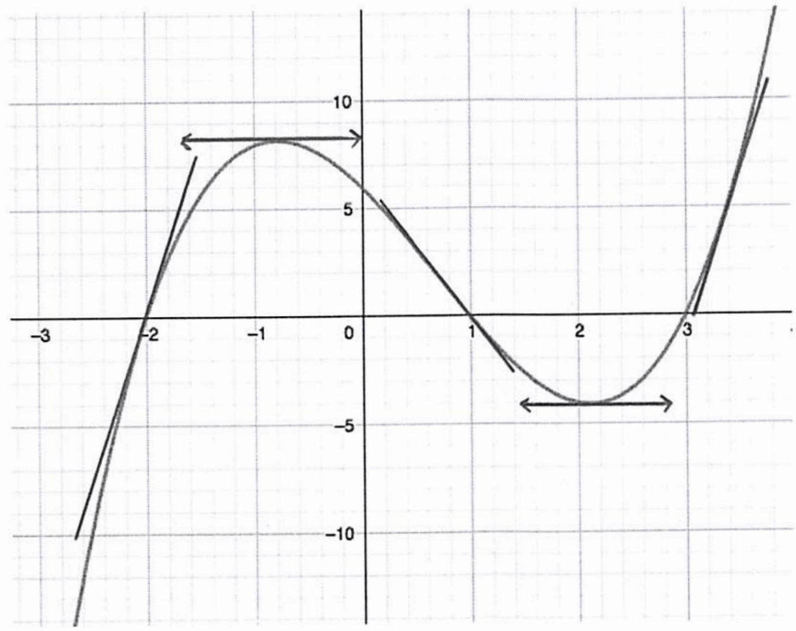
# 3.1 – Derivative of a function

In the last chapter, we used a limit to determine the slope of a tangent line. The slope of a tangent line is a precious information to have about a curve. It tells us if the curve is increasing or decreasing (sign of the slope) and it helps us find out maximum or minimum points.




The derivative of a function is another function that gives us for each value of $x$ in the domain the slope of the tangent line at that point. It is noted $f'$.

Definition of the **derivative of a function** $f$:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},}$$ provided the limit exists.

Anywhere that the derivative exists, we say that the function is **differentiable**.

Examples: Use the definition of the derivative to determine the derivative of the following functions:

a) $f(x) = 3x - 2$

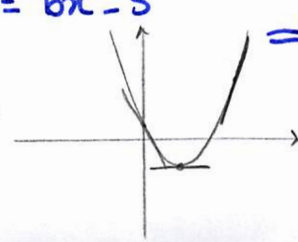
$$\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)-2-(3x-2)}{h} = \frac{3x+3h-2-3x+2}{h} = \frac{3h}{h}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 3 \quad \Rightarrow \quad f'(x) = 3$$ (look at the graph!)

b) $g(x) = 3x^2 - 5x + 1$

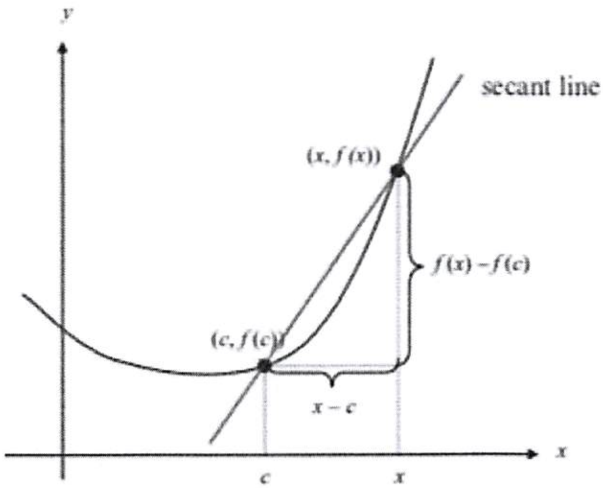
$$\frac{g(x+h)-g(x)}{h} = \frac{3(x+h)^2-5(x+h)+1-(3x^2-5x+1)}{h} = \frac{6xh+3h^2-5h}{h} = 6x+3h-5$$

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = 6x-5 \quad \Rightarrow \quad g'(x) = 6x-5$$



Note: Other notation for the derivative: $y'$, $\frac{dy}{dx}$, $\frac{d}{dx}f(x)$, $\frac{df}{dx}$

Alternate definition of the derivative: $\boxed{f'(x) = \lim_{c \to x} \frac{f(x)-f(c)}{x-c}}$ , provided the limit exists.




Example: Use the alternate definition of the derivative to determine the derivative of $f(x) = \sqrt{x}$

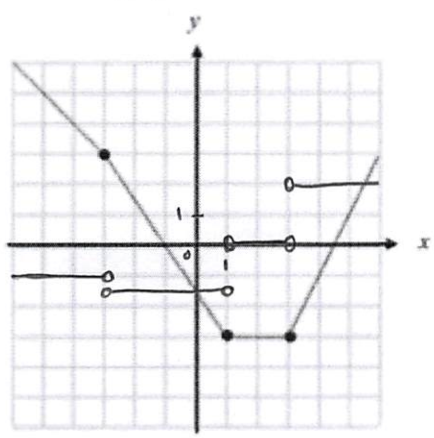
$x \in [0; +\infty)$
$c \in [0; +\infty)$
$x \neq c$

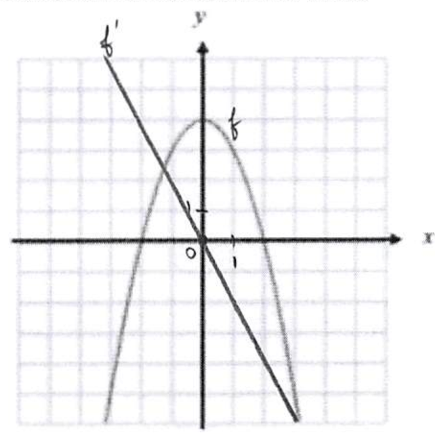
$$\frac{f(x)-f(c)}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})}{(x-c)(\sqrt{x}+\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}$$

$$\lim_{c \to x} \frac{f(x)-f(c)}{x-c} = \frac{1}{2\sqrt{x}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

The graphs of $f$ and $f$‘ are related but don’t look similar at all…

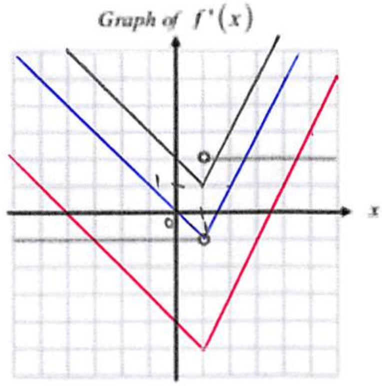
Examples: Given the graphs of these two functions, graph their derivatives on the same axis:



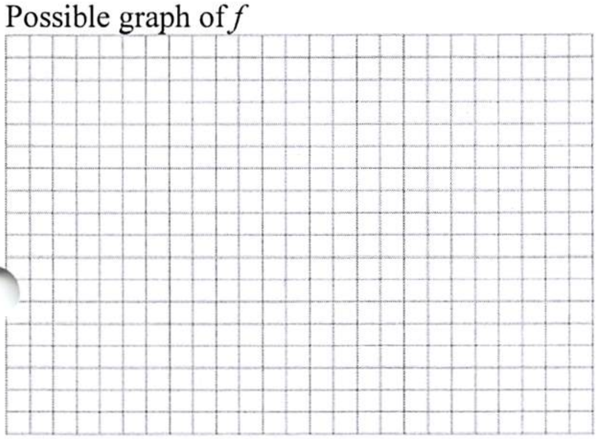



From the graph of the function it is possible to graph its derivative, but from the graph of a derivative, there are several possible functions…

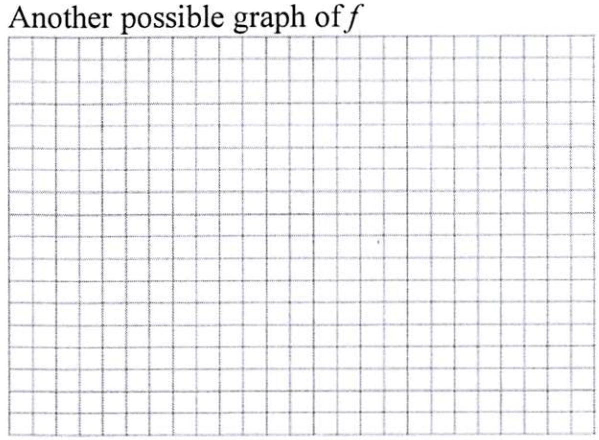
Example:



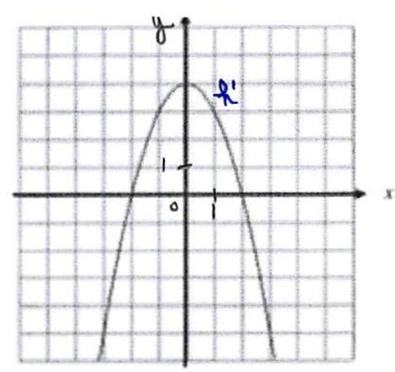

Possible graph of $f$



Another possible graph of $f$



<u>Another example</u>: Suppose the graph below is the graph of the derivative of $h$.




a) What is the value of $h'(0)$? What does this tell us about $h(x)$?

$h'(0) = 4$ when $x$ is 0, the slope of the graph of $h$ is 4.

b) Using the graph of $h'(x)$, how can we determine when the graph of $h(x)$ is going up? How about going down?

$h$ is going up if $h'$ is positive: on $(-2;2)$

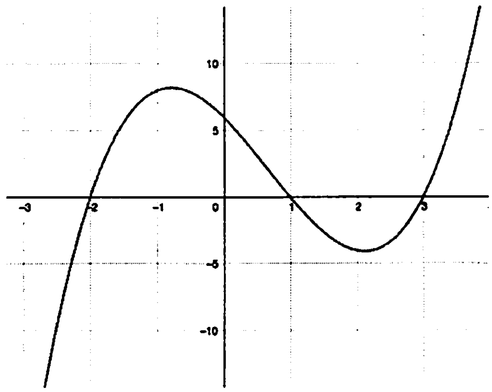
$h$ is going down if $h'$ is negative: on $(-\infty;-2)$ and $(2;+\infty)$

c) The graph of $h'(x)$ crosses the $x$-axis at $x = 2$ and $x = -2$. Describe the behavior of the graph of $h(x)$ at these points.

when $x = 2$ and $x = -2$, the graph of $h$ will have horizontal tangent lines. It's when $h$ changes direction.

We can already remember that:

| |
|---|
| When the slope of the tangent line (the derivative) is positive, the function is increasing.<br>When the slope of the tangent line (the derivative) is negative, the function is decreasing. |

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## 3.2 - Differentiability

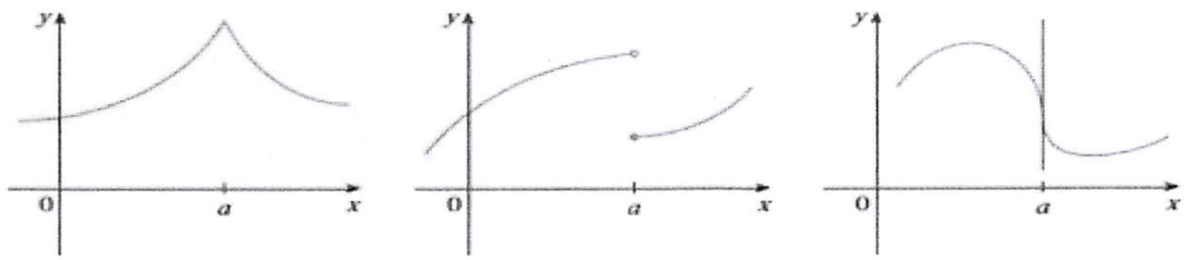
Reminder:
*Definition of the* ***derivative of a function*** *f:*

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},}$$ *provided the limit exists.*

*Anywhere that the derivative exists, we say that the function is* ***differentiable****.*

There are 3 types of situations when a function won't be differentiable at a point:

If a function **isn't continuous** at a certain point, it isn't differentiable at that point.
If the **tangent line** of a function is **vertical** at a certain point, the function isn't differentiable at that point.
If the tangent line on the right side of a point is different than the tangent line on the left side of that point, then the function isn't differentiable at that point… (**corner**)




(a) A corner (b) A discontinuity (c) A vertical tangent

Note: If a function is differentiable, then it is continuous. (true statement)
If a function is not continuous, then it is not differentiable (equivalent contrapositive statement)
But if a function is continuous, it doesn't mean that it is differentiable! (false converse statement)

⚠ If a statement is true, its contrapositive is also true, but its converse is not necessarily true…
Ex: Statement: if a figure is a square, then it has 4 sides
Contrapositive: if a figure doesn't have 4 sides, then it is not a square
Converse: if a figure has 4 sides, then it is a square.

*Example 7*: If $f$ is a function such that $\lim_{x\to -3}\frac{f(x)-f(-3)}{x+3}=2$, which of the following must be true?

A) The limit of $f(x)$ as $x$ approaches $-3$ does not exist. x

B) $f$ is not defined at $x=-3$. x

C) The derivative of $f$ at $x=-3$ is 2.

D) $f$ is continuous at $x=\underline{2}$. ≠ x (at -3)

E) $f(-3)=2$ x

On your Graphing Calculator:

To use your graphics calculator to find the derivative, use the `nDeriv(` function on the TI-83+. To access this function press [MATH], then 8 (or use [▲] and [▼] to go to `nDeriv(` and press [ENTER]). The `nDeriv(` function works as follows:

`nDeriv(function,variable,value)`

Where "*function*" is the function you want to find the derivative of, "*variable*" is the variable you are differentiating with respect to (usually $x$), and "*value*" is the point at which you want to find the derivative.

♬: Many times it is easier to type the function into `Y1`, and then enter `nDeriv(Y1,x,#)`.

*Example 10*: Use your calculator to find the derivative of $f(x)=x^2-3x+2$ at $x=-3$. Express your answer with the correct notation.

nDeriv $(x^2-3x+2, x, -3) = -9$ $\boxed{f'(-3)=-9}$

Note: Your calculator use another alternate definition of the derivative that doesn't work if the point considered is an end point of the domain…

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## 3.3 Rules for Differentiation

Determining the derivative of a function is a lot of work if we always go back to the definition of the derivative…
Instead, we're going to memorize the derivative of most usual functions as well as some rules regarding operations of functions $(\pm, \times, \div, \circ)$.
Remember that additions and subtractions are intuitive, but products and quotients are not…

*Rule #1 Derivative of a Constant Function*

If $c$ is any constant value, then $\frac{d}{dx}[c]=0$

This should not be too earth shattering to you, since the slope of a constant function is always 0!

*Example 1:* Let $f(x)=5$. Find $f'(x) = 0$

*Rule #2 Power Rule*

If $n$ is any number, then $\frac{d}{dx}[x^n]=n\cdot x^{n-1}$, provided $x^{n-1}$ exists.

The KEY to using the power rule is to get comfortable using exponent rules to write a function as a power of $x$.

*Example 2:* Let $f(x)=x^5$. Find $f'(x) = 5x^4$

*Example 3:* Let $f(x)=\sqrt[3]{x^2}$. Find $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

$f(x) = x^{2/3}$

*Example 4:* Let $f(x)=\frac{1}{x^4}$. Find $f'(x) = -4x^{-5} = -\frac{4}{x^5}$

$f(x) = x^{-4}$

*Rule 3: The Constant Multiple Rule*

If $u$ is a differentiable function of $x$ and $c$ is a constant, then

$$\frac{d}{dx}[cu]=c\frac{du}{dx}.$$

*Example 5*: Let $y=5x^7$. Find $\frac{dy}{dx} = 5\cdot(7x^6) = 35x^6$

*Example 6*: Let $g(x)=\frac{4}{5x^3}$. Find $g'(x) = \frac{4}{5}(-3x^{-4}) = -\frac{12}{5x^4}$

$= \frac{4}{5}x^{-3}$

*Rule 4: The Sum and Difference Rule*

If $u$ and $v$ are differentiable functions of $x$, then wherever $u$ and $v$ are differentiable

$$\frac{d}{dx}[u\pm v]=\frac{du}{dx}\pm\frac{dv}{dx}$$

*Example 7*: Let $y=x^3+4x^2-2x+7$. Find $y' = 3x^2+8x-2$

*Example 8*: Let $g(x)=\frac{3}{(-2x)^4}-\frac{x}{2}+\frac{1}{4}$. Find $g'(x) = \frac{3}{16}(-4x^{-5})-\frac{1}{2} = -\frac{3}{4x^5}-\frac{1}{2}$

$= \frac{3}{16x^4}-\frac{1}{2}x+\frac{1}{4}$

*Example 9*: Find the equation of the tangent line to the function $f(x)=4x^3-6x+5$ when $x=2$. $f(2)=25$

$f'(x)=12x^2-6 \quad f'(2)=42$

$y-25=42(x-2)$ $\quad y=42x-59$

*Example 10*: Let $h(x)=(x^2+1)(2x-5)$. Find $h'(x)$.

$h(x)=2x^3-5x^2+2x-5$

⚠ $h'(x) \neq 2x\cdot 2$

$h'(x)=6x^2-10x+2$

*Example 11*: The volume of a cube with sides of length $s$ is given by $V=s^3$. Find $\frac{dV}{ds}$ when $s=4$ centimeters.

$\frac{dV}{ds}=3s^2 \quad \left.\frac{dV}{ds}\right|_{s=4}=48$

never replace before taking the derivative!

*Rule 5: The Product Rule*

If $u$ and $v$ are differentiable functions of $x$, then

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

This is also written as

$$\frac{d}{dx}[uv] = uv' + vu'$$

Examples: Find the derivative without using the product rule, and then with the product rule:

a) $f(x) = \underbrace{(2x+3)}_{u}\underbrace{(4x^2-5)}_{v}$

$f'(x) = \underset{u'}{2}\ \cdot\ \underset{v}{(4x^2-5)} + \underset{v'}{8x}\ \cdot\ \underset{u}{(2x+3)}$

$= 8x^2 - 10 + 16x^2 + 24x$

$= 24x^2 + 24x - 10$

you could have expanded first...

$f(x) = 8x^3 + 12x^2 - 10x - 15$

$f'(x) = 24x^2 + 24x - 10$ ✓

b) $y = (3 + 2\sqrt{x})(5x^3 - 7)$

$u = 3 + 2\sqrt{x}$
$= 3 + 2x^{1/2}$

$u' = 2 \cdot \frac{1}{2}x^{-1/2}$
$= \frac{1}{\sqrt{x}}$

$y' = \frac{1}{\sqrt{x}}(5x^3 - 7) + 15x^2(3 + 2\sqrt{x})$

$= \frac{5x^3 - 7}{\sqrt{x}} + 45x^2 + 30x^2\sqrt{x}$

$= \frac{5x^3 - 7 + 45x^2\sqrt{x} + 30x^3}{\sqrt{x}} = \frac{35x^3 + 45x^2\sqrt{x} - 7}{\sqrt{x}}$

by expanding...

$y = 15x^3 + 10x^{7/2} - 14x^{1/2} - 21$

$y' = 45x^3 + 35x^{5/2} - 7x^{-1/2}$

*Rule 6: The Quotient Rule*

If $u$ and $v$ are differentiable functions of $x$, and $v \neq 0$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

This is also written as

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}.$$

Example 14: Find $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$ $= \frac{1(x^2+1)-2x(x)}{(x^2+1)^2}$

$u = x \longrightarrow u' = 1$

$v = x^2+1 \longrightarrow v' = 2x$

$= \frac{-x^2+1}{(x^2+1)^2}$

Example 15: Find $\frac{d}{dx}\left[\frac{5x^2}{x^3+1}\right]$ $= \frac{10x(x^3+1)-3x^2(5x^2)}{(x^3+1)^2}$

$u = 5x^2 \longrightarrow u' = 10x$

$v = x^3+1 \longrightarrow v' = 3x^2$

$= \frac{10x^4+10x-15x^4}{(x^3+1)^2}$

$= \frac{-5x^4+10x}{(x^3+1)^2} = \frac{-5x(x^3-2)}{(x^3+1)^2}$

**Higher Derivatives:**

We can also talk about the derivative of the derivative (we call it second derivative) and so on…

Here are all the notations:

| | | | | |
|---|---|---|---|---|
| *First derivative* | $y'$ | $f'(x)$ | $\frac{dy}{dx}$ | $\frac{d}{dx}[f(x)]$ |
| *Second derivative* | $y''$ | $f''(x)$ | $\frac{d^2y}{dx^2}$ | $\frac{d^2}{dx^2}[f(x)]$ |
| *Third derivative* | $y'''$ | $f'''(x)$ | $\frac{d^3y}{dx^3}$ | $\frac{d^3}{dx^3}[f(x)]$ |
| *Fourth derivative* | $y^{(4)}$ | $f^{(4)}(x)$ | $\frac{d^4y}{dx^4}$ | $\frac{d^4}{dx^4}[f(x)]$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| *$n^{th}$ derivative* | $y^{(n)}$ | $f^{(n)}(x)$ | $\frac{d^ny}{dx^n}$ | $\frac{d^n}{dx^n}[f(x)]$ |

*Example 16*: Find $\frac{d^4}{dx^4}\left[-5x^8+2x^6-9x^3+32x-1\right]$.

$$\frac{d}{dx}(-5x^8+2x^6-9x^3+32x-1) = -40x^7+12x^5-27x^2+32$$
$$\frac{d^2}{dx^2}(-5x^8+2x^6-9x^3+32x-1) = -280x^6+60x^4-54x$$
$$\frac{d^3}{dx^3}(-5x^8+2x^6-9x^3+32x-1) = -1680x^5+240x^3-54$$
$$\frac{d^4}{dx^4}f(x) = -8400x^4+720x^2$$

*Example 17*: Let $f(x)=\frac{x}{x-1}$. Find $f''(x)$.

$$f'(x) = \frac{x-1-x}{(x-1)^2} = -\frac{1}{(x-1)^2} = -\frac{1}{x^2-2x+1}$$

$$f''(x) = -\frac{0\cdot(x^2-2x+1)-(2x-2)(1)}{(x^2-2x+1)^2}$$
$$= \frac{2x-2}{(x-1)^4} = \frac{2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

*Example 18*: If $f^{(4)}(x)=2\sqrt{x}$, find $f^{(5)}(x)$.

$$f^{(5)}(x) = \frac{d}{dx}f^{(4)}(x) \qquad f^{(4)}(x) = 2x^{1/2}$$
$$= x^{-1/2}$$
$$= \frac{1}{\sqrt{x}}$$

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## 3.4 – Velocity and Other Rates of Change

*Instantaneous Rates of Change*

We have already seen that the instantaneous rate of change is the same as the slope of the tangent line and thus the derivative at that point. Unless we use the phrase "average rate of change", we will assume that in calculus the phrase "rate of change" refers to the instantaneous rate of change.

*Example 1*: The length of a rectangle is given by $2t + 1$ and its height is $\sqrt{t}$, where $t$ is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time, and indicate the units of measure for this rate.

$A = \sqrt{t}(2t+1) = 2t^{3/2} + t^{1/2}$ ($cm^2$)   $\frac{dA}{dt} = 3t^{1/2} + \frac{1}{2}t^{-1/2} = \frac{6t+1}{2\sqrt{t}}$ ($cm^2/s$)

Relationships between Position, Velocity, and Acceleration

The *displacement* of an object is the TOTAL CHANGE IN POSITION.

The *average velocity* of the object is described as TOTAL CHANGE IN POSITION (displacement) divided by the TOTAL CHANGE IN TIME. It can be thought of as the slope of the line connecting two points on a position function.

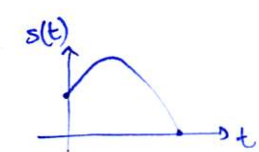
The *instantaneous velocity* of the object is the *derivative of the position function*. Unless term "average velocity" is used, we will assume velocity refers to instantaneous velocity. It is the slope of a tangent line to the position function.

Positive Velocity indicates movement in the positive direction.
Negative velocity indicates movement in the negative direction.

*Speed* is the absolute value of velocity. Thus speed is always positive value, whereas, ***velocity indicates direction***.

*Acceleration* is the rate of change in velocity, implying then that acceleration is the *derivative of velocity*. Since it is the derivative of velocity, it is also the *second derivative of position*.

*Example 2*: Bugs Bunny has been captured by Yosemite Sam and forced to "walk the plank". Instead of waiting for Yosemite Sam to finish cutting the board from underneath him, Bugs finally decides just to jump. Bugs' position, $s$, is given by $s(t) = -16t^2 + 16t + 320$, where $s$ is measured in feet and $t$ is measured in seconds.




a) What is Bugs' displacement from $t = 1$ to $t = 2$ seconds?

$s(2) - s(1) = 288 - 320 = -32$   (it went down 32 ft)

b) When will Bugs hit the ground?

$s(t) = 0 \Leftrightarrow -16t^2 + 16t + 320 = 0 \Leftrightarrow -16(t^2 - t - 20) = 0 \Leftrightarrow -16(t-5)(t+4) = 0$

$t = 5$ After 5s.   $\cancel{t = -4}$ not in the domain

c) What is Bugs' velocity at impact? (What are the units of this value?)

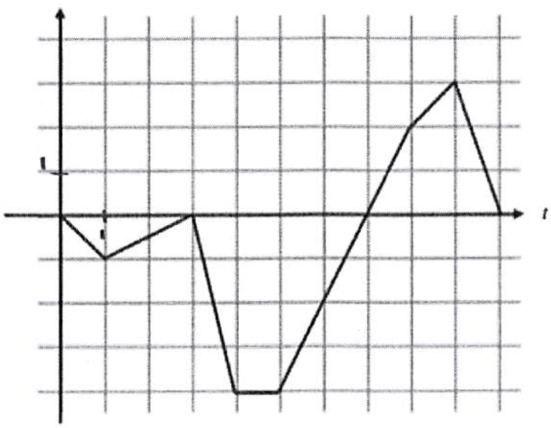
$s'(5) = ?$   $s'(t) = -32t + 16$   $s'(5) = -144$ he's going down 144 ft/s.

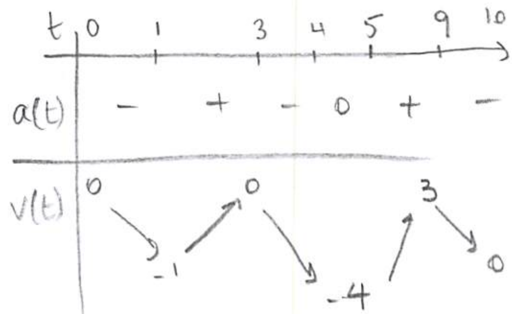
d) What is Bugs' speed at impact?   144 ft/s

e) Find Bugs' acceleration as a function of time. (What are the units of this value?)

$s''(t) = -32$ ft/s²

*Example 3*: Suppose the graph below shows the velocity of a particle moving along the $x$ – axis. Justify each response.




a) Which way does the particle move first?

left $(v<0)$

b) When does the particle stop?

when $t=3$
$t=7$
$t=10$
$(v=0)$

c) When does the particle change direction?

when $t=7$ (v changes sign)

d) When is the particle moving left?

$(0;7)$ (except at $t=3$ where it stops.)

e) When is the particle moving right?

$(7;10)$

f) When is the particle speeding up?

$(0;1)$ and $(3;4)$ and $(7;9)$ because v and a have the same sign.

g) When is the particle slowing down?

$(1;3)$ and $(5,7)$ and $(9,10)$ because v and a have opposite signs.

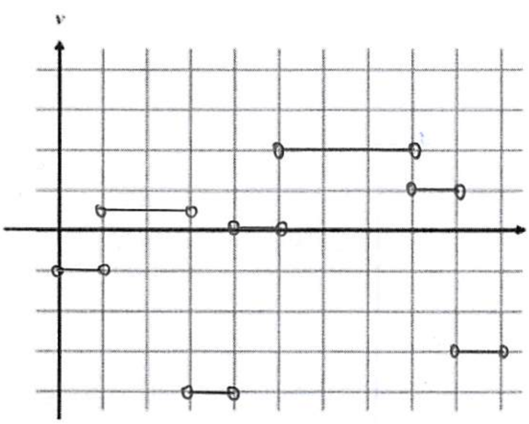
h) When is the particle moving the fastest?

on $(4;5)$ because $s(t)=|v(t)|=4$

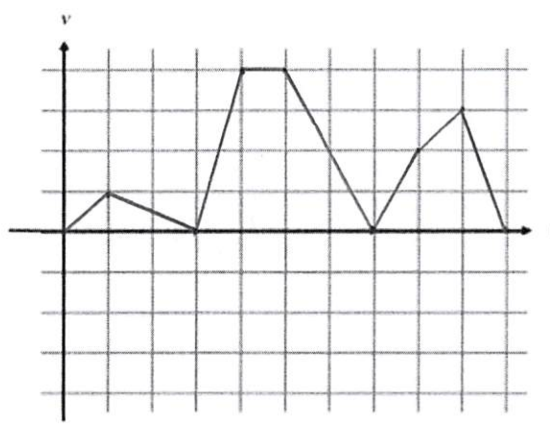
i) When is the particle moving at a constant speed?

on $[4;5]$

h) Graph the particle's acceleration for $0<t<10$.




j) Graph the particle's speed for $0\le t\le 10$.



Derivatives in Economics:

*Marginal Cost*: The marginal cost at $x$, given by $C'(x)$, is the approximate cost of the $(x+1)$st item.

*Marginal Revenue*: The marginal revenue at $x$, given by $R'(x)$, is the approximate revenue generated by the $(x+1)$st item.

*Marginal Profit*: The marginal profit at $x$, given by $P'(x)$, is the approximate profit generated by the $(x+1)$st item.

*Example 4*: Suppose that the daily cost, in dollars, of producing $x$ radios is $C(x)=0.002x^3+0.1x^2+42x+300$, and currently there are 40 radios produced daily.

a) What is the current daily cost? $C(40) = \$2268$

b) What would the actual additional daily cost of increasing production to 41 radios daily?

$C(41) = \$2327.942$ $\quad$ \$59.942 increase

c) What is the marginal cost of the 41st unit?

$C'(x) = 0.006x^2 + 0.2x + 42$ $\quad$ $C'(40) = \$59.6$

*Example 5*: Suppose the cost of producing $x$ units is given by $c(x)=4x^2+\dfrac{300}{x}$. What is the marginal cost of producing the 11th unit?

$c'(10)$?

$c'(x) = 8x - \dfrac{300}{x^2}$

$c'(10) = 77$

The 11th unit adds \$77 to the cost.

## 3.5 – Derivatives of Trigonometric Functions

*Derivatives of the Six Basic Trigonometric Functions*

$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Demos: (sin and cos using conjugate expressions and sinh/h + tan)

*Example 5*: Find the derivative of each function. Before you begin, state which rule(s) you are going to have to use. The product rule seems to be the rule that people forget to use … try not to be one of those people! ☺

a) $f(x) = x^2 \sin x$ $f'(x) = 2x.\sin x + x^2.\cos x$

Prod.

b) $f(x) = \frac{\cos x}{x}$ $f'(x) = \frac{-\sin x.x - \cos x}{x^2}$

quot.

$= -\frac{x.\sin x + \cos x}{x^2}$

c) $g(t) = \sqrt{t} + 4\sec t$ $g'(t) = \frac{1}{2\sqrt{t}} + 4\sec t.\tan t$

sum × kf

d) $h(\theta) = 5\sec\theta + \tan\theta$ $h'(\theta) = 5.\sec\theta.\tan\theta + \sec^2\theta$

kf + sum

e) $h(s) = \frac{1}{s} - 10\csc s$ $h'(s) = -\frac{1}{s^2} + 10\csc(s).\cot(s)$

sum + kf

f) $y = x\cot x$ $\frac{dy}{dx} = \cot x - x.\csc^2 x$

prod.

Example with the TI 83 or 84:

We will be using the `nDeriv(` function, except we will be using it to define a function under $Y_1$. Remember the syntax is

`nDeriv(function,variable,value)`

For our example, let's use $\cos x$. The difference will be that our function will be entered as a function of any variable other than $x$, and differentiated with respect to that same variable, and evaluated at the value of $x$ instead of an actual number.

So, we want to enter the following:

`Y₁=nDeriv(cos(T),T,X)`

choose a variable ≠ X

Step 1: Press [Y=].

Step 2: To enter `nDeriv(` ... Press [MATH] then 8: `nDeriv(`

Step 3: Enter the function using the [ALPHA] key to enter a variable other than $x$. I chose $T$, but it should work with any letter you choose.

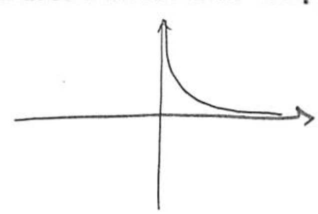
Step 4: After entering a comma, enter the same letter from step 3 as the variable you want to take the derivative with respect to. Again, I chose $T$, it's completely your choice.

Step 5: Enter the last comma, and then push the [X,T,Θ,n] button to enter the $X$. Do NOT use the [ALPHA] key to enter the $X$ or it will not work.

Step 6: Graph the function by pressing [GRAPH].

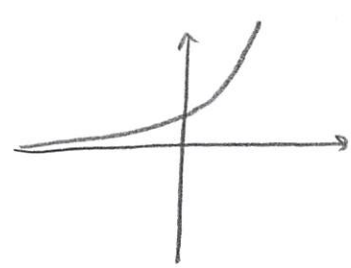
*Example 6*: Graph the derivative of $f(x) = \ln x$. What function does this look like? Graph your guess on the same screen.

$Y_1 = \text{nDeriv}(\ln T, T, x)$



*Example 7*: Graph the derivative of $f(x) = e^x$. What function does this look like? Graph your guess on the same screen.

$Y_1 = \text{nDeriv}(e^T, T, x)$



## 3.6 – The Chain Rule

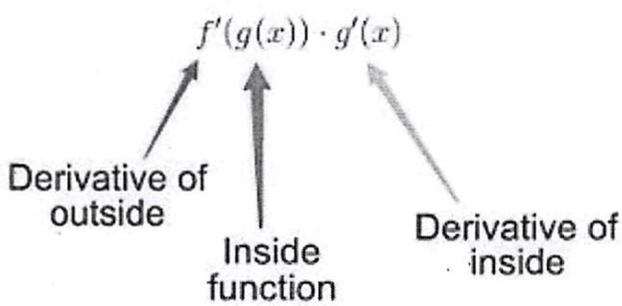
We can now differentiate many usual functions without having to go back to the definition. And we can also differentiate their products, quotients, …
We are now going to learn how to differentiate composite functions (when an usual function is "inside" an other usual function). For example: $f(x) = \sqrt{2x-3}$. We will call the method used "**The Chain Rule**".

**The Chain Rule** If $f$ and $g$ are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F$ is differentiable and $F'$ is given by the product

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$




Examples:

a) $f(x) = \dfrac{2}{3x+1}$

" $3x+1$ is inside $\frac{2}{x}$ "

$f'(x) = -\dfrac{2}{(3x+1)^2} \times 3$ $\quad f'(x) = -\dfrac{6}{(3x+1)^2}$

Quotient: $\dfrac{0(3x+1)-3(2)}{(3x+1)^2}$

b) $g(x) = (x^2+2)^3$

" $x^2+2$ is inside $x^3$ "

$g'(x) = 3(x^2+2)^2 \times 2x$

$g'(x) = 6x(x^2+2)^2$

) $h(x) = \sin(2x)$

*Example 3*: Find $k'(x)$, if $k(x) = \underbrace{(x^2+1)\overset{\text{composite}}{\sqrt{2x-3}}}_{\text{product}}$.

$$k'(x) = 2x\sqrt{2x-3} + (x^2+1)\frac{1}{2\sqrt{2x-3}} \times 2$$

$$= 2x\sqrt{2x-3} + \frac{2(x^2+1)}{2\sqrt{2x-3}} = \frac{2x(2x-3)+x^2+1}{\sqrt{2x-3}}$$

$$= \frac{5x^2-6x+1}{\sqrt{2x-3}}$$

*Example 4*: Find $\frac{dg}{dt}$, if $g = \left(\frac{t-2}{2t+1}\right)^9$

$$\frac{dg}{dt} = 9\left(\frac{t-2}{2t+1}\right)^8 \times \frac{2t+1-2(t-2)}{(2t+1)^2}$$

$$= \frac{45(t-2)^8}{(2t+1)^{10}}$$

*Example 5*: For each of the following, use the fact that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$ to find $f'(5)$, if possible. If it is not possible, state what additional information is required.

a) $f(x) = g(x)h(x)$

$$f'(x) = g'(x).h(x) + g(x)h'(x)$$

$$f'(5) = 6 . 3 + (-3)(-2) = 24$$

b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x)) . h'(x)$$

$$f'(5) = \underbrace{g'(3)}_{?} . h'(5) \longrightarrow \text{can't be found...}$$

c) $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{g'(x).h(x) - h'(x).g(x)}{[h(x)]^2} \qquad f'(5) = \frac{6.3-(-2).(-3)}{9} = \frac{4}{3}$$

d) $f(x) = [g(x)]^3$ ... ♫: Your book refers to $[g(x)]^3$ as $g^3(x)$

$$f'(x) = 3g^2(x) . g'(x)$$

$$f'(5) = 3g^2(5) . g'(5)$$

$$= 3 \times 9 \times 6$$

$$= 162.$$

## 3.7 – Implicit Differentiation

Equations that are solved for $y$ are called **explicit functions** (ex: $y = 3x - 5$).
Equations that are not solved for $y$ are called **implicit functions** (ex: $2x - 3y + 5 = 0$).

So far, we have only differentiated functions that were explicit.
We can also differentiate functions that are implicit by differentiating both sides of the equation and solving what we get for $y'$ (or $\frac{dy}{dx}$).
BE CAREFUL, when differentiating, don't forget that $y$ is a function of $x$…

Examples: Differentiate:

1) $2x - 3y + 5 = 0$.
→ $2 - 3\frac{dy}{dx} = 0$ $\quad 3\frac{dy}{dx} = 2$ $\quad \frac{dy}{dx} = \frac{2}{3}$

2) $x^2 + y^3 = 2x$
→ $2x + 3y^2 \cdot \frac{dy}{dx} = 2$ $\quad 3y^2\frac{dy}{dx} = 2 - 2x$ $\quad \frac{dy}{dx} = \frac{2-2x}{3y^2}$

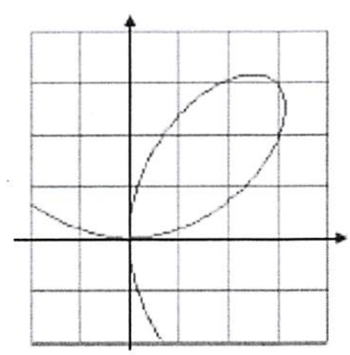
3) $3xy = \sqrt{y}$
→ $3\left(y + x\frac{dy}{dx}\right) = \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx}$

$3x\frac{dy}{dx} - \frac{1}{2\sqrt{y}}\frac{dy}{dx} = -3y$

$\frac{dy}{dx}\left(3x - \frac{1}{2\sqrt{y}}\right) = -3y$

$\frac{dy}{dx} = -\frac{3y}{3x - \frac{1}{2\sqrt{y}}}$

*Example 4*: Given the curve $x^3 + y^3 = 6xy$ (shown to the right).



a) Find $\frac{dy}{dx}$.

$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6\left(y + x\frac{dy}{dx}\right)$

$\frac{dy}{dx}\left(3y^2 - 6x\right) = 6y - 3x^2$

$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$

b) Find the equation of the tangent line and normal (perpendicular) line to the graph at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

$x$ $\quad y$

$\left.\frac{dy}{dx}\right|_{\text{point}} = \frac{6 \times \frac{8}{3} - 3\left(\frac{4}{3}\right)^2}{3\left(\frac{8}{3}\right)^2 - 6\left(\frac{4}{3}\right)} = \frac{4}{5}$

Tangent: $y - \frac{8}{3} = \frac{4}{5}\left(x - \frac{4}{3}\right)$ $\qquad$ Normal: $y - \frac{8}{3} = -\frac{5}{4}\left(x - \frac{4}{3}\right)$

*Example 5*: Find $\frac{dy}{dx}$ at (0, 0) of the function $\tan(x+y)=x$.

$$\sec^2(x+y)\cdot\left(1+\frac{dy}{dx}\right)=1$$

$$\frac{dy}{dx}\sec^2(x+y)=1-\sec^2(x+y)$$

$$\frac{dy}{dx}=\frac{1-\sec^2(x+y)}{\sec^2(x+y)}$$

$$\left.\frac{dy}{dx}\right|_{(0;0)}=\frac{1-\sec^2 0}{\sec^2 0}$$

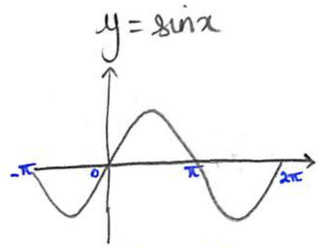
$$=\frac{0}{1}$$

$$=0$$

## 3.8 – Derivatives of Inverse Trigonometric Functions

Implicit differentiation allows us to determine the derivatives of Inverse trigonometric functions.

Example: Suppose $y = sin^{-1}x$. Determine $\frac{dy}{dx}$. (Be careful, domain must be restricted for $y$ to be a function…)




$D_R = \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$
so that $y = \sin^{-1}x$ is a function!

$\sin(\sin^{-1}x) = x$ (definition of inverse functions)

$\cos(\sin^{-1}x) \cdot \frac{d\sin^{-1}x}{dx} = 1$

$\frac{d\sin^{-1}x}{dx} = \frac{1}{\cos(\sin^{-1}x)}$

Note: $sin^{-1}x$ is the same thing as $\arcsin(x)$.

We won't reprove each formula, but we will commit them to memory:

***Derivatives of Inverse Trigonometric Functions*** **where** $u$ **is a function of** $x$.

1. $\frac{d}{dx}\left[\sin^{-1}(u)\right] = \frac{u'}{\sqrt{1-u^2}}$
2. $\frac{d}{dx}\left[\cos^{-1}(u)\right] = \frac{-u'}{\sqrt{1-u^2}}$
3. $\frac{d}{dx}\left[\tan^{-1}(u)\right] = \frac{u'}{1+u^2}$
4. $\frac{d}{dx}\left[\cot^{-1}(u)\right] = \frac{-u'}{1+u^2}$
5. $\frac{d}{dx}\left[\sec^{-1}(u)\right] = \frac{u'}{|u|\sqrt{u^2-1}}$
6. $\frac{d}{dx}\left[\csc^{-1}(u)\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$

Examples: Find the derivatives of:

a) $f(t) = sin^{-1}(t^2)$

$f'(t) = \frac{2t}{\sqrt{1-t^4}}$

b) $y = tan^{-1}\left(\sqrt{x-1}\right)$

$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x-1}}}{1 + (x-1)}$

$= \frac{1}{2x\sqrt{x-1}}$

**Derivative of an Inverse function**: $\boxed{\dfrac{df^{-1}}{dx}(x) = \dfrac{1}{f'(f^{-1}(x))}}$

Notation: If you evaluate a derivative at a certain point $x = a$, you can write $f'(a)$ or $\left.\frac{df}{dx}\right|_a$. You will see both notations in the textbook.

With that new notation, the previous formula becomes:

*Derivative of the inverse function at a point (p, q) ... this implies the point (q, p) is on the original function.*

To find the derivative of $f^{-1}$ at the point $(p, q)$ we find the reciprocal of the derivative of $f$ at the point $(q, p)$.

$$\left.\frac{df^{-1}}{dx}\right|_p = \frac{1}{\left.\frac{df}{dx}\right|_q}$$

*Example 5*: Let $f(x) = x^5 + 2x - 1$. Verify $(0, -1)$ is on the graph. Find $(f^{-1})'(-1)$.

$f(0) = -1$ ✓

$(f^{-1})'(-1) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$

$f'(x) = 5x^4 + 2$

*Example 6*: Let $f(x) = x^3 + 2x - 1$. Find $\left.\dfrac{df^{-1}}{dx}\right|_2 = \dfrac{1}{f'(1)} = \dfrac{1}{5}$

$f'(x) = 3x^2 + 2$

$f(x) = 2$

$\Leftrightarrow x^3 + 2x - 1 = 2$

$\Leftrightarrow x^3 + 2x - 3 = 0$

$(x-1)(x^2 + x + 3) = 0$

$\Delta = 1 - 4(1)(3) < 0$

no roots

$\boxed{x = 1}$

$f'(1) = 5$

# 3.9 – Derivatives of Exponential and Logarithmic Functions

Derivative of $f(x) = e^x$

$$\frac{d}{dx}\left[e^x\right] = e^x$$

Examples: Determine the derivatives of:

a) $f(x) = 3e^x + 5x + 3$

$f'(x) = 3e^x + 5$

b) $g(x) = e^x\sqrt{x^2 - 1}$

$g'(x) = e^x\sqrt{x^2-1} + \frac{2x}{2\sqrt{x^2-1}} \cdot e^x$

$= e^x\left(\frac{x^2-1+2}{\sqrt{x^2-1}}\right) = e^x \cdot \frac{x^2+1}{\sqrt{x^2-1}}$

c) $h(x) = e^{2x-1}$

$h'(x) = e^{2x-1} \cdot 2$

$= 2e^{2x-1}$

d) $k(x) = 3e^{\frac{-1}{x}}$

$k'(x) = 3 \cdot \frac{1}{x^2} \cdot e^{-1/x}$

Note: **Derivative of other exponential functions: $y = a^x$**

You can always rewrite an exponential function in base *e* by noticing that $\boxed{a = e^{\ln a}}$

Examples: Determine the derivatives of:

a) $f(x) = 3^x$

$f(x) = e^{x\ln 3}$

$f'(x) = \ln 3 \cdot e^{x\ln 3} = \ln 3 \cdot 3^x$

b) $g(x) = 5^{2x+1}$

$g(x) = e^{(2x+1)\ln 5}$

$g'(x) = 2\ln 5\, e^{(2x+1)\ln 5}$

$= 2\ln 5 \cdot 5^{2x+1}$

b) $h(x) = a^x$

$h(x) = e^{x\ln a}$

$h'(x) = \ln a \cdot e^{x\ln a}$

$h'(x) = \ln a \cdot a^x$

**Derivative of $f(x) = \ln x$**

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Examples: Determine the derivatives of:

a) $f(x) = \ln(3x - 5)$

$$f'(x) = \frac{3}{3x-5}$$

b) $g(x) = \frac{(2x+1)}{\ln(5x)}$

$$g'(x) = \frac{2\ln(5x) - \frac{5}{5x}(2x+1)}{(\ln(5x))^2}$$

$$= \frac{2x\ln 5x - 2x - 1}{x(\ln(5x))^2}$$

c) $h(x) = \ln(tanx)$

$$h'(x) = \frac{\sec^2 x}{\tan x} = \frac{1}{\cos^2 x} \cdot \frac{\cos x}{\sin x} = \frac{1}{\cos x \cdot \sin x}$$

<u>Note</u>: **Derivatives of other logarithmic functions: $y = log_a x$**

You can always rewrite a log expression in ln using: $log_a x = \frac{lnx}{lna}$

Examples: Determine the derivatives of:

a) $f(x) = log_4 x$

$$f(x) = \frac{\ln x}{\ln 4}$$

$$f'(x) = \frac{1}{\ln 4 . x}$$

b) $y = 5log_3(2x + 1)$ $= 5\frac{\ln(2x+1)}{\ln 3}$

$$\frac{dy}{dx} = \frac{5}{\ln 3} \cdot \frac{2}{2x+1}$$

c) $y = log_a x$

$$y = \frac{\ln x}{\ln a}$$

$$\frac{dy}{dx} = \frac{1}{x\ln a}$$

**Logarithmic Differentiation:**

When expressions are very complicated with many products and exponent rules, it is sometimes easier to compose the function by ln in order to benefit from the log laws…

Example: Determine the derivatives of: $y = \dfrac{\sqrt[3]{5x}\cdot e^{2x^2}}{5x^3}$

usual way : $y' = \dfrac{\left(\frac{1}{3}(5x)^{-2/3} \times 5 \times e^{2x^2} + 4x \times e^{2x^2} \times \sqrt[3]{5x}\right) \times 5x^3 - 15x^2 \times \sqrt[3]{5x} \times e^{2x^2}}{(5x^3)^2}$

$= \ldots$

Logarithmic Differentiation : $\ln y = \frac{1}{3}\ln(5x) + 2x^2 - \ln 5 - 3\ln x$

$\dfrac{y'}{y} = \frac{1}{3} \times \dfrac{5}{5x} + 4x - \dfrac{3}{x}$

$\dfrac{y'}{y} = \dfrac{1}{3x} + \dfrac{12x^2}{3x} - \dfrac{9}{3x}$

$\dfrac{y'}{y} = \dfrac{12x^2 - 8}{3x}$

$$\boxed{y' = \frac{12x^2-8}{3x} \times \frac{\sqrt[3]{5x}\,.\,e^{2x^2}}{5x^3}}$$

## Logarithmic Differentiation:

When expressions are very complicated with many products and exponent rules, it is sometimes easier to compose the function by ln in order to benefit from the log laws…

Example: Determine the derivatives of: $y = \frac{x\sqrt{x^2+1}}{(2x+1)^{\frac{2}{3}}}$

logarithmic differentiation

$$\ln y = \ln x + \ln\sqrt{x^2+1} - \frac{2}{3}\ln(2x+1)$$

$$\ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(2x+1)$$

diff.

$$\frac{y'}{y} = \frac{1}{x} + \frac{2x}{2(x^2+1)} - \frac{2}{3}\cdot\frac{2}{2x+1}$$

$$\frac{y'}{y} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{4}{3(2x+1)}$$

$$\frac{y'}{y} = \frac{3(x^2+1)(2x+1) + 3x^2(2x+1) - 4x(x^2+1)}{3x(x^2+1)(2x+1)}$$

$$\frac{y'}{y} = \frac{3(2x^3+x^2+2x+1) + 6x^3 + 3x^2 - 4x^3 - 4x}{3x(x^2+1)(2x+1)}$$

$$\frac{y'}{y} = \frac{8x^3+6x^2+2x+3}{3x(x^2+1)(2x+1)}$$

$$y' = \frac{8x^3+6x^2+2x+3}{3\cancel{x}(x^2+1)(2x+1)} \times \frac{\cancel{x}\sqrt{x^2+1}}{(2x+1)^{2/3}}$$

$$y' = \frac{8x^3+6x^2+2x+3}{3\sqrt{x^2+1}\,(2x+1)^{5/3}}$$

classic method

$$\frac{dy}{dx} = \frac{\left(\sqrt{x^2+1} + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\cdot x\right)(2x+1)^{2/3} - \frac{2}{3}(2x+1)^{-\frac{1}{3}}\cdot 2\cdot x\sqrt{x^2+1}}{(2x+1)^{4/3}}$$

$$= \frac{\frac{x^2+1+x^2}{\sqrt{x^2+1}}\cdot(2x+1)^{2/3} - \frac{4x(x^2+1)}{3(2x+1)^{1/3}\sqrt{x^2+1}}}{(2x+1)^{4/3}}$$

$$= \frac{\frac{3(2x^2+1)(2x+1) - 4x(x^2+1)}{3\sqrt{x^2+1}\cdot(2x+1)^{1/3}}}{(2x+1)^{4/3}}$$

$$= \frac{12x^3+6x^2+6x+3-4x^3-4x}{3\sqrt{x^2+1}\cdot(2x+1)^{5/3}}$$

$$= \frac{8x^3+6x^2+2x+3}{3\sqrt{x^2+1}\cdot(2x+1)^{5/3}}$$