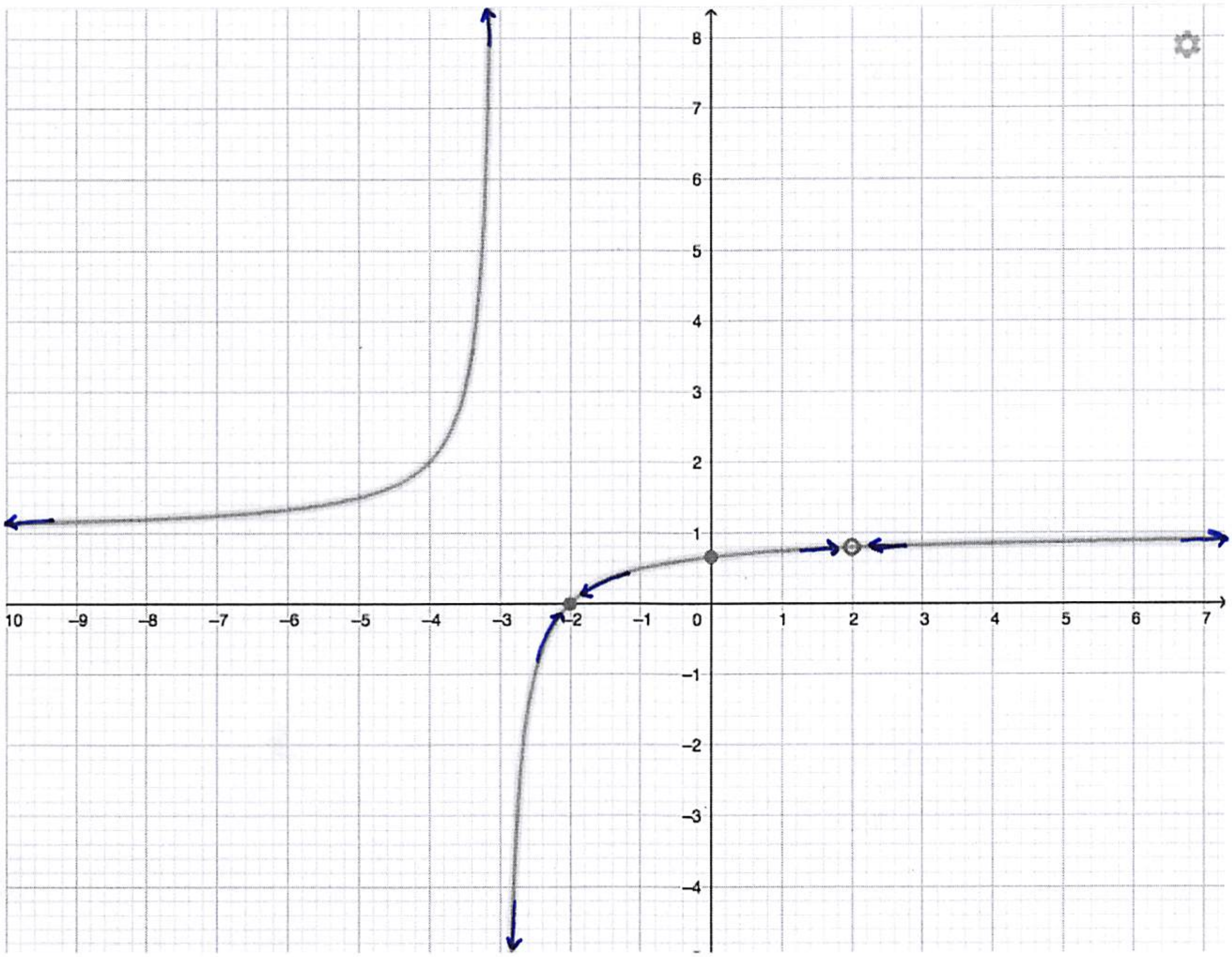
# 2.1 - Limits

Let's have a look at the graph of: $f(x) = \frac{x^2-4}{x^2+x-6} = \frac{(x+2)(x-2)}{(x+3)(x-2)}$:



$\lim_{x \to 2} f(x) = 0.8$

$\lim_{x \to -2} f(x) = 0$

$\lim_{x \to -3} f(x) =$ DNE ($+$ or $-\infty$)
it depends on the side we're approaching $-3$ from

$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = 1$

**Informal definition of a limit:**
The limit of a function is the *y*-value that the function approaches when the *x*-value gets closer and closer to a certain value (from both sides when relevant).

Note: ∞ is not a real number. Therefore, when the limit equals ∞ we consider that the limit doesn't exist, however, the information describes well the behaviour of the function.

**Formal definition of a limit at a finite point:**

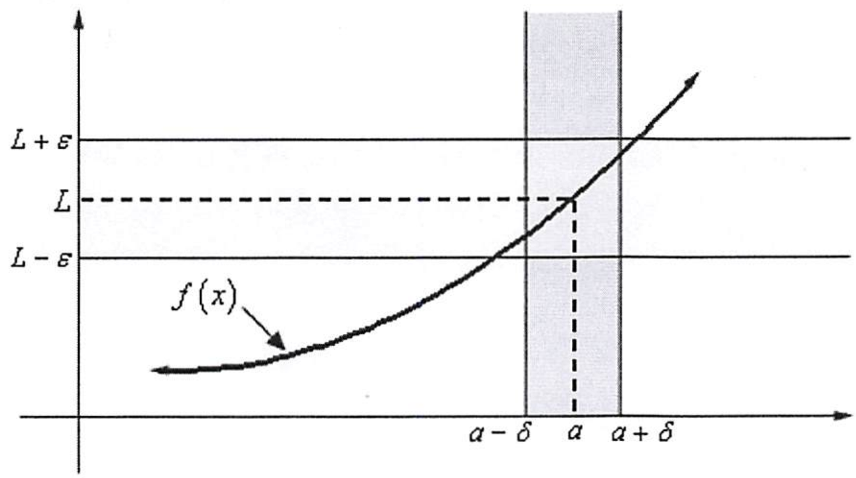
Let $f(x)$ be a function defined on an open interval that contains a value $a$, except possibly at $a$.

Then, we say that $\lim_{x \to a} f(x) = L$

if for every number $\varepsilon > 0$, there is some number $\delta > 0$ such that:

$$|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Graphically, it means:



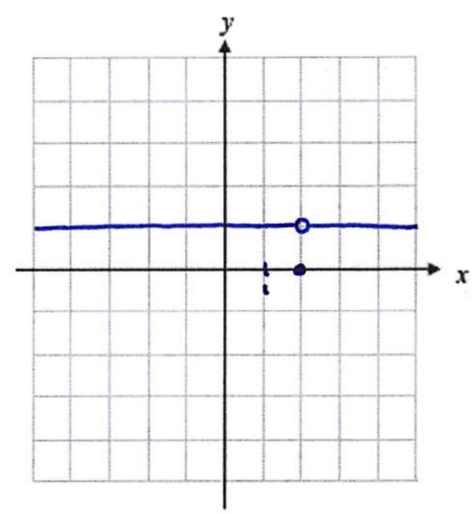

We can determine limits by looking at a graph or by working on the equation…

Examples:

1)

Use the graph to find $\lim_{x \to 2} g(x)$, where $g$ is defined as

$$g(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$



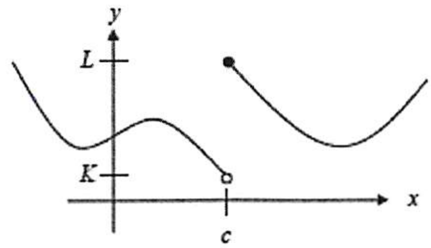

$\lim_{x \to 2} g(x) = 1$

2)

Let $f(x) = 3x - 5$, determine $\lim_{x \to -3} f(x)$

$\lim_{x \to -3} f(x) = -14$

**One sided limits:**
Sometimes, the function doesn't approach the same value from the right side and from the left side…




In that case, we consider that the limit as $x$ approaches $c$ doesn't exist.
However, we can still describe this behavior by writing:

$\lim_{x \to c^+} f(x) = L$ … "the limit of $f(x)$ as $x$ approaches $c$ from the right is $L$"

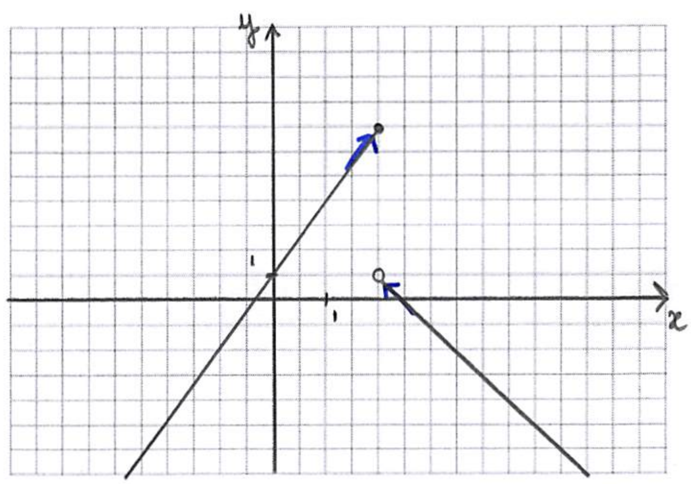
$\lim_{x \to c^-} f(x) = K$ … "the limit of $f(x)$ as $x$ approaches $c$ from the left is $K$"

Thus, we can say that the limit of a function as $x$ approaches any number $c$ exists if and only if the limit as $x$ approaches $c$ from the right is equal to the limit as $x$ approaches $c$ from the left. Using limit notation we have

$$\boxed{\lim_{x \to c} f(x) \text{ exists} \Leftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)}$$

Example: Let $f(x) = \begin{cases} 5 - 2x & if\ x > 2 \\ 3x + 1 & if\ x \le 2 \end{cases}$

a) Graph $f(x)$




b) Find $\lim_{x \to 2^+} f(x) = 1$

c) Find $\lim_{x \to 2^-} f(x) = 7$

d) What can you say about $\lim_{x \to 2} f(x)$?

DNE

*When Limits Do Not Exist*

If there does not exist a <u>number</u> $L$ satisfying the condition in the definition, then we say the $\lim_{x \to c} f(x)$ does not exist.

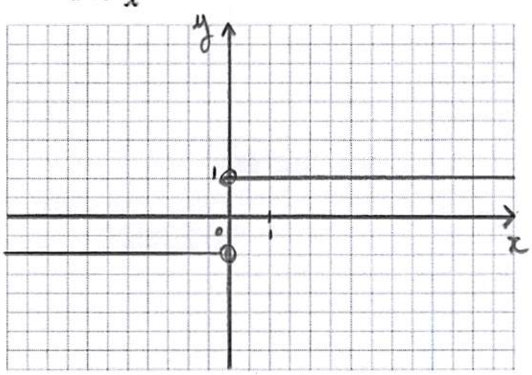
Limits typically fail for three reasons:

1. $f(x)$ approaches a different number from the right side of $c$ than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as $x$ approaches $c$. $\lim_{x \to c} f(x) = \pm\infty$
3. $f(x)$ oscillates between two fixed values as $x$ approaches $c$.

Examples: Investigate the existence of the following limits:

(a) $\lim_{x \to 0} \frac{|x|}{x}$

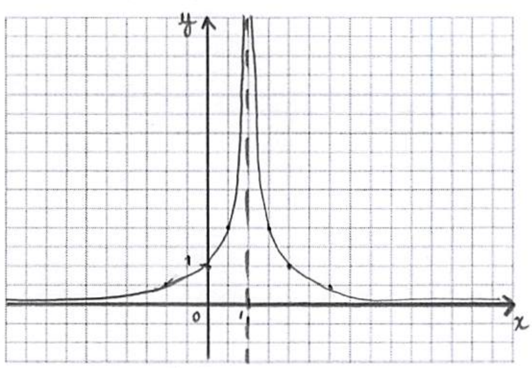
$D = \mathbb{R} \setminus \{0\}$




$y = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

DNE (reason 1)

(b) $\lim_{x \to 1} \frac{1}{(x-1)^2}$



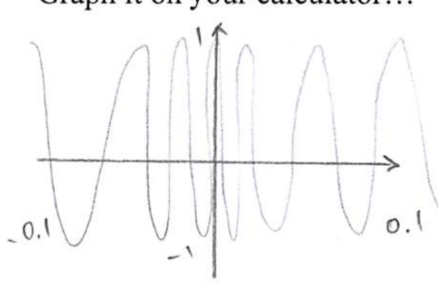

DNE but we can write

(reason 2)

$\lim_{x \to 1} \frac{1}{(x-1)^2} = +\infty$

(c) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

Graph it on your calculator…




DNE (reason 3)

**Determining limits from equations:**

Usually, to determine limits, we start by trying to substitute the value directly and see what we get by using the intuitive rules:

Let $b$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

$\lim_{x\to c} b = b$

$\lim_{x\to c} x = c$

$\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$

$\lim_{x\to c}[f(x)\cdot g(x)] = LK$

$\lim_{x\to c}[b\cdot f(x)] = bL$

$\lim_{x\to c}[f(x)]^{r/s} = L^{r/s}$
provided $r$ and $s$ are integers and $s \neq 0$

$\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ ; provided $K \neq 0$

Examples:

a) $\lim_{x\to 1}(-x^2+1) = 0$

b) $\lim_{x\to 3}\dfrac{\sqrt{x+1}}{x-4} = -2$

c) $\lim_{h\to 0}(3h^2+2h) = 0$

d) $\lim_{h\to 0}(3x^2-2xh+5h) = 3x^2$

However, when we are dealing with limits that equal 0 or $\infty$, we sometimes get what we call: **indeterminate forms** like:

$$\boxed{\frac{0}{0} \quad or \quad \frac{\infty}{\infty} \quad or \quad 0\times\infty \quad or \quad \infty-\infty}$$

Examples:

1) $\lim_{x\to -1}\dfrac{2x^2-x-3}{x+1}$ "$\frac{0}{0}$"

2) $\lim_{x\to 0}\dfrac{\frac{1}{x}}{\frac{1}{x^2}}$ "$\frac{\infty}{\infty}$"

3) $\lim_{x\to 2}(x-2)\times\dfrac{1}{x^2-4x+4}$ "$0\times\frac{1}{0}$" i.e "$0\times\infty$"

4) $\lim_{x\to -1}\left(\dfrac{1}{x+1}-\dfrac{1}{3x+3}\right)$ "$\infty-\infty$"

In those cases, we can't conclude with a direct substitution. You need to find a way to transform the expression to know "who wins" …

The answer can be a number, 0, $\infty$ or DNE (does not exist)

1) $\frac{2x^2-x-3}{x+1} = \frac{(x+1)(2x-3)}{x+1} = 2x-3 \quad \lim_{x\to -1} \frac{2x^2-x-3}{x+1} = -5$

2) $\frac{\frac{1}{x}}{\frac{1}{x^2}} = \frac{1}{x} \times \frac{x^2}{1} = x \quad \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{1}{x^2}} = 0$

3) $(x-2) \times \frac{1}{x^2-4x+4} = \frac{x-2}{(x-2)^2} = \frac{1}{x-2}$ if $x>2$, $(x-2)>0$; if $x<2$, $(x-2)<0$

$$\left.\begin{array}{l}\lim_{x\to 2^+} (x-2)\times\frac{1}{x^2-4x+4} = +\infty \\ \lim_{x\to 2^-} (x-2)\times\frac{1}{x^2-4x+4} = -\infty\end{array}\right\} \lim_{x\to 2} (x-2)\times\frac{1}{x^2-4x+4} \text{ DNE}$$

4) $\frac{1}{x+1} - \frac{1}{3x+3} = \frac{3}{3(x+1)} - \frac{1}{3(x+1)}$

$$= \frac{2}{3(x+1)} \left.\begin{array}{l}\xrightarrow[x\to -1^+]{} +\infty \\ \xrightarrow[x\to -1^-]{} -\infty\end{array}\right\} \lim_{x\to -1}\left(\frac{1}{x+1} - \frac{1}{3x+3}\right) \text{ DNE}$$

<u>Note</u>: - It is important to always write the word lim in front of your expression as long as there still are some variables left. If you want to work on transforming the expression, you can start a different line and not write lim anywhere… until you take the limit.

- When an expression approaches 0, depending if it's coming from the right or the left, the "not quite 0" has a different sign. Therefore, in a "$\frac{1}{0}$" type of limit, you need to determine if it's a "positive" zero or a "negative" zero to conclude about a + or a - ∞.

More examples:

5) $\lim_{x\to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$ "$\frac{0}{0}$"

$$\frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \frac{\frac{4}{4(x+4)} - \frac{x+4}{4(x+4)}}{x}$$

$$= \frac{\frac{-x}{4(x+4)}}{x}$$

$$= -\frac{1}{4(x+4)}$$

$$\lim_{x\to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \frac{-1}{16}$$

6) $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$ $\qquad \frac{\sqrt{x+1}-2}{x-3}=\frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}$

"$\frac{0}{0}$"

$$=\frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}=\frac{1}{\sqrt{x+1}+2} \qquad \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\frac{1}{4}$$

7) $\lim_{x\to -1}\frac{1}{x+1}-\frac{2}{(x+1)(x+3)}$ $\qquad \frac{1}{x+1}-\frac{2}{(x+1)(x+3)}=\frac{x+3}{(x+1)(x+3)}-\frac{2}{(x+1)(x+3)}$

"$\infty-\infty$"

$$=\frac{x+1}{(x+1)(x+3)}=\frac{1}{x+3} \qquad \lim_{x\to -1}\frac{1}{x+1}-\frac{2}{(x+1)(x+3)}=\frac{1}{2}$$

→ see indeterminate forms summary.

Another "famous limit that you need to remember is:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}}$$

It is actually more general than that and you can remember that:

$$\lim_{☺\to 0}\frac{\sin ☺}{☺}$$

Examples:

1) $\lim_{x\to 0}\frac{\sin(3x)}{3x}=1$

$3x \xrightarrow[x\to 0]{} 0$

2) $\lim_{x\to 0}\frac{\sin(5x)}{3x}=\lim_{x\to 0}\frac{5}{3}\cdot\cancel{\frac{\sin(5x)}{5x}}^{1}=\frac{5}{3}$

$\frac{0}{0}$

3) $\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim_{x\to 0}\frac{\sin x}{\cos x}\times\frac{1}{x}$

$$=\lim_{x\to 0}\cancel{\frac{\sin x}{x}}^{1}\times\frac{1}{\cancel{\cos x}^{1}}=1$$

Sometimes, we can't work directly on the function. Another option would be to use the **squeezing theorem** (or sandwich theorem):

**The Sandwich Theorem**

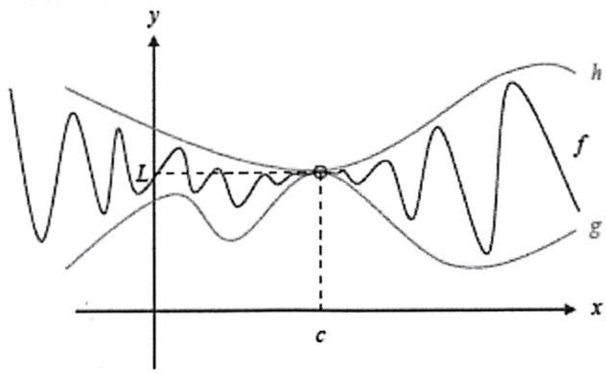
If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about $c$, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

In other words, if we "sandwich" the function $f$ between two other functions $g$ and $h$ that both have the same limit as $x$ approaches $c$, then $f$ is "forced" to have the same limit too.




Examples:

1) proof of $\lim_{x \to 0} \frac{\sin x}{x}$ (ask for worksheet)
2) Prove that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all $x \ne 0$

$\overset{\nearrow 0}{-x^2} \le x^2 \sin\left(\frac{1}{x}\right) \le \overset{\nearrow 0}{x^2}$ for all $x \ne 0$

The sandwich theorem applies and gives us that

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Hwk: worksheet (except for questions 3 – 5; 16 from day 1)
+ textbook p 65 # 1 – 10; p 66 # 5 – 44; 51 – 62; 65 – 74

## 2.2 – Limits involving infinity

In this section, we continue to investigate limits when $x$ approaches infinity (it can be done from one side only) and also when the function approaches infinity as $x$ approaches a certain value. These situations are linked with the concepts of asymptotes:

*Definition:* *Horizontal Asymptote*

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

*Definition:* *Vertical Asymptote*

The line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

***This occurs whenever there is a value of $x$ that gives you a 0 in the denominator (but not the numerator).

We already have mentioned the **indeterminate forms** like:

$$\frac{0}{0} \quad or \quad \frac{\infty}{\infty} \quad or \quad 0\times\infty \quad or \quad \infty-\infty$$

And we remember that in a "$\frac{1}{0}$" type of limit, you need to determine if it's a "positive" zero or a "negative" zero to conclude about a + or a - ∞.

**Limits as $x$ approaches infinity:**

We have learned in chapter 9 that when the function is a **rational function**, its end behavior **as $x$ approaches infinity** is the same than the end behavior of its leading term…

Applications:

a) $\lim\limits_{x\to\infty} \frac{2x+5}{3x^2-6x+1} = \lim\limits_{x\to\infty} \frac{2x}{3x^2} = \lim\limits_{x\to\infty} \frac{2}{3x} = 0$

"$\frac{\infty}{\infty}$"

b) $\lim\limits_{x\to\infty} \frac{2x^2-3x+5}{x^2+1} = \lim\limits_{x\to\infty} \frac{2x^2}{x^2} = 2$

"$\frac{\infty}{\infty}$"

c) $\lim\limits_{x\to\infty} \frac{x^4+x^3+9}{3x-3} = \lim\limits_{x\to\infty} \frac{x^4}{3x} = \lim\limits_{x\to\infty} \frac{1}{3}x^3 = \infty$

"$\frac{\infty}{\infty}$"

Note: This is only true as $x$ approaches infinity!!

When determining limits towards infinity involving non-rational functions, you can try to use "direct substitution", and if it's indeterminate the squeezing theorem or try to **factor the strongest term**…

Examples:

1) $\lim_{x\to\infty} \frac{\sin(x)}{x}$

$-1 \le \sin x \le 1$ for all $x$

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$ SQ $\Rightarrow \lim_{x\to\infty}\frac{\sin x}{x} = 0$

2) $\lim_{x\to\infty} \frac{5+2^x}{3-2^x}$

$\frac{5+2^x}{3-2^x} = \frac{\cancel{2^x}\left(\frac{5}{2^x}+1\right)}{\cancel{2^x}\left(\frac{3}{2^x}-1\right)}$

$\lim_{x\to\infty}\frac{5}{2^x} = 0$

$\lim_{x\to\infty}\frac{3}{2^x} = 0$

$\therefore \lim_{x\to\infty}\frac{5+2^x}{3-2^x} = -1$

3) $\lim_{x\to-\infty} \frac{5+2^x}{3-2^x}$

$\lim_{x\to-\infty} 2^x = 0$ $\therefore \lim_{x\to-\infty}\frac{5+2^x}{3-2^x} = \frac{5}{3}$

**<u>Infinite limits as *x* approaches *a*:</u>**

Examples:

Determine the vertical asymptotes of each function and describe the function's behavior around it.

a) $f(x) = \frac{x^2-1}{2x+4} = \frac{\overset{-1}{(x+1)}\overset{-3}{(x-1)}}{2(x+2)}$

v. asymptote : $x = -2$

$\lim_{x\to-2^+} f(x) = +\infty$ $\lim_{x\to-2^-} f(x) = -\infty$

b) $g(x) = \frac{1-x}{2x^2-5x-3} = \frac{1-x}{(x-3)(2x+1)}$

v. asymptotes: $x = 3$ and $x = -\frac{1}{2}$

$\lim_{x\to3^+} g(x) = -\infty$ $\lim_{x\to3^-} g(x) = +\infty$ $\lim_{x\to-\frac{1}{2}^+} g(x) = -\infty$ $\lim_{x\to-\frac{1}{2}^-} g(x) = +\infty$

c) $h(x) = \frac{x-2}{3x^2-5x-2} = \frac{x-2}{(x-2)(3x+1)} = \frac{1}{3x+1}$

v. asymptote : $x = -\frac{1}{3}$

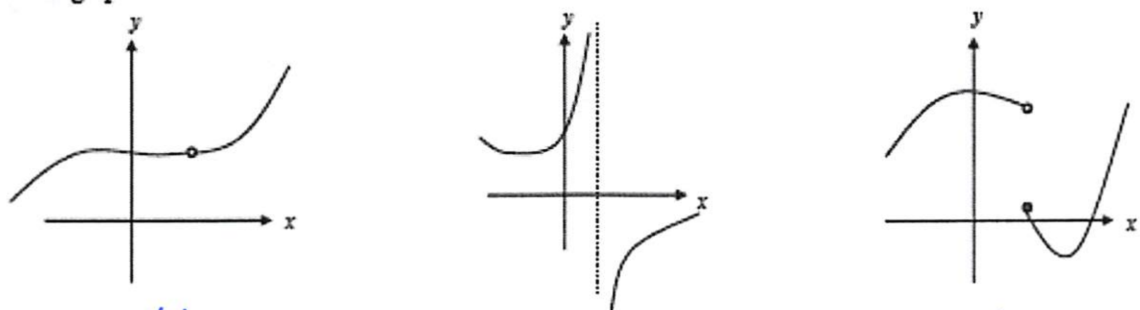
(there is a hole @ 2)

$\lim_{x\to-\frac{1}{3}^+} h(x) = +\infty$ $\lim_{x\to-\frac{1}{3}^-} h(x) = -\infty$

Hwk: worksheet + p 76 # 1 – 54;

## 2.3 – Continuity

In non – technical terms, a function is continuous if you can draw the function "without ever lifting your pencil". THIS IS NOT A DEFINITION YOU SHOULD USE AFTER TODAY!!! The following graphs demonstrate three types of discontinuous graphs.




graph 1 graph 2 graph 3

*Discontinuities: Removable versus Non-Removable*

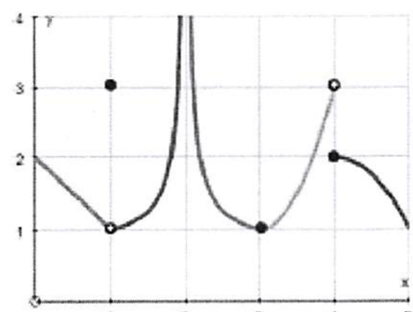
To say a function is discontinuous is not sufficient. We would like to know what type of discontinuity exists. If the function is not continuous, but I could make it continuous by appropriately defining or redefining $f(c)$, then we say that $f$ has a **removable discontinuity**. Otherwise, we say $f$ has a **non-removable discontinuity**.

Once again, informally we say that $f$ has a removable discontinuity if there is a "hole" in the function, but $f$ has a non-removable discontinuity if there is a "jump" or a vertical asymptote.

*Example 1*: Which (if any) of the three graphs above have a removable discontinuity?

graph 1 only

*Example 2*: Find the points (intervals) at which the function below is continuous, and the points at which it is discontinuous.




continuous on $[0;1) \cup (1;2) \cup (2;4) \cup (4;5]$

discontinuous at 1, 2 and 4.

**Continuity at a point c:**

A function $f$ is continuous at a point c iff:

$$\boxed{\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)}$$

Example: Determine if $h$ is continuous at 1.

$$h(x) = \begin{cases} -2x+3 & ;x<1 \\ x^2 & ;x\geq 1 \end{cases}$$

$\lim_{x\to 1^-} h(x) = -2(1)+3 = 1$

$\lim_{x\to 1^+} h(x) = 1^2 = 1$

$h(1) = 1$

yes it is!

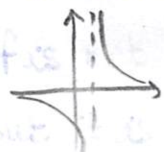
**Note:** This definition assumes that c is not an end value of a closed domain…we can easily see what it would be for the end values…(one sided limit available only…)

**Continuity of a function:**
A function is continuous if it is continuous at every point in its domain.

Examples: Determine if the following functions are continuous or not.

(a) $f(x)=\dfrac{1}{x-1}$

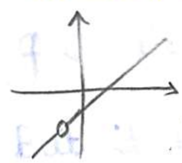
$D=\mathbb{R}\setminus\{1\}$



$f$ is continuous on $D$

(b) $g(x)=\dfrac{2x^2+x-6}{x+2} = \dfrac{(x+2)(2x-3)}{x+2} = 2x-3$

$D=\mathbb{R}\setminus\{-2\}$



$f$ is continuous on $D$

Note: $k(x)=\begin{cases} g(x) & \text{if } x\neq -2 \\ -7 & \text{if } x=-2 \end{cases}$

would be continuous on $\mathbb{R}$

(c) $h(x)=\begin{cases} -2x+3 & ;x<1 \\ x^2 & ;x\geq 1 \end{cases}$ $\quad D=\mathbb{R}$ ✓

$h$ is continuous over $(-\infty;1)$ (linear)
$h$ is continuous over $(1;+\infty)$ (quad)
Is it continuous at 1? $\lim\limits_{x\to 1^-} h(x) = \lim\limits_{x\to 1^+} h(x) = h(1) = 1 \Rightarrow$ yes.
$\Rightarrow$ yes, $f$ is continuous on $D$

Examples: Determine the value of the constant $a$ or $k$ so that the following functions are continuous on their domain.

$g(x)=\begin{cases} x^2+7 & \text{if } x\geq 1 \\ x+a & \text{if } x<1 \end{cases}$

$g$ is continuous on $(1;+\infty)$ for all $a$
$g$ is continuous on $(-\infty;1)$ for all $a$

$g(1)=\lim\limits_{x\to 1^+} g(x) = 8 \qquad \lim\limits_{x\to 1^-} g(x) = 1+a$

so $g$ will be continuous if $1+a=8$
$\boxed{a=7}$

$h(x)=\begin{cases} \dfrac{x^4-1}{x-1} & \text{if } x\neq 1 \\ k & \text{if } x=1 \end{cases}$ $\qquad \dfrac{x^4-1}{x-1} = \dfrac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{x-1}}$

$\lim\limits_{x\to 1} h(x) = 4 \;\therefore$
$h$ is continuous on $\mathbb{R}$ if $k=4$

**Properties of Continuous functions:**

1) If $f$ and $g$ are continuous at $x = c$, then the following combinations are continuous at $x = c$:
   a) sums: $f + g$
   b) Difference: $f - g$
   c) Products: $f.g$
   d) Constant multiples: $k.f$ for any number $k$
   e) ⚠Quotients : $\frac{f}{g}$ provided that $g(c) \neq 0$

2) If $f$ is continuous at $c$ and $g$ is continuous at $f(c)$, then the composite $g \circ f$ is continuous at $c$.

Example: Show that $y = \left|\frac{x.sinx}{x^2+2}\right|$ is continuous

$D = \mathbb{R}$ ✓

let $f(x) = x$
$g(x) = \sin x$
$h(x) = x^2+2$
$k(x) = |x|$

* $f, g, h$ are continuous for all $x$.

$y = k\left(\frac{f \times g}{h}(x)\right)$

and $h(x) \neq 0$ for all $x$

$\therefore$ $\frac{f \times g}{h}$ is continuous for all $x$

* $k$ is continuous for all $x$

$\therefore$ this function is continuous.

**The Intermediate Value Theorem (IVT)**

If $f$ is continuous on the closed interval [a, b] then $f$ takes on every value between $f(a)$ and $f(b)$.

Suppose $k$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in [a, b] such that $f(c) = k$.

♫: The Intermediate value theorem tells you that at least one $c$ exists, but it does not give you a method for finding $c$. This theorem is an example of an *existence theorem*.

In the Intermediate Value Theorem …

a) What are the necessary requirements in order to apply this theorem?

f has to be continuous on a closed interval

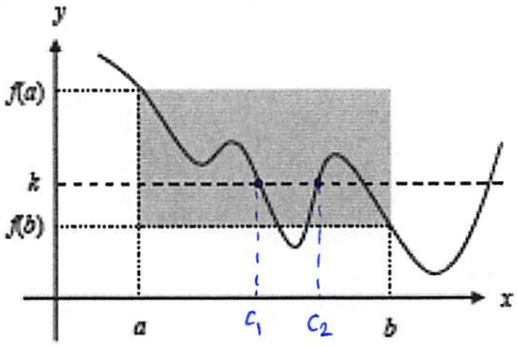
b) $k$ is on which axis?

the y-axis.

c) $c$ is on which axis?

the x-axis.

Consider the function $f$ below.




- Is $f$ continuous on [a, b] ? yes
- Is $k$ between $f(a)$ and $f(b)$ ? yes
- In this example, if $a < c < b$, then there are 2 $c$'s such that $f(c) = k$.
- Label the $c$'s on the graph as $c_1, c_2, \ldots$

Example: Let $f(x) = \frac{x^2}{x-1}$. Verify that the Intermediate Value Theorem applies to the interval $\left[\frac{5}{2}, 4\right]$ and explain why the IVT guarantees an $x$-value of $c$ where $f(c) = \cancel{6}$ 5.

- $f$ is continuous on $\left[\frac{5}{2}; 4\right]$ ✓
  closed interval ✓
- $f\left(\frac{5}{2}\right) = \frac{25}{6} \approx 4.166$
  $f(4) = \frac{16}{3} \approx 5.33$

$5 \in \left[\frac{25}{6}; \frac{16}{3}\right]$ so there is a $c \in \left[\frac{5}{2}; 4\right]$ where $f(c) = 5$

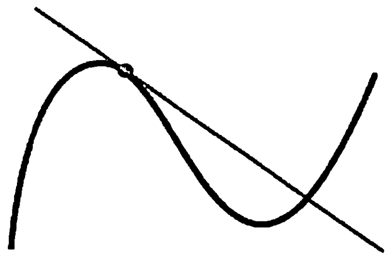
the IVT applies and states that

Hwk: worksheet + textbook p 84 # 1 – 36; 47 – 50;

## 2.4 – Rates of Change and Tangent Lines

Calculus will be used a lot for optimization. When given a curve, it is going to be important to see "how fast" the curve is increasing or decreasing. For a straight line, this information is given by the slope, but for a curve, it's not as clear…

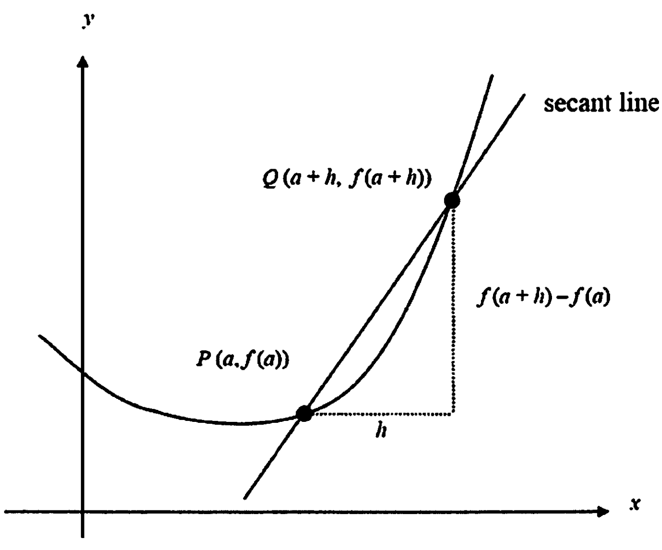
The idea is the consider that the curve is best approximated at a point by its tangent line at this point.



There is no good geometric definition of a **tangent line** to a curve at a certain point. We usually say that it's the best linear approximation of the curve at this point, and that it touches the curve only once around the point of tangency, but it's not rigorously true…

In order to determine the equation of a tangent line at a certain point P on a curve, we need to determine its slope.

The slope of a tangent line of a curve is the limit of the slopes of the secant lines close to that point…




The **slope of the tangent line** of a curve $y = f(x)$ at a point $P(a, f(a))$ is the number:

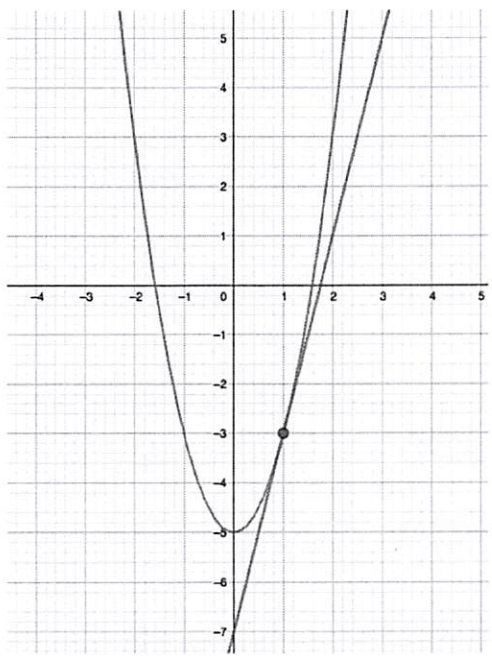
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Provided the limit exists…

Your goal is to simplify this quotient until you have cancelled algebraically the $h$ in the denominator in order to determine the limit…

Examples:

a) $f(x) = 2x^2 - 5$ slope of the tangent line at point $P(1; -3)$.



$$\frac{f(1+h) - f(1)}{h} = \frac{2(1+h)^2 - 5 - (-3)}{h}$$

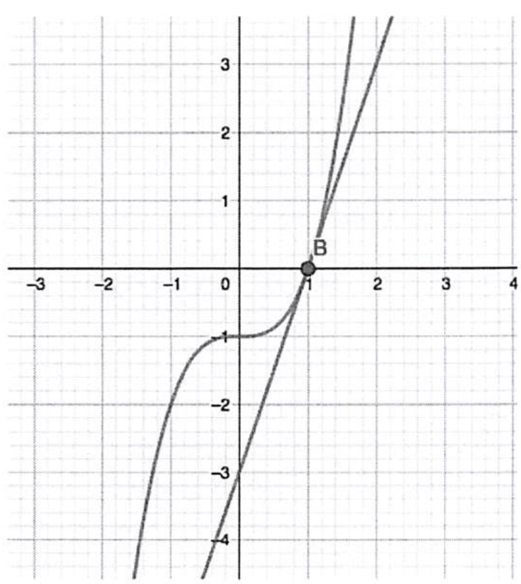
$$= \frac{2(h^2+2h+1) - 2}{h}$$

$$= \frac{2h^2 + 4h}{h}$$

$$= 2h + 4 \xrightarrow[h \to 0]{} 4$$

the slope of the tangent line at $(1; -3)$ is 4.

b) $g(x) = x^3 - 1$ slope of the tangent line at point $P(1; 0)$.




$$\frac{g(1+h) - g(1)}{h} = \frac{(1+h)^3 - 1 - 0}{h}$$

$$= \frac{h^3 + 3h^2 + 3h + 1 - 1}{h}$$

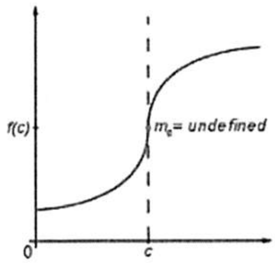
$$= h^2 + 3h + 3 \xrightarrow[h \to 0]{} 3$$

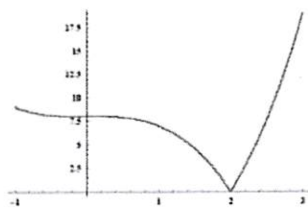
the slope … is 3.

If the limit doesn't exist, it can be because the curve has a vertical tangent line or because the function isn't continuous at that point or …

> If $f$ is continuous at $a$ and
> $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\pm\infty$$
> the vertical line, $x = a$ is a **vertical tangent line** to the graph of $f$.

Types of graphs that have a vertical tangent line:



<u>Important connection: **Average and Instantaneous velocity:**</u>

In pre – calculus courses, you used the formula $d = rt$ to determine the speed of an object. What you found was the object's average speed. A moving body's **average speed** during an interval of time is found by dividing the total distance covered by the elapsed time. (Speed is always positive … Velocity indicates direction and can be negative.) We are going to find the **average velocity**.

If an object is dropped from an initial height of $h_0$, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, $s$ feet) of an object that has fallen for $t$ seconds.

*Example 1*: Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.

a) What is the position function for Wile E. Coyote?

$$s(t) = -16t^2 + 900$$

b) Find Wile E.'s average velocity for the first 3 seconds.

$$\frac{s(3)-s(0)}{3-0} = \frac{756-900}{3} = -48 \text{ ft/s}$$

c) Find Wile E.'s average velocity between $t = 2$ and $t = 3$ seconds.

$$\frac{s(3)-s(2)}{3-2} = \frac{756-836}{1} = -80 \text{ ft/s}$$

d) Find Wile E.'s velocity at the instant $t = 3$ seconds.

$$\lim_{h\to 0}\frac{s(3+h)-s(3)}{h} = \lim_{h\to 0}(-16h-96) = -96 \text{ ft/s}$$

Hwk: worksheet 2.1 # 3, 4, 5, 16 - 2.4 # 11

$$s(3+h) = -16(3+h)^2+900$$
$$= -16h^2-96h+756$$
$$s(3) = 756$$

**Normal line to a curve:**

*Normal Line*

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

Example: Let $f(x) = \frac{1}{x+1}$

a) Find the slope of the curve at $x = a$.

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{a+h+1} - \frac{1}{a+1}}{h}$$

$$= \frac{\frac{a+1-(a+h+1)}{(a+1)(a+h+1)}}{h}$$

$$= -\frac{1}{(a+1)(a+h+1)} \xrightarrow[h \to 0]{} -\frac{1}{(a+1)^2}$$

b) Find the slope of the curve at $x = 2$.

$$\frac{f(2+h)-f(2)}{h} \longrightarrow -\frac{1}{9}$$

c) Write the equation of the tangent line to the curve at $x = 2$.

$y - y_p = m(x - x_p)$

when $x_p = 2$, then $y_p = \frac{1}{3}$

$$y - \frac{1}{3} = -\frac{1}{9}(x-2)$$

$$y = -\frac{1}{9}x + \frac{5}{9}$$

d) Write the equation of the normal line to the curve at $x = 2$.

$m_N = 9$ same point $(2; \frac{1}{3})$

$$y - \frac{1}{3} = 9(x-2)$$

$$y = 9x - \frac{53}{3}$$