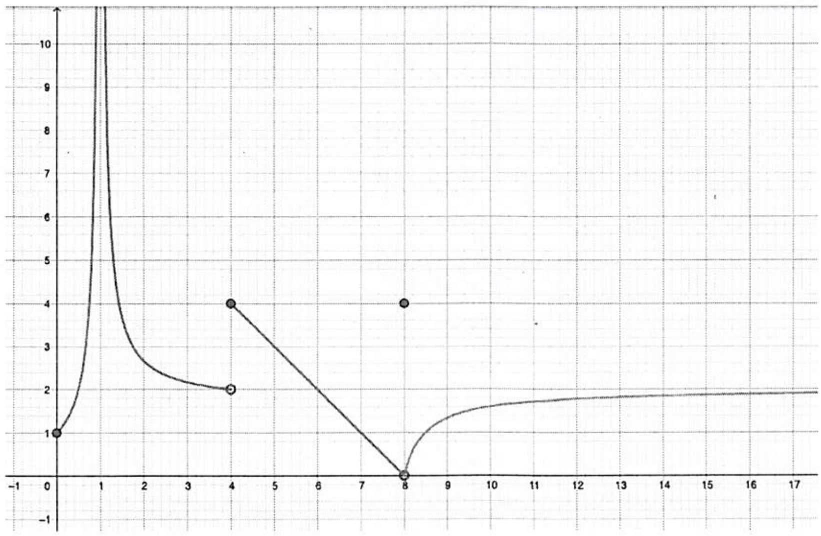
Name: Solutions

AP Calc

Chapter 2 TEST
Graphing Calculators not allowed

1. Consider the following graph of a function f:



a) Determine the equations of all of its asymptotes. [1]

$x = 1$ $y = 2$

b) What is the domain of f? [1]

$D = [0,1) \cup (1, +\infty)$

c) Where is the function discontinuous on its domain? [1]

at $x = 4$ and $x = 8$

d) Determine the following limits and values if they exist: [4]

i) $\lim_{x \to 1} f(x) = +\infty$ (DNE)

ii) $\lim_{x \to 8} f(x) = 0$

iii) $\lim_{x \to 4^+} f(x) = 4$

iv) $\lim_{x \to 4^-} f(x) = 2$

v) $f(4) = 4$

vi) $f(8) = 4$

vii) $\lim_{x \to 4} f(x)$ DNE

viii) $\lim_{x \to \infty} f(x) = 2$

2. A particle is moving along the x-axis so that at a time t (in seconds), it is at a position $x = 3t^2 - 12t + 1$ (in metres). [3]

a) Find its average velocity in the first 3 seconds.

$$\frac{x(3)-x(0)}{3-0} = \frac{27-36+1-1}{3} = -\frac{8}{3} \text{ m/s}$$

b) Find its instantaneous velocity at time $t = 1$.

$$\frac{x(1+h)-x(1)}{h} = \frac{3(1+h)^2-12(1+h)+1-(3-12+1)}{h}$$

$$= \frac{3h^2-6h}{h} = 3h-6 \xrightarrow[h\to 0]{} \boxed{-6 \text{ m/s}}$$

c) In which direction is it moving at time $t = 1$?

velocity is negative $\Rightarrow$ to the left

d) On which side of the origin is it at time $t = 1$?

$x(1) = -8$ (negative) $\Rightarrow$ on the left side of the origin

3. Soit $f(x) = \sqrt{x}$. [4]

a) Determine its slope at $x = a$.

$$\frac{f(a+h)-f(a)}{h} = \frac{\sqrt{a+h}-\sqrt{a}}{h} = \frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})} = \frac{1}{\sqrt{a+h}+\sqrt{a}} \xrightarrow[h\to 0]{} \boxed{\frac{1}{2\sqrt{a}}}$$

b) Determine its slope at $x = 4$.

replace a by 4 $\Rightarrow \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$

c) Determine the equation of the tangent line at $x = 4$.

$f(4) = 2$ tangent line : $y-2 = \frac{1}{4}(x-4)$

$$\boxed{y = \frac{1}{4}x + 1}$$

d) Determine the equation of the normal line at $x = 4$.

slope : -4 $\Rightarrow y-2 = -4(x-4)$

$$\boxed{y = -4x + 18}$$

4. Use the IVT to prove that the equation $x^3 - 4x = 1 - x^2$ is solvable. [2]

Let $f(x) = x^3 + x^2 - 4x - 1$

f is continuous on $[-1, 0]$

$f(-1) = 3 > 0$

$f(0) = -1 < 0$

IVT applies.

$f(x) = 0$ has at least one solution on $(-1, 0)$

5. Determine the following limits [16]

a) $\lim_{t\to -4} \frac{t^2}{4-t} = \frac{16}{8} = 2$

b) $\lim_{x\to\infty} (\ln x - x) = \lim_{x\to\infty} x\Big(\underbrace{\overset{\nearrow 0}{\frac{\ln x}{x}} - 1}_{\longrightarrow -1}\Big) = -\infty$

c) $\lim_{x\to -2} \frac{x^2+2x}{x^2-4} = \lim_{x\to -2} \frac{x\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \lim_{x\to -2} \frac{x}{x-2} = \frac{-2}{-4} = \frac{1}{2}$

d) $\lim_{h\to 2} \frac{1}{4-h^2}$ DNE $\quad \lim_{h\to 2^+} \frac{1}{4-h^2} = -\infty \quad \lim_{h\to 2^-} \frac{1}{4-h^2} = +\infty$

e) $\lim_{x\to 2} \frac{|x-2|}{x-2}$ DNE $\quad \lim_{x\to 2^+} \frac{|x-2|}{x-2} = 1 \quad \lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1$

f) $\lim_{t\to 0} \frac{t}{\sqrt{4+t}-\sqrt{4-t}} = \lim_{t\to 0} \frac{t\left(\sqrt{4+t}+\sqrt{4-t}\right)}{4+t-(4-t)}$

$= \lim_{t\to 0} \frac{\sqrt{4+t}+\sqrt{4-t}}{2} = 2$

g) $\lim_{s\to 0} \frac{(s+1)^2-(s-1)^2}{s} = \lim_{s\to 0} \frac{s^2+2s+1-(s^2-2s+1)}{s}$

$= \lim_{s\to 0} \frac{4s}{s} = 4$

h) $\lim_{x\to 0} \frac{\sin(2\pi x)}{\sin(3\pi x)}$ $= \lim_{x\to 0} \frac{\sin(2\pi x)}{2\pi x} \cdot \frac{3\pi x}{\sin(3\pi x)} \cdot \frac{2}{3} = \frac{2}{3}$

i) $\lim_{x\to 1^-} \frac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$ $= \lim_{x\to 1^-} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{\sqrt{2}}$

j) $\lim_{x\to\infty} \frac{x^3-5x^2+7}{8+2x-5x^3}$ $= \lim_{x\to\infty} \frac{x^3}{-5x^3} = -\frac{1}{5}$

k) $\lim_{x\to\infty} \frac{e^x-x^2}{e^x+x^5}$ $= \lim_{x\to\infty} \frac{\cancel{e^x}(1-\frac{x^2}{e^x})}{\cancel{e^x}(1+\frac{x^5}{e^x})} = 1$

l) $\lim_{x\to-\infty} \frac{x^2+3}{x^3+2}$ $= \lim_{x\to-\infty} \frac{x^2}{x^3} = \lim_{x\to-\infty} \frac{1}{x} = 0$

m) $\lim_{x\to 3} \frac{1}{3-x}$ DNE $\lim_{x\to 3^+} \frac{1}{3-x} = -\infty$ $\lim_{x\to 3^-} \frac{1}{3-x} = +\infty$

n) Let $\lfloor x \rfloor$ be the greatest integer function.

$\lim_{x\to 3^+} \lfloor x \rfloor = 3$ $\lim_{x\to 3^-} \lfloor x \rfloor = 2$ $\lim_{x\to 3} \lfloor x \rfloor$ DNE

p) $\lim_{x\to\infty} \frac{4\sin x}{x}$

$-1 \le \sin x \le 1$

$-\frac{4}{x} \le \frac{4\sin x}{x} \le \frac{4}{x}$ if $x>0$

sandwich theorem: $\lim_{x\to\infty} \frac{4\sin x}{x} = 0$

q) $\lim_{x\to-\infty}[x+\sqrt{x^2-4x+1}]$ $= \lim_{x\to-\infty} \dfrac{x^2-(x^2-4x+1)}{x-\sqrt{x^2-4x+1}}$

$= \lim_{x\to-\infty} \dfrac{4x-1}{x-\sqrt{x^2\left(1-\frac{4}{x}+\frac{1}{x^2}\right)}}$

$= \lim_{x\to-\infty} \dfrac{x(4-1/x)}{x+x\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}}$

$= \lim_{x\to-\infty} \dfrac{\cancel{x}\left(4-\frac{1}{x}\right)}{\cancel{x}\left(1+\sqrt{1-4/x+1/x^2}\right)} = \boxed{2}$

6. For each function, determine k such that it is continuous on its domain. [3]

a) $f(x)=\begin{cases} \dfrac{e^x\tan x}{2x} & , \ if\ -\frac{\pi}{2}<x<0 \\ k & , \ if\ x\geq 0 \end{cases}$

$\lim_{x\to0^-}\dfrac{e^x\tan x}{2x} = \lim_{x\to0^-} \overset{\to 1}{e^x}\cdot\dfrac{\overset{1}{\cancel{\sin x}}}{\underbrace{2\cos x}_{\to 2}\cdot\cancel{x}} = \frac{1}{2}$

$\Rightarrow \boxed{k=\frac{1}{2}}$

b) $g(x)=\begin{cases} x-k\,, & if\ x<3 \\ 1-kx\,, & if\ x\geq 3 \end{cases}$

$\left.\begin{array}{l}\lim_{x\to3^-}(x-k)=3-k \\ \lim_{x\to3^+}(1-kx)=1-3k\end{array}\right\}$

$3-k=1-3k$

$2k=-2$

$\boxed{k=-1}$

7. TRUE or FALSE? [3]
If TRUE, no explanation is needed.
If FALSE, give a counterexample (expression or graph)
a) If $\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}(f(x)+g(x))$ does not exist.

TRUE

b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}(f(x)+g(x))$ does not exist.

FALSE ex: $f(x)=\frac{1}{x}$ $g(x)=-\frac{1}{x}$ $a=0$

c) If f is continuous at a, then so is $|f|$

TRUE

d) If $|f|$ is continuous at a, then so is f.

FALSE ex: $f(x)=\begin{cases}1 & \text{if } x\geq 0\\ -1 & \text{if } x<0\end{cases}$

$|f(x)|=1$ for all x.