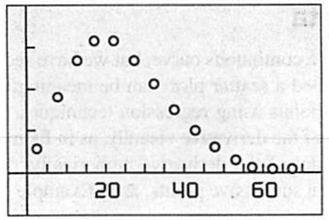


[–5, 75] by [–0.01, 0.04]

Figure 3.7 A scatter plot of the derivative data in Table 3.2. (Example 5)

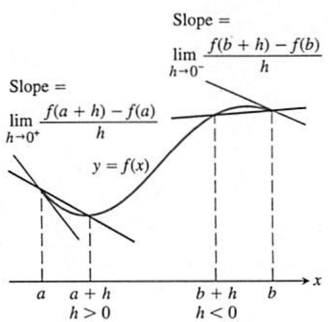


Figure 3.8 Derivatives at endpoints are one-sided limits.

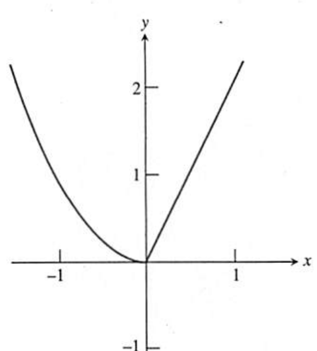


Figure 3.9 A function with different one-sided derivatives at $x = 0$. (Example 6)

A scatter plot of the derivative data in Table 3.2 is shown in Figure 3.7.

From the derivative plot, we can see that the rate of change peaks near $x = 20$. You can impress your friends with your "psychic powers" by predicting a shared birthday in a room of just 25 people (since you will be right about 57% of the time), but the derivative warns you to be cautious: a few less people can make quite a difference. On the other hand, going from 40 people to 100 people will not improve your chances much at all.

Now try Exercise 2

Generating shared birthday probabilities: If you know a little about probability, you might try generating the probabilities in Table 3.1. Extending the Idea Exercise 45 at the end of this section shows how to generate them on a calculator.

One-Sided Derivatives

A function $y = f(x)$ is **differentiable on a closed interval** $[a, b]$ if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \quad \text{[the \textbf{right-hand derivative at} } a\text{]}$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \quad \text{[the \textbf{left-hand derivative at} } b\text{]}$$

exist at the endpoints. In the right-hand derivative, h is positive and $a + h$ approaches a from the right. In the left-hand derivative, h is negative and $b + h$ approaches b from the left (Figure 3.8).

Right-hand and left-hand derivatives may be defined at any point of a function's domain.

The usual relationship between one-sided and two-sided limits holds for derivatives. Theorem 3, Section 2.1, allows us to conclude that a function has a (two-sided) derivative at a point if and only if the function's right-hand and left-hand derivatives are defined and equal at that point.

EXAMPLE 6 One-Sided Derivatives can Differ at a Point

Show that the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there (Figure 3.9).

$$y = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0 \end{cases}$$

SOLUTION

We verify the existence of the left-hand derivative:

$$\lim_{h\to 0^-} \frac{(0+h)^2 - 0^2}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = 0.$$

We verify the existence of the right-hand derivative:

$$\lim_{h\to 0^+} \frac{2(0+h) - 0^2}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Since the left-hand derivative equals zero and the right-hand derivative equals 2, the derivatives are not equal at $x = 0$. The function does not have a derivative at 0.

Now try Exercise 3

Quick Review 3.1 *(For help, go to Sections 2.1 and 2.4.)*

In Exercises 1–4, evaluate the indicated limit algebraically.

1. $\lim_{h\to 0} \frac{(2+h)^2 - 4}{h}$
2. $\lim_{x\to 2^+} \frac{x+3}{2}$
3. $\lim_{y\to 0^-} \frac{|y|}{y}$
4. $\lim_{x\to 4} \frac{2x-8}{\sqrt{x}-2}$
5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.
6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

In Exercises 7–10, let

$$f(x) = \begin{cases} x + 2, & x \le 1 \\ (x-1)^2, & x > 1. \end{cases}$$

7. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.
8. Find $\lim_{h\to 0^+} f(1 + h)$.
9. Does $\lim_{x\to 1} f(x)$ exist? Explain.
10. Is f continuous? Explain.

Section 3.1 Exercises

In Exercises 1–4, use the definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

to find the derivative of the given function at the indicated point.

1. $f(x) = 1/x$, $a = 2$
2. $f(x) = x^2 + 4$, $a = 1$
3. $f(x) = 3 - x^2$, $a = -1$
4. $f(x) = x^3 + x$, $a = 0$

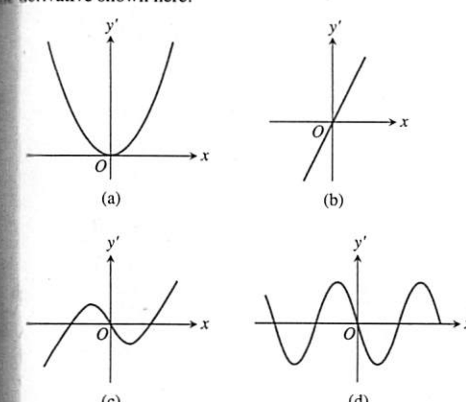
In Exercises 5–8, use the definition

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

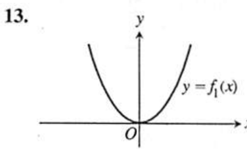
to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x$, $a = 2$
6. $f(x) = x^2 + 4$, $a = 1$
7. $f(x) = \sqrt{x+1}$, $a = 3$
8. $f(x) = 2x + 3$, $a = -1$
9. Find $f'(x)$ if $f(x) = 3x - 12$.
10. Find dy/dx if $y = 7x$.
11. Find $\frac{d}{dx}(x^2)$.
12. Find $\frac{d}{dx} f(x)$ if $f(x) = 3x^2$.

In Exercises 13–16, match the graph of the function with the graph of the derivative shown here:



13.



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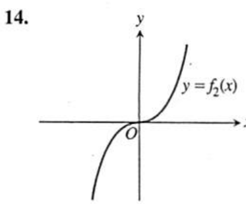

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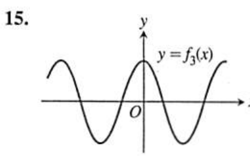
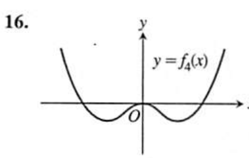


16.



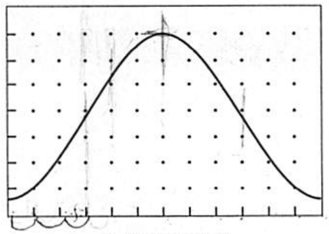
17. If $f(2) = 3$ and $f'(2) = 5$, find an equation of **(a)** the *tangent* line, and **(b)** the *normal* line to the graph of $y = f(x)$ at the point where $x = 2$.

18. Find the derivative of the function $y = 2x^2 - 13x + 5$ and use it to find an equation of the line tangent to the curve at $x = 3$.

19. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = x^3$ at the point (1, 1).

20. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = \sqrt{x}$ at $x = 4$.

21. ***Daylight in Fairbanks*** The viewing window below shows the number of hours of daylight in Fairbanks, Alaska, on each day for a typical 365-day period from January 1 to December 31. Answer the following questions by estimating slopes on the graph in hours per day. For the purposes of estimation, assume that each month has 30 days.



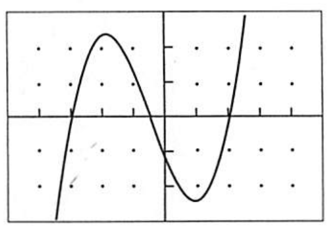
[0, 365] by [0, 24]

(a) On about what date is the amount of daylight increasing at the fastest rate? What is that rate?

(b) Do there appear to be days on which the rate of change in the amount of daylight is zero? If so, which ones?

(c) On what dates is the rate of change in the number of daylight hours positive? negative?

22. ***Graphing f′ from f*** Given the graph of the function f below, sketch a graph of the *derivative* of f.



[−5, 5] by [−3, 3]

23. The graphs in Figure 3.10a show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.10b shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 3.

(a) What is the value of the derivative of the rabbit population in Figure 3.10 when the number of rabbits is largest? smallest?

(b) What is the size of the rabbit population in Figure 3.10 when its derivative is largest? smallest?

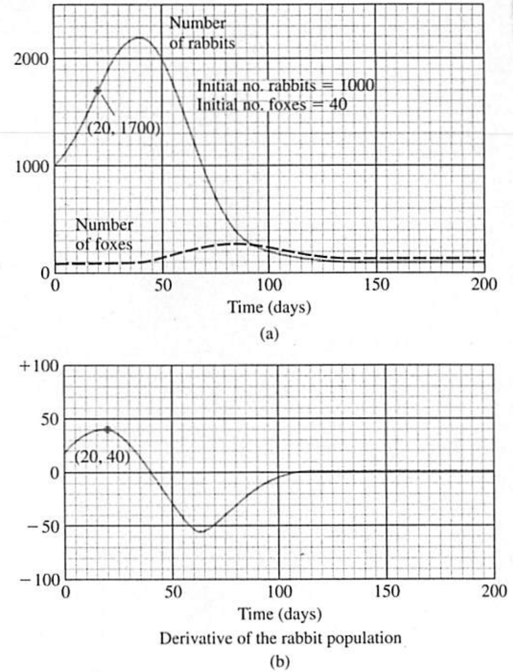
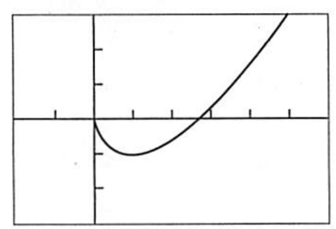


Figure 3.10 Rabbits and foxes in an arctic predator-prey food chain. *Source: Differentiation* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, MA, 1975, p. 86.

24. Shown below is the graph of $f(x) = x \ln x - x$. From what you know about the graphs of functions (i) through (v), pick out the one that is the *derivative* of f for $x > 0$.



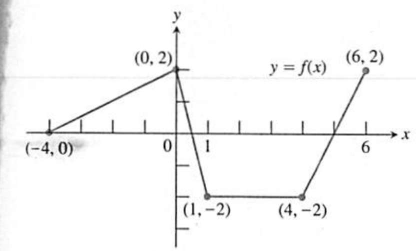
[−2, 6] by [−3, 3]

i. $y = \sin x$ **ii.** $y = \ln x$ **iii.** $y = \sqrt{x}$
iv. $y = x^2$ **v.** $y = 3x - 1$

25. From what you know about the graphs of functions (i) through (v), pick out the one that is *its own derivative*.

i. $y = \sin x$ **ii.** $y = x$ **iii.** $y = \sqrt{x}$
iv. $y = e^x$ **v.** $y = x^2$

26. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end.



(a) Graph the function's derivative.

(b) At what values of x between $x = -4$ and $x = 6$ is the function not differentiable?

27. ***Graphing f from f′*** Sketch the graph of a continuous function f with $f(0) = -1$ and

$$f'(x) = \begin{cases} 1, & x < -1 \\ -2, & x > -1. \end{cases}$$

28. ***Graphing f from f′*** Sketch the graph of a continuous function f with $f(0) = 1$ and

$$f'(x) = \begin{cases} 2, & x < 2 \\ -1, & x > 2. \end{cases}$$

In Exercises 29 and 30, use the data to answer the questions.

29. ***A Downhill Skier*** Table 3.3 gives the approximate distance traveled by a downhill skier after t seconds for $0 \le t \le 10$. Use the method of Example 5 to sketch a graph of the derivative; then answer the following questions:

(a) What does the derivative represent?

(b) In what units would the derivative be measured?

(c) Can you guess an equation of the derivative by considering its graph?

Table 3.3 Skiing Distances

Time t (seconds)	Distance Traveled (feet)
0	0
1	3.3
2	13.3
3	29.9
4	53.2
5	83.2
6	119.8
7	163.0
8	212.9
9	269.5
10	332.7

30. ***A Whitewater River*** Bear Creek, a Georgia river known to kayaking enthusiasts, drops more than 770 feet over one stretch of 3.24 miles. By reading a contour map, one can estimate the elevations (y) at various distances (x) downriver from the start of the kayaking route (Table 3.4).

Table 3.4 Elevations along Bear Creek

Distance Downriver (miles)	River Elevation (feet)
0.00	1577
0.56	1512
0.92	1448
1.19	1384
1.30	1319
1.39	1255
1.57	1191
1.74	1126
1.98	1062
2.18	998
2.41	933
2.64	869
3.24	805

(a) Sketch a graph of elevation (y) as a function of distance downriver (x).

(b) Use the technique of Example 5 to get an approximate graph of the derivative, dy/dx.

(c) The average change in elevation over a given distance is called a *gradient*. In this problem, what units of measure would be appropriate for a gradient?

(d) In this problem, what units of measure would be appropriate for the derivative?

(e) How would you identify the most dangerous section of the river (ignoring rocks) by analyzing the graph in (a)? Explain.

(f) How would you identify the most dangerous section of the river by analyzing the graph in (b)? Explain.

31. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^2 + x, & x \le 1 \\ 3x - 2, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

32. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

33. **Writing to Learn** Graph $y = \sin x$ and $y = \cos x$ in the same viewing window. Which function could be the derivative of the other? Defend your answer in terms of the behavior of the graphs.

34. In Example 2 of this section we showed that the derivative of $y = \sqrt{x}$ is a function with domain $(0, \infty)$. However, the function $y = \sqrt{x}$ itself has domain $[0, \infty)$, so it could have a right-hand derivative at $x = 0$. Prove that it does not.

35. **Writing to Learn** Use the concept of the derivative to define what it might mean for two parabolas to be parallel. Construct equations for two such parallel parabolas and graph them. Are the parabolas "everywhere equidistant," and if so, in what sense?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

36. True or False If $f(x) = x^2 + x$, then $f'(x)$ exists for every real number x. Justify your answer.

37. True or False If the left-hand derivative and the right-hand derivative of f exist at $x = a$, then $f'(a)$ exists. Justify your answer.

38. Multiple Choice Let $f(x) = 4 - 3x$. Which of the following is equal to $f'(-1)$?

(A) -7 (B) 7 (C) -3 (D) 3 (E) does not exist

39. Multiple Choice Let $f(x) = 1 - 3x^2$. Which of the following is equal to $f'(1)$?

(A) -6 (B) -5 (C) 5 (D) 6 (E) does not exist

In Exercises 40 and 41, let

$$f(x) = \begin{cases} x^2 - 1, & x < 0 \\ 2x - 1, & x \geq 0. \end{cases}$$

40. Multiple Choice Which of the following is equal to the left-hand derivative of f at $x = 0$?

(A) -2 (B) 0 (C) 2 (D) ∞ (E) $-\infty$

41. Multiple Choice Which of the following is equal to the right-hand derivative of f at $x = 0$?

(A) -2 (B) 0 (C) 2 (D) ∞ (E) $-\infty$

Explorations

42. Let $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x, & x > 1. \end{cases}$

(a) Find $f'(x)$ for $x < 1$.

(b) Find $f'(x)$ for $x > 1$.

(c) Find $\lim_{x\to 1^-} f'(x)$.

(d) Find $\lim_{x\to 1^+} f'(x)$.

(e) Does $\lim_{x\to 1} f'(x)$ exist? Explain.

(f) Use the definition to find the left-hand derivative of f at $x = 1$ if it exists.

(g) Use the definition to find the right-hand derivative of f at $x = 1$ if it exists.

(h) Does $f'(1)$ exist? Explain.

43. Group Activity Using graphing calculators, have each person in your group do the following:

(a) pick two numbers a and b between 1 and 10;

(b) graph the function $y = (x - a)(x + b)$;

(c) graph the *derivative* of your function (it will be a line with slope 2);

(d) find the y-intercept of your derivative graph.

(e) Compare your answers and determine a simple way to predict the y-intercept, given the values of a and b. Test your result.

Extending the Ideas

44. Find the unique value of k that makes the function

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ 3x + k, & x > 1 \end{cases}$$

differentiable at $x = 1$.

45. *Generating the Birthday Probabilities* Example 5 of this section concerns the probability that, in a group of n people, at least two people will share a common birthday. You can generate these probabilities on your calculator for values of n from 1 to 365.

Step 1: Set the values of N and P to zero:

```
0→N:0→P
                    0
```

Step 2: Type in this single, multi-step command:

```
N+1→N:1-(1-P)(366
-N)/365→P: {N,P}
```

Now each time you press the ENTER key, the command will print a new value of N (the number of people in the room) alongside P (the probability that at least two of them share a common birthday):

```
              {1 0}
     {2 .002739726}
     {3 .0082041659}
     {4 .0163559125}
     {5 .0271355737}
     {6 .0404624836}
     {7 .0562357031}
```

If you have some experience with probability, try to answer the following questions without looking at the table:

(a) If there are three people in the room, what is the probability that they all have *different* birthdays? (Assume that there are 365 possible birthdays, all of them equally likely.)

(b) If there are three people in the room, what is the probability that at least two of them share a common birthday?

(c) Explain how you can use the answer in part (b) to find the probability of a shared birthday when there are *four* people in the room. (This is how the calculator statement in Step 2 generates the probabilities.)

(d) Is it reasonable to assume that all calendar dates are equally likely birthdays? Explain your answer.

3.2 Differentiability

What you'll learn about

- How $f'(a)$ Might Fail to Exist
- Differentiability Implies Local Linearity
- Derivatives on a Calculator
- Differentiability Implies Continuity
- Intermediate Value Theorem for Derivatives

... and why

Graphs of differentiable functions can be approximated by their tangent lines at points where the derivative exists.

How rough can the graph of a continuous function be?

The graph of the absolute value function fails to be differentiable at a single point. If you graph $y = \sin^{-1}(\sin(x))$ on your calculator, you will see a continuous function with an *infinite* number of points of nondifferentiability. But can a continuous function fail to be differentiable at *every* point?

The answer, surprisingly enough, is yes, as Karl Weierstrass showed in 1872. One of his formulas (there are many like it) was

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite (but converging) sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, so to speak, the formula produces a function whose graph is too bumpy in the limit to have a tangent anywhere!

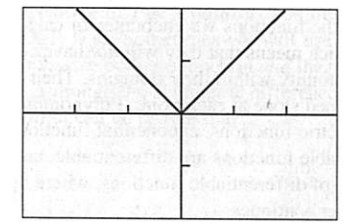
How $f'(a)$ Might Fail to Exist

A function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines,

$$\frac{f(x) - f(a)}{x - a},$$

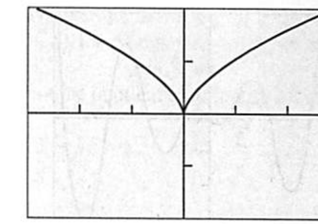
fail to approach a limit as x approaches a. Figures 3.11–3.14 illustrate four different instances where this occurs. For example, a function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

1. a *corner*, where the one-sided derivatives differ; Example: $f(x) = |x|$



[–3, 3] by [–2, 2]

Figure 3.11 There is a "corner" at $x = 0$.

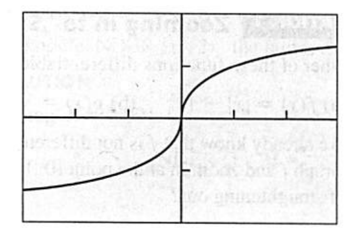


[–3, 3] by [–2, 2]

Figure 3.12 There is a "cusp" at $x = 0$.

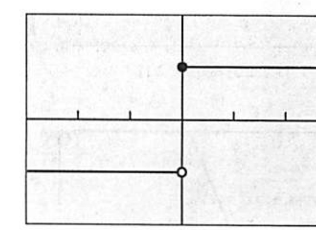
2. a *cusp*, where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other (an extreme case of a corner); Example: $f(x) = x^{2/3}$

3. a *vertical tangent*, where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides (in this example, ∞); Example: $f(x) = \sqrt[3]{x}$



[–3, 3] by [–2, 2]

Figure 3.13 There is a vertical tangent line at $x = 0$.



[–3, 3] by [–2, 2]

Figure 3.14 There is a discontinuity at $x = 0$.

4. a *discontinuity* (which will cause one or both of the one-sided derivatives to be nonexistent). Example: The *Unit Step Function*

$$U(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

In this example, the left-hand derivative fails to exist:

$$\lim_{h\to 0^-} \frac{(-1) - (1)}{h} = \lim_{h\to 0^-} \frac{-2}{h} = \infty.$$

Quick Review 3.2 *(For help, go to Sections 1.2 and 2.1.)*

In Exercises 1–5, tell whether the limit could be used to define $f'(a)$ (assuming that f is differentiable at a).

1. $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

2. $\lim_{h\to 0} \dfrac{f(a+h)-f(h)}{h}$

3. $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$

4. $\lim_{x\to a} \dfrac{f(a)-f(x)}{a-x}$

5. $\lim_{h\to 0} \dfrac{f(a+h)+f(a-h)}{h}$

6. Find the domain of the function $y = x^{4/3}$.

7. Find the domain of the function $y = x^{3/4}$.

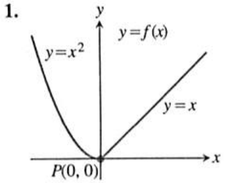
8. Find the range of the function $y = |x-2| + 3$.

9. Find the slope of the line $y - 5 = 3.2(x + \pi)$.

10. If $f(x) = 5x$, find
$$\frac{f(3+0.001) - f(3-0.001)}{0.002}.$$

Section 3.2 Exercises

In Exercises 1–4, compare the right-hand and left-hand derivatives to show that the function is not differentiable at the point P. Find all points where f is not differentiable.

1.


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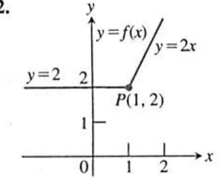

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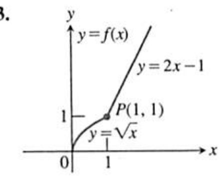

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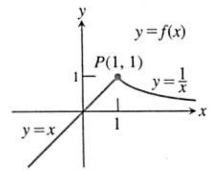
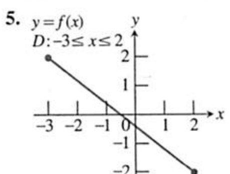


In Exercises 5–10, the graph of a function over a closed interval D is given. At what domain points does the function appear to be

(a) differentiable? (b) continuous but not differentiable?

(c) neither continuous nor differentiable?

5.


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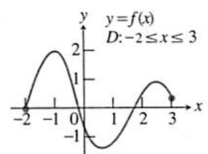

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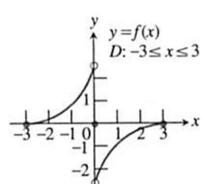

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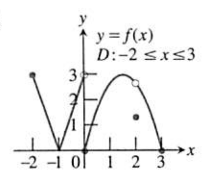

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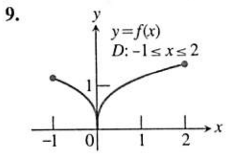

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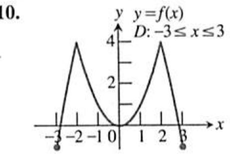


In Exercises 11–16, the function fails to be differentiable at $x = 0$. Tell whether the problem is a corner, a cusp, a vertical tangent, or a discontinuity.

11. $y = \begin{cases} \tan^{-1} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

12. $y = x^{4/5}$

13. $y = x + \sqrt{x^2} + 2$

14. $y = 3 - \sqrt[3]{x}$

15. $y = 3x - 2|x| - 1$

16. $y = \sqrt[3]{|x|}$

In Exercises 17–26, find the numerical derivative of the given function at the indicated point. Use $h = 0.001$. Is the function differentiable at the indicated point?

17. $f(x) = 4x - x^2,\ x = 0$

18. $f(x) = 4x - x^2,\ x = 3$

19. $f(x) = 4x - x^2,\ x = 1$

20. $f(x) = x^3 - 4x,\ x = 0$

21. $f(x) = x^3 - 4x,\ x = -2$

22. $f(x) = x^3 - 4x,\ x = 2$

23. $f(x) = x^{2/3},\ x = 0$

24. $f(x) = |x - 3|,\ x = 3$

25. $f(x) = x^{2/5},\ x = 0$

26. $f(x) = x^{4/5},\ x = 0$

Group Activity In Exercises 27–30, use NDER to graph the derivative of the function. If possible, identify the derivative function by looking at the graph.

27. $y = -\cos x$

28. $y = 0.25x^4$

29. $y = \dfrac{x|x|}{2}$

30. $y = -\ln|\cos x|$

In Exercises 31–36, find all values of x for which the function is differentiable.

31. $f(x) = \dfrac{x^3 - 8}{x^2 - 4x - 5}$

32. $h(x) = \sqrt[3]{3x - 6} + 5$

33. $P(x) = \sin(|x|) - 1$

34. $Q(x) = 3\cos(|x|)$

35. $g(x) = \begin{cases} (x+1)^2, & x \le 0 \\ 2x + 1, & 0 < x < 3 \\ (4 - x)^2, & x \ge 3 \end{cases}$

36. $C(x) = x|x|$

37. Show that the function
$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$
is not the derivative of any function on the interval $-1 \le x \le 1$.

38. **Writing to Learn** Recall that the numerical derivative (NDER) can give meaningless values at points where a function is not differentiable. In this exercise, we consider the numerical derivatives of the functions $1/x$ and $1/x^2$ at $x = 0$.

(a) Explain why neither function is differentiable at $x = 0$.

(b) Find NDER at $x = 0$ for each function.

(c) By analyzing the definition of the symmetric difference quotient, explain why NDER returns wrong responses that are so different from each other for these two functions.

39. Let f be the function defined as
$$f(x) = \begin{cases} 3 - x, & x < 1 \\ ax^2 + bx, & x \ge 1 \end{cases}$$
where a and b are constants.

(a) If the function is continuous for all x, what is the relationship between a and b?

(b) Find the unique values for a and b that will make f both continuous and differentiable.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

40. **True or False** If f has a derivative at $x = a$, then f is continuous at $x = a$. Justify your answer.

41. **True or False** If f is continuous at $x = a$, then f has a derivative at $x = a$. Justify your answer.

42. **Multiple Choice** Which of the following is true about the graph of $f(x) = x^{4/5}$ at $x = 0$?

(A) It has a corner.

(B) It has a cusp.

(C) It has a vertical tangent.

(D) It has a discontinuity.

(E) $f(0)$ does not exist.

43. **Multiple Choice** Let $f(x) = \sqrt[3]{x - 1}$. At which of the following points is $f'(a) \neq \text{NDER}(f, x, a)$?

(A) $a = 1$ (B) $a = -1$ (C) $a = 2$ (D) $a = -2$ (E) $a = 0$

Exercises 44 and 45, let
$$f(x) = \begin{cases} 2x + 1, & x \le 0 \\ x^2 + 1, & x > 0. \end{cases}$$

44. **Multiple Choice** Which of the following is equal to the left-hand derivative of f at $x = 0$?

(A) $2x$ (B) 2 (C) 0 (D) $-\infty$ (E) ∞

45. **Multiple Choice** Which of the following is equal to the right-hand derivative of f at $x = 0$?

(A) $2x$ (B) 2 (C) 0 (D) $-\infty$ (E) ∞

Explorations

46. (a) Enter the expression "$x < 0$" into Y1 of your calculator using "<" from the TEST menu. Graph Y1 in DOT MODE in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.

(b) Describe the graph in part (a).

(c) Enter the expression "$x \ge 0$" into Y1 of your calculator using "≥" from the TEST menu. Graph Y1 in DOT MODE in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.

(d) Describe the graph in part (c).

47. ***Graphing Piecewise Functions on a Calculator*** Let
$$f(x) = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0. \end{cases}$$

(a) Enter the expression "(X²)(X≤0)+(2X)(X>0)" into Y1 of your calculator and draw its graph in the window $[-4.7, 4.7]$ by $[-3, 5]$.

(b) Explain why the values of Y1 and $f(x)$ are the same.

(c) Enter the numerical derivative of Y1 into Y2 of your calculator and draw its graph in the same window. Turn off the graph of Y1.

(d) Use TRACE to calculate NDER(Y1, x, −0.1), NDER(Y1, x, 0), and NDER(Y1, x, 0.1). Compare with Section 3.1, Example 6.

Extending the Ideas

48. ***Oscillation*** There is another way that a function might fail to be differentiable, and that is by *oscillation*. Let
$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) Show that f is continuous at $x = 0$.

(b) Show that
$$\frac{f(0+h) - f(0)}{h} = \sin \frac{1}{h}.$$

(c) Explain why
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$
does not exist.

(d) Does f have either a left-hand or right-hand derivative at $x = 0$?

(e) Now consider the function
$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$
Use the definition of the derivative to show that g is differentiable at $x = 0$ and that $g'(0) = 0$.

EXAMPLE 7 Using the Power Rule

Find an equation for the line tangent to the curve

$$y = \frac{x^2 + 3}{2x}$$

at the point (1, 2). Support your answer graphically.

SOLUTION

We could find the derivative by the Quotient Rule, but it is easier to first simplify the function as a sum of two powers of x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2x} + \frac{3}{2x}\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}x + \frac{3}{2}x^{-1}\right)$$
$$= \frac{1}{2} - \frac{3}{2}x^{-2}$$

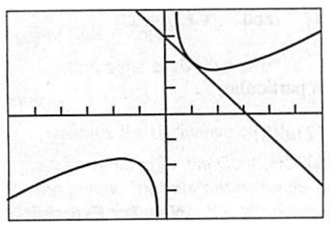
The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[\frac{1}{2} - \frac{3}{2}x^{-2}\right]_{x=1} = \frac{1}{2} - \frac{3}{2} = -1.$$

The line through (1, 2) with slope $m = -1$ is

$$y - 2 = (-1)(x - 1)$$
$$y = -x + 1 + 2$$
$$y = -x + 3.$$

We graph $y = (x^2 + 3)/2x$ and $y = -x + 3$ (Figure 3.20), observing that the line appears to be tangent to the curve at (1, 2). Thus, we have graphical support that our computations are correct. *Now try Exercise 2*



[–6, 6] by [–4, 4]

Figure 3.20 The line $y = -x + 3$ appears to be tangent to the graph of

$$y = \frac{x^2 + 3}{2x}$$

at the point (1, 2). (Example 7)

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is called the *first derivative* of y with respect to x. The first derivative may itself be a differentiable function of x. If so, its derivative,

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

is called the *second derivative* of y with respect to x. If y'' ("y double-prime") is differentiable, its derivative,

$$y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3},$$

is called the *third derivative* of y with respect to x. The names continue as you might expect they would, except that the multiple-prime notation begins to lose its usefulness after about three primes. We use

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} \quad \text{"y super n"}$$

to denote the **nth derivative** of y with respect to x. (We also use d^ny/dx^n.) Do not confuse $y^{(n)}$ with the nth power of y, which is y^n.

Technology Tip

HIGHER ORDER DERIVATIVES WITH NDER

Some graphers will allow the *nesting* of the NDER function,

NDER2 f = NDER(NDER f),

but such nesting, in general, is safe only to the second derivative. Beyond that, the error buildup in the algorithm makes the results unreliable.

EXAMPLE 8 Finding Higher Order Derivatives

Find the first four derivatives of $y = x^3 - 5x^2 + 2$.

SOLUTION

The first four derivatives are:

First derivative:	$y' = 3x^2 - 10x$;
Second derivative:	$y'' = 6x - 10$;
Third derivative:	$y''' = 6$;
Fourth derivative:	$y^{(4)} = 0$.

This function has derivatives of all orders, the fourth and higher order derivatives all being zero. *Now try Exercise 33.*

EXAMPLE 9 Finding Instantaneous Rate of Change

An orange farmer currently has 200 trees yielding an average of 15 bushels of oranges per tree. She is expanding her farm at the rate of 15 trees per year, while improved husbandry is improving her average annual yield by 1.2 bushels per tree. What is the current (instantaneous) rate of increase of her total annual production of oranges?

SOLUTION

Let the functions t and y be defined as follows.

$t(x)$ = the number of trees x years from now.

$y(x)$ = yield per tree x years from now.

Then $p(x) = t(x)y(x)$ is the total production of oranges in year x. We know the following values.

$$t(0) = 200, \quad y(0) = 15$$
$$t'(0) = 15, \quad y'(0) = 1.2$$

We need to find $p'(0)$, where $p = ty$.

$$p'(0) = t(0)y'(0) + y(0)t'(0)$$
$$= (200)(1.2) + (15)(15)$$
$$= 465$$

The rate we seek is 465 bushels per year. *Now try Exercise 51.*

Quick Review 3.3 *(For help, go to Sections 1.2 and 3.1.)*

In Exercises 1–6, write the expression as a sum of powers of x.

1. $(x^2 - 2)(x^{-1} + 1)$

2. $\left(\frac{x}{x^2 + 1}\right)^{-1}$

3. $3x^2 - \frac{2}{x} + \frac{5}{x^2}$

4. $\frac{3x^4 - 2x^3 + 4}{2x^2}$

5. $(x^{-1} + 2)(x^{-2} + 1)$

6. $\frac{x^{-1} + x^{-2}}{x^{-3}}$

7. Find the positive roots of the equation
$$2x^3 - 5x^2 - 2x + 6 = 0$$
and evaluate the function $y = 500x^6$ at each root. Round your answers to the nearest integer, but only in the final step.

8. If $f(x) = 7$ for all real numbers x, find
   (a) $f(10)$. (b) $f(0)$.
   (c) $f(x + h)$. (d) $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$.

9. Find the derivatives of these functions with respect to x.
   (a) $f(x) = \pi$ (b) $f(x) = \pi^2$ (c) $f(x) = \pi^{15}$

10. Find the derivatives of these functions with respect to x using the definition of the derivative.
   (a) $f(x) = \frac{x}{\pi}$ (b) $f(x) = \frac{\pi}{x}$

Section 3.3 Exercises

In Exercises 1–6, find dy/dx.

1. $y = -x^2 + 3$
2. $y = \frac{x^3}{3} - x$
3. $y = 2x + 1$
4. $y = x^2 + x + 1$
5. $y = \frac{x^3}{3} + \frac{x^2}{2} + x$
6. $y = 1 - x + x^2 - x^3$

In Exercises 7–12, find the horizontal tangents of the curve.

7. $y = x^3 - 2x^2 + x + 1$
8. $y = x^3 - 4x^2 + x + 2$
9. $y = x^4 - 4x^2 + 1$
10. $y = 4x^3 - 6x^2 - 1$
11. $y = 5x^3 - 3x^5$
12. $y = x^4 - 7x^3 + 2x^2 + 15$

13. Let $y = (x + 1)(x^2 + 1)$. Find dy/dx **(a)** by applying the Product Rule, and **(b)** by multiplying the factors first and then differentiating.

14. Let $y = (x^2 + 3)/x$. Find dy/dx **(a)** by using the Quotient Rule, and **(b)** by first dividing the terms in the numerator by the denominator and then differentiating.

In Exercises 15–22, find dy/dx. Support your answer graphically.

15. $(x^3 + x + 1)(x^4 + x^2 + 1)$
16. $(x^2 + 1)(x^3 + 1)$
17. $y = \frac{2x + 5}{3x - 2}$
18. $y = \frac{x^2 + 5x - 1}{x^2}$
19. $y = \frac{(x - 1)(x^2 + x + 1)}{x^3}$
20. $y = (1 - x)(1 + x^2)^{-1}$
21. $y = \frac{x^2}{1 - x^3}$
22. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

23. Suppose u and v are functions of x that are differentiable at $x = 0$, and that $u(0) = 5$, $u'(0) = -3$, $v(0) = -1$, $v'(0) = 2$. Find the values of the following derivatives at $x = 0$.
   (a) $\frac{d}{dx}(uv)$ (b) $\frac{d}{dx}\left(\frac{u}{v}\right)$
   (c) $\frac{d}{dx}\left(\frac{v}{u}\right)$ (d) $\frac{d}{dx}(7v - 2u)$

24. Suppose u and v are functions of x that are differentiable at $x = 2$ and that $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$. Find the values of the following derivatives at $x = 2$.
   (a) $\frac{d}{dx}(uv)$ (b) $\frac{d}{dx}\left(\frac{u}{v}\right)$
   (c) $\frac{d}{dx}\left(\frac{v}{u}\right)$ (d) $\frac{d}{dx}(3u - 2v + 2uv)$

25. Which of the following numbers is the slope of the line tangent to the curve $y = x^2 + 5x$ at $x = 3$?
   i. 24 ii. $-5/2$ iii. 11 iv. 8

26. Which of the following numbers is the slope of the line $3x - 2y + 12 = 0$?
   i. 6 ii. 3 iii. 3/2 iv. 2/3

In Exercises 27 and 28, find an equation for the line tangent to the curve at the given point.

27. $y = \frac{x^3 + 1}{2x}$, $x = 1$
28. $y = \frac{x^4 + 2}{x^2}$, $x = -1$

In Exercises 29–32, find dy/dx.

29. $y = 4x^{-2} - 8x + 1$
30. $y = \frac{x^{-4}}{4} - \frac{x^{-3}}{3} + \frac{x^{-2}}{2} - x^{-1} + 3$
31. $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$
32. $y = 2\sqrt{x} - \frac{1}{\sqrt{x}}$

In Exercises 33–36, find the first four derivatives of the function.

33. $y = x^4 + x^3 - 2x^2 + x - 5$
34. $y = x^2 + x + 3$
35. $y = x^{-1} + x^2$
36. $y = \frac{x + 1}{x}$

In Exercises 37–42, support your answer graphically.

37. Find an equation of the line perpendicular to the tangent to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$.

38. Find the tangents to the curve $y = x^3 + x$ at the points where the slope is 4. What is the smallest slope of the curve? At what value of x does the curve have this slope?

39. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

40. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

41. Find the tangents to *Newton's serpentine*,
$$y = \frac{4x}{x^2 + 1},$$
at the origin and the point $(1, 2)$.

42. Find the tangent to the *witch of Agnesi*,
$$y = \frac{8}{4 + x^2},$$
at the point $(2, 1)$.

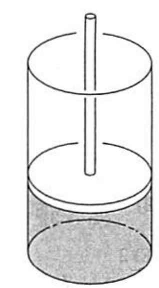
43. Use the definition of derivative (given in Section 3.1, Equation 1) to show that
   (a) $\frac{d}{dx}(x) = 1$.
   (b) $\frac{d}{dx}(-u) = -\frac{du}{dx}$.

44. Use the Product Rule to show that
$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$
for any constant c.

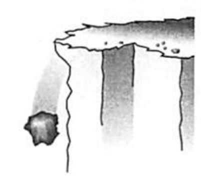
45. Devise a rule for $\frac{d}{dx}\left(\frac{1}{f(x)}\right)$.

When we work with functions of a single variable in mathematics, we often call the independent variable x and the dependent variable y. Applied fields use many different letters, however. Here are some examples.

46. ***Cylinder Pressure*** If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form
$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$
in which a, b, n, and R are constants. Find dP/dV.



47. ***Free Fall*** When a rock falls from rest near the surface of the earth, the distance it covers during the first few seconds is given by the equation
$$s = 4.9t^2.$$
In this equation, s is the distance in meters and t is the elapsed time in seconds. Find ds/dt and d^2s/dt^2.



Group Activity In Exercises 48–52, *work in groups of two or three* to solve the problems.

48. ***The Body's Reaction to Medicine*** The reaction of the body to a dose of medicine can often be represented by an equation of the form
$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$
where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

   Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to medicine. In Chapter 4, we shall see how to find the amount of medicine to which the body is most sensitive. *Source: Some Mathematical Models in Biology*, Revised Edition, December 1967, PB-202 364, p. 221; distributed by N.T.I.S., U.S. Department of Commerce.

49. **Writing to Learn** Recall that the area A of a circle with radius r is πr^2 and that the circumference C is $2\pi r$. Notice that $dA/dr = C$. Explain in terms of geometry why the instantaneous rate of change of the area with respect to the radius should equal the circumference.

50. **Writing to Learn** Recall that the volume V of a sphere of radius r is $(4/3)\pi r^3$ and that the surface area A is $4\pi r^2$. Notice that $dV/dr = A$. Explain in terms of geometry why the instantaneous rate of change of the volume with respect to the radius should equal the surface area.

51. ***Orchard Farming*** An apple farmer currently has 156 trees yielding an average of 12 bushels of apples per tree. He is expanding his farm at a rate of 13 trees per year, while improved husbandry is improving his average annual yield by 1.5 bushels per tree. What is the current (instantaneous) rate of increase of his total annual production of apples? Answer in appropriate units of measure.

52. ***Picnic Pavilion Rental*** The members of the Blue Boar society always divide the pavilion rental fee for their picnics equally among the members. Currently there are 65 members and the pavilion rents for \$250. The pavilion cost is increasing at a rate of \$10 per year, while the Blue Boar membership is increasing at a rate of 6 members per year. What is the current (instantaneous) rate of change in each member's share of the pavilion rental fee? Answer in appropriate units of measure.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

53. **True or False** $\frac{d}{dx}(\pi^3) = 3\pi^2$. Justify your answer.

54. **True or False** The graph of $f(x) = 1/x$ has no horizontal tangents. Justify your answer.

55. **Multiple Choice** Let $y = uv$ be the product of the functions u and v. Find $y'(1)$ if $u(1) = 2$, $u'(1) = 3$, $v(1) = -1$, and $v'(1) = 1$.
(A) -4 (B) -1 (C) 1 (D) 4 (E) 7

56. **Multiple Choice** Let $f(x) = x - \frac{1}{x}$. Find $f''(x)$.
(A) $1 + \frac{1}{x^2}$ (B) $1 - \frac{1}{x^2}$ (C) $\frac{2}{x^3}$
(D) $-\frac{2}{x^3}$ (E) does not exist

57. **Multiple Choice** Which of the following is $\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$?
(A) $\frac{2}{(x-1)^2}$ (B) 0 (C) $-\frac{x^2+1}{x^2}$
(D) $2x - \frac{1}{x^2} - 1$ (E) $-\frac{2}{(x-1)^2}$

58. **Multiple Choice** Assume $f(x) = (x^2 - 1)(x^2 + 1)$. Which of the following gives the number of horizontal tangents of f?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Extending the Ideas

59. ***Leibniz's Proof of the Product Rule*** Here's how Leibniz explained the Product Rule in a letter to his colleague John Wallis:

It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x + dx$, dx is rejected. Similarly we cannot have xdx and $dxdx$ standing together, as xdx is incomparably greater than $dxdx$. Hence if we are to differentiate uv, we write

$$d(uv) = (u + du)(v + dv) - uv$$
$$= uv + vdu + udv + dudv - uv$$
$$= vdu + udv.$$

Answer the following questions about Leibniz's proof.

(a) What does Leibniz mean by a quantity being "rejected"?

(b) What happened to $dudv$ in the last step of Leibniz's proof?

(c) Divide both sides of Leibniz's formula

$$d(uv) = vdu + udv$$

by the differential dx. What formula results?

(d) Why would the critics of Leibniz's time have objected to dividing both sides of the equation by dx?

(e) Leibniz had a similar simple (but not-so-clean) proof of the Quotient Rule. Can you reconstruct it?

Quick Quiz for AP* Preparation: Sections 3.1–3.3

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** Let $f(x) = |x + 1|$. Which of the following statements about f are true?
I. f is continuous at $x = -1$.
II. f is differentiable at $x = -1$.
III. f has a corner at $x = -1$.
(A) I only (B) II only (C) III only
(D) I and III only (E) I and II only

2. **Multiple Choice** If the line normal to the graph of f at the point (1, 2) passes through the point $(-1, 1)$, then which of the following gives the value of $f'(1) = ?$
(A) -2 (B) 2 (C) $-1/2$ (D) 1/2 (E) 3

3. **Multiple Choice** Find dy/dx if $y = \frac{4x-3}{2x+1}$.
(A) $\frac{10}{(4x-3)^2}$ (B) $-\frac{10}{(4x-3)^2}$ (C) $\frac{10}{(2x+1)^2}$
(D) $-\frac{10}{(2x+1)^2}$ (E) 2

4. **Free Response** Let $f(x) = x^4 - 4x^2$.
(a) Find all the points where f has horizontal tangents.
(b) Find an equation of the tangent line at $x = 1$.
(c) Find an equation of the normal line at $x = 1$.

3.4 Velocity and Other Rates of Change

What you'll learn about
- Instantaneous Rates of Change
- Motion along a Line
- Sensitivity to Change
- Derivatives in Economics

. . . and why

Derivatives give the rates at which things change in the world.

Instantaneous Rates of Change

In this section we examine some applications in which derivatives as functions are used to represent the rates at which things change in the world around us. It is natural to think of change as change with respect to time, but other variables can be treated in the same way. For example, a physician may want to know how change in dosage affects the body's response to a drug. An economist may want to study how the cost of producing steel varies with the number of tons produced.

If we interpret the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

as the average rate of change of the function f over the interval from x to $x + h$, we can interpret its limit as h approaches 0 to be the rate at which f is changing at the point x.

DEFINITION Instantaneous Rate of Change

The **(instantaneous) rate of change** of f with respect to x at a is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time. The word, however, is frequently omitted in practice. When we say *rate of change,* we mean *instantaneous rate of change.*

EXAMPLE 1 Enlarging Circles

(a) Find the rate of change of the area A of a circle with respect to its radius r.

(b) Evaluate the rate of change of A at $r = 5$ and at $r = 10$.

(c) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

SOLUTION

The area of a circle is related to its radius by the equation $A = \pi r^2$.

(a) The (instantaneous) rate of change of A with respect to r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \cdot 2r = 2\pi r.$$

(b) At $r = 5$, the rate is 10π (about 31.4). At $r = 10$, the rate is 20π (about 62.8).

Notice that the rate of change gets bigger as r gets bigger. As can be seen in Figure 3.21, the same change in radius brings about a bigger change in area as the circles grow radially away from the center.

(c) The appropriate units for dA/dr are square inches (of area) per inch (of radius).

Now try Exercise 1.

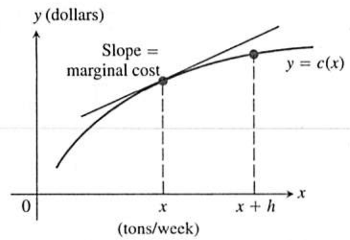


Figure 3.30 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons per week is $c(x + h) - c(x)$.

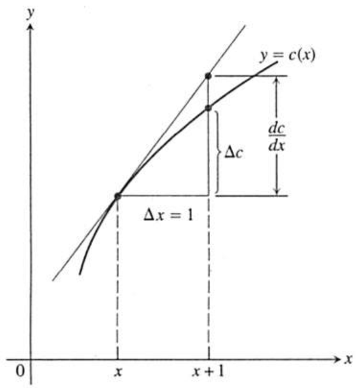


Figure 3.31 Because dc/dx is the slope of the tangent at x, the marginal cost dc/dx approximates the extra cost Δc of producing $\Delta x = 1$ more unit.

In a manufacturing operation, *the cost of production* $c(x)$ is a function of x, the numb of units produced. The *marginal cost of production* is the rate of change of cost with r spect to the level of production, so it is dc/dx.

Suppose $c(x)$ represents the dollars needed to produce x tons of steel in one week, costs more to produce $x + h$ tons per week, and the cost difference divided by h is th average cost of producing each additional ton.

$$\frac{c(x+h) - c(x)}{h} = \begin{cases} \text{the average cost of each of the} \\ \text{additional } h \text{ tons produced} \end{cases}$$

The limit of this ratio as $h \to 0$ is the **marginal cost** of producing more steel per we when the current production is x tons (Figure 3.30).

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of pr ducing one more unit,

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x. This approximation is acceptable if th slope of c does not change quickly near x, for then the difference quotient is close to its lim dc/dx even if $\Delta x = 1$ (Figure 3.31). The approximation works best for large values of x.

EXAMPLE 7 Marginal Cost and Marginal Revenue

Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 10 radiators are produced, and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. Find the marginal cost and **marginal revenue.**

SOLUTION

The marginal cost of producing one more radiator a day when 10 are being produced is $c'(10)$.

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195 \text{ dollars}$$

The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12,$$

so,

$$r'(10) = 3(100) - 6(10) + 12 = 252 \text{ dollars.}$$

Now try Exercises 27 and 2

Quick Review 3.4 *(For help, go to Sections 1.2, 3.1, and 3.3.)*

In Exercises 1–10, answer the questions about the graph of the quadratic function $y = f(x) = -16x^2 + 160x - 256$ by analyzing the equation algebraically. Then support your answers graphically.

1. Does the graph open upward or downward?
2. What is the y-intercept?
3. What are the x-intercepts?
4. What is the range of the function?
5. What point is the vertex of the parabola?
6. At what x-values does $f(x) = 80$?
7. For what x-value does $dy/dx = 100$?
8. On what interval is $dy/dx > 0$?
9. Find $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.
10. Find d^2y/dx^2 at $x = 7$.

Section 3.4 Exercises

1. (a) Write the volume V of a cube as a function of the side length s.
   (b) Find the (instantaneous) rate of change of the volume V with respect to a side s.
   (c) Evaluate the rate of change of V at $s = 1$ and $s = 5$.
   (d) If s is measured in inches and V is measured in cubic inches, what units would be appropriate for dV/ds?
2. (a) Write the area A of a circle as a function of the circumference C.
   (b) Find the (instantaneous) rate of change of the area A with respect to the circumference C.
   (c) Evaluate the rate of change of A at $C = \pi$ and $C = 6\pi$.
   (d) If C is measured in inches and A is measured in square inches, what units would be appropriate for dA/dC?
3. (a) Write the area A of an equilateral triangle as a function of the side length s.
   (b) Find the (instantaneous) rate of change of the area A with respect to a side s.
   (c) Evaluate the rate of change of A at $s = 2$ and $s = 10$.
   (d) If s is measured in inches and A is measured in square inches, what units would be appropriate for dA/ds?
4. A square of side length s is inscribed in a circle of radius r.
   (a) Write the area A of the square as a function of the radius r of the circle.
   (b) Find the (instantaneous) rate of change of the area A with respect to the radius r of the circle.
   (c) Evaluate the rate of change of A at $r = 1$ and $r = 8$.
   (d) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

Group Activity In Exercises 5 and 6, the coordinates s of a moving body for various values of t are given. **(a)** Plot s versus t on coordinate paper, and sketch a smooth curve through the given points. **(b)** Assuming that this smooth curve represents the motion of the body, estimate the velocity at $t = 1.0$, $t = 2.5$, and $t = 3.5$.

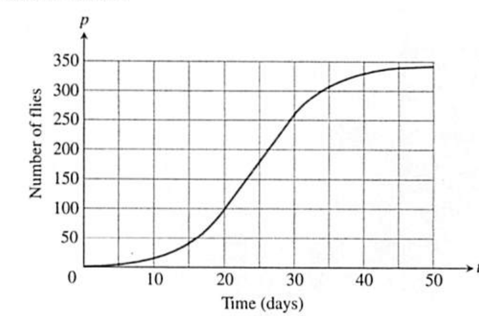
5.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	12.5	26	36.5	44	48.5	50	48.5	44	36.5

6.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	3.5	−4	−8.5	−10	−8.5	−4	3.5	14	27.5

7. **Group Activity** ***Fruit Flies*** *(Example 2, Section 2.4 continued)* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.
   (a) Use the graphical technique of Section 3.1, Example 3, to graph the derivative of the fruit fly population introduced in Section 2.4. The graph of the population is reproduced below. What units should be used on the horizontal and vertical axes for the derivative's graph?
   (b) During what days does the population seem to be increasing fastest? slowest?



8. ***Draining a Tank*** The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

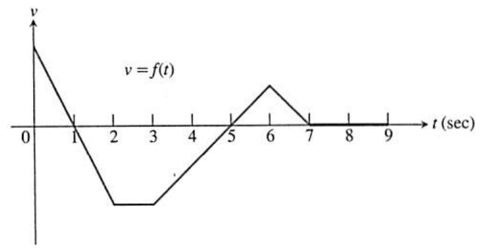
9. ***Particle Motion*** The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.

(a) When does the particle move forward? move backward? speed up? slow down?

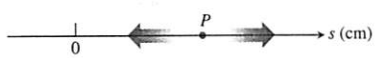
(b) When is the particle's acceleration positive? negative? zero?

(c) When does the particle move at its greatest speed?

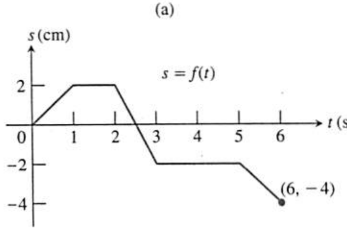
(d) When does the particle stand still for more than an instant?



10. ***Particle Motion*** A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



(a)

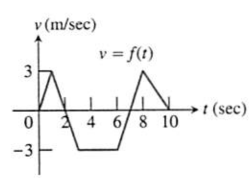


(b)

(a) When is P moving to the left? moving to the right? standing still?

(b) Graph the particle's velocity and speed (where defined).

11. ***Particle Motion*** The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



(a) When does the body reverse direction?

(b) When (approximately) is the body moving at a constant speed?

(c) Graph the body's speed for $0 \le t \le 10$.

(d) Graph the acceleration, where defined.

12. ***Thoroughbred Racing*** A racehorse is running a 10-furlong race. (A furlong is 220 yards, although we will use furlongs and seconds as our units in this exercise.) As the horse passes each furlong marker (F), a steward records the time elapsed (t) since the beginning of the race, as shown in the table below:

F	0	1	2	3	4	5	6	7	8	9	10
t	0	20	33	46	59	73	86	100	112	124	135

(a) How long does it take the horse to finish the race?

(b) What is the average speed of the horse over the first 5 furlongs?

(c) What is the approximate speed of the horse as it passes the 3-furlong marker?

(d) During which portion of the race is the horse running the fastest?

(e) During which portion of the race is the horse accelerating the fastest?

13. ***Lunar Projectile Motion*** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

(a) Find the rock's velocity and acceleration as functions of time. (The acceleration in this case is the acceleration of gravity on the moon.)

(b) How long did it take the rock to reach its highest point?

(c) How high did the rock go?

(d) When did the rock reach half its maximum height?

(e) How long was the rock aloft?

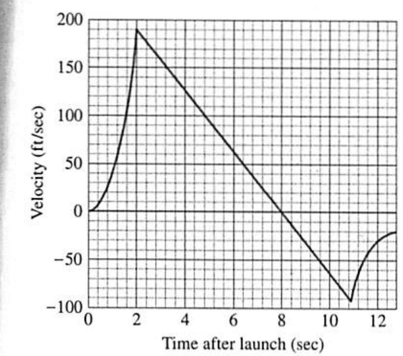
14. ***Free Fall*** The equations for free fall near the surfaces of Mars and Jupiter (s in meters, t in seconds) are: Mars, $s = 1.86t^2$; Jupiter, $s = 11.44t^2$. How long would it take a rock falling from rest to reach a velocity of 16.6 m/sec (about 60 km/h) on each planet?

15. ***Projectile Motion*** On Earth, in the absence of air, the rock in Exercise 13 would reach a height of $s = 24t - 4.9t^2$ meters in t seconds. How high would the rock go?

16. ***Speeding Bullet*** A bullet fired straight up from the moon's surface would reach a height of $s = 832t - 2.6t^2$ ft after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long would it take the bullet to get back down in each case?

17. ***Parametric Graphing*** Devise a grapher simulation of the problem situation in Exercise 16. Use it to support the answers obtained analytically.

18. ***Launching a Rocket*** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts downward. The parachute slows the rocket to keep it from breaking when it lands. This graph shows velocity data from the flight.



Use the graph to answer the following.

(a) How fast was the rocket climbing when the engine stopped?

(b) For how many seconds did the engine burn?

(c) When did the rocket reach its highest point? What was its velocity then?

(d) When did the parachute pop out? How fast was the rocket falling then?

(e) How long did the rocket fall before the parachute opened?

(f) When was the rocket's acceleration greatest? When was the acceleration constant?

19. ***Particle Motion*** A particle moves along a line so that its position at any time $t \ge 0$ is given by the function

$$s(t) = t^2 - 3t + 2,$$

where s is measured in meters and t is measured in seconds.

(a) Find the displacement during the first 5 seconds.

(b) Find the average velocity during the first 5 seconds.

(c) Find the instantaneous velocity when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) At what values of t does the particle change direction?

(f) Where is the particle when s is a minimum?

20. ***Particle Motion*** A particle moves along a line so that its position at any time $t \ge 0$ is given by the function $s(t) = -t^3 + 7t^2 - 14t + 8$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

21. ***Particle Motion*** A particle moves along a line so that its position at any time $t \ge 0$ is given by the function $s(t) = (t - 2)^2(t - 4)$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

22. ***Particle Motion*** A particle moves along a line so that its position at any time $t \ge 0$ is given by the function $s(t) = t^3 - 6t^2 + 8t + 2$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

23. ***Particle Motion*** The position of a body at time t sec is $s = t^3 - 6t^2 + 9t$ m. Find the body's acceleration each time the velocity is zero.

24. ***Finding Speed*** A body's velocity at time t sec is $v = 2t^3 - 9t^2 + 12t - 5$ m/sec. Find the body's speed each time the acceleration is zero.

25. ***Draining a Tank*** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

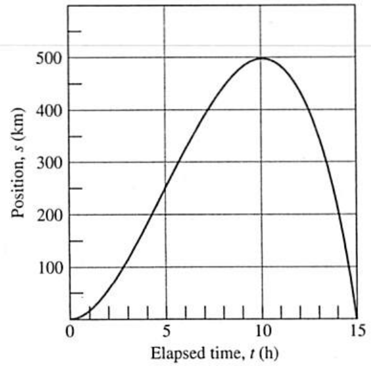
$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.}$$

(a) Find the rate dy/dt (m/h) at which the water level is changing at time t.

(b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?

(c) Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

26. ***Moving Truck*** The graph here shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 hours later at $t = 15$.



(a) Use the technique described in Section 3.1, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.

(b) Suppose $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2, and compare your graphs with those in part (a).

27. ***Marginal Cost*** Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

(a) Find the average cost of producing 100 washing machines.

(b) Find the marginal cost when 100 machines are produced.

(c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

28. ***Marginal Revenue*** Suppose the weekly revenue in dollars from selling x custom-made office desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$

(a) Draw the graph of r. What values of x make sense in this problem situation?

(b) Find the marginal revenue when x desks are sold.

(c) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing sales from 5 desks a week to 6 desks a week.

(d) **Writing to Learn** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

29. ***Finding Profit*** The monthly profit (in thousands of dollars) of a software company is given by

$$P(x) = \frac{10}{1 + 50 \cdot 2^{5-0.1x}},$$

where x is the number of software packages sold.

(a) Graph $P(x)$.

(b) What values of x make sense in the problem situation?

(c) Use NDER to graph $P'(x)$. For what values of x is P relatively sensitive to changes in x?

(d) What is the profit when the marginal profit is greatest?

(e) What is the marginal profit when 50 units are sold? 100 units, 125 units, 150 units, 175 units, and 300 units?

(f) What is $\lim_{x\to\infty} P(x)$? What is the maximum profit possible?

(g) **Writing to Learn** Is there a practical explanation to the maximum profit answer? Explain your reasoning.

30. In Step 1 of Exploration 2, at what time is the particle at the point (5, 2)?

31. **Group Activity** The graphs in Figure 3.32 show as functions of time t the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

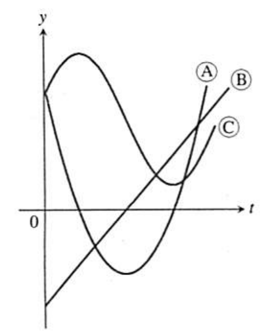


Figure 3.32 The graphs for Exercise 31.

32. **Group Activity** The graphs in Figure 3.33 show as functions of time t the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

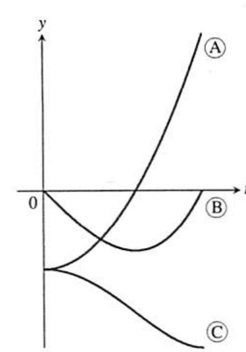


Figure 3.33 The graphs for Exercise 32.

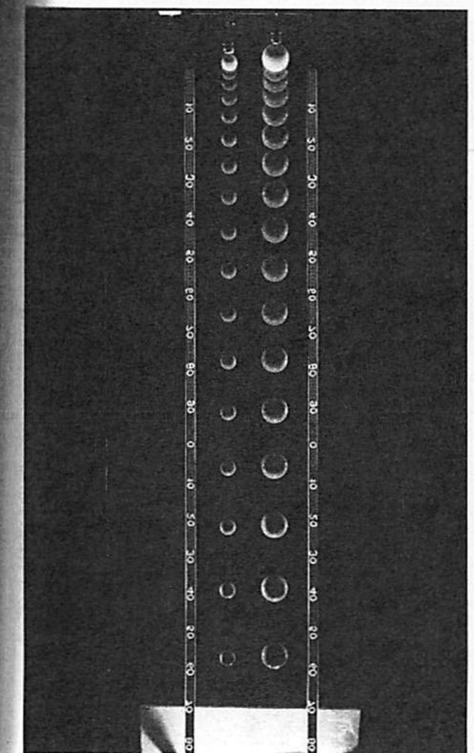


Figure 3.34 Two balls falling from rest. (Exercise 38)

33. ***Pisa by Parachute*** *(continuation of Exercise 18)* A few years ago, Mike McCarthy parachuted 179 ft from the top of the Tower of Pisa. Make a rough sketch to show the shape of the graph of his downward velocity during the jump.

34. ***Inflating a Balloon*** The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

(a) At what rate does the volume change with respect to the radius when $r = 2$ ft?

(b) By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

35. ***Volcanic Lava Fountains*** Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? in miles per hour? [*Hint:* If v_0 is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0t - 16t^2$ feet. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.]

36. **Writing to Learn** Suppose you are looking at a graph of velocity as a function of time. How can you estimate the acceleration at a given point in time?

37. ***Particle Motion*** The position (x-coordinate) of a particle moving on the line $y = 2$ is given by $x(t) = 2t^3 - 13t^2 + 22t - 5$ where t is time in seconds.

(a) Describe the motion of the particle for $t \ge 0$.

(b) When does the particle speed up? slow down?

(c) When does the particle change direction?

(d) When is the particle at rest?

(e) Describe the velocity and speed of the particle.

(f) When is the particle at the point (5, 2)?

38. ***Falling Objects*** The multiflash photograph in Figure 3.34 shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.

(a) How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

(b) How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

(c) About how fast was the light flashing (flashes per second)?

39. **Writing to Learn** Explain how the Sum and Difference Rule (Rule 4 in Section 3.3) can be used to derive a formula for *marginal profit* in terms of marginal revenue and marginal cost.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

40. **True or False** The speed of a particle at $t = a$ is given by the value of the velocity at $t = a$. Justify your answer.

41. **True or False** The acceleration of a particle is the second derivative of the position function. Justify your answer.

42. **Multiple Choice** Find the instantaneous rate of change of $f(x) = x^2 - 2/x + 4$ at $x = -1$.

(A) -7 (B) -4 (C) 0 (D) 4 (E) 7

43. **Multiple Choice** Find the instantaneous rate of change of the volume of a cube with respect to a side length x.

(A) x (B) $3x$ (C) $6x$ (D) $3x^2$ (E) x^3

In Exercises 44 and 45, a particle moves along a line so that its position at any time $t \ge 0$ is given by $s(t) = 2 + 7t - t^2$.

44. **Multiple Choice** At which of the following times is the particle moving to the left?

(A) $t = 0$ (B) $t = 1$ (C) $t = 2$ (D) $t = 7/2$ (E) $t = 4$

45. **Multiple Choice** When is the particle at rest?

(A) $t = 1$ (B) $t = 2$ (C) $t = 7/2$ (D) $t = 4$ (E) $t = 5$

Explorations

46. ***Bacterium Population*** When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while but then stopped growing and began to decline. The size of the population at time t (hours) was $b(t) = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at $t = 0$, $t = 5$, and $t = 10$ hours.

47. *Finding f from f′* Let $f'(x) = 3x^2$.

(a) Compute the derivatives of $g(x) = x^3$, $h(x) = x^3 - 2$, and $t(x) = x^3 + 3$.

(b) Graph the numerical derivatives of g, h, and t.

(c) Describe a *family* of functions, $f(x)$, that have the property that $f'(x) = 3x^2$.

(d) Is there a function f such that $f'(x) = 3x^2$ and $f(0) = 0$? If so, what is it?

(e) Is there a function f such that $f'(x) = 3x^2$ and $f(0) = 3$? If so, what is it?

48. *Airplane Takeoff* Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. If the aircraft will become airborne when its speed reaches 200 km/h, how long will it take to become airborne, and what distance will it have traveled by that time?

Extending the Ideas

49. *Even and Odd Functions*

(a) Show that if f is a differentiable even function, then f' is an odd function.

(b) Show that if f is a differentiable odd function, then f' is an even function.

50. *Extended Product Rule* Derive a formula for the derivative of the product fgh of three differentiable functions.

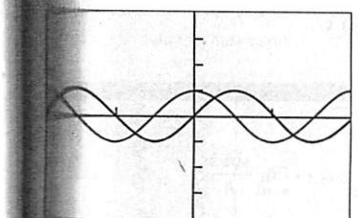
3.5 Derivatives of Trigonometric Functions

What you'll learn about

- Derivative of the Sine Function
- Derivative of the Cosine Function
- Simple Harmonic Motion
- Jerk
- Derivatives of the Other Basic Trigonometric Functions

. . . and why

The derivatives of sines and cosines play a key role in describing periodic change.



[−2π, 2π] by [−4, 4]

Figure 3.35 Sine and its derivative. What is the derivative? (Exploration 1)

Derivative of the Sine Function

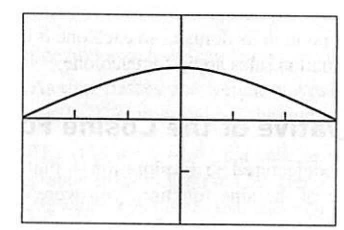
Trigonometric functions are important because so many of the phenomena we want information about are periodic (heart rhythms, earthquakes, tides, weather). It is known that continuous periodic functions can always be expressed in terms of sines and cosines, so the derivatives of sines and cosines play a key role in describing periodic change. This section introduces the derivatives of the six basic trigonometric functions.

EXPLORATION 1 Making a Conjecture with NDER

In the window $[-2\pi, 2\pi]$ by $[-4, 4]$, graph $y_1 = \sin x$ and $y_2 = \text{NDER}(\sin x)$ (Figure 3.35).

1. When the graph of $y_1 = \sin x$ is increasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
2. When the graph of $y_1 = \sin x$ is decreasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
3. When the graph of $y_1 = \sin x$ stops increasing and starts decreasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
4. At the places where $\text{NDER}(\sin x) = \pm 1$, what appears to be the slope of the graph of $y_1 = \sin x$?
5. Make a conjecture about what function the derivative of sine might be. Test your conjecture by graphing your function and $\text{NDER}(\sin x)$ in the same viewing window.
6. Now let $y_1 = \cos x$ and $y_2 = \text{NDER}(\cos x)$. Answer questions (1) through (5) *without* looking at the graph of $\text{NDER}(\cos x)$ until you are ready to test your conjecture about what function the derivative of cosine might be.

If you conjectured that the derivative of the sine function is the cosine function, then you are right. We will confirm this analytically, but first we appeal to technology one more time to evaluate two limits needed in the proof (see Figure 3.36 below and Figure 3.37 on the next page):



[−3, 3] by [−2, 2]

(a)

X	Y1
−.03	.99985
−.02	.99993
−.01	.99998
0	ERROR
.01	.99998
.02	.99993
.03	.99985

Y1 = sin(X)/X

(b)

Figure 3.36 (a) Graphical and (b) tabular support that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.

Quick Review 3.5 *(For help, go to Sections 1.6, 3.1, and 3.4.)*

1. Convert 135 degrees to radians.
2. Convert 1.7 radians to degrees.
3. Find the exact value of $\sin(\pi/3)$ without a calculator.
4. State the domain and the range of the cosine function.
5. State the domain and the range of the tangent function.
6. If $\sin a = -1$, what is $\cos a$?
7. If $\tan a = -1$, what are two possible values of $\sin a$?
8. Verify the identity:
$$\frac{1-\cos h}{h} = \frac{\sin^2 h}{h(1+\cos h)}.$$
9. Find an equation of the line tangent to the curve $y = 2x^3 - 7x^2 + 10$ at the point (3, 1).
10. A particle moves along a line with velocity $v = 2t^3 - 7t^2 + 10$ for time $t \geq 0$. Find the acceleration of the particle at $t = 3$.

Section 3.5 Exercises

In Exercises 1–10, find dy/dx. Use your grapher to support your analysis if you are unsure of your answer.

1. $y = 1 + x - \cos x$
2. $y = 2\sin x - \tan x$
3. $y = \frac{1}{x} + 5\sin x$
4. $y = x \sec x$
5. $y = 4 - x^2 \sin x$
6. $y = 3x + x\tan x$
7. $y = \frac{4}{\cos x}$
8. $y = \frac{x}{1+\cos x}$
9. $y = \frac{\cot x}{1+\cot x}$
10. $y = \frac{\cos x}{1+\sin x}$

In Exercises 11 and 12, a weight hanging from a spring (see Figure 3.38) bobs up and down with position function $s = f(t)$ (s in meters, t in seconds). What are its velocity and acceleration at time t? Describe its motion.

11. $s = 5\sin t$
12. $s = 7\cos t$

In Exercises 13–16, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds).

(a) Find the body's velocity, speed, and acceleration at time t.

(b) Find the body's velocity, speed, and acceleration at time $t = \pi/4$.

(c) Describe the motion of the body.

13. $s = 2 + 3\sin t$
14. $s = 1 - 4\cos t$
15. $s = 2\sin t + 3\cos t$
16. $s = \cos t - 3\sin t$

In Exercises 17–20, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds). Find the jerk at time t.

17. $s = 2\cos t$
18. $s = 1 + 2\cos t$
19. $s = \sin t - \cos t$
20. $s = 2 + 2\sin t$
21. Find equations for the lines that are tangent and normal to the graph of $y = \sin x + 3$ at $x = \pi$.
22. Find equations for the lines that are tangent and normal to the graph of $y = \sec x$ at $x = \pi/4$.
23. Find equations for the lines that are tangent and normal to the graph of $y = x^2 \sin x$ at $x = 3$.
24. Use the definition of the derivative to prove that $(d/dx)(\cos x) = -\sin x$. (You will need the limits found at the beginning of this section.)
25. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\frac{d}{dx}\tan x = \sec^2 x$ (b) $\frac{d}{dx}\sec x = \sec x \tan x$

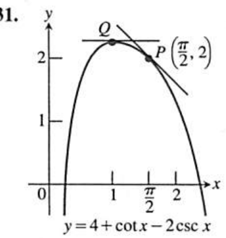
26. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\frac{d}{dx}\cot x = -\csc^2 x$ (b) $\frac{d}{dx}\csc x = -\csc x \cot x$

27. Show that the graphs of $y = \sec x$ and $y = \cos x$ have horizontal tangents at $x = 0$.
28. Show that the graphs of $y = \tan x$ and $y = \cot x$ have no horizontal tangents.
29. Find equations for the lines that are tangent and normal to the curve $y = \sqrt{2}\cos x$ at the point $(\pi/4, 1)$.
30. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$.

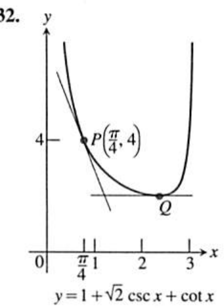
In Exercises 31 and 32, find an equation for (a) the tangent to the curve at P and (b) the horizontal tangent to the curve at Q.

31.



$y = 4 + \cot x - 2\csc x$

32.



$y = 1 + \sqrt{2}\csc x + \cot x$

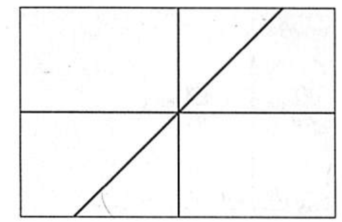
Group Activity In Exercises 33 and 34, a body is moving in simple harmonic motion with position $s = f(t)$ (s in meters, t in seconds).

(a) Find the body's velocity, speed, acceleration, and jerk at time t.

(b) Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

(c) Describe the motion of the body.

33. $s = 2 - 2\sin t$
34. $s = \sin t + \cos t$
35. Find y'' if $y = \csc x$.
36. Find y'' if $y = \theta\tan\theta$.
37. **Writing to Learn** Is there a value of b that will make
$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$
continuous at $x = 0$? differentiable at $x = 0$? Give reasons for your answers.
38. Find $\frac{d^{999}}{dx^{999}}(\cos x)$.
39. Find $\frac{d^{725}}{dx^{725}}(\sin x)$.
40. ***Local Linearity*** This is the graph of the function $y = \sin x$ close to the origin. Since $\sin x$ is differentiable, this graph resembles a line. Find an equation for this line.



41. ***(Continuation of Exercise 40)*** For values of x close to 0, the linear equation found in Exercise 40 gives a good approximation of $\sin x$.

(a) Use this fact to estimate $\sin(0.12)$.

(b) Find $\sin(0.12)$ with a calculator. How close is the approximation in part (a)?

42. Use the identity $\sin 2x = 2\sin x\cos x$ to find the derivative of $\sin 2x$. Then use the identity $\cos 2x = \cos^2 x - \sin^2 x$ to express that derivative in terms of $\cos 2x$.
43. Use the identity $\cos 2x = \cos x\cos x - \sin x\sin x$ to find the derivative of $\cos 2x$. Express the derivative in terms of $\sin 2x$.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

In Exercises 44 and 45, a spring is bobbing up and down on the end of a spring according to $s(t) = -3\sin t$.

44. **True or False** The spring is traveling upward at $t = 3\pi/4$. Justify your answer.
45. **True or False** The velocity and speed of the particle are the same at $t = \pi/4$. Justify your answer.
46. **Multiple Choice** Which of the following is an equation of the tangent line to $y = \sin x + \cos x$ at $x = \pi$?

(A) $y = -x + \pi - 1$ (B) $y = -x + \pi + 1$
(C) $y = -x - \pi + 1$ (D) $y = -x - \pi - 1$
(E) $y = x - \pi + 1$

47. **Multiple Choice** Which of the following is an equation of the normal line to $y = \sin x + \cos x$ at $x = \pi$?

(A) $y = -x + \pi - 1$ (B) $y = x - \pi - 1$ (C) $y = x - \pi + 1$
(D) $y = x + \pi + 1$ (E) $y = x + \pi - 1$

48. **Multiple Choice** Find y'' if $y = x\sin x$.

(A) $-x\sin x$ (B) $x\cos x + \sin x$ (C) $-x\sin x + 2\cos x$
(D) $x\sin x$ (E) $-\sin x + \cos x$

49. **Multiple Choice** A body is moving in simple harmonic motion with position $s = 3 + \sin t$. At which of the following times is the velocity zero?

(A) $t = 0$ (B) $t = \pi/4$ (C) $t = \pi/2$
(D) $t = \pi$ (E) none of these

Exploration

50. ***Radians vs. Degrees*** What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

(a) With your grapher in degree mode, graph
$$f(h) = \frac{\sin h}{h}$$
and estimate $\lim_{h\to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit should be $\pi/180$?

(b) With your grapher in degree mode, estimate
$$\lim_{h\to 0}\frac{\cos h - 1}{h}.$$

(c) Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

(d) Derive the formula for the derivative of $\cos x$ using degree-mode limits.

(e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

Extending the Ideas

51. Use analytic methods to show that
$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$
[*Hint:* Multiply numerator and denominator by $(\cos h + 1)$.]
52. Find A and B in $y = A\sin x + B\cos x$ so that $y'' - y = \sin x$.

EXAMPLE 7 Finding Slope

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

SOLUTION

(a) $$\frac{dy}{dx} = 5\sin^4 x \cdot \frac{d}{dx}\sin x \qquad \text{Power Chain Rule with } u = \sin x,\ n = 5$$
$$= 5\sin^4 x \cos x$$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

(b) $$\frac{dy}{dx} = \frac{d}{dx}(1 - 2x)^{-3}$$
$$= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \qquad \text{Power Chain Rule with } u = (1 - 2x),\ n = -3$$
$$= -3(1 - 2x)^{-4} \cdot (-2)$$
$$= \frac{6}{(1 - 2x)^4}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers. *Now try Exercise 5*

EXAMPLE 8 Radians Versus Degrees

It is important to remember that the formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Figure 3.44.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure.

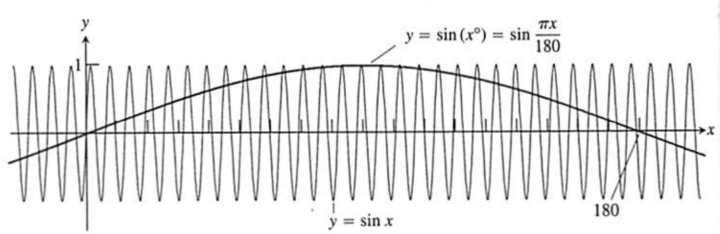


Figure 3.44 $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$. (Example 8)

Quick Review 3.6 *(For help, go to Sections 1.2 and 1.6.)*

In Exercises 1–5, let $f(x) = \sin x$, $g(x) = x^2 + 1$, and $h(x) = 7x$. Write a simplified expression for the composite function.

1. $f(g(x))$
2. $f(g(h(x)))$
3. $(g \circ h)(x)$
4. $(h \circ g)(x)$
5. $f\left(\frac{g(x)}{h(x)}\right)$

In Exercises 6–10, let $f(x) = \cos x$, $g(x) = \sqrt{x + 2}$, and $h(x) = 3x^2$. Write the given function as a composite of two or more of f, g, and h. For example, $\cos 3x^2$ is $f(h(x))$.

6. $\sqrt{\cos x + 2}$
7. $\sqrt{3\cos^2 x + 2}$
8. $3\cos x + 6$
9. $\cos 27x^4$
10. $\cos\sqrt{2 + 3x^2}$

Section 3.6 Exercises

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

1. $y = \sin(3x + 1)$, $u = 3x + 1$
2. $y = \sin(7 - 5x)$, $u = 7 - 5x$
3. $y = \cos(\sqrt{3}x)$, $u = \sqrt{3x}$
4. $y = \tan(2x - x^3)$, $u = 2x - x^3$
5. $y = \left(\frac{\sin x}{1 + \cos x}\right)^2$, $u = \frac{\sin x}{1 + \cos x}$
6. $y = 5\cot\left(\frac{2}{x}\right)$, $u = \frac{2}{x}$
7. $y = \cos(\sin x)$, $u = \sin x$
8. $y = \sec(\tan x)$, $u = \tan x$

In Exercises 9–12, an object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of t.

9. $s = \cos\left(\frac{\pi}{2} - 3t\right)$
10. $s = t\cos(\pi - 4t)$
11. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$
12. $s = \sin\left(\frac{3\pi}{2}t\right) + \cos\left(\frac{7\pi}{4}t\right)$

In Exercises 13–24, find dy/dx. If you are unsure of your answer, use NDER to support your computation.

13. $y = (x + \sqrt{x})^{-2}$
14. $y = (\csc x + \cot x)^{-1}$
15. $y = \sin^{-5} x - \cos^3 x$
16. $y = x^3(2x - 5)^4$
17. $y = \sin^3 x \tan 4x$
18. $y = 4\sqrt{\sec x + \tan x}$
19. $y = \frac{3}{\sqrt{2x + 1}}$
20. $y = \frac{x}{\sqrt{1 + x^2}}$
21. $y = \sin^2(3x - 2)$
22. $y = (1 + \cos 2x)^2$
23. $y = (1 + \cos^2 7x)^3$
24. $y = \sqrt{\tan 5x}$

In Exercises 25–28 find $dr/d\theta$.

25. $r = \tan(2 - \theta)$
26. $r = \sec 2\theta \tan 2\theta$
27. $r = \sqrt{\theta \sin \theta}$
28. $r = 2\theta\sqrt{\sec \theta}$

In Exercises 29–32, find y''.

29. $y = \tan x$
30. $y = \cot x$
31. $y = \cot(3x - 1)$
32. $y = 9\tan(x/3)$

In Exercises 33–38, find the value of $(f \circ g)'$ at the given value of x.

33. $f(u) = u^5 + 1$, $u = g(x) = \sqrt{x}$, $x = 1$
34. $f(u) = 1 - \frac{1}{u}$, $u = g(x) = \frac{1}{1 - x}$, $x = -1$
35. $f(u) = \cot\frac{\pi u}{10}$, $u = g(x) = 5\sqrt{x}$, $x = 1$
36. $f(u) = u + \frac{1}{\cos^2 u}$, $u = g(x) = \pi x$, $x = \frac{1}{4}$
37. $f(u) = \frac{2u}{u^2 + 1}$, $u = g(x) = 10x^2 + x + 1$, $x = 0$
38. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2$, $u = g(x) = \frac{1}{x^2} - 1$, $x = -1$

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 39 and 40.

39. Find dy/dx if $y = \cos(6x + 2)$ by writing y as a composite with
    (a) $y = \cos u$ and $u = 6x + 2$.
    (b) $y = \cos 2u$ and $u = 3x + 1$.
40. Find dy/dx if $y = \sin(x^2 + 1)$ by writing y as a composite with
    (a) $y = \sin(u + 1)$ and $u = x^2$.
    (b) $y = \sin u$ and $u = x^2 + 1$.

In Exercises 41–48, find the equation of the line tangent to the curve at the point defined by the given value of t.

41. $x = 2\cos t$, $y = 2\sin t$, $t = \pi/4$
42. $x = \sin 2\pi t$, $y = \cos 2\pi t$, $t = -1/6$
43. $x = \sec^2 t - 1$, $y = \tan t$, $t = -\pi/4$
44. $x = \sec t$, $y = \tan t$, $t = \pi/6$
45. $x = t$, $y = \sqrt{t}$, $t = 1/4$
46. $x = 2t^2 + 3$, $y = t^4$, $t = -1$
47. $x = t - \sin t$, $y = 1 - \cos t$, $t = \pi/3$
48. $x = \cos t$, $y = 1 + \sin t$, $t = \pi/2$

49. Let $x = t^2 + t$, and let $y = \sin t$.

(a) Find dy/dx as a function of t.

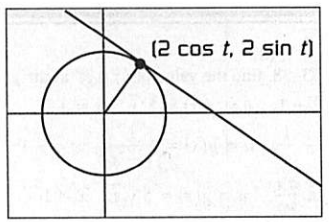
(b) Find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$ as a function of t.

(c) Find $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ as a function of t.

Use the Chain Rule and your answer from part (b).

(d) Which of the expressions in parts (b) and (c) is d^2y/dx^2?

50. A circle of radius 2 and center (0, 0) can be parametrized by the equations $x = 2\cos t$ and $y = 2\sin t$. Show that for any value of t, the line tangent to the circle at $(2\cos t, 2\sin t)$ is perpendicular to the radius.



51. Let $s = \cos\theta$. Evaluate ds/dt when $\theta = 3\pi/2$ and $d\theta/dt = 5$.

52. Let $y = x^2 + 7x - 5$. Evaluate dy/dt when $x = 1$ and $dx/dt = 1/3$.

53. What is the largest value possible for the slope of the curve $y = \sin(x/2)$?

54. Write an equation for the tangent to the curve $y = \sin mx$ at the origin.

55. Find the lines that are tangent and normal to the curve $y = 2\tan(\pi x/4)$ at $x = 1$. Support your answer graphically.

56. ***Working with Numerical Values*** Suppose that functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $2f(x)$ at $x = 2$ (b) $f(x) + g(x)$ at $x = 3$

(c) $f(x)\cdot g(x)$ at $x = 3$ (d) $f(x)/g(x)$ at $x = 2$

(e) $f(g(x))$ at $x = 2$ (f) $\sqrt{f(x)}$ at $x = 2$

(g) $1/g^2(x)$ at $x = 3$ (h) $\sqrt{f^2(x) + g^2(x)}$ at $x = 2$

57. ***Extension of Example 8*** Show that $\dfrac{d}{dx}\cos(x°)$ is $-\dfrac{\pi}{180}\sin(x°)$.

58. ***Working with Numerical Values*** Suppose that the function f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $5f(x) - g(x)$, $x = 1$ (b) $f(x)g^3(x)$, $x = 0$

(c) $\dfrac{f(x)}{g(x) + 1}$, $x = 1$ (d) $f(g(x))$, $x = 0$

(e) $g(f(x))$, $x = 0$ (f) $(g(x) + f(x))^{-2}$, $x = 1$

(g) $f(x + g(x))$, $x = 0$

59. ***Orthogonal Curves*** Two curves are said to cross at right angles if their tangents are perpendicular at the crossing point. The technical word for "crossing at right angles" is **orthogonal**. Show that the curves $y = \sin 2x$ and $y = -\sin(x/2)$ are orthogonal at the origin. Draw both graphs and both tangents in a square viewing window.

60. **Writing to Learn** Explain why the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

is not simply the well-known rule for multiplying fractions.

61. ***Running Machinery Too Fast*** Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

Figure 3.45 The internal forces in the engine get so large that they tear the engine apart when the velocity is too great.

62. **Group Activity** ***Temperatures in Fairbanks, Alaska.*** The graph in Figure 3.46 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

(a) On what day is the temperature increasing the fastest?

(b) About how many degrees per day is the temperature increasing when it is increasing at its fastest?

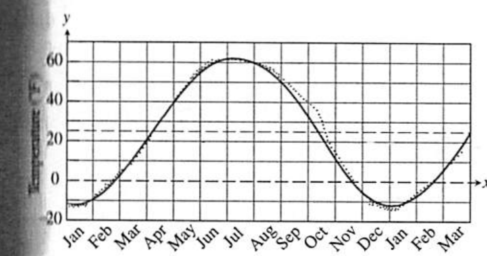


Figure 3.46 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 62).

63. ***Particle Motion*** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

64. ***Constant Acceleration*** Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

65. ***Falling Meteorite*** The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

66. ***Particle Acceleration*** A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

67. ***Temperature and the Period of a Pendulum*** For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

68. **Writing to Learn** ***Chain Rule*** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f\circ g)(x) = |x|^2 = x^2 \quad\text{and}\quad (g\circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

69. ***Tangents*** Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

70. **True or False** $\dfrac{d}{dx}(\sin x) = \cos x$, if x is measured in degrees or radians. Justify your answer.

71. **True or False** The slope of the normal line to the curve $x = 3\cos t$, $y = 3\sin t$ at $t = \pi/4$ is -1. Justify your answer.

72. **Multiple Choice** Which of the following is dy/dx if $y = \tan(4x)$?

(A) $4\sec(4x)\tan(4x)$ (B) $\sec(4x)\tan(4x)$ (C) $4\cot(4x)$

(D) $\sec^2(4x)$ (E) $4\sec^2(4x)$

73. **Multiple Choice** Which of the following is dy/dx if $y = \cos^2(x^3 + x^2)$?

(A) $-2(3x^2 + 2x)$

(B) $-(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(C) $-2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(D) $2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

(E) $2(3x^2 + 2x)$

In Exercises 74 and 75, use the curve defined by the parametric equations $x = t - \cos t$, $y = -1 + \sin t$.

74. **Multiple Choice** Which of the following is an equation of the tangent line to the curve at $t = 0$?

(A) $y = x$ (B) $y = -x$ (C) $y = x + 2$

(D) $y = x - 2$ (E) $y = -x - 2$

75. **Multiple Choice** At which of the following values of t is $dy/dx = 0$?

(A) $t = \pi/4$ (B) $t = \pi/2$ (C) $t = 3\pi/4$

(D) $t = \pi$ (E) $t = 2\pi$

Explorations

76. ***The Derivative of sin 2x*** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x+h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

77. ***The Derivative of cos (x^2)*** Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on screen, graph

$$y = \frac{\cos[(x+h)^2] - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

Extending the Ideas

78. ***Absolute Value Functions*** Let u be a differentiable functi of x.

(a) Show that $\frac{d}{dx}|u| = u'\frac{u}{|u|}$.

(b) Use part (a) to find the derivatives of $f(x) = |x^2 - 9|$ and $g(x) = |x| \sin x$.

79. ***Geometric and Arithmetic Mean*** The geometric mean of u and v is $G = \sqrt{uv}$ and the arithmetic mean is $A = (u+v)/2$. Show that if $u = x$, $v = x + c$, c a real number, then

$$\frac{dG}{dx} = \frac{A}{G}.$$

Quick Quiz for AP* Preparation: Sections 3.4–3.6

You should solve the following problems without using a graphing calculator.

1. **Multiple Choice** Which of the following gives dy/dx for $y = \sin^4(3x)$?

(A) $4\sin^3(3x)\cos(3x)$ (B) $12\sin^3(3x)\cos(3x)$
(C) $12\sin(3x)\cos(3x)$ (D) $12\sin^3(3x)$
(E) $-12\sin^3(3x)\cos(3x)$

2. **Multiple Choice** Which of the following gives y'' for $y = \cos x + \tan x$?

(A) $-\cos x + 2\sec^2 x \tan x$ (B) $\cos x + 2\sec^2 x \tan x$
(C) $-\sin x + \sec^2 x$ (D) $-\cos x + \sec^2 x \tan x$
(E) $\cos x + \sec^2 x \tan x$

3. **Multiple Choice** Which of the following gives dy/dx for the parametric curve $x = 3\sin t$, $y = 2\cos t$?

(A) $-\frac{3}{2}\cot t$ (B) $\frac{3}{2}\cot t$ (C) $-\frac{2}{3}\tan t$ (D) $\frac{2}{3}\tan t$ (E) tan

4. **Free Response** A particle moves along a line so that its position at any time $t \ge 0$ is given by $s(t) = -t^2 + t + 2$, wher s is measured in meters and t is measured in seconds.

(a) What is the initial position of the particle?

(b) Find the velocity of the particle at any time t.

(c) When is the particle moving to the right?

(d) Find the acceleration of the particle at any time t.

(e) Find the speed of the particle at the moment when $s(t) = 0$.

3.7 Implicit Differentiation

hat you'll learn about

- mplicitly Defined Functions
- enses, Tangents, and Normal ines
- erivatives of Higher Order
- ational Powers of Differentiable unctions

and why

mplicit differentiation allows us find derivatives of functions at are not defined or written plicitly as a function of a single riable.

Implicitly Defined Functions

The graph of the equation $x^3 + y^3 - 9xy = 0$ (Figure 3.47) has a well-defined slope nearly every point because it is the union of the graphs of the functions $y = f_1($ $y = f_2(x)$, and $y = f_3(x)$, which are differentiable except at O and A. But how do we fi the slope when we cannot conveniently solve the equation to find the functions? The a swer is to treat y as a differentiable function of x and differentiate both sides of the equ tion with respect to x, using the differentiation rules for sums, products, and quotients, a the Chain Rule. Then solve for dy/dx in terms of x and y *together* to obtain a formula th calculates the slope at any point (x, y) on the graph from the values of x and y.

The process by which we find dy/dx is called **implicit differentiation.** The phrase derives from the fact that the equation

$$x^3 + y^3 - 9xy = 0$$

defines the functions f_1, f_2, and f_3 implicitly (i.e., hidden inside the equation), without giving us *explicit* formulas to work with.

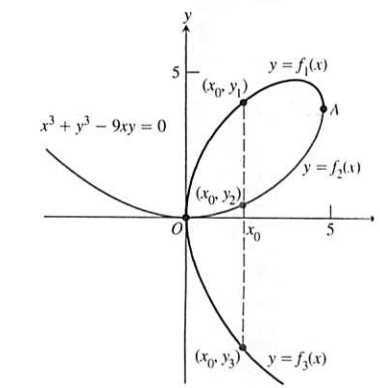


Figure 3.47 The graph of $x^3 + y^3 - 9xy = 0$ (called a *folium*). Although not the graph of a function, it is the union of the graphs of three separate functions. This particular curve dates to Descartes in 1638.

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x$.

SOLUTION

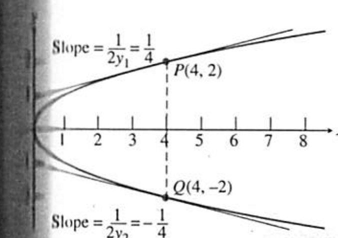
To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating y as a differentiable function of x and applying the Chain Rule:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1 \qquad \frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Now try Exercise 3



ure 3.48 The derivative found in mple 1 gives the slope for the tangent at both P and Q, because it is a func- of y.

In the previous example we differentiated with respect to x, and yet the derivative w obtained appeared as a function of y. Not only is this acceptable, it is actually quite usefu Figure 3.48, for example, shows that the curve has two different tangent lines when $x =$ one at the point $(4, 2)$ and the other at the point $(4, -2)$. Since the formula for dy/dx d pends on y, our single formula gives the slope in both cases.

Implicit differentiation will frequently yield a derivative that is expressed in terms both x and y, as in Example 2.

EXAMPLE 6 Using the Rational Power Rule

(a) $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

Notice that $\sqrt{x}$ is defined at $x = 0$, but $1/(2\sqrt{x})$ is not.

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}(x^{-1/3}) = \dfrac{2}{3x^{1/3}}$

The original function is defined for all real numbers, but the derivative is undefined at $x = 0$. Recall Figure 3.12, which showed that this function's graph has a *cusp* at $x = 0$.

(c) $\dfrac{d}{dx}(\cos x)^{-1/5} = -\dfrac{1}{5}(\cos x)^{-6/5} \cdot \dfrac{d}{dx}(\cos x)$

$= -\dfrac{1}{5}(\cos x)^{-6/5}(-\sin x)$

$= \dfrac{1}{5}\sin x(\cos x)^{-6/5}$

Now try Exercise 3

Quick Review 3.7 *(For help, go to Section 1.2 and Appendix A.5.)*

In Exercises 1–5, sketch the curve defined by the equation and find two functions y_1 and y_2 whose graphs will combine to give the curve.

1. $x - y^2 = 0$ **2.** $4x^2 + 9y^2 = 36$

3. $x^2 - 4y^2 = 0$ **4.** $x^2 + y^2 = 9$

5. $x^2 + y^2 = 2x + 3$

In Exercises 6–8, solve for y' in terms of y and x.

6. $x^2y' - 2xy = 4x - y$

7. $y'\sin x - x\cos x = xy' + y$

8. $x(y^2 - y') = y'(x^2 - y)$

In Exercises 9 and 10, find an expression for the function using rational powers rather than radicals.

9. $\sqrt{x}(x - \sqrt[3]{x})$ **10.** $\dfrac{x + \sqrt[3]{x^2}}{\sqrt{x^3}}$

Section 3.7 Exercises

In Exercises 1–8, find dy/dx.

1. $x^2y + xy^2 = 6$ **2.** $x^3 + y^3 = 18xy$

3. $y^2 = \dfrac{x-1}{x+1}$ **4.** $x^2 = \dfrac{x-y}{x+y}$

5. $x = \tan y$ **6.** $x = \sin y$

7. $x + \tan(xy) = 0$ **8.** $x + \sin y = xy$

In Exercises 9–12, find dy/dx and find the slope of the curve at the indicated point.

9. $x^2 + y^2 = 13$, $(-2, 3)$

10. $x^2 + y^2 = 9$, $(0, 3)$

11. $(x-1)^2 + (y-1)^2 = 13$, $(3, 4)$

12. $(x+2)^2 + (y+3)^2 = 25$, $(1, -7)$

In Exercises 13–16, find where the slope of the curve is defined.

13. $x^2y - xy^2 = 4$ **14.** $x = \cos y$

15. $x^3 + y^3 = xy$ **16.** $x^2 + 4xy + 4y^2 - 3x = 6$

In Exercises 17–26, find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

17. $x^2 + xy - y^2 = 1$, $(2, 3)$

18. $x^2 + y^2 = 25$, $(3, -4)$

19. $x^2y^2 = 9$, $(-1, 3)$

20. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$

21. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$

22. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$

23. $2xy + \pi\sin y = 2\pi$, $(1, \pi/2)$

24. $x\sin 2y = y\cos 2x$, $(\pi/4, \pi/2)$

25. $y = 2\sin(\pi x - y)$, $(1, 0)$

26. $x^2\cos^2 y - \sin y = 0$, $(0, \pi)$

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

27. $x^2 + y^2 = 1$ **28.** $x^{2/3} + y^{2/3} = 1$

29. $y^2 = x^2 + 2x$ **30.** $y^2 + 2y = 2x + 1$

In Exercises 31–42, find dy/dx.

31. $y = x^{9/4}$ **32.** $y = x^{-3/5}$

33. $y = \sqrt[3]{x}$ **34.** $y = \sqrt[4]{x}$

35. $y = (2x+5)^{-1/2}$ **36.** $y = (1-6x)^{2/3}$

37. $y = x\sqrt{x^2+1}$ **38.** $y = \dfrac{x}{\sqrt{x^2+1}}$

39. $y = \sqrt{1 - \sqrt{x}}$ **40.** $y = 3(2x^{-1/2} + 1)^{-1/3}$

41. $y = 3(\csc x)^{3/2}$ **42.** $y = [\sin(x+5)]^{5/4}$

Which of the following could be true if $f''(x) = x^{-1/3}$?

(a) $f(x) = \dfrac{3}{2}x^{2/3} - 3$ **(b)** $f(x) = \dfrac{9}{10}x^{5/3} - 7$

(c) $f'''(x) = -\dfrac{1}{3}x^{-4/3}$ **(d)** $f'(x) = \dfrac{3}{2}x^{2/3} + 6$

Which of the following could be true if $g''(t) = 1/t^{3/4}$?

(a) $g'(t) = 4\sqrt[4]{t} - 4$ **(b)** $g'''(t) = -4/\sqrt[4]{t}$

(c) $g(t) = t - 7 + (16/5)t^{5/4}$ **(d)** $g'(t) = (1/4)t^{1/4}$

The Eight Curve **(a)** Find the slopes of the figure-eight-shaped curve

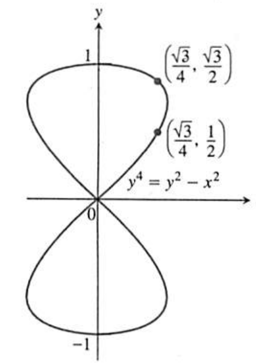
$$y^4 = y^2 - x^2$$

at the two points shown on the graph that follows.

(b) Use parametric mode and the two pairs of parametric equations

$$x_1(t) = \sqrt{t^2 - t^4}, \quad y_1(t) = t,$$
$$x_2(t) = -\sqrt{t^2 - t^4}, \quad y_2(t) = t,$$

to graph the curve. Specify a window and a parameter interval.



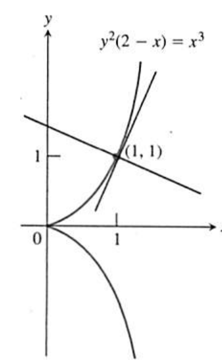
The Cissoid of Diocles (dates from about 200 B.C.)

(a) Find equations for the tangent and normal to the cissoid of Diocles,

$$y^2(2 - x) = x^3,$$

at the point (1, 1) as pictured below.

(b) Explain how to reproduce the graph on a grapher.



47. **(a)** Confirm that $(-1, 1)$ is on the curve defined by $x^3y^2 = \cos(\pi y)$.

(b) Use part (a) to find the slope of the line tangent to the curve at $(-1, 1)$.

48. **Grouping Activity**

(a) Show that the relation

$$y^3 - xy = -1$$

cannot be a function of x by showing that there is more than one possible y-value when $x = 2$.

(b) On a small enough square with center (2, 1), the part of the graph of the relation within the square will define a function $y = f(x)$. For this function, find $f'(2)$ and $f''(2)$.

49. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

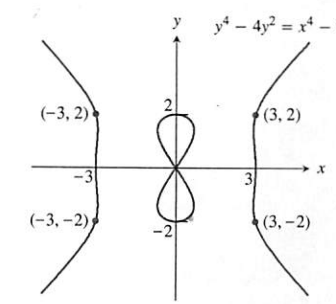
50. Find points on the curve $x^2 + xy + y^2 = 7$ **(a)** where the tangent is parallel to the x-axis and **(b)** where the tangent is parallel to the y-axis. (In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?)

51. ***Orthogonal Curves*** Two curves are *orthogonal* at a point of intersection if their tangents at that point cross at right angles. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal at (1, 1) and (1, −1). Use parametric mode to draw the curves and to show the tangent lines.

52. The position of a body moving along a coordinate line at time t is $s = (4 + 6t)^{3/2}$, with s in meters and t in seconds. Find the body's velocity and acceleration when $t = 2$ sec.

53. The velocity of a falling body is $v = 8\sqrt{s - t} + 1$ feet per second at the instant t(sec) the body has fallen s feet from its starting point. Show that the body's acceleration is 32 ft/sec².

54. ***The Devil's Curve (Gabriel Cramer [the Cramer of Cramer's Rule], 1750)*** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



55. ***The Folium of Descartes*** (See Figure 3.47 on page 157)

(a) Find the slope of the folium of Descartes, $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

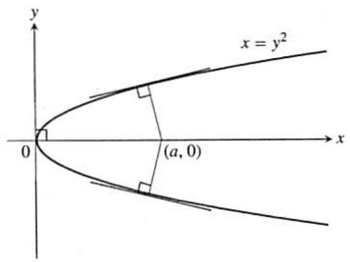
(b) At what point other than the origin does the folium have a horizontal tangent?

(c) Find the coordinates of the point A in Figure 3.47, where the folium has a vertical tangent.

56. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

57. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

58. Show that if it is possible to draw these three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown here, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

59. True or False The slope of $xy^2 + x = 1$ at $(1/2, 1)$ is 2. Justify your answer.

60. True or False The derivative of $y = \sqrt[3]{x}$ is $\frac{1}{3x^{2/3}}$. Justify your answer.

In Exercises 61 and 62, use the curve $x^2 - xy + y^2 = 1$.

61. Multiple Choice Which of the following is equal to dy/dx?

(A) $\frac{y-2x}{2y-x}$ (B) $\frac{y+2x}{2y-x}$

(C) $\frac{2x}{x-2y}$ (D) $\frac{2x+y}{x-2y}$

(E) $\frac{y+2x}{x}$

62. Multiple Choice Which of the following is equal to $\frac{d^2y}{dx^2}$?

(A) $-\frac{6}{(2y-x)^3}$ (B) $\frac{10y^2 - 10x^2 - 10xy}{(2y-x)^3}$

(C) $\frac{8x^2 - 4xy + 8y^2}{(x-2y)^3}$ (D) $\frac{10x^2 + 10y^2}{(x-2y)^3}$

(E) $\frac{2}{x}$

63. Multiple Choice Which of the following is equal to dy/dx if $y = x^{3/4}$?

(A) $\frac{3x^{1/3}}{4}$ (B) $\frac{4x^{1/4}}{3}$ (C) $\frac{3x^{1/4}}{4}$ (D) $\frac{4}{3x^{1/4}}$ (E) $\frac{3}{4x^{1/4}}$

64. Multiple Choice Which of the following is equal to the slope of the tangent to $y^2 - x^2 = 1$ at $(1, \sqrt{2})$?

(A) $-\frac{1}{\sqrt{2}}$ (B) $-\sqrt{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$ (E) 0

Extending the Ideas

65. *Finding Tangents*

(a) Show that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) has equation

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

(b) Find an equation for the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1).

66. *End Behavior Model* Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Show that

(a) $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$.

(b) $g(x) = (b/a)|x|$ is an end behavior model for

$$f(x) = (b/a)\sqrt{x^2 - a^2}.$$

(c) $g(x) = -(b/a)|x|$ is an end behavior model for

$$f(x) = -(b/a)\sqrt{x^2 - a^2}.$$

3.8 Derivatives of Inverse Trigonometric Functions

What you'll learn about

- Derivatives of Inverse Functions
- Derivative of the Arcsine
- Derivative of the Arctangent
- Derivative of the Arcsecant
- Derivatives of the Other Three

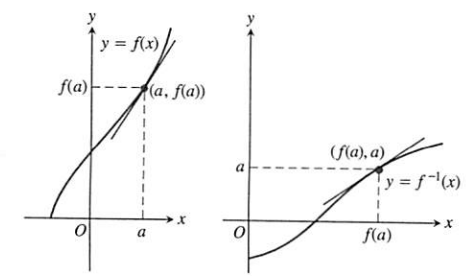
. . . and why

The relationship between the graph of a function and its inverse allows us to see the relationship between their derivatives.

Derivatives of Inverse Functions

In Section 1.5 we learned that the graph of the inverse of a function f can be obtained by reflecting the graph of f across the line $y = x$. If we combine that with our understanding of what makes a function differentiable, we can gain some quick insights into the differentiability of inverse functions.

As Figure 3.52 suggests, the reflection of a continuous curve with no cusps or corners will be another continuous curve with no cusps or corners. Indeed, if there is a tangent line to the graph of f at the point $(a, f(a))$, then that line will reflect across $y = x$ to become a tangent line to the graph of f^{-1} at the point $(f(a), a)$. We can even see geometrically that the *slope* of the reflected tangent line (when it exists and is not zero) will be the *reciprocal* of the slope of the original tangent line, since a change in y becomes a change in x in the reflection, and a change in x becomes a change in y.



The slopes are reciprocal: $\left.\frac{df^{-1}}{dx}\right|_{f(a)} = \frac{1}{\left.\frac{df}{dx}\right|_a}$

Figure 3.52 The graphs of a function and its inverse. Notice that the tangent lines have reciprocal slopes.

All of this serves as an introduction to the following theorem, which we will assume as we proceed to find derivatives of inverse functions. Although the essentials of the proof are illustrated in the geometry of Figure 3.52, a careful analytic proof is more appropriate for an advanced calculus text and will be omitted here.

THEOREM 3 Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f has an inverse and f^{-1} is differentiable at every point of the interval $f(I)$.

Quick Review 3.8 *(For help, go to Sections 1.2, 1.5, and 1.6.)*

In Exercises 1–5, give the *domain* and *range* of the function, and evaluate the function at $x = 1$.

1. $y = \sin^{-1} x$
2. $y = \cos^{-1} x$
3. $y = \tan^{-1} x$
4. $y = \sec^{-1} x$
5. $y = \tan(\tan^{-1} x)$

In Exercises 6–10, find the inverse of the given function.

6. $y = 3x - 8$
7. $y = \sqrt[3]{x + 5}$
8. $y = \dfrac{8}{x}$
9. $y = \dfrac{3x - 2}{x}$
10. $y = \arctan(x/3)$

Section 3.8 Exercises

In Exercises 1–8, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$
2. $y = \cos^{-1}(1/x)$
3. $y = \sin^{-1}\sqrt{2}t$
4. $y = \sin^{-1}(1 - t)$
5. $y = \sin^{-1}\dfrac{3}{t^2}$
6. $y = s\sqrt{1 - s^2} + \cos^{-1} s$
7. $y = x\sin^{-1} x + \sqrt{1 - x^2}$
8. $y = \dfrac{1}{\sin^{-1}(2x)}$

In Exercises 9–12, a particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t)$. Find the velocity at the indicated value of t.

9. $x(t) = \sin^{-1}\left(\dfrac{t}{4}\right)$, $t = 3$
10. $x(t) = \sin^{-1}\left(\dfrac{\sqrt{t}}{4}\right)$, $t = 4$
11. $x(t) = \tan^{-1} t$, $t = 2$
12. $x(t) = \tan^{-1}(t^2)$, $t = 1$

In Exercises 13–22, find the derivatives of y with respect to the appropriate variable.

13. $y = \sec^{-1}(2s + 1)$
14. $y = \sec^{-1} 5s$
15. $y = \csc^{-1}(x^2 + 1)$, $x > 0$
16. $y = \csc^{-1} x/2$
17. $y = \sec^{-1}\dfrac{1}{t}$, $0 < t < 1$
18. $y = \cot^{-1}\sqrt{t}$
19. $y = \cot^{-1}\sqrt{t - 1}$
20. $y = \sqrt{s^2 - 1} - \sec^{-1} s$
21. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x$, $x > 1$
22. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$

In Exercises 23–26, find an equation for the tangent to the graph of y at the indicated point.

23. $y = \sec^{-1} x$, $x = 2$
24. $y = \tan^{-1} x$, $x = 2$
25. $y = \sin^{-1}\left(\dfrac{x}{4}\right)$, $x = 3$
26. $y = \tan^{-1}(x^2)$, $x = 1$

27. (a) Find an equation for the line tangent to the graph of $y = \tan x$ at the point $(\pi/4, 1)$.
    (b) Find an equation for the line tangent to the graph of $y = \tan^{-1} x$ at the point $(1, \pi/4)$.

28. Let $f(x) = x^5 + 2x^3 + x - 1$.
    (a) Find $f(1)$ and $f'(1)$.
    (b) Find $f^{-1}(3)$ and $(f^{-1})'(3)$.

29. Let $f(x) = \cos x + 3x$.
    (a) Show that f has a differentiable inverse.
    (b) Find $f(0)$ and $f'(0)$.
    (c) Find $f^{-1}(1)$ and $(f^{-1})'(1)$.

30. **Group Activity** Graph the function $f(x) = \sin^{-1}(\sin x)$ in the viewing window $[-2\pi, 2\pi]$ by $[-4, 4]$. Then answer the following questions:
    (a) What is the domain of f?
    (b) What is the range of f?
    (c) At which points is f not differentiable?
    (d) Sketch a graph of $y = f'(x)$ without using NDER or computing the derivative.
    (e) Find $f'(x)$ algebraically. Can you reconcile your answer with the graph in part (d)?

31. **Group Activity** A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x = \arctan t$.
    (a) Prove that the particle is always moving to the right.
    (b) Prove that the particle is always decelerating.
    (c) What is the limiting position of the particle as t approaches infinity?

In Exercises 32–34, use the inverse function–inverse cofunction identities to derive the formula for the derivative of the function.

32. arccosine
33. arccotangent
34. arccosecant

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

35. **True or False** The domain of $y = \sin^{-1} x$ is $-1 \leq x \leq 1$. Justify your answer.

36. **True or False** The domain of $y = \tan^{-1} x$ is $-1 \leq x \leq 1$. Justify your answer.

37. **Multiple Choice** Which of the following is $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{2}\right)$?
    (A) $-\dfrac{2}{\sqrt{4 - x^2}}$ (B) $-\dfrac{1}{\sqrt{4 - x^2}}$ (C) $\dfrac{2}{4 + x^2}$
    (D) $\dfrac{2}{\sqrt{4 - x^2}}$ (E) $\dfrac{1}{\sqrt{4 - x^2}}$

38. **Multiple Choice** Which of the following is $\dfrac{d}{dx}\tan^{-1}(3x)$?
    (A) $-\dfrac{3}{1 + 9x^2}$ (B) $-\dfrac{1}{1 + 9x^2}$ (C) $\dfrac{1}{1 + 9x^2}$
    (D) $\dfrac{3}{1 + 9x^2}$ (E) $\dfrac{3}{\sqrt{1 - 9x^2}}$

39. **Multiple Choice** Which of the following is $\dfrac{d}{dx}\sec^{-1}(x^2)$?
    (A) $\dfrac{2}{x\sqrt{x^4 - 1}}$ (B) $\dfrac{2}{x\sqrt{x^2 - 1}}$ (C) $\dfrac{2}{x\sqrt{1 - x^4}}$
    (D) $\dfrac{2}{x\sqrt{1 - x^2}}$ (E) $\dfrac{2x}{\sqrt{1 - x^4}}$

40. **Multiple Choice** Which of the following is the slope of the tangent line to $y = \tan^{-1}(2x)$ at $x = 1$?
    (A) $-2/5$ (B) $1/5$ (C) $2/5$ (D) $5/2$ (E) 5

Explorations

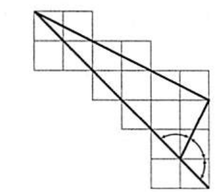
In Exercises 41–46, find (a) the right end behavior model, (b) the left end behavior model, and (c) any horizontal tangents for the function if they exist.

41. $y = \tan^{-1} x$
42. $y = \cot^{-1} x$
43. $y = \sec^{-1} x$
44. $y = \csc^{-1} x$
45. $y = \sin^{-1} x$
46. $y = \cos^{-1} x$

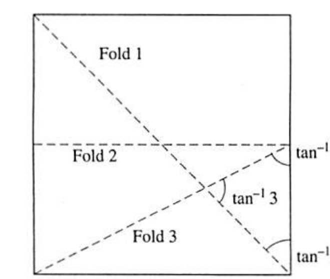
Extending the Ideas

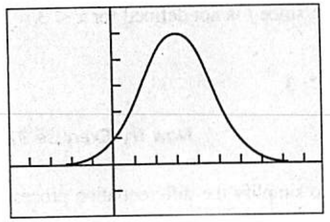
47. ***Identities*** Confirm the following identities for $x > 0$.
    (a) $\cos^{-1} x + \sin^{-1} x = \pi/2$
    (b) $\tan^{-1} x + \cot^{-1} x = \pi/2$
    (c) $\sec^{-1} x + \csc^{-1} x = \pi/2$

48. ***Proof Without Words*** The figure gives a proof without words that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



49. ***(Continuation of Exercise 48)*** Here is a way to construct $\tan^{-1} 1$, $\tan^{-1} 2$, and $\tan^{-1} 3$ by folding a square of paper. Try it and explain what is going on.





[–5, 10] by [–10, 30]

Figure 3.59 The graph of dP/dt, the rate of spread of the flu in Example 8. The graph of P is shown in Figure 3.58.

(b) To find the rate at which the flu spreads, we find dP/dt. To find dP/dt, we need to invoke the Chain Rule twice:

$$\frac{dP}{dt} = \frac{d}{dt}(100(1 + e^{3-t})^{-1}) = 100 \cdot (-1)(1 + e^{3-t})^{-2} \cdot \frac{d}{dt}(1 + e^{3-t})$$

$$= -100(1 + e^{3-t})^{-2} \cdot (0 + e^{3-t} \cdot \frac{d}{dt}(3 - t))$$

$$= -100(1 + e^{3-t})^{-2}(e^{3-t} \cdot (-1))$$

$$= \frac{100e^{3-t}}{(1 + e^{3-t})^2}$$

At $t = 3$, then, $dP/dt = 100/4 = 25$. The flu is spreading to 25 students per day.

(c) We could estimate when the flu is spreading the fastest by seeing where the graph of $y = P(t)$ has the steepest upward slope, but we can answer both the "when" and the "what" parts of this question most easily by finding the maximum point on the graph of the derivative (Figure 3.59).

We see by tracing on the curve that the maximum rate occurs at about 3 days, when (as we have just calculated) the flu is spreading at a rate of 25 students per day.

Now try Exercise 51.

Quick Review 3.9 *(For help, go to Sections 1.3 and 1.5.)*

1. Write $\log_5 8$ in terms of natural logarithms.
2. Write 7^x as a power of e.

In Exercises 3–7, simplify the expression using properties of exponents and logarithms.

3. $\ln(e^{\tan x})$
4. $\ln(x^2 - 4) - \ln(x + 2)$
5. $\log_2(8^{x-5})$
6. $(\log_4 x^{15})/(\log_4 x^{12})$
7. $3\ln x - \ln 3x + \ln(12x^2)$

In Exercises 8–10, solve the equation algebraically using logarithms. Give an *exact* answer, such as $(\ln 2)/3$, and also an approximate answer to the nearest hundredth.

8. $3^x = 19$
9. $5^t \ln 5 = 18$
10. $3^{x+1} = 2^x$

Section 3.9 Exercises

In Exercises 1–28, find dy/dx. Remember that you can use NDER to support your computations.

1. $y = 2e^x$
2. $y = e^{2x}$
3. $y = e^{-x}$
4. $y = e^{-5x}$
5. $y = e^{2x/3}$
6. $y = e^{-x/4}$
7. $y = xe^2 - e^x$
8. $y = x^2e^x - xe^x$
9. $y = e^{\sqrt{x}}$
10. $y = e^{(x^2)}$
11. $y = 8^x$
12. $y = 9^{-x}$
13. $y = 3^{\csc x}$
14. $y = 3^{\cot x}$
15. $y = \ln(x^2)$
16. $y = (\ln x)^2$
17. $y = \ln(1/x)$
18. $y = \ln(10/x)$
19. $y = \ln(\ln x)$
20. $y = x\ln x - x$
21. $y = \log_4 x^2$
22. $y = \log_5 \sqrt{x}$
23. $y = \log_2(1/x)$
24. $y = 1/\log_2 x$
25. $y = \ln 2 \cdot \log_2 x$
26. $y = \log_3(1 + x\ln 3)$
27. $y = \log_{10} e^x$
28. $y = \ln 10^x$

29. At what point on the graph of $y = 3^x + 1$ is the tangent line parallel to the line $y = 5x - 1$?
30. At what point on the graph of $y = 2e^x - 1$ is the tangent line perpendicular to the line $y = -3x + 2$?
31. A line with slope m passes through the origin and is tangent to $y = \ln(2x)$. What is the value of m?
32. A line with slope m passes through the origin and is tangent to $y = \ln(x/3)$. What is the value of m?

In Exercises 33–36, find dy/dx.

33. $y = x^{\pi}$
34. $y = x^{1+\sqrt{2}}$
35. $y = x^{-\sqrt{2}}$
36. $y = x^{1-e}$

In Exercises 37–42, find $f'(x)$ and state the domain of f'.

37. $f(x) = \ln(x + 2)$
38. $f(x) = \ln(2x + 2)$
39. $f(x) = \ln(2 - \cos x)$
40. $f(x) = \ln(x^2 + 1)$
41. $f(x) = \log_2(3x + 1)$
42. $f(x) = \log_{10}\sqrt{x + 1}$

Group Activity In Exercises 43–48, use the technique of logarithmic differentiation to find dy/dx.

43. $y = (\sin x)^x, \quad 0 < x < \pi/2$
44. $y = x^{\tan x}, \quad x > 0$
45. $y = \sqrt[5]{\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}}$
46. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$
47. $y = x^{\ln x}$
48. $y = x^{(1/\ln x)}$

49. Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

50. Find an equation for a line that is normal to the graph of $y = xe^x$ and goes through the origin.

51. ***Spread of a Rumor*** The spread of a rumor in a certain school is modeled by the equation

$$P(t) = \frac{300}{1 + 2^{4-t}},$$

where $P(t)$ is the total number of students who have heard the rumor t days after the rumor first started to spread.

(a) Estimate the initial number of students who first heard the rumor.

(b) How fast is the rumor spreading after 4 days?

(c) When will the rumor spread at its maximum rate? What is that rate?

52. ***Spread of Flu*** The spread of flu in a certain school is modeled by the equation

$$P(t) = \frac{200}{1 + e^{5-t}},$$

where $P(t)$ is the total number of students infected t days after the flu first started to spread.

(a) Estimate the initial number of students infected with this flu.

(b) How fast is the flu spreading after 4 days?

(c) When will the flu spread at its maximum rate? What is that rate?

53. ***Radioactive Decay*** The amount A (in grams) of radioactive plutonium remaining in a 20-gram sample after t days is given by the formula

$$A = 20 \cdot (1/2)^{t/140}.$$

At what rate is the plutonium decaying when $t = 2$ days? Answer in appropriate units.

54. For any positive constant k, the derivative of $\ln(kx)$ is $1/x$. Prove this fact

(a) by using the Chain Rule.

(b) by using a property of logarithms and differentiating.

55. Let $f(x) = 2^x$.

(a) Find $f'(0)$.

(b) Use the definition of the derivative to write $f'(0)$ as a limit.

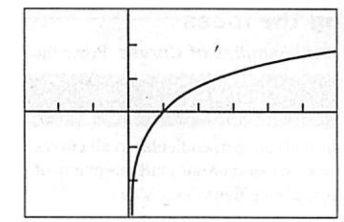
(c) Deduce the exact value of

$$\lim_{h \to 0} \frac{2^h - 1}{h}.$$

(d) What is the exact value of

$$\lim_{h \to 0} \frac{7^h - 1}{h}?$$

56. **Writing to Learn** The graph of $y = \ln x$ looks as though it might be approaching a horizontal asymptote. Write an argument based on the graph of $y = e^x$ to explain why it does not.



[–3, 6] by [–3, 3]

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

57. **True or False** The derivative of $y = 2^x$ is 2^x. Justify your answer.

58. **True or False** The derivative of $y = e^{2x}$ is $2(\ln 2)e^{2x}$. Justify your answer.

59. **Multiple Choice** If a flu is spreading at the rate of

$$P(t) = \frac{150}{1 + e^{4-t}},$$

which of the following is the initial number of persons infected?

(A) 1 (B) 3 (C) 7 (D) 8 (E) 75

60. **Multiple Choice** Which of the following is the domain of $f'(x)$ if $f(x) = \log_2(x + 3)$?

(A) $x < -3$ (B) $x \le 3$ (C) $x \ne -3$ (D) $x > -3$ (E) $x \ge -3$

61. **Multiple Choice** Which of the following gives dy/dx if $y = \log_{10}(2x - 3)$?

(A) $\dfrac{2}{(2x-3)\ln 10}$ (B) $\dfrac{2}{2x-3}$ (C) $\dfrac{1}{(2x-3)\ln 10}$ (D) $\dfrac{1}{2x-3}$ (E) $\dfrac{1}{2x}$

62. **Multiple Choice** Which of the following gives the slope of the tangent line to the graph of $y = 2^{1-x}$ at $x = 2$?

(A) $-\dfrac{1}{2}$ (B) $\dfrac{1}{2}$ (C) -2 (D) 2 (E) $-\dfrac{\ln 2}{2}$

Exploration

63. Let $y_1 = a^x$, $y_2 = \text{NDER}\ y_1$, $y_3 = y_2/y_1$, and $y_4 = e^{y_3}$.

(a) Describe the graph of y_4 for $a = 2, 3, 4, 5$. Generalize your description to an arbitrary $a > 1$.

(b) Describe the graph of y_3 for $a = 2, 3, 4, 5$. Compare a table of values for y_3 for $a = 2, 3, 4, 5$ with $\ln a$. Generalize your description to an arbitrary $a > 1$.

(c) Explain how parts (a) and (b) support the statement

$$\frac{d}{dx}a^x = a^x \quad \text{if and only if} \quad a = e.$$

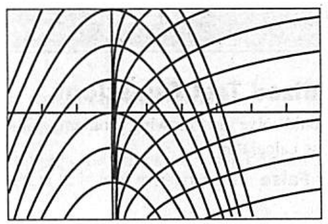
(d) Show algebraically that $y_1 = y_2$ if and only if $a = e$.

Extending the Ideas

64. *Orthogonal Families of Curves* Prove that all curves in the family

$$y = -\frac{1}{2}x^2 + k$$

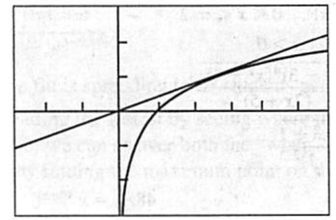
(k any constant) are perpendicular to all curves in the family $y = \ln x + c$ (c any constant) at their points of intersection. (See accompanying figure.)



[–3, 6] by [–3, 3]

65. *Which is Bigger, π^e or e^π?* Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though, by using the result from Example 3 of this section.

Recall from that example that the line through the origin tangent to the graph of $y = \ln x$ has slope $1/e$.



[–3, 6] by [–3, 3]

(a) Find an equation for this tangent line.

(b) Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.

(c) Show that $\ln(x^e) < x$ for all positive $x \neq e$.

(d) Conclude that $x^e < e^x$ for all positive $x \neq e$.

(e) So which is bigger, π^e or e^π?

Quick Quiz for AP* Preparation: Sections 3.7–3.9

You may use a graphing calculator to solve the following problems.

1. Multiple Choice Which of the following gives dy/dx at $x = 1$ if $x^3 + 2xy = 9$?

(A) 11/2 (B) 5/2 (C) 3/2 (D) −5/2 (E) −11/2

2. Multiple Choice Which of the following gives dy/dx if $y = \cos^3(3x - 2)$?

(A) $-9\cos^2(3x-2)\sin(3x-2)$

(B) $-3\cos^2(3x-2)\sin(3x-2)$

(C) $9\cos^2(3x-2)\sin(3x-2)$

(D) $-9\cos^2(3x-2)$

(E) $-3\cos^2(3x-2)$

3. Multiple Choice Which of the following gives dy/dx if $y = \sin^{-1}(2x)$?

(A) $-\dfrac{2}{\sqrt{1-4x^2}}$ (B) $-\dfrac{1}{\sqrt{1-4x^2}}$ (C) $\dfrac{2}{\sqrt{1-4x^2}}$

(D) $\dfrac{1}{\sqrt{1-4x^2}}$ (E) $\dfrac{2x}{1+4x^2}$

4. Free Response A curve in the xy-plane is defined by $xy^2 - x^3y = 6$.

(a) Find dy/dx.

(b) Find an equation for the tangent line at each point on the curve with x-coordinate 1.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

lculus at Work

rk at Ramsey County Hospital and her community hospitals in the Minapolis area, both with patients and in oratory. I have wanted to be a physince I was about 12 years old, and I an attending medical school when I 30 years old. I am now working in the of internal medicine.

Cardiac patients are common in my , especially in the diagnostic stages. of the machines that is sometimes used in the emergency room to diagnose problems is called a Swan-Ganz catheter, named after its inventors Harold James Swan and William Ganz. The catheter is inserted into the pulmonary artery and then is hooked up to a cardiac monitor. A program measures cardiac output by looking at changes of slope in the curve. This information alerts me to left-sided heart failure.

Lupe Bolding, M.D.
Ramsey County Hospital
Minneapolis, MN

apter 3 Key Terms

eration (p. 130)
velocity (p. 128)
Rule (p. 149)
ant Multiple Rule (p. 117)
ative of a Constant Function (p. 116)
ative of f at a (p. 99)
entiable function (p. 99)
entiable on a closed interval (p. 104)
lacement (p. 128)
fall constants (p. 130)
licit differentiation (p. 157)
antaneous rate of change (p. 127)
antaneous velocity (p. 128)
mediate Value Theorem for Derivatives (p. 113)
inverse function–inverse cofunction identities (p. 168)
jerk (p. 144)
left-hand derivative (p. 104)
local linearity (p. 110)
logarithmic differentiation (p. 177)
marginal cost (p. 134)
marginal revenue (p. 134)
nth derivative (p. 122)
normal to the surface (p. 159)
numerical derivative (NDER) (p. 111)
orthogonal curves (p. 154)
orthogonal families (p. 180)
Power Chain Rule (p. 151)
Power Rule for Arbitrary Real Powers (p. 176)
Power Rule for Negative Integer Powers of x (p. 121)
Power Rule for Positive Integer Powers of x (p. 116)
Power Rule for Rational Powers of x (p. 161)
Product Rule (p. 119)
Quotient Rule (p. 120)
right-hand derivative (p. 104)
sensitivity to change (p. 133)
simple harmonic motion (p. 143)
speed (p. 129)
Sum and Difference Rule (p. 117)
symmetric difference quotient (p. 111)
velocity (p. 128)

apter 3 Review Exercises

collection of exercises marked in red could be used as a chapter

ercises 1–30, find the derivative of the function.

1. $y = x^5 - \frac{1}{8}x^2 + \frac{1}{4}x$
2. $y = 3 - 7x^3 + 3x^7$
3. $y = 2\sin x\cos x$
4. $y = \dfrac{2x+1}{2x-1}$
5. $= \cos(1 - 2t)$
6. $s = \cot\dfrac{2}{t}$
7. $y = \sqrt{x} + 1 + \dfrac{1}{\sqrt{x}}$
8. $y = x\sqrt{2x+1}$
9. $r = \sec(1 + 3\theta)$
10. $r = \tan^2(3 - \theta^2)$
11. $y = x^2\csc 5x$
12. $y = \ln\sqrt{x}$
13. $y = \ln(1 + e^x)$
14. $y = xe^{-x}$
15. $y = e^{(1+\ln x)}$
16. $y = \ln(\sin x)$
17. $r = \ln(\cos^{-1}x)$
18. $r = \log_2(\theta^2)$
19. $s = \log_5(t - 7)$
20. $s = 8^{-t}$
21. $y = x^{\ln x}$
22. $y = \dfrac{(2x)2^x}{\sqrt{x^2+1}}$
23. $y = e^{\tan^{-1}x}$
24. $y = \sin^{-1}\sqrt{1-u^2}$
25. $y = t\sec^{-1}t - \frac{1}{2}\ln t$
26. $y = (1 + t^2)\cot^{-1}2t$
27. $y = z\cos^{-1}z - \sqrt{1-z^2}$
28. $y = 2\sqrt{x-1}\csc^{-1}\sqrt{x}$

29. $y = \csc^{-1}(\sec x),\ 0 \le x \le 2\pi$

30. $r = \left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

In Exercises 31–34, find all values of x for which the function is differentiable.

31. $y = \ln x^2$

32. $y = \sin x - x\cos x$

33. $y = \sqrt{\dfrac{1 - x}{1 + x^2}}$

34. $y = (2x - 7)^{-1}(x + 5)$

In Exercises 35–38, find dy/dx.

35. $xy + 2x + 3y = 1$

36. $5x^{4/5} + 10y^{6/5} = 15$

37. $\sqrt{xy} = 1$

38. $y^2 = \dfrac{x}{x + 1}$

In Exercises 39–42, find d^2y/dx^2 by implicit differentiation.

39. $x^3 + y^3 = 1$

40. $y^2 = 1 - \dfrac{2}{x}$

41. $y^3 + y = 2\cos x$

42. $x^{1/3} + y^{1/3} = 4$

In Exercises 43 and 44, find all derivatives of the function.

43. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

44. $y = \dfrac{x^5}{120}$

In Exercises 45–48, find an equation for the **(a)** tangent and **(b)** normal to the curve at the indicated point.

45. $y = \sqrt{x^2 - 2x},\quad x = 3$

46. $y = 4 + \cot x - 2\csc x,\quad x = \pi/2$

47. $x^2 + 2y^2 = 9,\quad (1, 2)$

48. $x + \sqrt{xy} = 6,\quad (4, 1)$

In Exercises 49–52, find an equation for the line tangent to the curve at the point defined by the given value of t.

49. $x = 2\sin t,\quad y = 2\cos t,\quad t = 3\pi/4$

50. $x = 3\cos t,\quad y = 4\sin t,\quad t = 3\pi/4$

51. $x = 3\sec t,\quad y = 5\tan t,\quad t = \pi/6$

52. $x = \cos t,\quad y = t + \sin t,\quad t = -\pi/4$

53. **Writing to Learn**

(a) Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

(b) Is f continuous at $x = 1$? Explain.

(c) Is f differentiable at $x = 1$? Explain.

54. **Writing to Learn** For what values of the constant m is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

(a) continuous at $x = 0$? Explain.

(b) differentiable at $x = 0$? Explain.

In Exercises 55–58, determine where the function is **(a)** differentiable, **(b)** continuous but not differentiable, and **(c)** neither continuous nor differentiable.

55. $f(x) = x^{4/5}$

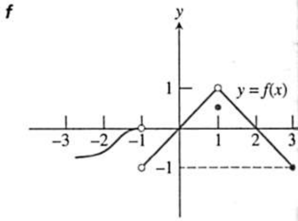
56. $g(x) = \sin(x^2 + 1)$

57. $f(x) = \begin{cases} 2x - 3, & -1 \le x < 0 \\ x - 3, & 0 \le x \le 4 \end{cases}$

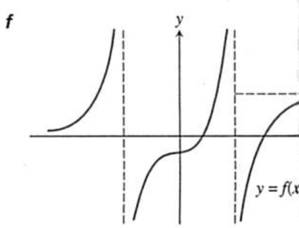
58. $g(x) = \begin{cases} \dfrac{x - 1}{x}, & -2 \le x < 0 \\ \dfrac{x + 1}{x}, & 0 \le x \le 2 \end{cases}$

In Exercises 59 and 60, use the graph of f to sketch the graph of f'.

59. ***Sketching f′ from f***

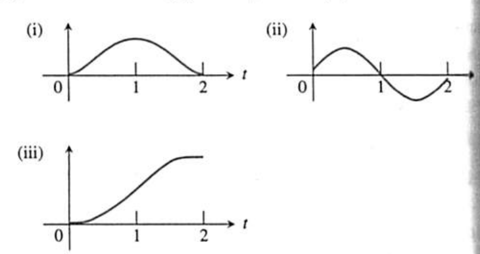


60. ***Sketching f′ from f***



61. ***Recognizing Graphs*** The following graphs show the distance traveled, velocity, and acceleration for each second of a 2-minute automobile trip. Which graph shows

(a) distance ? (b) velocity? (c) acceleration?



62. ***Sketching f from f′*** Sketch the graph of a continuous function f with $f(0) = 5$ and

$$f'(x) = \begin{cases} -2, & x < 2 \\ -0.5, & x > 2. \end{cases}$$

63. ***Sketching f from f′*** Sketch the graph of a continuous function f with $f(-1) = 2$ and

$$f'(x) = \begin{cases} -2, & x < 1 \\ 1, & 1 < x < 4 \\ -1, & 4 < x < 6. \end{cases}$$

Which of the following statements could be true if $f''(x) = x^{1/3}$?

i. $f(x) = \dfrac{9}{28}x^{7/3} + 9$

ii. $f'(x) = \dfrac{9}{28}x^{7/3} - 2$

iii. $f'(x) = \dfrac{3}{4}x^{4/3} + 6$

iv. $f(x) = \dfrac{3}{4}x^{4/3} - 4$

A. i only

B. iii only

C. ii and iv only

D. i and iii only

Derivative from Data The following data give the coordinates of a moving body for various values of t.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4
s (ft)	10	38	58	70	74	70	58	38	10

(a) Make a scatter plot of the (t, s) data and sketch a smooth curve through the points.

(b) Compute the average velocity between consecutive points of the table.

(c) Make a scatter plot of the data in part (b) using the midpoints of the t values to represent the data. Then sketch a smooth curve through the points.

(d) **Writing to Learn** Why does the curve in part (c) approximate the graph of ds/dt?

Working with Numerical Values Suppose that a function f and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	−2
1	−3	1/5

Find the first derivative of the following combinations at the given value of x.

(a) $\sqrt{x}f(x),\quad x = 1$

(b) $\sqrt{f(x)},\quad x = 0$

(c) $f(\sqrt{x}),\quad x = 1$

(d) $f(1 - 5\tan x),\quad x = 0$

(e) $\dfrac{f(x)}{2 + \cos x},\quad x = 0$

(f) $10\sin\left(\dfrac{\pi x}{2}\right)f^2(x),\quad x = 1$

Working with Numerical Values Suppose that functions f and g and their first derivatives have the following values at $x = -1$ and $x = 0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−1	0	−1	2	1
0	−1	−3	−2	4

Find the first derivative of the following combinations at the given value of x.

(a) $3f(x) - g(x),\quad x = -1$

(b) $f^2(x)g^3(x),\quad x = 0$

(c) $g(f(x)),\quad x = -1$

(d) $f(g(x)),\quad x = -1$

(e) $\dfrac{f(x)}{g(x) + 2},\quad x = 0$

(f) $g(x + f(x)),\quad x = 0$

68. Find the value of dw/ds at $s = 0$ if $w = \sin(\sqrt{r} - 2)$ and $r = 8\sin(s + \pi/6)$.

69. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

70. ***Particle Motion*** The position at time $t \ge 0$ of a particle moving along the s-axis is

$$s(t) = 10\cos(t + \pi/4).$$

(a) Give parametric equations that can be used to simulate the motion of the particle.

(b) What is the particle's initial position $(t = 0)$?

(c) What points reached by the particle are farthest to the left and right of the origin?

(d) When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

71. ***Vertical Motion*** On Earth, if you shoot a paper clip 64 ft straight up into the air with a rubber band, the paper clip will be $s(t) = 64t - 16t^2$ feet above your hand at t sec after firing.

(a) Find ds/dt and d^2s/dt^2.

(b) How long does it take the paper clip to reach its maximum height?

(c) With what velocity does it leave your hand?

(d) On the moon, the same force will send the paper clip to a height of $s(t) = 64t - 2.6t^2$ ft in t sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?

72. ***Free Fall*** Suppose two balls are falling from rest at a certain height in centimeters above the ground. Use the equation $s = 490t^2$ to answer the following questions.

(a) How long does it take the balls to fall the first 160 cm? What is their average velocity for the period?

(b) How fast are the balls falling when they reach the 160-cm mark? What is their acceleration then?

73. ***Filling a Bowl*** If a hemispherical bowl of radius 10 in. is filled with water to a depth of x in., the volume of water is given by $V = \pi[10 - (x/3)]x^2$. Find the rate of increase of the volume per inch increase of depth.

74. ***Marginal Revenue*** A bus will hold 60 people. The fare charged (p dollars) is related to the number x of people who use the bus by the formula $p = [3 - (x/40)]^2$.

(a) Write a formula for the total revenue per trip received by the bus company.

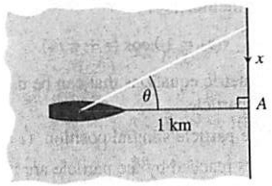
(b) What number of people per trip will make the marginal revenue equal to zero? What is the corresponding fare?

(c) **Writing to Learn** Do you think the bus company's fare policy is good for its business?

75. *Searchlight* The figure shows a boat 1 km offshore sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

(a) How fast is the light moving along the shore when it reaches point A?

(b) How many revolutions per minute is 0.6 rad/sec?



76. *Horizontal Tangents* The graph of $y = \sin(x - \sin x)$ appears to have horizontal tangents at the x-axis. Does it?

77. *Fundamental Frequency of a Vibrating Piano String* We measure the frequencies at which wires vibrate in cycles (trips back and forth) per sec. The unit of measure is a *hertz:* 1 cycle per sec. Middle A on a piano has a frequency 440 hertz. For any given wire, the fundamental frequency y is a function of four variables:

r: the radius of the wire;

l: the length;

d: the density of the wire;

T: the tension (force) holding the wire taut.

With r and l in centimeters, d in grams per cubic centimeter, and T in dynes (it takes about 100,000 dynes to lift an apple), the fundamental frequency of the wire is

$$y = \frac{1}{2rl}\sqrt{\frac{T}{\pi d}}.$$

If we keep all the variables fixed except one, then y can be alternatively thought of as four different functions of one variable, $y(r)$, $y(l)$, $y(d)$, and $y(T)$. How would changing each variable affect the string's fundamental frequency? To find out, calculate $y'(r)$, $y'(l)$, $y'(d)$, and $y'(T)$.

78. *Spread of Measles* The spread of measles in a certain school is given by

$$P(t) = \frac{200}{1 + e^{5-t}},$$

where t is the number of days since the measles first appeared, and $P(t)$ is the total number of students who have caught the measles to date.

(a) Estimate the initial number of students infected with measles.

(b) About how many students in all will get the measles?

(c) When will the rate of spread of measles be greatest? What is this rate?

79. Graph the function $f(x) = \tan^{-1}(\tan 2x)$ in the window $[-\pi, \pi]$ by $[-4, 4]$. Then answer the following questions.

(a) What is the domain of f?

(b) What is the range of f?

(c) At which points is f not differentiable?

(d) Describe the graph of f'.

80. If $x^2 - y^2 = 1$, find d^2y/dx^2 at the point $(2, \sqrt{3})$.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

81. A particle moves along the x-axis so that at any time $t \geq 0$ its position is given by $x(t) = t^3 - 12t + 5$.

(a) Find the velocity of the particle at any time t.

(b) Find the acceleration of the particle at any time t.

(c) Find all values of t for which the particle is at rest.

(d) Find the speed of the particle when its acceleration is zero.

(e) Is the particle moving toward the origin or away from the origin when $t = 3$? Justify your answer.

82. Let $y = \dfrac{e^x + e^{-x}}{2}$.

(a) Find $\dfrac{dy}{dx}$.

(b) Find $\dfrac{d^2y}{dx^2}$.

(c) Find an equation of the line tangent to the curve at $x = 1$.

(d) Find an equation of the line normal to the curve at $x = 1$.

(e) Find any points where the tangent line is horizontal.

83. Let $f(x) = \ln(1 - x^2)$.

(a) State the domain of f.

(b) Find $f'(x)$.

(c) State the domain of f'.

(d) Prove that $f''(x) < 0$ for all x in the domain of f.