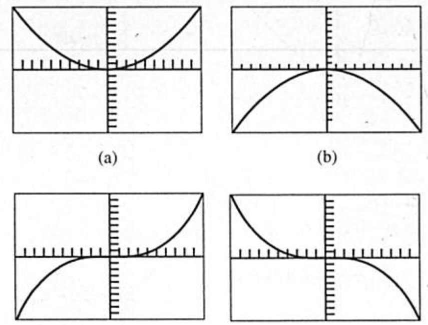
31. $f(x) = \cot x$ **32.** $f(x) = \sec x$

33. $f(x) = \frac{\tan x}{\sin x}$ **34.** $f(x) = \frac{\cot x}{\cos x}$

In Exercises 35–38, match the function with the graph of its end behavior model.

35. $y = \frac{2x^3 - 3x^2 + 1}{x + 3}$ **36.** $y = \frac{x^5 - x^4 + x + 1}{2x^2 + x - 3}$

37. $y = \frac{2x^4 - x^3 + x^2 - 1}{2 - x}$ **38.** $y = \frac{x^4 - 3x^3 + x^2 - 1}{1 - x^2}$



(a) (b) (c) (d)

In Exercises 39–44, **(a)** find a power function end behavior model for f. **(b)** Identify any horizontal asymptotes.

39. $f(x) = 3x^2 - 2x + 1$ **40.** $f(x) = -4x^3 + x^2 - 2x - 1$

41. $f(x) = \frac{x - 2}{2x^2 + 3x - 5}$ **42.** $f(x) = \frac{3x^2 - x + 5}{x^2 - 4}$

43. $f(x) = \frac{4x^3 - 2x + 1}{x - 2}$ **44.** $f(x) = \frac{-x^4 + 2x^2 + x - 3}{x^2 - 4}$

In Exercises 45–48, find **(a)** a simple basic function as a right end behavior model and **(b)** a simple basic function as a left end behavior model for the function.

45. $y = e^x - 2x$ **46.** $y = x^2 + e^{-x}$

47. $y = x + \ln|x|$ **48.** $y = x^2 + \sin x$

In Exercises 49–52, use the graph of $y = f(1/x)$ to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

49. $f(x) = xe^x$ **50.** $f(x) = x^2 e^{-x}$

51. $f(x) = \frac{\ln|x|}{x}$ **52.** $f(x) = x \sin\frac{1}{x}$

In Exercises 53 and 54, find the limit of $f(x)$ as **(a)** $x\to-\infty$, **(b)** $x\to\infty$, **(c)** $x\to 0^-$, and **(d)** $x\to 0^+$.

53. $f(x) = \begin{cases} 1/x, & x < 0 \\ -1, & x \geq 0 \end{cases}$

54. $f(x) = \begin{cases} \frac{x - 2}{x - 1}, & x \leq 0 \\ 1/x^2, & x > 0 \end{cases}$

Group Activity In Exercises 55 and 56, sketch a graph of a function $y = f(x)$ that satisfies the stated conditions. Include any asymptotes.

55. $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 5^-} f(x) = \infty$, $\lim_{x\to 5^+} f(x) = \infty$,

$\lim_{x\to\infty} f(x) = -1$, $\lim_{x\to -2^+} f(x) = -\infty$,

$\lim_{x\to -2^-} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$

56. $\lim_{x\to 2} f(x) = -1$, $\lim_{x\to 4^+} f(x) = -\infty$, $\lim_{x\to 4^-} f(x) = \infty$,

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 2$

57. Group Activity ***End Behavior Models*** Suppose that $g_1(x)$ is a right end behavior model for $f_1(x)$ and that $g_2(x)$ is a right end behavior model for $f_2(x)$. Explain why this makes $g_1(x)/g_2(x)$ a right end behavior model for $f_1(x)/f_2(x)$.

58. Writing to Learn Let L be a real number, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = \infty$ or $-\infty$. Can $\lim_{x\to c} (f(x) + g(x))$ be determined? Explain.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

59. True or False It is possible for a function to have more than one horizontal asymptote. Justify your answer.

60. True or False If $f(x)$ has a vertical asymptote at $x = c$, then either $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \infty$ or $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = -\infty$. Justify your answer.

61. Multiple Choice $\lim_{x\to 2^-} \frac{x}{x - 2} =$

(A) $-\infty$ (B) ∞ (C) 1 (D) $-1/2$ (E) -1

62. Multiple Choice $\lim_{x\to 0} \frac{\cos(2x)}{x} =$

(A) 1/2 (B) 1 (C) 2 (D) cos 2 (E) does not exist

63. Multiple Choice $\lim_{x\to 0} \frac{\sin(3x)}{x} =$

(A) 1/3 (B) 1 (C) 3 (D) sin 3 (E) does not exist

64. Multiple Choice Which of the following is an end behavior for

$$f(x) = \frac{2x^3 - x^2 + x + 1}{x^3 - 1}?$$

(A) x^3 (B) $2x^3$ (C) $1/x^3$ (D) 2 (E) 1/2

Exploration

65. ***Exploring Properties of Limits*** Find the limits of f, g, and fg as $x\to c$.

(a) $f(x) = \frac{1}{x}$, $g(x) = x$, $c = 0$

(b) $f(x) = -\frac{2}{x^3}$, $g(x) = 4x^3$, $c = 0$

(c) $f(x) = \frac{3}{x - 2}$, $g(x) = (x - 2)^3$, $c = 2$

(d) $f(x) = \frac{5}{(3 - x)^4}$, $g(x) = (x - 3)^2$, $c = 3$

(e) **Writing to Learn** Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$. Based on your observations in parts (a)–(d), what can you say about $\lim_{x\to c} (f(x) \cdot g(x))$?

Extending the Ideas

66. ***The Greatest Integer Function***

(a) Show that

$$\frac{x - 1}{x} < \frac{\text{int } x}{x} \leq 1 \ (x > 0) \text{ and } \frac{x - 1}{x} > \frac{\text{int } x}{x} \geq 1 \ (x < 0).$$

(b) Determine $\lim_{x\to\infty} \frac{\text{int } x}{x}$.

(c) Determine $\lim_{x\to-\infty} \frac{\text{int } x}{x}$.

67. ***Sandwich Theorem*** Use the Sandwich Theorem to confirm the limit as $x\to\infty$ found in Exercise 3.

68. Writing to Learn Explain why there is no value L for which $\lim_{x\to\infty} \sin x = L$.

In Exercises 69–71, find the limit. Give a convincing argument that the value is correct.

69. $\lim_{x\to\infty} \frac{\ln x^2}{\ln x}$

70. $\lim_{x\to\infty} \frac{\ln x}{\log x}$

71. $\lim_{x\to\infty} \frac{\ln(x + 1)}{\ln x}$

Quick Quiz for AP* Preparation: Sections 2.1 and 2.2

You should solve the following problems without using a graphing calculator.

1. Multiple Choice Find $\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3}$, if it exists.

(A) -1 (B) 1 (C) 2 (D) 5 (E) does not exist

2. Multiple Choice Find $\lim_{x\to 2^+} f(x)$, if it exists, where

$$f(x) = \begin{cases} 3x + 1, & x < 2 \\ \frac{5}{x + 1}, & x \geq 2 \end{cases}$$

(A) 5/3 (B) 13/3 (C) 7 (D) ∞ (E) does not exist

3. Multiple Choice Which of the following lines is a horizontal asymptote for

$$f(x) = \frac{3x^3 - x^2 + x - 7}{2x^3 + 4x - 5}?$$

(A) $y = \frac{3}{2}x$ (B) $y = 0$ (C) $y = 2/3$ (D) $y = 7/5$ (E) $y = 3/2$

4. Free Response Let $f(x) = \frac{\cos x}{x}$.

(a) Find the domain and range of f.

(b) Is f even, odd, or neither? Justify your answer.

(c) Find $\lim_{x\to\infty} f(x)$.

(d) Use the Sandwich Theorem to justify your answer to part (c).

Quick Review 2.3 *(For help, go to Sections 1.2 and 2.1.)*

1. Find $\lim_{x \to -1} \dfrac{3x^2 - 2x + 1}{x^3 + 4}$.

2. Let $f(x) = \text{int } x$. Find each limit.

(a) $\lim_{x \to -1^-} f(x)$ (b) $\lim_{x \to -1^+} f(x)$ (c) $\lim_{x \to -1} f(x)$ (d) $f(-1)$

3. Let $f(x) = \begin{cases} x^2 - 4x + 5, & x < 2 \\ 4 - x, & x \ge 2. \end{cases}$

Find each limit.

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$ (d) $f(2)$

In Exercises 4–6, find the remaining functions in the list of functions: $f, g, f \circ g, g \circ f$.

4. $f(x) = \dfrac{2x - 1}{x + 5}$, $g(x) = \dfrac{1}{x} + 1$

5. $f(x) = x^2$, $(g \circ f)(x) = \sin x^2$, domain of $g = [0, \infty)$

6. $g(x) = \sqrt{x - 1}$, $(g \circ f)(x) = 1/x$, $x > 0$

7. Use factoring to solve $2x^2 + 9x - 5 = 0$.

8. Use graphing to solve $x^3 + 2x - 1 = 0$.

In Exercises 9 and 10, let

$$f(x) = \begin{cases} 5 - x, & x \le 3 \\ -x^2 + 6x - 8, & x > 3. \end{cases}$$

9. Solve the equation $f(x) = 4$.

10. Find a value of c for which the equation $f(x) = c$ has no solution.

Section 2.3 Exercises

In Exercises 1–10, find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

1. $y = \dfrac{1}{(x + 2)^2}$ **2.** $y = \dfrac{x + 1}{x^2 - 4x + 3}$

3. $y = \dfrac{1}{x^2 + 1}$ **4.** $y = |x - 1|$

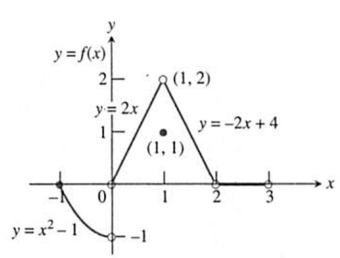
5. $y = \sqrt{2x + 3}$ **6.** $y = \sqrt[3]{2x - 1}$

7. $y = |x|/x$ **8.** $y = \cot x$

9. $y = e^{1/x}$ **10.** $y = \ln(x + 1)$

In Exercises 11–18, use the function f defined and graphed below to answer the questions.

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$



11. (a) Does $f(-1)$ exist?

(b) Does $\lim_{x \to -1^+} f(x)$ exist?

(c) Does $\lim_{x \to -1^+} f(x) = f(-1)$?

(d) Is f continuous at $x = -1$?

12. (a) Does $f(1)$ exist?

(b) Does $\lim_{x \to 1} f(x)$ exist?

(c) Does $\lim_{x \to 1} f(x) = f(1)$?

(d) Is f continuous at $x = 1$?

13. (a) Is f defined at $x = 2$? (Look at the definition of f.)

(b) Is f continuous at $x = 2$?

14. At what values of x is f continuous?

15. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

16. What new value should be assigned to $f(1)$ to make the new function continuous at $x = 1$?

17. Writing to Learn Is it possible to extend f to be continuous at $x = 0$? If so, what value should the extended function have there? If not, why not?

18. Writing to Learn Is it possible to extend f to be continuous at $x = 3$? If so, what value should the extended function have there? If not, why not?

In Exercises 19–24, (a) find each point of discontinuity. (b) Which of the discontinuities are removable? not removable? Give reasons for your answers.

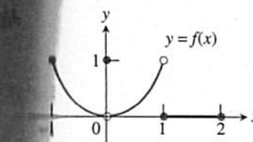
19. $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

20. $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

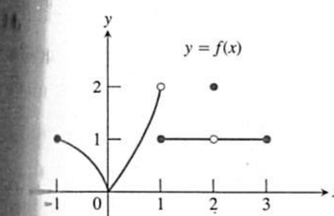
21. $f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

22. $f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

23.



24.



In Exercises 25–30, give a formula for the extended function that is continuous at the indicated point.

25. $f(x) = \dfrac{x^2 - 9}{x + 3}$, $x = -3$ **26.** $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$, $x = 1$

27. $f(x) = \dfrac{\sin x}{x}$, $x = 0$ **28.** $f(x) = \dfrac{\sin 4x}{x}$, $x = 0$

29. $f(x) = \dfrac{x - 4}{\sqrt{x} - 2}$, $x = 4$

30. $f(x) = \dfrac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}$, $x = 2$

In Exercises 31 and 32, explain why the given function is continuous.

31. $f(x) = \dfrac{1}{x - 3}$ **32.** $g(x) = \dfrac{1}{\sqrt{x - 1}}$

In Exercises 33–36, use Theorem 7 to show that the given function is continuous.

33. $f(x) = \sqrt{\left(\dfrac{x}{x + 1}\right)}$ **34.** $f(x) = \sin(x^2 + 1)$

35. $f(x) = \cos(\sqrt[3]{1 - x})$ **36.** $f(x) = \tan\left(\dfrac{x^2}{x^2 + 4}\right)$

Group Activity In Exercises 37–40, verify that the function is continuous and state its domain. Indicate which theorems you are using, and which functions you are assuming to be continuous.

37. $y = \dfrac{1}{\sqrt{x + 2}}$ **38.** $y = x^2 + \sqrt[3]{4 - x}$

39. $y = |x^2 - 4x|$ **40.** $y = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \ne 1 \\ 2, & x = 1 \end{cases}$

In Exercises 41–44, sketch a possible graph for a function f that has the stated properties.

41. $f(3)$ exists but $\lim_{x \to 3} f(x)$ does not.

42. $f(-2)$ exists, $\lim_{x \to -2^+} f(x) = f(-2)$, but $\lim_{x \to -2} f(x)$ does not exist.

43. $f(4)$ exists, $\lim_{x \to 4} f(x)$ exists, but f is not continuous at $x = 4$.

44. $f(x)$ is continuous for all x except $x = 1$, where f has a nonremovable discontinuity.

45. *Solving Equations* Is any real number exactly 1 less than its fourth power? Give any such values accurate to 3 decimal places.

46. *Solving Equations* Is any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

47. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \ge 3 \end{cases}$$

is continuous.

48. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} 2x + 3, & x \le 2 \\ ax + 1, & x > 2 \end{cases}$$

is continuous.

49. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \ge -1 \end{cases}$$

is continuous.

50. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} x^2 + x + a, & x < 1 \\ x^3, & x \ge 1 \end{cases}$$

is continuous.

51. Writing to Learn Explain why the equation $e^{-x} = x$ has at least one solution.

52. *Salary Negotiation* A welder's contract promises a 3.5% salary increase each year for 4 years and Luisa has an initial salary of \$36,500.

(a) Show that Luisa's salary is given by

$$y = 36{,}500(1.035)^{\text{int } t},$$

where t is the time, measured in years, since Luisa signed the contract.

(b) Graph Luisa's salary function. At what values of t is it continuous?

53. *Airport Parking* Valuepark charge \$1.10 per hour or fraction of an hour for airport parking. The maximum charge per day is \$7.25.

(a) Write a formula that gives the charge for x hours with $0 \le x \le 24$. (*Hint:* See Exercise 52.)

(b) Graph the function in part (a). At what values of x is it continuous?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

54. True or False A continuous function cannot have a point of discontinuity. Justify your answer.

55. True or False It is possible to extend the definition of a function f at a jump discontinuity $x = a$ so that f is continuous at $x = a$. Justify your answer.

56. Multiple Choice On which of the following intervals is $f(x) = \frac{1}{\sqrt{x}}$ not continuous?

(A) $(0, \infty)$ (B) $[0, \infty)$ (C) $(0, 2)$

(D) $(1, 2)$ (E) $[1, \infty)$

57. Multiple Choice Which of the following points is not a point of discontinuity of $f(x) = \sqrt{x - 1}$?

(A) $x = -1$ (B) $x = -1/2$ (C) $x = 0$

(D) $x = 1/2$ (E) $x = 1$

58. Multiple Choice Which of the following statements about the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -x + 3, & 1 < x < 2 \end{cases}$$

is not true?

(A) $f(1)$ does not exist.

(B) $\lim_{x\to 0^+} f(x)$ exists.

(C) $\lim_{x\to 2^-} f(x)$ exists.

(D) $\lim_{x\to 1} f(x)$ exists.

(E) $\lim_{x\to 1} f(x) \neq f(1)$

59. Multiple Choice Which of the following points of discontinuity of

$$f(x) = \frac{x(x-1)(x-2)^2(x+1)^2(x-3)^2}{x(x-1)(x-2)(x+1)^2(x-3)^3}$$

is not removable?

(A) $x = -1$ (B) $x = 0$ (C) $x = 1$

(D) $x = 2$ (E) $x = 3$

Exploration

60. Let $f(x) = \left(1 + \frac{1}{x}\right)^x$.

(a) Find the domain of f. (b) Draw the graph of f.

(c) **Writing to Learn** Explain why $x = -1$ and $x = 0$ are points of discontinuity of f.

(d) **Writing to Learn** Are either of the discontinuities in part (c) removable? Explain.

(e) Use graphs and tables to estimate $\lim_{x\to\infty} f(x)$.

Extending the Ideas

61. *Continuity at a Point* Show that $f(x)$ is continuous at $x = a$ if and only if

$$\lim_{h\to 0} f(a + h) = f(a).$$

62. *Continuity on Closed Intervals* Let f be continuous and never zero on $[a, b]$. Show that either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

63. *Properties of Continuity* Prove that if f is continuous on an interval, then so is $|f|$.

64. *Everywhere Discontinuous* Give a convincing argument that the following function is not continuous at any real number.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

2.4 Rates of Change and Tangent Lines

What you'll learn about

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

. . . and why

The tangent line determines the direction of a body's motion at every point along its path.

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

Marjorie Lee Browne (1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

SOLUTION

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval $[1, 3]$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies *(Drosophila)* grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

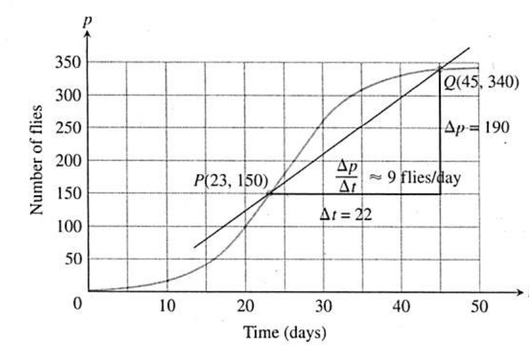


Figure 2.27 Growth of a fruit fly population in a controlled experiment.
Source: Elements of Mathematical Biology. (Example 2)

EXAMPLE 2 Growing Drosophila in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\textit{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day},$$

or about 9 flies per day.

continued

Quick Review 2.4 *(For help, go to Section 1.1.)*

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B.

1. $A(-5, 2),\ B(3, 5)$ **2.** $A(1, 3),\ B(a, b)$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3),\ (5, -1)$ **4.** $(-3, -1),\ (3, 3)$

In Exercises 5–9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= 3/2$

6. through $(1, 6)$ and $(4, -1)$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$

9. through $(-1, 3)$ and parallel to $2x + 3y = 5$

10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$?

Section 2.4 Exercises

In Exercises 1–6, find the average rate of change of the function over each interval.

1. $f(x) = x^3 + 1$
(a) $[2, 3]$ (b) $[-1, 1]$

2. $f(x) = \sqrt{4x + 1}$
(a) $[0, 2]$ (b) $[10, 12]$

3. $f(x) = e^x$
(a) $[-2, 0]$ (b) $[1, 3]$

4. $f(x) = \ln x$
(a) $[1, 4]$ (b) $[100, 103]$

5. $f(x) = \cot t$
(a) $[\pi/4, 3\pi/4]$ (b) $[\pi/6, \pi/2]$

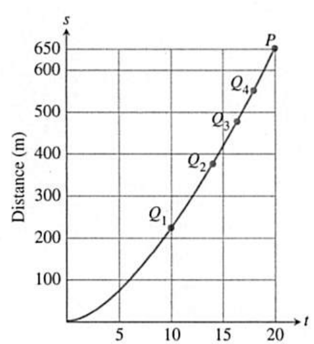
6. $f(x) = 2 + \cos t$
(a) $[0, \pi]$ (b) $[-\pi, \pi]$

In Exercises 7 and 8, a distance-time graph is shown.

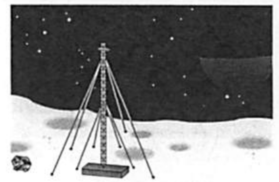
(a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What is the appropriate unit for these slopes?

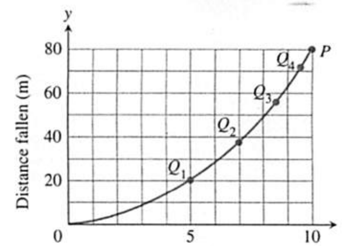
(b) Estimate the speed at point P.

7. ***Accelerating from a Standstill*** The figure shows the distance-time graph for a 1994 Ford® Mustang Cobra™ accelerating from a standstill.



8. ***Lunar Data*** The accompanying figure shows a distance-time graph for a wrench that fell from the top platform of a communication mast on the moon to the station roof 80 m below.





In Exercises 9–12, at the indicated point find

(a) the slope of the curve,

(b) an equation of the tangent, and

(c) an equation of the normal.

(d) Then draw a graph of the curve, tangent line, and normal line in the same square viewing window.

9. $y = x^2$ at $x = -2$ **10.** $y = x^2 - 4x$ at $x = 1$

11. $y = \dfrac{1}{x - 1}$ at $x = 2$ **12.** $y = x^2 - 3x - 1$ at $x = 0$

In Exercises 13 and 14, find the slope of the curve at the indicated point.

13. $f(x) = |x|$ at (a) $x = 2$ (b) $x = -3$

14. $f(x) = |x - 2|$ at $x = 1$

In Exercises 15–18, determine whether the curve has a tangent at the indicated point. If it does, give its slope. If not, explain why not.

15. $f(x) = \begin{cases} 2 - 2x - x^2, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$ at $x = 0$

16. $f(x) = \begin{cases} -x, & x < 0 \\ x^2 - x, & x \geq 0 \end{cases}$ at $x = 0$

17. $f(x) = \begin{cases} 1/x, & x \leq 2 \\ \dfrac{4 - x}{4}, & x > 2 \end{cases}$ at $x = 2$

18. $f(x) = \begin{cases} \sin x, & 0 \leq x < 3\pi/4 \\ \cos x, & 3\pi/4 \leq x \leq 2\pi \end{cases}$ at $x = 3\pi/4$

In Exercises 19–22, (a) find the slope of the curve at $x = a$.
(b) **Writing to Learn** Describe what happens to the tangent at $x = a$ as a changes.

19. $y = x^2 + 2$

20. $y = 2/x$

21. $y = \dfrac{1}{x - 1}$

22. $y = 9 - x^2$

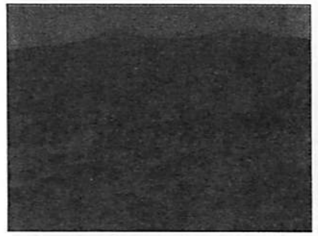
23. ***Free Fall*** An object is dropped from the top of a 100-m tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

24. ***Rocket Launch*** At t sec after lift-off, the height of a rocket is $3t^3$ ft. How fast is the rocket climbing after 10 sec?

25. ***Area of Circle*** What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.?

26. ***Volume of Sphere*** What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.?

27. ***Free Fall on Mars*** The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec.



28. ***Free Fall on Jupiter*** The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec.

29. ***Horizontal Tangent*** At what point is the tangent to $f(x) = x^2 + 4x - 1$ horizontal?

30. ***Horizontal Tangent*** At what point is the tangent to $f(x) = 3 - 4x - x^2$ horizontal?

31. ***Finding Tangents and Normals***

(a) Find an equation for each tangent to the curve $y = 1/(x - 1)$ that has slope -1. (See Exercise 21.)

(b) Find an equation for each normal to the curve $y = 1/(x - 1)$ that has slope 1.

32. ***Finding Tangents*** Find the equations of all lines tangent to $y = 9 - x^2$ that pass through the point $(1, 12)$.

33. Table 2.2 gives the amount of federal spending in billions of dollars for national defense for several years.

Table 2.2 National Defense Spending

Year	National Defense Spending ($ billions)
1990	299.3
1995	272.1
1999	274.9
2000	294.5
2001	305.5
2002	348.6
2003	404.9

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the average rate of change in spending from 1990 to 1995.

(b) Find the average rate of change in spending from 2000 to 2001.

(c) Find the average rate of change in spending from 2002 to 2003.

(d) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Find the quadratic regression equation for the data and superimpose its graph on a scatter plot of the data.

(e) Compute the average rates of change in parts (a), (b), and (c) using the regression equation.

(f) Use the regression equation to find how fast the spending was growing in 2003.

(g) **Writing to Learn** Explain why someone might be hesitant to make predictions about the rate of change of national defense spending based on this equation.

34. Table 2.3 gives the amount of federal spending in billions of dollars for agriculture for several years.

Table 2.3 Agriculture Spending

Year	Agriculture Spending ($ billions)
1990	12.0
1995	9.8
1999	23.0
2000	36.6
2001	26.4
2002	22.0
2003	22.6

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot of the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 2000, Q_2 the point corresponding to 2001, and Q_3 the point corresponding to 2002. Find the slope of the secant line PQ_i for $i = 1, 2, 3$.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

35. True or False If the graph of a function has a tangent line at $x = a$, then the graph also has a normal line at $x = a$. Justify your answer.

36. True or False The graph of $f(x)=|x|$ has a tangent line at $x = 0$. Justify your answer.

37. Multiple Choice If the line L tangent to the graph of a function f at the point $(2, 5)$ passes through the point $(-1, -3)$, what is the slope of L?

(A) $-3/8$ (B) $3/8$ (C) $-8/3$ (D) $8/3$ (E) undefined

38. Multiple Choice Find the average rate of change of $f(x) = x^2 + x$ over the interval $[1, 3]$.

(A) -5 (B) $1/5$ (C) $1/4$ (D) 4 (E) 5

39. Multiple Choice Which of the following is an equation of the tangent to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = -2x$ (B) $y = 2x$ (C) $y = -2x + 4$
(D) $y = -x + 3$ (E) $y = x + 3$

40. Multiple Choice Which of the following is an equation of the normal to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = \frac{1}{2}x + \frac{3}{2}$ (B) $y = -\frac{1}{2}x$ (C) $y = \frac{1}{2}x + 2$
(D) $y = -\frac{1}{2}x + 2$ (E) $y = 2x + 5$

Explorations

In Exercises 41 and 42, complete the following for the function.

(a) Compute the difference quotient

$$\frac{f(1 + h) - f(1)}{h}.$$

(b) Use graphs and tables to estimate the limit of the difference quotient in part (a) as $h \to 0$.

(c) Compare your estimate in part (b) with the given number.

(d) **Writing to Learn** Based on your computations, do you think the graph of f has a tangent at $x = 1$? If so, estimate its slope. If not, explain why not.

41. $f(x) = e^x$, e **42.** $f(x) = 2^x$, $\ln 4$

Group Activity In Exercises 43–46, the curve $y = f(x)$ has a **vertical tangent** at $x = a$ if

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \infty$$

or if

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = -\infty.$$

In each case, the right- and left-hand limits are required to be the same: both $+\infty$ or both $-\infty$.

Use graphs to investigate whether the curve has a vertical tangent at $x = 0$.

43. $y = x^{2/5}$ **44.** $y = x^{3/5}$
45. $y = x^{1/3}$ **46.** $y = x^{2/3}$

Extending the Ideas

In Exercises 47 and 48, determine whether the graph of the function has a tangent at the origin. Explain your answer.

47. $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

48. $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

49. *Sine Function* Estimate the slope of the curve $y = \sin x$ at $x = 1$. (*Hint:* See Exercises 41 and 42.)

Quick Quiz for AP* Preparation: Sections 2.3 and 2.4

You may use a calculator with these problems.

1. Multiple Choice Which of the following values is the average rate of $f(x) = \sqrt{x + 1}$ over the interval $(0, 3)$?

(A) -3 (B) -1 (C) $-1/3$ (D) $1/3$ (E) 3

2. Multiple Choice Which of the following statements is false for the function

$$f(x) = \begin{cases} \frac{3}{4}x, & 0 \le x < 4 \\ 2, & x = 4 \\ -x + 7, & 4 < x \le 6 \\ 1, & 6 < x < 8 \end{cases}?$$

(A) $\lim_{x\to 4} f(x)$ exists (B) $f(4)$ exists
(C) $\lim_{x\to 6} f(x)$ exists (D) $\lim_{x\to 8^-} f(x)$ exists
(E) f is continuous at $x = 4$

3. Multiple Choice Which of the following is an equation for the tangent line to $f(x) = 9 - x^2$ at $x = 2$?

(A) $y = \frac{1}{4}x + \frac{9}{2}$ (B) $y = -4x + 13$
(C) $y = -4x - 3$ (D) $y = 4x - 3$
(E) $y = 4x + 13$

4. Free Response Let $f(x) = 2x - x^2$.

(a) Find $f(3)$. (b) Find $f(3 + h)$.

(c) Find $\frac{f(3 + h) - f(3)}{h}$.

(d) Find the instantaneous rate of change of f at $x = 3$.

Chapter 2 Key Terms

average rate of change (p. 87)
average speed (p. 59)
connected graph (p. 83)
Constant Multiple Rule for Limits (p. 61)
continuity at a point (p. 78)
continuous at an endpoint (p. 79)
continuous at an interior point (p. 79)
continuous extension (p. 81)
continuous function (p. 81)
continuous on an interval (p. 81)
difference quotient (p. 90)
Difference Rule for Limits (p. 61)
discontinuous (p. 79)
end behavior model (p. 74)
free fall (p. 91)
horizontal asymptote (p. 70)
infinite discontinuity (p. 80)
instantaneous rate of change (p. 91)
instantaneous speed (p. 91)
intermediate value property (p. 83)
Intermediate Value Theorem for Continuous Functions (p. 83)
jump discontinuity (p. 80)
left end behavior model (p. 74)
left-hand limit (p. 64)
limit of a function (p. 60)
normal to a curve (p. 91)
oscillating discontinuity (p. 80)
point of discontinuity (p. 79)
Power Rule for Limits (p. 71)
Product Rule for Limits (p. 61)
Properties of Continuous Functions (p. 82)
Quotient Rule for Limits (p. 61)
removable discontinuity (p. 80)
right end behavior model (p. 74)
right-hand limit (p. 64)
Sandwich Theorem (p. 65)
secant to a curve (p. 87)
slope of a curve (p. 89)
Sum Rule for Limits (p. 61)
tangent line to a curve (p. 88)
two-sided limit (p. 64)
vertical asymptote (p. 72)
vertical tangent (p. 94)

Chapter 2 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, find the limits.

1. $\lim_{x\to -2} (x^3 - 2x^2 + 1)$ **2.** $\lim_{x\to -2} \frac{x^2 + 1}{3x^2 - 2x + 5}$

3. $\lim_{x\to 4} \sqrt{1 - 2x}$ **4.** $\lim_{x\to 5} \sqrt[4]{9 - x^2}$

5. $\lim_{x\to 0} \frac{\frac{1}{2 + x} - \frac{1}{2}}{x}$ **6.** $\lim_{x\to \pm\infty} \frac{2x^2 + 3}{5x^2 + 7}$

7. $\lim_{x\to \pm\infty} \frac{x^4 + x^3}{12x^3 + 128}$ **8.** $\lim_{x\to 0} \frac{\sin 2x}{4x}$

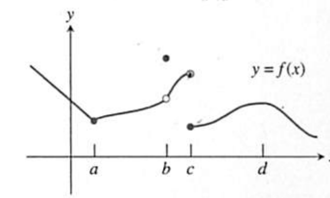
9. $\lim_{x\to 0} \frac{x \csc x + 1}{x \csc x}$ **10.** $\lim_{x\to 0} e^x \sin x$

11. $\lim_{x\to 7/2^+} \operatorname{int}(2x - 1)$ **12.** $\lim_{x\to 7/2^-} \operatorname{int}(2x - 1)$

13. $\lim_{x\to \infty} e^{-x} \cos x$ **14.** $\lim_{x\to \infty} \frac{x + \sin x}{x + \cos x}$

In Exercises 15–20, determine whether the limit exists on the basis of the graph of $y = f(x)$. The domain of f is the set of real numbers.

15. $\lim_{x\to d} f(x)$ **16.** $\lim_{x\to c^+} f(x)$
17. $\lim_{x\to c^-} f(x)$ **18.** $\lim_{x\to c} f(x)$
19. $\lim_{x\to b} f(x)$ **20.** $\lim_{x\to a} f(x)$



In Exercises 21–24, determine whether the function f used in Exercises 15–20 is continuous at the indicated point.

21. $x = a$ **22.** $x = b$
23. $x = c$ **24.** $x = d$

In Exercises 25 and 26, use the graph of the function with domain $-1 \le x \le 3$.

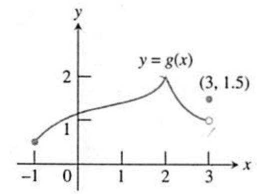
25. Determine

(a) $\lim_{x\to 3^-} g(x)$. (b) $g(3)$.

(c) whether $g(x)$ is continuous at $x = 3$.

(d) the points of discontinuity of $g(x)$.

(e) **Writing to Learn** whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not.



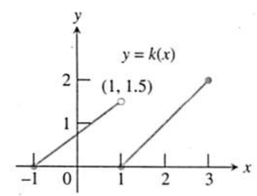
26. Determine

(a) $\lim_{x\to 1^-} k(x)$. (b) $\lim_{x\to 1^+} k(x)$. (c) $k(1)$.

(d) whether $k(x)$ is continuous at $x = 1$.

(e) the points of discontinuity of $k(x)$.

(f) **Writing to Learn** whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not.



In Exercises 27 and 28, **(a)** find the vertical asymptotes of the graph of $y = f(x)$, and **(b)** describe the behavior of $f(x)$ to the left and right of any vertical asymptote.

27. $f(x) = \dfrac{x+3}{x+2}$ **28.** $f(x) = \dfrac{x-1}{x^2(x+2)}$

In Exercises 29 and 30, answer the questions for the piecewise-defined function.

29. $f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = -1$, 0, and 1.

(b) Does f have a limit as x approaches -1? 0? 1? If so, what is it? If not, why not?

(c) Is f continuous at $x = -1$? 0? 1? Explain.

30. $f(x) = \begin{cases} |x^3 - 4x|, & x < 1 \\ x^2 - 2x - 2, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = 1$.

(b) Does f have a limit as $x \to 1$? If so, what is it? If not, why not?

(c) At what points is f continuous?

(d) At what points is f discontinuous?

In Exercises 31 and 32, find all points of discontinuity of the function.

31. $f(x) = \dfrac{x+1}{4-x^2}$ **32.** $g(x) = \sqrt[3]{3x+2}$

In Exercises 33–36, find **(a)** a power function end behavior model and **(b)** any horizontal asymptotes.

33. $f(x) = \dfrac{2x+1}{x^2-2x+1}$ **34.** $f(x) = \dfrac{2x^2+5x-1}{x^2+2x}$

35. $f(x) = \dfrac{x^3-4x^2+3x+3}{x-3}$ **36.** $f(x) = \dfrac{x^4-3x^2+x-1}{x^3-x+1}$

In Exercises 37 and 38, find **(a)** a right end behavior model and **(b)** a left end behavior model for the function.

37. $f(x) = x + e^x$ **38.** $f(x) = \ln|x| + \sin x$

Group Activity In Exercises 39 and 40, what value should be assigned to k to make f a continuous function?

39. $f(x) = \begin{cases} \dfrac{x^2+2x-15}{x-3}, & x \ne 3 \\ k, & x = 3 \end{cases}$

40. $f(x) = \begin{cases} \dfrac{\sin x}{2x}, & x \ne 0 \\ k, & x = 0 \end{cases}$

Group Activity In Exercises 41 and 42, sketch a graph of a function f that satisfies the given conditions.

41. $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-\infty} f(x) = \infty$,

$\lim_{x\to 3^+} f(x) = \infty$, $\lim_{x\to 3^-} f(x) = -\infty$

42. $\lim_{x\to 2} f(x)$ does not exist, $\lim_{x\to 2^+} f(x) = f(2) = 3$

43. ***Average Rate of Change*** Find the average rate of change of $f(x) = 1 + \sin x$ over the interval $[0, \pi/2]$.

44. ***Rate of Change*** Find the instantaneous rate of change of the volume $V = (1/3)\pi r^2 H$ of a cone with respect to the radius r at $r = a$ if the height H does not change.

45. ***Rate of Change*** Find the instantaneous rate of change of the surface area $S = 6x^2$ of a cube with respect to the edge length x at $x = a$.

46. ***Slope of a Curve*** Find the slope of the curve $y = x^2 - x - 2$ at $x = a$.

47. ***Tangent and Normal*** Let $f(x) = x^2 - 3x$ and $P = (1, f(1))$. Find **(a)** the slope of the curve $y = f(x)$ at P, **(b)** an equation of the tangent at P, and **(c)** an equation of the normal at P.

48. ***Horizontal Tangents*** At what points, if any, are the tangents to the graph of $f(x) = x^2 - 3x$ horizontal? (See Exercise 47.)

49. ***Bear Population*** The number of bears in a federal wildlife reserve is given by the population equation

$$p(t) = \frac{200}{1 + 7e^{-0.1t}},$$

where t is in years.

(a) **Writing to Learn** Find $p(0)$. Give a possible interpretation of this number.

(b) Find $\lim_{t\to\infty} p(t)$.

(c) **Writing to Learn** Give a possible interpretation of the result in part (b).

50. ***Taxi Fares*** Bluetop Cab charges \$3.20 for the first mile and \$1.35 for each additional mile or part of a mile.

(a) Write a formula that gives the charge for x miles with $0 \le x \le 20$.

(b) Graph the function in (a). At what values of x is it discontinuous?

51. Table 2.4 gives the population of Florida for several years.

Table 2.4 Population of Florida

Year	Population (in thousands)
1998	15,487
1999	15,759
2000	15,983
2001	16,355
2002	16,692
2003	17,019

Source: U.S. Census Bureau, *Statistical Abstract of the United States; 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot for the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 1998, Q_2 the point corresponding to 1999, . . . , and Q_5 the point corresponding to 2002. Find the slope of the secant the PQ_i for $i = 1, 2, 3, 4, 5$.

(c) Predict the rate of change of population in 2003.

(d) Find a linear regression equation for the data, and use it to calculate the rate of the population in 2003.

52. ***Limit Properties*** Assume that

$$\lim_{x\to c} [f(x) + g(x)] = 2,$$

$$\lim_{x\to c} [f(x) - g(x)] = 1,$$

and that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Find $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$.

AP* Examination Preparation

You should solve the following problems without using a graphing calculation.

53. Free Response Let $f(x) = \dfrac{x}{|x^2 - 9|}$.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote of the graph of f.

(c) Write an equation for each horizontal asymptote of the graph of f.

(d) Is f odd, even, or neither? Justify your answer.

(e) Find all values of x for which f is discontinuous and classify each discontinuity as removable or nonremovable.

54. Free Response Let $f(x) = \begin{cases} x^2 - a^2x & \text{if } x < 2, \\ 4 - 2x^2 & \text{if } x \ge 2. \end{cases}$

(a) Find $\lim_{x\to 2^-} f(x)$.

(b) Find $\lim_{x\to 2^+} f(x)$.

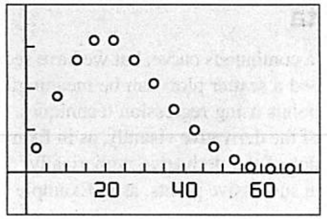
(c) Find all values of a that make f continuous at 2. Justify your answer.

55. Free Response Let $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3}$.

(a) Find all zeros of f.

(b) Find a right end behavior model $g(x)$ for f.

(c) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.



[–5, 75] by [–0.01, 0.04]

Figure 3.7 A scatter plot of the derivative data in Table 3.2. (Example 5)

A scatter plot of the derivative data in Table 3.2 is shown in Figure 3.7.

From the derivative plot, we can see that the rate of change peaks near $x = 20$. You can impress your friends with your "psychic powers" by predicting a shared birthday in a room of just 25 people (since you will be right about 57% of the time), but the derivative warns you to be cautious: a few less people can make quite a difference. On the other hand, going from 40 people to 100 people will not improve your chances much at all.

Now try Exercise 2

Generating shared birthday probabilities: If you know a little about probability, you might try generating the probabilities in Table 3.1. Extending the Idea Exercise 45 at the end of this section shows how to generate them on a calculator.

One-Sided Derivatives

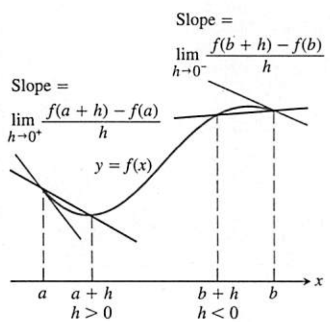


Figure 3.8 Derivatives at endpoints are one-sided limits.

A function $y = f(x)$ is **differentiable on a closed interval** $[a, b]$ if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \quad \text{[the right-hand derivative at } a\text{]}$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \quad \text{[the left-hand derivative at } b\text{]}$$

exist at the endpoints. In the right-hand derivative, h is positive and $a + h$ approaches a from the right. In the left-hand derivative, h is negative and $b + h$ approaches b from the left (Figure 3.8).

Right-hand and left-hand derivatives may be defined at any point of a function's domain.

The usual relationship between one-sided and two-sided limits holds for derivatives. Theorem 3, Section 2.1, allows us to conclude that a function has a (two-sided) derivative at a point if and only if the function's right-hand and left-hand derivatives are defined and equal at that point.

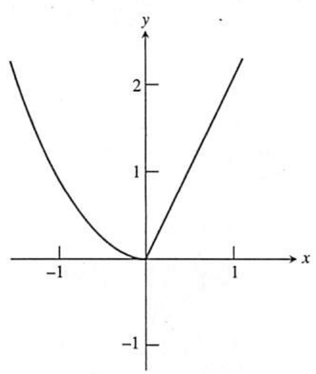


Figure 3.9 A function with different one-sided derivatives at $x = 0$. (Example 6)

EXAMPLE 6 One-Sided Derivatives can Differ at a Point

Show that the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there (Figure 3.9).

$$y = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0 \end{cases}$$

SOLUTION

We verify the existence of the left-hand derivative:

$$\lim_{h\to 0^-} \frac{(0+h)^2 - 0^2}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = 0.$$

We verify the existence of the right-hand derivative:

$$\lim_{h\to 0^+} \frac{2(0+h) - 0^2}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Since the left-hand derivative equals zero and the right-hand derivative equals 2, the derivatives are not equal at $x = 0$. The function does not have a derivative at 0.

Now try Exercise 3

Quick Review 3.1 *(For help, go to Sections 2.1 and 2.4.)*

In Exercises 1–4, evaluate the indicated limit algebraically.

1. $\lim_{h\to 0} \frac{(2+h)^2 - 4}{h}$
2. $\lim_{x\to 2^+} \frac{x+3}{2}$
3. $\lim_{y\to 0^-} \frac{|y|}{y}$
4. $\lim_{x\to 4} \frac{2x-8}{\sqrt{x}-2}$
5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.
6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

In Exercises 7–10, let

$$f(x) = \begin{cases} x+2, & x \le 1 \\ (x-1)^2, & x > 1. \end{cases}$$

7. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.
8. Find $\lim_{h\to 0^+} f(1+h)$.
9. Does $\lim_{x\to 1} f(x)$ exist? Explain.
10. Is f continuous? Explain.

Section 3.1 Exercises

In Exercises 1–4, use the definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

to find the derivative of the given function at the indicated point.

1. $f(x) = 1/x,\ a = 2$
2. $f(x) = x^2 + 4,\ a = 1$
3. $f(x) = 3 - x^2,\ a = -1$
4. $f(x) = x^3 + x,\ a = 0$

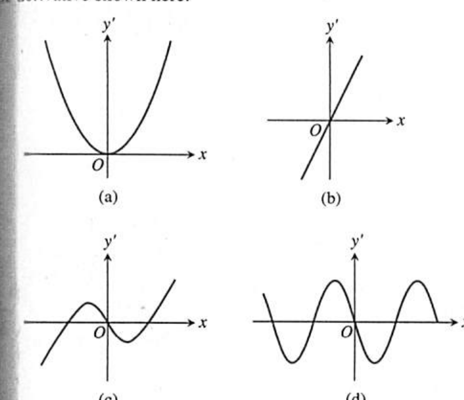
In Exercises 5–8, use the definition

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x,\ a = 2$
6. $f(x) = x^2 + 4,\ a = 1$
7. $f(x) = \sqrt{x+1},\ a = 3$
8. $f(x) = 2x + 3,\ a = -1$
9. Find $f'(x)$ if $f(x) = 3x - 12$.
10. Find dy/dx if $y = 7x$.
11. Find $\frac{d}{dx}(x^2)$.
12. Find $\frac{d}{dx} f(x)$ if $f(x) = 3x^2$.

In Exercises 13–16, match the graph of the function with the graph of the derivative shown here:



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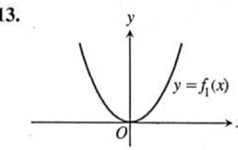

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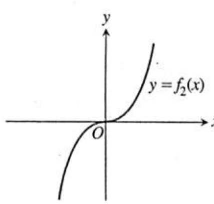

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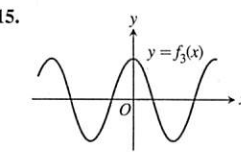

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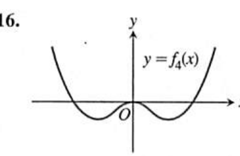

17. If $f(2) = 3$ and $f'(2) = 5$, find an equation of **(a)** the *tangent* line, and **(b)** the *normal* line to the graph of $y = f(x)$ at the point where $x = 2$.