2.1 – Worksheet – Day 1

All work must be shown and justified in this course for full credit. Insupported answers may receive NO credit.

1. The only way to guarantee the existence of a limit is to algebraically prove it. Describe the different ways you can investigate the existence of a limit.

- substitute the value in the equation
- look at the graph (from both sides when relevant)

2. Using words, explain what is meant by the expression $\lim_{q \to c} f(q) = T$.

when the q values are approaching the value c (from both sides), the values of the function are approaching T.

3. Assume $\lim_{x \to b} f(x) = 7$ and $\lim_{x \to b} g(x) = -3$. Determine:

a) $\lim_{x \to b}(f(x) + g(x)) = 4$

b) $\lim_{x \to b}(f(x).g(x)) = -21$

c) $\lim_{x \to b}(4g(x)) = -12$

d) $\lim_{x \to b}\left(\frac{f(x)}{g(x)}\right) = -\frac{7}{3}$

4. When asked to evaluate the limit of a function, what should be done first?

* substitution of the value in the equation

5. Evaluate the following limits by using direct substitution.

a) $\lim_{x \to 7} sec\left(\frac{\pi x}{6}\right) = sec\left(\frac{7\pi}{6}\right) = -\frac{2}{\sqrt{3}}$

$= -\frac{2\sqrt{3}}{3}$

b) $\lim_{x \to 4} \sqrt[3]{x+4} = \sqrt[3]{4+4} = \sqrt[3]{8} = 2$

c) $\lim_{x \to 1/2} 3x^2(2x-1) = 3\left(\frac{1}{2}\right)^2\left(2 \times \frac{1}{2} - 1\right)$

$= 0$

d) $\lim_{y \to 2} \frac{y^2+5y+6}{y+2} = \frac{20}{4} = 5$

e) $\lim_{x \to -2} (x-6)^{2/3} = (-8)^{2/3}$

$= \sqrt[3]{-8}^{\,2} = (-2)^2 = 4$

f) $\lim_{x \to 2} \sqrt{x+3} = \sqrt{5}$

6. Explain why you <u>cannot</u> use direct substitution to determine each of the following limits.

a) $\lim\limits_{x\to-2}\sqrt{x-2}$

$D=[2;+\infty)$
-2 is not on the domain

b) $\lim\limits_{x\to0}\frac{1}{x^2}$

"$\frac{1}{0}$" is undefined

c) $\lim\limits_{x\to0}\frac{(4+x)^2-16}{x}$

"$\frac{0}{0}$" is undefined

7. If a limit does not exist, there are 3 possible reasons why. List them all three.

→ different limits on the left and right sides
→ the function goes to infinity (increasing or decreasing without bounds)
→ oscillates between 2 fixed values.

8. Find each limit or explain why the limit does not exist.

a) $\lim\limits_{x\to2} f(x)$, if $f(x)=\begin{cases} \ln x \text{ for } 0<x\le 2 \\ x^2\ln x \text{ for } 2<x\le 4\end{cases}$

$\lim\limits_{x\to2^-} f(x) = \ln 2 \neq \lim\limits_{x\to2^+} f(x)\ 4\ln 2$

DNE

b) $\lim\limits_{x\to2^+} f(x)$, if $f(x)=\begin{cases} 3x+1, & x<2 \\ \frac{5}{x+1}, & x\ge 2\end{cases}$

$\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^+}\frac{5}{x+1} = \frac{5}{3}$

c) $\lim\limits_{x\to1}\frac{x^2-4}{x-1}$

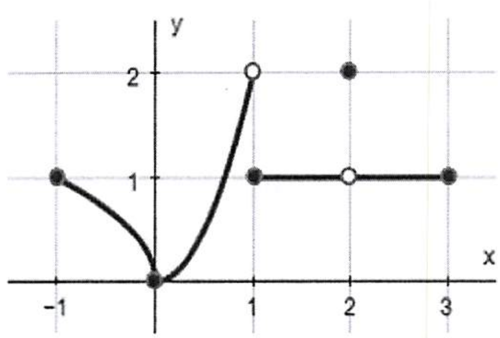
$\left.\begin{array}{l} \lim\limits_{x\to1^+}\frac{x^2-4 \,\to -3}{x-1\,\to 0^+} = -\infty \\ \lim\limits_{x\to1^-}\frac{x^2-4\,\to -3}{x-1\,\to 0^-} = +\infty \end{array}\right\}$ DNE

d) $\lim\limits_{x\to2}\frac{x+1}{(x-2)^2}$

$\lim\limits_{x\to2}\frac{x+1\,\to 3}{(x-2)^2\,\to 0^+} = +\infty$ (DNE but should be written)

no matter which side you're coming from.

9. Determine whether each statement about the graph below is True or False.



a) $\lim\limits_{x\to-1^+} f(x)=1$ ✓

b) $\lim\limits_{x\to2} f(x)$ does not exist X

c) $\lim\limits_{x\to2} f(x)=2$ X

d) $\lim\limits_{x\to1^-} f(x)=2$ ✓

e) $\lim\limits_{x\to1^+} f(x)=1$ ✓

f) $\lim\limits_{x\to1} f(x)$ does not exist ✓

g) $\lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^-} f(x)$ ✓

h) $\lim\limits_{x\to c} f(x)$ exists at every c in $(-1, 1)$ ✓

i) $\lim\limits_{x\to c} f(x)$ exists at every c in $(1, 3)$ ✓

10. Use the graph of $f(x)$ to estimate the limits and value of the function, or explain why the limit does not exist.

a) $\lim_{x\to 1^+} f(x) = 2$

b) $\lim_{x\to 1^-} f(x) = -1$

c) $\lim_{x\to 1} f(x)$ DNE
different left vs right

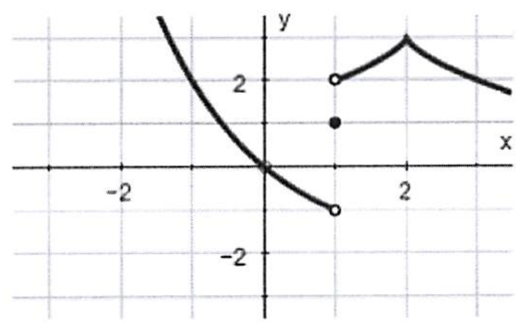
d) $f(1) = 1$

e) $\lim_{x\to 2^+} f(x) = 3$

f) $\lim_{x\to 2^-} f(x) = 3$

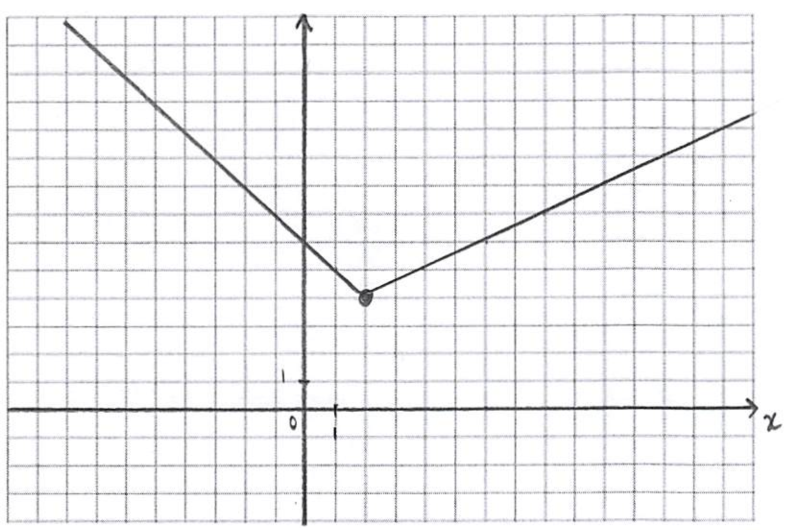
g) $\lim_{x\to 2} f(x) = 3$

h) $f(2) = 3$



11. For each of the following functions, (i) draw the graph, (ii) determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$, and (iii) explain what the value of $\lim_{x\to c} f(x)$ is or explain why it doesn't exist.

a) $c=2,\ f(x)=\begin{cases} 6-x, & \text{if } x<2 \\ 4, & \text{if } x=2 \\ \frac{x}{2}+3, & \text{if } x>2 \end{cases}$



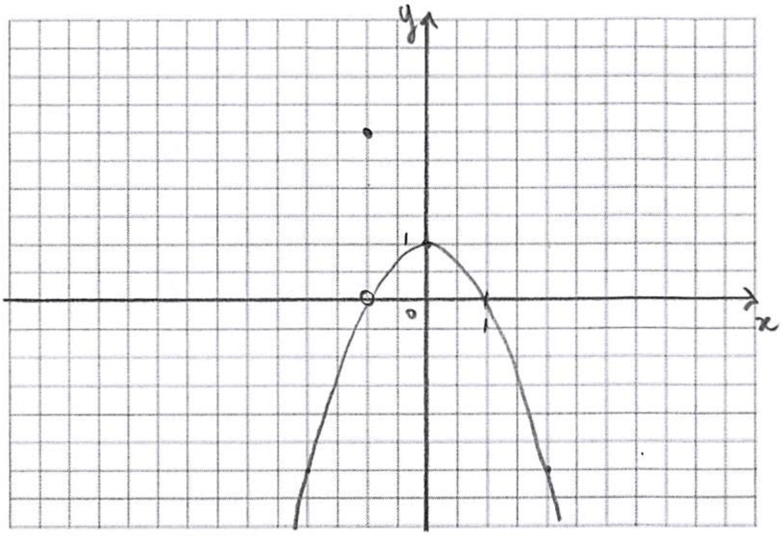
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (6-x) = 4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \left(\frac{x}{2}+3\right) = 4$

$\Rightarrow \lim_{x\to 2} f(x) = 4$

(same value on both sides)

b) $c=-1,\ f(x)=\begin{cases} 1-x^2, & \text{if } x\neq -1 \\ 3, & \text{if } x=-1 \end{cases}$



$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-}(1-x^2) = 0$

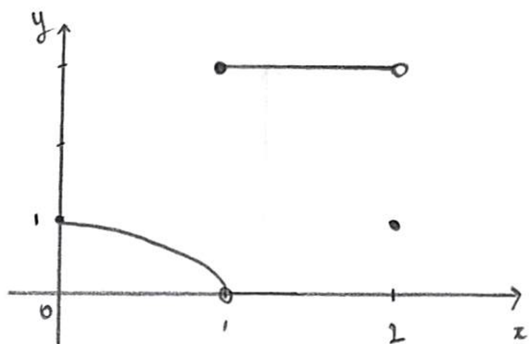
$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+}(1-x^2) = 0$

$\Rightarrow \lim\limits_{x\to -1} f(x) = 0$

(same value)

12.

Suppose $f(x)=\begin{cases} \sqrt{1-x^2}, & \text{if } 0\leq x<1 \\ 3, & \text{if } 1\leq x<2 \\ 1, & \text{if } x=2 \end{cases}$. Draw a graph of $f(x)$, then answer the following questions.



a) At what points c in the domain of f does $\lim\limits_{x\to c} f(x)$ exist?

$(0;1)\cup(1;2)$

b) At what point(s) c does only the left-hand limit exist?

$\{2\}$

c) At what point(s) c does only the right-hand limit exist?

$\{0\}$

and at $\{1\}$ both sided limits exist...

AP Calculus
2.1 Worksheet Day 2

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. When evaluating limits, what does it mean if direct substitution gives you $\frac{7}{0}$? That it could be + or − ∞ and could depend on the side... You need to determine the sign of the 0...
2. When evaluating limits, what does it mean if direct substitution gives you $\frac{0}{0}$? It could be anything... you need to work on the indeterminate form (often by factoring...)
3. What are the methods (options) for dealing with the result $\frac{0}{0}$?
   - factoring
   - conjugate
4. Evaluate the following limits algebraically.

a) $\lim_{x\to 3}\frac{x^2-x-6}{x-3} = \lim_{x\to 3}\frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}}$

$= 5$

b) $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0}\frac{\frac{2}{2(2+x)}-\frac{2+x}{2(2+x)}}{x}$

$= \lim_{x\to 0}\frac{\frac{-x}{2(2+x)}}{x}$

$= \lim_{x\to 0} -\frac{1}{2(2+x)} = -\frac{1}{4}$

c) $\lim_{x\to 0}\frac{\sqrt{2x+1}-1}{x}$

$= \lim_{x\to 0}\frac{(\sqrt{2x+1}-1)(\sqrt{2x+1}+1)}{x(\sqrt{2x+1}+1)}$

$= \lim_{x\to 0}\frac{2x+1-1}{x(\sqrt{2x+1}+1)} = \lim_{x\to 0}\frac{2}{\sqrt{2x+1}+1} = 1$

d) $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4}\frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(x-4)(\sqrt{x+5}+3)}$

$= \lim_{x\to 4}\frac{x+5-9}{(x-4)(\sqrt{x+5}+3)}$

$= \lim_{x\to 4}\frac{1}{\sqrt{x+5}+3} = \frac{1}{6}$

e) $\lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{x-1}{(x+1)(x-1)}$

$= \lim_{x\to 1}\frac{1}{x+1}$

$= \frac{1}{2}$

f) $\lim_{x\to 0}\frac{(4+x)^2-16}{x} = \lim_{x\to 0}\frac{x^2+8x+16-16}{x}$

$= \lim_{x\to 0}\frac{x^2+8x}{x}$

$= \lim_{x\to 0}(x+8) = 8$

g) $\lim_{t\to 2}\frac{t^2-3t+2}{t^2-4}$

$= \lim_{t\to 2}\frac{(t-2)(t-1)}{(t-2)(t+2)}$

$= \lim_{t\to 2}\frac{t-1}{t+2}$

$= \frac{1}{4}$

h) $\lim_{x\to 0}\frac{(2+x)^3-8}{x} = \lim_{x\to 0}\frac{x^3+6x^2+12x+8-8}{x}$

$= \lim_{x\to 0}(x^2+6x+12)$

$= 12$

One of the limits you should know is $\lim_{x \to 0} \frac{\sin x}{x} = $ __1__. This limit ONLY works when the denominator matches the inside of the sine function. If they do not match, you cannot change the inside of a sine function without a trig identity. Your goal will be to correctly show the algebra in order to use this limit.

5. Evaluate each of the following limits analytically. *Be sure to show ALL steps in your evaluation.*

a) $\lim_{x \to 0} \frac{\sin x}{5x} = \lim_{x \to 0} \frac{1}{5} \cdot \cancel{\frac{\sin x}{x}}^{\,1}$

$= \frac{1}{5}$

b) $\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{x \to 0} \cancel{\frac{\sin(5x)}{5x}}^{\,1} \times 5$

$= 5$

c) $\lim_{x \to \pi/4} \frac{\sin(x - \pi/4)}{x - \pi/4} = 1$

because

$\lim_{x \to \pi/4} (x - \frac{\pi}{4}) = 0$

d) $\lim_{x \to 0} \frac{3 \sin 4x}{\sin 3x} = \lim_{x \to 0} \frac{3x}{\sin 3x} \cdot \frac{\sin 4x}{4x} \times 4$

$= 4$

6. Evaluate each of the following by combining properties of limits and your algebra skills.

a) $\lim_{x \to 0} \frac{x + \sin x}{x} = \lim_{x \to 0} \left(1 + \frac{\sin x}{x}\right)$

$= 2$

b) $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{\cos x} \times \frac{1}{x}$

$= \lim_{x \to 0} \cancel{\frac{\sin x}{x}}^{\,1} \times \cancel{\frac{1}{\cos x}}^{\,1}$

$= 1$

c) $\lim_{x \to 0} \frac{\sin x}{2x^2 - x} = \lim_{x \to 0} \frac{\sin x}{x(2x-1)}$

$= \lim_{x \to 0} \cancel{\frac{\sin x}{x}}^{\,1} \cdot \cancel{\frac{1}{2x-1}}^{\,-1}$

$= -1$

d) $\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \cancel{\frac{\sin x}{x}}^{\,1} \cdot \cancel{\sin x}^{\,0}$

$= 0$

7. Consider $\lim_{x \to 0} \frac{f(x) - f(0)}{x^2} =$

a) If you use direct substitution, what result do you get? "$\frac{0}{0}$" indeterminate form.

b) Evaluate the limit if $f(x) = 2x^2 + 1$.

$\lim_{x \to 0} \frac{2x^2 + 1 - 1}{x^2} = \lim_{x \to 0} \frac{2x^2}{x^2} = 2.$

8. If $a \neq 0$, then $\lim_{x \to a} \frac{x^2 - a^2}{x^4 - a^4} = \lim_{x \to a} \frac{x^2 - a^2}{(x^2 + a^2)(x^2 - a^2)}$

$= \lim_{x \to a} \frac{1}{x^2 + a^2}$

$= \frac{1}{2a^2}$

9. Evaluate the following limits analytically (all mixed up):

a) $\lim_{x\to 0}\frac{\frac{3}{4+x}-\frac{3}{4}}{x} = \lim_{x\to 0}\frac{\frac{12}{4(4+x)}-\frac{3(4+x)}{4(4+x)}}{x}$

$= \lim_{x\to 0}\left(-\frac{3}{4(4+x)}\right) = -\frac{3}{16}$

b) $\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2} = \lim_{x\to 0}\frac{x^2(5x+8)}{x^2(3x^2-16)}$

$= \lim_{x\to 0}\frac{5x+8}{3x^2-16} = -\frac{1}{2}$

c) $\lim_{x\to 3}\frac{\sqrt{x+1}}{x-4} = \frac{\sqrt{4}}{-1} = -2$

(direct substitution ⇒ no indeterminate form)

d) $\lim_{x\to 0}\frac{x^2-3x}{x} = \lim_{x\to 0}\frac{x(x-3)}{x}$

$= \lim_{x\to 0}(x-3)$

$= -3$

e) $\lim_{x\to 1}\frac{x}{x^2-x}$ DNE → $+\infty$ on the right side, → $-\infty$ on the left side

"$\frac{1}{0}$" sign of the 0...

$\lim_{x\to 1^-} x^2-x = \lim_{x\to 1^-} x(x-1) = 0^-$

$\lim_{x\to 1^+} x(x-1) = 0^+$

f) $\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\frac{\sin 2x}{2x}\times 2$

$= 2$

g) $\lim_{x\to 0}\frac{\sin 7x}{3x} = \lim_{x\to 0}\cancel{\frac{\sin 7x}{7x}}^{1}\times\frac{7\cancel{x}}{3\cancel{x}}$

$= \frac{7}{3}$

h) $\lim_{x\to 4}\frac{x^2-5x+4}{x^2-2x-8} = \lim_{x\to 4}\frac{(x-4)(x-1)}{(x-4)(x+2)}$

$= \lim_{x\to 4}\frac{x-1}{x+2}$

$= \frac{1}{2}$

12. Evaluate $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$.

♫: h is going to 0 … not x … so treat this as if h is the variable … your final answer will have a x in it.

$\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$

13. Suppose $g(x)=\begin{cases}2-x, & \text{if } x\le 1\\ \frac{x}{2}+1, & \text{if } x>1\end{cases}$.

a) $\lim_{x\to 1^-} g(x) =$

$= \lim_{x\to 1^-}(2-x)$

$= 1$

b) $\lim_{x\to 1^+} g(x) =$

$= \lim_{x\to 1^+}\left(\frac{x}{2}+1\right)$

$= \frac{3}{2}$

c) $\lim_{x\to 1} g(x) =$ DNE

d) $g(1) =$ 1

AP Calculus
2.2 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Answer the following questions:

a) How do you find horizontal asymptotes? $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

b) Which of the parent functions have horizontal asymptotes? List the function(s) and asymptote(s)

exponential : $y = b^x$ asymptote : $y=0$
reciprocal : $\frac{1}{x}$ $y=0$
$1/x^2$ $y=0$

c) How do you find vertical asymptotes?

x values for which denom $= 0$ and num $\neq 0$ (which leads to $\lim_{x\to c} f(x) = \pm\infty$)

d) Which of the parent functions have vertical asymptotes? List the function(s) and asymptote(s)

logarithms : $y = \log_b x$ —— $x=0$
$1/x$ $x=0$
$1/x^2$ $x=0$

e) When must you look for oblique (slanted) asymptotes? How do you find them?

when $\lim_{x\to\pm\infty} f(x) = \pm\infty$ for ex : rational functions ...

2. For each of the following, find (i) $\lim_{x\to\infty} f(x)$ and (ii) $\lim_{x\to-\infty} f(x)$. Then (iii) identify all horizontal asymptotes, if any.

a) $f(x)=\dfrac{x-2}{2x^2+3x-5}$

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{2x^2} = 0$

horiz. asymptote: $y=0$

b) $f(x)=\dfrac{4x^3-2x+1}{x^2-2x+1}$

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{4x^3}{x^2} = +\infty$

$\lim_{x\to-\infty} f(x) = -\infty$

NO horiz. asymp.

c) $f(x)=\dfrac{3x^2-x+5}{x^2-4}$

$\lim_{x\to\pm\infty} f(x) = 3$

horiz. asympt : $y=3$

d) $f(x)=\dfrac{e^{-x}}{x} = \dfrac{1}{xe^x}$

$\lim_{x\to+\infty} f(x) = 0$

"$\frac{\infty}{-\infty}$" $\lim_{x\to-\infty} f(x) = -\infty$ (famous limit)

h. asymptote : $y=0$ towards $+\infty$

e) $f(x)=\dfrac{|x|}{x} = \begin{cases} 1 & \text{if } x\geq 0 \\ -1 & \text{if } x<0 \end{cases}$

$\lim_{x\to+\infty} f(x) = 1$

$\lim_{x\to-\infty} f(x) = -1$

h. asymptotes : $y=1$ towards $+\infty$
(weird situat°) $y=-1$ towards $-\infty$

f) $f(x)=\dfrac{\sin x}{2x^2+x}$

$-\frac{1}{2x^2+x} \leq f(x) \leq \frac{1}{2x^2+x}$

$\lim_{x\to\pm\infty} f(x) = 0$

H.A $y=0$

3. One of the functions in 2a – 2c has a slanted (oblique) asymptote. Explain why, and then find the asymptote.

2-b) $\lim_{x\to+\infty} f(x) = +\infty$ $\lim_{x\to-\infty} f(x) = -\infty$.

$f(x) = 4x+8 + \cancel{\dfrac{10x-7}{x^2-2x+1}} \to 0$ (result of the long division)

slanted asymptote : $y = 4x+8$

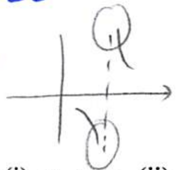
4. For each of the following, (i) find the vertical asymptotes of the graph of $f(x)$ and (ii) describe the behavior of the graph of $f(x)$ to the left and right of each asymptote.

a) $f(x)=\dfrac{1}{x-3}$

v. asympt : $x=3$

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to 3^+} f(x) = +\infty$



b) $f(x)=\dfrac{1}{x^2-4} = \dfrac{1}{(x+2)(x-2)}$

v. asympt: $x=-2$ & $x=2$

$\lim_{x\to -2^-} f(x) = +\infty$ $\quad \lim_{x\to -2^+} f(x) = -\infty$

$\lim_{x\to 2^-} f(x) = -\infty$ $\quad \lim_{x\to 2^+} f(x) = +\infty$

c) $f(x)=\dfrac{1-x}{2x^2-5x-3} = \dfrac{1-x}{(2x+1)(x-3)}$

v. asympt : $x=-\frac{1}{2}$ $\quad x=3$

$\lim_{x\to -\frac{1}{2}^-} f(x) = +\infty$ $\quad \lim_{x\to -\frac{1}{2}^+} f(x) = -\infty$

$\lim_{x\to 3^-} f(x) = +\infty$ $\quad \lim_{x\to 3^+} f(x) = -\infty$

5. Find the limit of $g(x)$ as (i) $x\to\infty$, (ii) $x\to-\infty$, (iii) $x\to 0^-$, and (iv) $x\to 0^+$

a) $g(x)=\begin{cases} \frac{1}{x} & \text{if } x<0 \\ \frac{2x-3}{x+1} & \text{if } x\geq 0 \end{cases}$

$\lim_{x\to +\infty} g(x) = \lim_{x\to+\infty} \dfrac{2x-3}{x+1} = 2$

$\lim_{x\to -\infty} g(x) = \lim_{x\to-\infty} \dfrac{1}{x} = 0$

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \dfrac{1}{x} = -\infty$ $\quad \lim_{x\to 0^+} g(x) = -3$

b) $g(x)=\begin{cases} \frac{3x}{x+1} & \text{if } x\leq 0 \\ \frac{1}{x^2} & \text{if } x>0 \end{cases}$

$\lim_{x\to -\infty} g(x) = \lim_{x\to-\infty} \dfrac{3x}{x+1} = 3$

$\lim_{x\to +\infty} g(x) = \lim_{x\to+\infty} \dfrac{1}{x^2} = 0$

$\lim_{x\to 0^-} g(x) = 0$ $\quad \lim_{x\to 0^+} g(x) = +\infty$

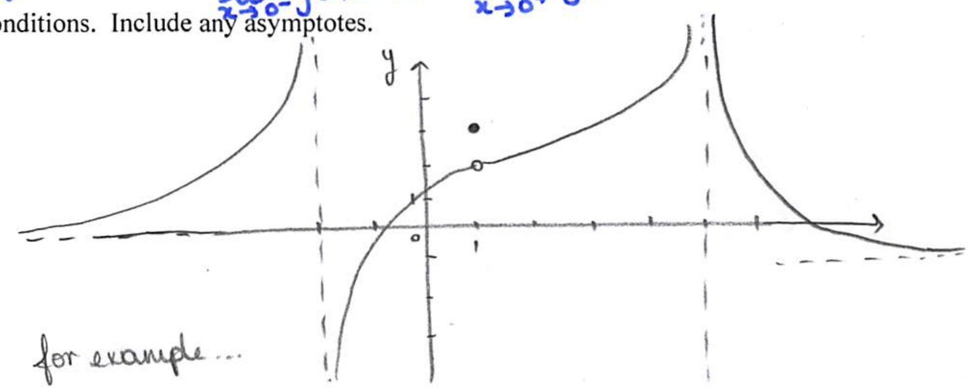
6. Sketch a **function** that satisfies the stated conditions. Include any asymptotes.

$\lim_{x\to 1} f(x)=2$ $\qquad \lim_{x\to 5^-} f(x)=\infty$

$\lim_{x\to 5^+} f(x)=\infty$ $\qquad \lim_{x\to \infty} f(x)=-1$

$\lim_{x\to -\infty} f(x)=0$ $\qquad \lim_{x\to -2^-} f(x)=\infty$

$\lim_{x\to -2^+} f(x)=-\infty$



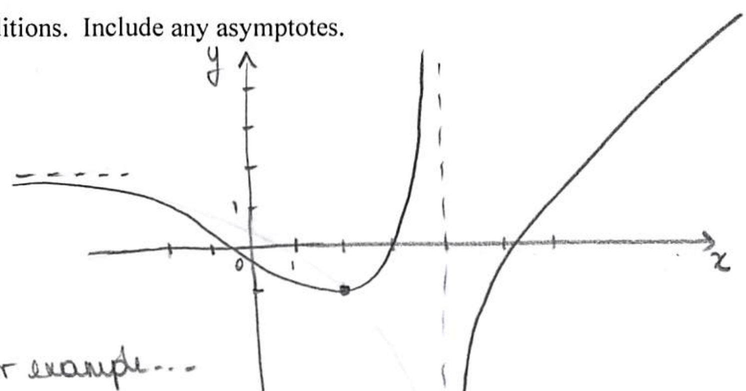
for example ...

7. Sketch a **function** that satisfies the stated conditions. Include any asymptotes.

$\lim_{x\to 2} f(x)=-1$ $\qquad \lim_{x\to 4^+} f(x)=-\infty$

$\lim_{x\to 4^-} f(x)=\infty$ $\qquad \lim_{x\to \infty} f(x)=\infty$

$\lim_{x\to -\infty} f(x)=2$

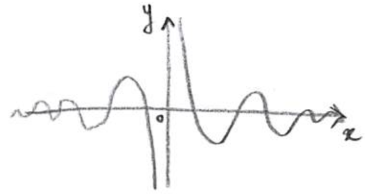


for example ...

8. Explain why there is no value L for which $\lim_{x\to\infty} \sin x = L$.

$y = \sin x$ keeps oscillating between -1 and 1 forever.

9. Let $f(x)=\dfrac{\cos x}{x}$.



a) Find the domain and range of f.

$D = \mathbb{R} \setminus \{0\}$ $\qquad$ $R = \mathbb{R}$

b) Is f even, odd, or neither? Justify your response.

$f(-x) = \dfrac{\cos(-x)}{-x} = -\dfrac{\cos x}{x} = -f(x)$

$\Rightarrow f$ is ODD.

c) Find $\lim\limits_{x\to\infty} f(x)$. Give a reason for your answer.

$-1 \leq \cos x \leq 1$

$-\dfrac{1}{x} \leq \dfrac{\cos x}{x} \leq \dfrac{1}{x}$ $\quad (x>0)$ ← which makes sense towards $+\infty$

$\therefore \lim\limits_{x\to+\infty} f(x) = 0$

10. If k is a positive integer, then $\lim\limits_{x\to\infty}\dfrac{x^k}{e^x} = \ ?$ Explain your answer.

[Try letting $k = 2$ … what about $k = 10$? … what about $k = 1000$?]

$\lim\limits_{x\to+\infty} \dfrac{x^k}{e^x} = 0$ (famous limit)

11. ***Investigate*** using tables and graphs to determine the value of each limit: $\lim\limits_{x\to\infty}\dfrac{3x-2}{\sqrt{2x^2+1}}$ and $\lim\limits_{x\to-\infty}\dfrac{3x-2}{\sqrt{2x^2+1}}$

end behaviour model →

$\lim\limits_{x\to+\infty} \dfrac{3x-2}{\sqrt{2x^2+1}} = \lim\limits_{x\to+\infty} \dfrac{3x}{\sqrt{2x^2}} = \lim\limits_{x\to+\infty} \dfrac{3x}{|x|\sqrt{2}} = \dfrac{3}{\sqrt{2}}$

$\lim\limits_{x\to-\infty} \dfrac{3x-2}{\sqrt{2x^2+1}} = \lim\limits_{x\to-\infty} \dfrac{3x}{\sqrt{2x^2}} = \lim\limits_{x\to-\infty} \dfrac{3x}{|x|\sqrt{2}} = -\dfrac{3}{\sqrt{2}}$

remember: $\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$ ⚠ $\sqrt{x^2} = |x|$

12. Evaluate each of the following limits using all methods learned from this chapter.

a) $\lim\limits_{x\to\infty}\left(\cancel{\dfrac{2}{x}}+1\right)\left(\cancel{\dfrac{5x^2-1}{x^2}}\right) = 5$ $\quad$ (1) (5)

b) $\lim\limits_{n\to\infty}\dfrac{3n^3-5n}{n^3-2n^2+1} = \lim\limits_{n\to+\infty}\dfrac{3n^3}{n^3} = 3$

c) $\lim\limits_{x\to\infty}\left(5-\cancel{\dfrac{2}{x^2}}\right) = 5$ $\quad$ (0)

d) $\lim\limits_{x\to\pi/2} \sec x$ $\quad$ "$\dfrac{1}{0}$"

$\lim\limits_{x\to\frac{\pi}{2}^-} \sec x = +\infty$

$\lim\limits_{x\to\frac{\pi}{2}^+} \sec x = -\infty$

$\Big\}$ $\lim\limits_{x\to\pi/2} \sec x$ DNE

e) $\lim\limits_{x\to\infty} \cancel{e^{-x}} \underbrace{\cos x}_{\text{no lim}}$ $\quad$ (0)

$-1 \leq \cos x \leq 1$

$(e^{-x} > 0)$ $\quad -e^{-x} \leq e^{-x}\cos x \leq e^{-x}$ $\quad$ ($-e^{-x} \to 0$, $e^{-x} \to 0$)

$\therefore$ squeezing theorem:

$\lim\limits_{x\to\infty} e^{-x}\cos x = 0$

f) $\lim\limits_{x\to 7/2^+} \operatorname{int}(2x-1) = 6$ $\quad$ ($2x-1 \to 6^+$)

(Note: $\lim\limits_{x\to\frac{7}{2}^-} \operatorname{int}(2x-1) = 5$) $\quad$ ($2x-1 \to 6^-$)

3/4

g) $\lim_{x\to\infty} \frac{\cos\left(\cancel{\frac{1}{x}}\right)^{\to 0}}{1+\cancel{\frac{1}{x}}^{\to 0}} = 1$

h) $\lim_{n\to\infty} \frac{4n^3}{n^2+10000n} = \lim_{n\to+\infty} \frac{4n^3}{n^2} = +\infty$ (DNE)

i) $\lim_{x\to 0} \frac{\sin 2x}{4x} = \lim_{x\to 0} \cancel{\frac{\sin 2x}{2x}}^{\to 1} \times \frac{1}{2}$

$= \frac{1}{2}$

j) $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0} \frac{\frac{2-(2+x)}{2(2+x)}}{x}$

$= \lim_{x\to 0} -\frac{1}{2(2+x)} = -\frac{1}{4}$

k) $\lim_{x\to\infty} \frac{x\sin x + 2\sin x}{2x^2} = 0$

$= \lim_{x\to+\infty} \frac{\sin x(x+2)}{2x^2}$

$-(x+2) \le \sin x(x+2) \le x+2$

$-\cancel{\frac{x+2}{2x^2}}^{\to 0} \le \frac{\sin x(x+2)}{2x^2} \le \cancel{\frac{x+2}{2x^2}}^{\to 0}$

l) $\lim_{x\to -2} \frac{x^2+1}{3x^2-2x+5} = \frac{(-2)^2+1}{3(-2)^2-2(-2)+5}$

$= \frac{5}{21}$

AP Calculus
2.3 Worksheet

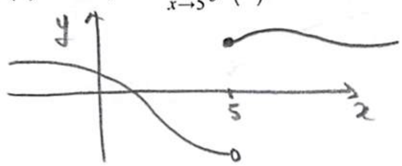
All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What is the definition of continuity? continuity at c if $\underbrace{\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)}_{\text{when c is not a limit of the domain}} = f(c)$

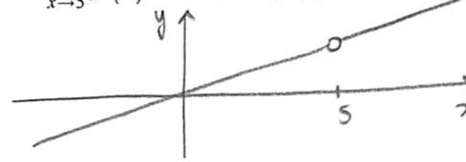
$c \in D$

2. Sketch a possible graph for each function described.

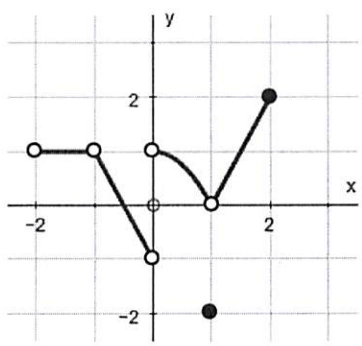
a) $f(5)$ exists, but $\lim_{x\to 5} f(x)$ does not exist.



b) The $\lim_{x\to 5} f(x)$ exists, but $f(5)$ does not exist.



3. Use the function $g(x)$ defined and graphed below to answer the following questions.



$$g(x)=\begin{cases} 1 & \text{if } -2<x<-1 \\ -2x-1 & \text{if } -1<x<0 \\ 1-x^2 & \text{if } 0<x<1 \\ -2 & \text{if } x=1 \\ 2x-2 & \text{if } 1<x\le 2 \end{cases}$$

a) Does $g(1)$ exist? yes

$g(1)=-2$

b) Does $\lim_{x\to 1} g(x)$ exist? yes

$\lim_{x\to 1} g(x)=0$

c) Does $\lim_{x\to 1} g(x)=g(1)$? NO

d) Is g continuous at $x=1$? NO

(from c))

e) Is g defined at $x=-1$? NO

f) Is g continuous at $x=-1$? NO

g) For what values of x is g continuous?

$(-2;-1)$; $(-1;0)$; $(0;1)$ & $(1;2]$

h) What value should be assigned to $g(-1)$ to make the extended function continuous at $x=-1$?

$g(-1) \stackrel{?}{=} 1$

i) What new value should be assigned to $g(1)$ to make the new function continuous at $x=1$?

$g(1) \stackrel{?}{=} 0$

j) Is it possible to extend g to be continuous at $x=0$? If so, what value should the extended function have there? If not, why not?

No, $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$

1/4

4. Let $f(x)=\begin{cases} x^2-1 & ;x<3 \\ 2ax & ;x\geq 3 \end{cases}$. Find a value of a so that the function f is continuous.

f is continuous over $(-\infty;3)$
f is continuous over $(3;+\infty)$

Using the definition of continuity, justify your response.

continuity at 3:

$$\lim_{x\to 3^-} f(x) = 8 \qquad \lim_{x\to 3^+} f(x) = f(3) = 6a$$

$6a = 8$

$\boxed{a = \frac{4}{3}}$

5. Let $f(x)=\begin{cases} 2x+3 & ;x\leq 2 \\ kx+1 & ;x>2 \end{cases}$. Find a value of k so that the function f is continuous.

Using the definition of continuity, justify your response.

continuity at 2:

$$\lim_{x\to 2^-} f(x) = f(2) = 7 \qquad \lim_{x\to 2^+} f(x) = 2k+1$$

$2k+1 = 7$

$\boxed{k=3}$

6. Let $f(x)=\begin{cases} x^2-a^2x & ;x<2 \\ 4-2x^2 & ;x\geq 2 \end{cases}$. Find all values of a that make f continuous at 2.

Using the definition of continuity, justify your response.

$$\lim_{x\to 2^-} f(x) = -2a^2+4 \qquad f(2) = \lim_{x\to 2^+} f(x) = -4$$

$-2a^2+4 = -4$

$2a^2 = 8$

$a^2 = 4$

$\boxed{a = \pm 2}$

7. If $f(x)=\begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & \text{if } x\neq 2 \\ k+3 & \text{if } x=2 \end{cases}$, and if f is continuous at $x = 2$, then k = ?

Using the definition of continuity, justify your response.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(\sqrt{2x+5}-\sqrt{x+7})(\sqrt{2x+5}+\sqrt{x+7})}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})} = \lim_{x\to 2} \frac{2x+5-x-7}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{2x+5}+\sqrt{x+7}} = \frac{1}{6}$$

$f(2) = k+3 = \frac{1}{6}$ $\qquad \boxed{k = -\frac{17}{6}}$

8. If the function f is continuous for all real numbers and if $f(x)=\dfrac{x^2-4}{x+2}$, when $x\neq -2$, then $f(-2)$ =

Use the definition of continuity to justify your response.

$$f(x) = \frac{(x+2)(x-2)}{x+2} = x-2 \quad , x \neq 2$$

(removable discontinuity)

$$\lim_{x\to -2} f(x) = -4 \qquad \therefore \boxed{f(-2) = -4}$$

2/4

9. Let f be the function defined by the following:

$$f(x)=\begin{cases}\sin x, & x<0\\ x^2, & 0\le x<1\\ 2-x, & 1\le x<2\\ x-3, & x\ge 2\end{cases}$$

For what values of x is f NOT continuous? Use the definition of continuity to explain why.

- $\lim\limits_{x\to 0^-} f(x)=0 \qquad \lim\limits_{x\to 0^+} f(x)=f(0)=0$
- $\lim\limits_{x\to 1^-} f(x)=1 \qquad \lim\limits_{x\to 1^+} f(x)=f(1)=1$
- $\lim\limits_{x\to 2^-} f(x)=0 \qquad \lim\limits_{x\to 2^+} f(x)=f(2)=-1 \implies$ $\boxed{f \text{ is not continuous at } 2}$

10. Determine the points of discontinuity and identify their type for each of the following functions:

a) $y=\dfrac{1}{(x+2)^2}$

$\lim\limits_{x\to -2} y=+\infty$

vertical asymptote : $x=-2$

(not removable)

b) $y=\dfrac{x-1}{x^2-4x+3} = \dfrac{x-1}{(x-1)(x-3)}$

- $\lim\limits_{x\to 1} y=-\frac{1}{2} \implies$ hole $(1;-\frac{1}{2})$ (removable)
- $\left.\begin{array}{l}\lim\limits_{x\to 3^+} y=+\infty\\ \lim\limits_{x\to 3^-} y=-\infty\end{array}\right\}$ v. asymptote $x=3$ (not removable)

c) $y=\dfrac{|x|}{x} = \begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$

$\implies$ jump at 0.

11. Write an extended function so that the given function is continuous at the indicated point.

a) $h(x)=\dfrac{\sin(5x)}{x}$ at $x=0$

$\lim\limits_{x\to 0} h(x) = \lim\limits_{x\to 0} \dfrac{\sin 5x}{5x}\times 5 = 5$

$h_E(x)=\begin{cases}\dfrac{\sin 5x}{x} & \text{if } x\neq 0\\ 5 & \text{if } x=0\end{cases}$

b) $k(x)=\dfrac{x-4}{\sqrt{x}-2}$ at $x=4$

$\lim\limits_{x\to 4} k(x) = \lim\limits_{x\to 4} \dfrac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}} = 4$

$k_E(x)=\begin{cases}\dfrac{x-4}{\sqrt{x}-2} & \text{if } x\neq 4\\ 4 & \text{if } x=4\end{cases}$

12. Let f be the function given by $f(x)=\dfrac{(x-1)(x^2-4)}{x^2-a}$. For what positive values of a is f continuous for all real numbers?

A) None B) 1 only C) 2 only D) 4 only E) 1 and 4

let $a>0$

$f(x)=\dfrac{(x-1)(x+2)(x-2)}{(x+a)(x-a)}$

if $a\neq 1$ and $a\neq 2$: 2 v. asympt.

if $a=2$: 2 holes

if $a=1$: 1 hole & 1 v. asymptote.

3/4

13. Let $g(x)=\frac{x^2+5x+6}{x^2+7x+10} = \frac{(x+2)(x+3)}{(x+2)(x+5)} = \frac{x+3}{x+5}$

a) Find the domain of $g(x)$.

$D = \mathbb{R} \setminus \{-5; -2\}$

b) Find the $\lim_{x \to c} g(x)$ for all values of c where $g(x)$ is not defined.

$\lim_{x \to -2} g(x) = \frac{1}{3}$

$\left.\begin{array}{l}\lim_{x \to -5^-} g(x) = +\infty \\ \lim_{x \to -5^+} g(x) = -\infty\end{array}\right\}$ $\lim_{x \to -5} g(x)$ DNE

c) Find any horizontal asymptotes and justify your response.

$\lim_{x \to \pm\infty} g(x) = 1 \Rightarrow y = 1$ is the only H.A

d) Find any vertical asymptotes and justify your response.

$x = -5$ is the only V.A $\quad \lim_{x \to -5^-} g(x) = +\infty$ and $\lim_{x \to -5^+} g(x) = -\infty$

e) Write an extension to the function so that $g(x)$ is continuous at $x = -2$.
Use the definition of continuity to justify your response.

$$g_E(x) = \begin{cases} \frac{x^2+5x+6}{x^2+7x+10} & \text{if } x \neq -2 \text{ and } x \neq -5 \\ \frac{1}{3} & \text{if } x = -2 \end{cases}$$

14. Without using a picture, give a written explanation of why the function $f(x)=x^2-4x+3$ has a zero in the interval [2, 4].

$f(2) = -1$
$f(4) = 3$

and f is continuous over $[2;4]$

since $0 \in [-1;3]$ then IVT guarantees the existence of at least one zero in $[2;4]$

15. Without using a picture, give a written explanation of why the function $f(x)=x^2+2x-3$ must equal 3 at least once in the interval [0, 2].

$\left.\begin{array}{l}\left.\begin{array}{l} f(0) = -3 \\ f(2) = 5 \end{array}\right\} 3 \in [-3;5] \\ f \text{ is continuous over } [0;2] \end{array}\right\}$ IVT guarantees the existence of such a point over $[0;2]$

16. Let $h(x)=\begin{cases} 3x^2-4, & \text{if } x \le 2 \\ 5+4x, & \text{if } x > 2 \end{cases}$.

a) What is $h(0)$?

$h(0) = -4$

b) What is $h(4)$?

$h(4) = 21$

c) On the interval [0, 4], there is no value of x such that $h(x) = 10$ even though $h(0) < 10$ and $h(4) > 10$. Explain why this result does not contradict the IVT.

$\left.\begin{array}{l}\lim_{x \to 2^-} h(x) = h(2) = 8 \\ \lim_{x \to 2^+} h(x) = 13\end{array}\right\}$ f is not continuous at 2.

$\therefore$ f is not continuous over $[0;4]$

$\therefore$ IVT cannot be applied.

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AP Calculus
2.4 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What is a difference quotient? $\frac{f(a+h)-f(a)}{h}$

2. How do you find the slope of a curve (aka slope of the tangent line to a curve) when $x = a$?

$$\text{slope} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

3. What is a *normal line*?

a line that intersects the graph at the point of tangency, and is perpendicular to the tangent line.

4. What is the difference between the AVERAGE RATE OF CHANGE and INSTANTANEOUS RATE OF CHANGE?

AVERAGE: $\frac{f(a)-f(b)}{a-b}$ (slope of the secant line)

INSTANTANEOUS: $\lim_{a\to b} \frac{f(a)-f(b)}{a-b}$ or $\lim_{h\to 0} \frac{f(b+h)-f(b)}{h}$ (slope of the tangent line)

5. Find the average rate of change of each function over the indicated interval.

a) $h(x) = e^x$ on $[-2, 0]$

$$\frac{h(-2)-h(0)}{-2-0} = \frac{e^{-2}-1}{-2} = \boxed{\frac{1-e^{-2}}{2}}$$

b) $k(x) = 2 + \sin x$ on $[-\pi/2, \pi/2]$

$$\frac{k(\frac{\pi}{2}) - k(-\frac{\pi}{2})}{\frac{\pi}{2} - (-\frac{\pi}{2})} = \frac{3-1}{\pi} = \boxed{\frac{2}{\pi}}$$

c) $f(x) = x^2 - x$ on $[1, 3]$

$$\frac{f(3)-f(1)}{3-1} = \frac{6-0}{2} = \boxed{3}$$

6. Let $f(x) = x^3$.

a) Write and simplify an expression for $f(a+h)$.

$$f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$$

b) Find the slope of the curve at $x = a$.

$$\frac{f(a+h)-f(a)}{h} = \frac{a^3+3a^2h+3ah^2+h^3-a^3}{h} = 3a^2+3ah+h^2$$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \boxed{3a^2}$$

c) When does the slope equal 12?

$3a^2 = 12 \Leftrightarrow a^2 = 4 \Leftrightarrow a = \pm 2$ (2 different points)

d) Write the equation of the tangent line to the curve at $x = 4$ $m = 48$ point $(4; 64)$

$\Rightarrow y - 64 = 48(x-4)$ or $\boxed{y = 48x - 128}$

e) Write the equation of the normal line to the curve at $x = 4$

$m = -\frac{1}{48}$ point $(4; 64)$

$\Rightarrow \boxed{y - 64 = -\frac{1}{48}(x-4)}$

1/3

7. Let $g(x)=\sqrt{x}$

a) Find the average rate of change from $x = 4$ to $x = 9$.

$$\frac{g(9)-g(4)}{9-4} = \frac{3-2}{5} = \boxed{\frac{1}{5}}$$

b) Find the instantaneous rate of change at $x = 9$.

$$\frac{g(9+h)-g(9)}{h} = \frac{\sqrt{9+h}-3}{h} = \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} = \frac{9+h-9}{h(\sqrt{9+h}+3)}$$

$$= \frac{1}{\sqrt{9+h}+3}$$

$$\lim_{h\to 0}\frac{g(9+h)-g(9)}{h} = \boxed{\frac{1}{6}}$$

c) Write the equation of the tangent line when $x = 9$

$m=\frac{1}{6}$ point $(9;3)$

$$\boxed{y-3=\frac{1}{6}(x-9)}$$

d) Write the equation of the normal line when $x = 9$.

$m=-6$ point $(9;3)$

$$\boxed{y-3=-6(x-9)}$$

8. Let $y=\dfrac{1}{x-1}$. Find the slope of the curve at $x = 2$. Using this slope, write the equation of the tangent line and the equation of the normal line at that point.

$$\frac{f(2+h)-f(2)}{h} = \frac{\frac{1}{2+h-1}-\frac{1}{2-1}}{h} = \frac{\frac{1}{h+1}-1}{h} = \frac{\frac{1-(h+1)}{h+1}}{h} = -\frac{1}{h+1}$$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = -1$$

slope of tangent: -1
slope of normal: 1
$\}$ point $(2;1)$

tangent: $\boxed{y-1=-(x-2)}$ normal: $\boxed{y-1=x-2}$

9. Let $y=x^2-3x-2$. Find the slope of the curve at $x = 0$. Using this slope, write the equation of the tangent line and the equation of the normal line at that point.

$$\frac{f(0+h)-f(0)}{h} = \frac{h^2-3h-2-(-2)}{h} = h-3 \xrightarrow[h\to 0]{} -3$$ point $(0;-2)$

tangent: $\boxed{y+2=-3x}$

normal: $\boxed{y+2=\frac{1}{3}x}$

2/3

10. Find an equation of the tangent line to the graph of $f(x)=\frac{3}{x}$ at $x=1$.

point : $(1;3)$

$$\frac{f(1+h)-f(1)}{h} = \frac{\frac{3}{1+h}-3}{h} = \frac{\frac{3-3(1+h)}{1+h}}{h} = -\frac{3}{1+h} \xrightarrow[h\to 0]{} -3$$

slope of the tangent line

$$\boxed{y-3=-3(x-1)}$$

11. An object is dropped from the top of a 150-m tower. It's height above the ground after t seconds is $150-4.9t^2$ m. How fast is the object falling 2 seconds after it is dropped?

$$\frac{f(2+h)-f(2)}{h} = \frac{150-4.9(2+h)^2-(150-4.9\times 2^2)}{h} = \frac{-4.9(4+4h+h^2)+4.9\times 4}{h}$$

$$= \frac{-19.6h-4.9h^2}{h} = -19.6-4.9h \xrightarrow[h\to 0]{} -19.6$$

$\boxed{\text{It is falling } 19.6 \text{ m/s}}$

12. What is the rate of change of the area of a circle with respect to the radius when the radius is 4 in?

$$A=\pi r^2$$

$$\frac{A(4+h)-A(4)}{h} = \frac{\pi(4+h)^2-16\pi}{h} = \frac{\pi(16+8h+h^2)-16\pi}{h}$$

$$= 8\pi+\pi h \xrightarrow[h\to 0]{} \boxed{8\pi \text{ in}^2/\text{in}}$$

or in

13. At what point is the tangent line to $k(x)=x^2-6x+1$ horizontal?

horizontal line $\Rightarrow$ slope $=0$

$$\frac{k(a+h)-k(a)}{h} = \frac{(a+h)^2-6(a+h)+1-(a^2-6a+1)}{h}$$

$$= \frac{\cancel{a^2}+2ah+h^2\cancel{-6a}-6h\cancel{+1}\cancel{-a^2}\cancel{+6a}\cancel{-1}}{h}$$

$$= 2a+h-6 \xrightarrow[h\to 0]{} 2a-6$$

slope at $x=a$

$2a-6=0$

$a=3 \Rightarrow \boxed{\text{point } (3;-8)}$

3/3

2.4 – Worksheet – add on

1. How do you find the average speed of an object?

$$\frac{\text{position 2} - \text{position 1}}{\text{time 2} - \text{time 1}}$$

2. Suppose an object moves along the x-axis with its position function given by $x(t) = 5t^2 + 7t$, where t is measured in seconds.
   a) What is the average speed from $t = 2$ to $t = 4$ seconds?

$$\frac{x(4) - x(2)}{4 - 2} = \frac{108 - 34}{2} = 37 \text{ unit/s}$$

   b) How fast is the object moving at exactly $t = 4$ seconds?

$$\frac{x(4+h) - x(4)}{h} = \frac{5(4+h)^2 + 7(4+h) - (5(4)^2 + 7(4))}{h}$$

$$= \frac{\cancel{80} + 40h + 5h^2 + \cancel{28} + 7h - \cancel{80} - \cancel{28}}{h}$$

$$= 47 + 5h \longrightarrow \boxed{47 \text{ units/s}}$$

3. An rover on another planet drops an object off a cliff. The object falls $y = gt^2$ m in t sec, where g is a constant. Five seconds after the object was dropped it lands 30m below.
   a) Find the value of g.

It fell 30m in 5s $\quad 30 = g(5)^2 \Rightarrow \boxed{g = 1.2}$

   b) Find the average speed for the fall.

$$\frac{30 - 0}{5 - 0} = \boxed{6 \text{ m/s}}$$

   c) With what speed did the rock hit the bottom?

$$\frac{f(5+h) - f(5)}{h} = \frac{1.2(5+h)^2 - 1.2 \times 25}{h}$$

$$= \frac{1.2(25 + 10h + h^2) - 1.2 \times 25}{h}$$

$$= 12 + 1.2h \xrightarrow[h \to 0]{} \boxed{12 \text{ m/s}}$$