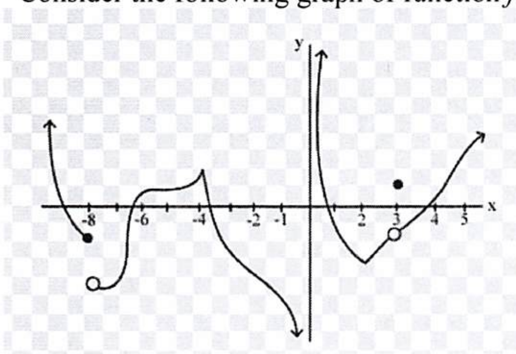
Chapter 3 TEST

1. Consider the following graph of function *f*. [3]



a) Determine where the function is discontinuous.

at $x=-8$ and $x=3$

b) Determine the intervals where the function is increasing.

$(-8,-4)$, $(2,3)$ and $(3,+\infty)$

c) Determine where the function is not differentiable.

at $x=-8$, $x=-4$, $x=2$ and $x=3$

2. a) Determine the slope of $y = x^3 + 1$ at the point $x = a$. [1]

$\frac{dy}{dx} = 3x^2 \Rightarrow \boxed{3a^2}$

b) Find the equations of the tangent lines to its graph that have a slope 3. [2]

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

at $x=1$: $P_1(1,2)$

$\Rightarrow y-2 = 3(x-1)$ $\boxed{y=3x-1}$

at $x=-1$: $P_2(-1,0)$

$\Rightarrow y = 3(x+1)$ $\boxed{y=3x+3}$

3. Find all points on the curve $y = \ln(x^2)$ where the tangent line is perpendicular to the line $y = 4x - 3$. [2]

slope : $-\frac{1}{4}$

$\frac{dy}{dx} = \frac{1}{x^2} \cdot 2x$

$= \frac{2}{x}$

$\frac{2}{x} = -\frac{1}{4}$

$x = -8$

$\boxed{\text{Point} : (-8, \ln 64)}$

4. a) Let $f(x) = \cancel{\frac{x}{x^2+4}}\ \frac{1}{x-2}$
Use the definition of the derivative to determine $f'(x)$. [2]

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h-2} - \frac{1}{x-2}}{h} = \frac{-h}{h(x-2)(x+h-2)}$$

$$= \frac{1}{h}\cdot\frac{x-2-(x+h-2)}{(x-2)(x+h-2)} = -\frac{1}{(x-2)(x+h-2)} \xrightarrow{h\to 0} \boxed{-\frac{1}{(x-2)^2}}$$

b) Let $g(x) = \sqrt{x+6}$.
Use the alternate definition of the derivative to determine $g'(3)$. [2]

$$\frac{g(x)-g(3)}{x-3} = \frac{\sqrt{x+6}-3}{x-3} = \frac{1}{\sqrt{x+6}+3} \xrightarrow{x\to 3} \boxed{\frac{1}{6}}$$

$$= \frac{x+6-9}{(x-3)(\sqrt{x+6}+3)}$$

5. Determine the derivatives of the following functions: [13]

a) $y = \frac{1}{3}x^3 - x$ $\quad \frac{dy}{dx} = \boxed{x^2-1}$

b) $g(x) = \frac{2-x}{2+x}$ $\quad g'(x) = \frac{-(2+x)-(2-x)}{(2+x)^2} = \boxed{-\frac{4}{(2+x)^2}}$

c) $f(x) = \frac{1}{\sqrt{1+x^2}}$ $\quad f'(x) = -\frac{1}{2}(1+x^2)^{-3/2}\cdot 2x$

$f(x) = (1+x^2)^{-1/2}$ $\quad = \boxed{-\frac{x}{(1+x^2)^{3/2}}}$

d) $h(x) = \frac{1+x+x^2+x^3}{x^4}$

$h(x) = x^{-4}+x^{-3}+x^{-2}+x^{-1}$ $\quad h'(x) = -4x^{-5}-3x^{-4}-2x^{-3}-x^{-2}$

$$= \boxed{\frac{-4-3x-2x^2-x^3}{x^5}}$$

e) $j(x) = (4 - x^{2/5})^{-5/2}$

$$j'(x) = -\frac{5}{2}(4-x^{2/5})^{-7/2}\cdot\left(-\frac{2}{5}x^{-1/5}\right)$$

$$= \boxed{x^{-1/5}(4-x^{2/5})^{-7/2}}$$

f) $k(x) = \sqrt{2 + \tan^2 x}$

$k'(x) = \frac{1}{2\sqrt{2+\tan^2 x}} \cdot 2\tan x \cdot \sec^2 x$ $\quad k'(x) = \frac{\tan x \cdot \sec^2 x}{\sqrt{2+\tan^2 x}}$

g) $l(x) = x^2 e^{\frac{x}{2}}$

$l'(x) = 2x \cdot e^{x/2} + \frac{1}{2} e^{x/2} \cdot x^2$ $\quad l'(x) = x e^{x/2}\left(2 + \frac{1}{2}x\right)$

h) $m(x) = \ln(3x - 2)$

$m'(x) = \frac{1}{3x-2} \cdot 3$ $\quad m'(x) = \frac{3}{3x-2}$

i) $y = \frac{e^x + e^{-x}}{2}$ $\quad y = \frac{1}{2}\left(e^x + e^{-x}\right)$

$\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$

j) $n(x) = 5^{2x+1}$

$n'(x) = 5^{2x+1} \cdot \ln 5 \cdot 2$ $\quad n'(x) = 2\ln 5 \cdot 5^{2x+1}$

k) $q(x) = x\sin^{-1}x$

$q'(x) = \sin^{-1}x + x \cdot \frac{1}{\sqrt{1-x^2}}$ $\quad q'(x) = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

l) $r(x) = \tan^{-1}(x^2 - 2)$

$r'(x) = \frac{1}{1+(x^2-2)^2} \cdot 2x$ $\quad r'(x) = \frac{2x}{1+(x^2-2)^2}$

m) $p(x) = x^{\sqrt{x}}$

$\ln p(x) = \sqrt{x}\ln x$

$\frac{p'(x)}{p(x)} = \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}$

$p'(x) = \frac{\sqrt{x}}{2x}\left(\ln x + 2\right) x^{\sqrt{x}}$

6. a) Determine $f'(2)$ if $f(x) = \frac{4}{x^2}$ $f(x) = 4x^{-2}$ [1]

$f'(x) = -8x^{-3}$ $f'(x) = -\frac{8}{x^3}$

$f'(2) = -\frac{8}{2^3}$ $\boxed{f'(2) = -1}$

b) Express the derivative of function $y = [f(\sqrt{x})]^2$ in terms of f and f'. [1]

$$\frac{dy}{dx} = 2f(\sqrt{x}) \cdot f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}} f(\sqrt{x}) f'(\sqrt{x})$$

c) Express the derivative of function $y = \frac{f(x)-g(x)}{f(x)+g(x)}$ in terms of f, g, f' and g'. [1]

$$\frac{dy}{dx} = \frac{(f'(x)-g'(x))(f(x)+g(x)) - (f'(x)+g'(x))(f(x)-g(x))}{(f(x)+g(x))^2}$$

$$= \frac{2f'(x)g(x) - 2g'(x)f(x)}{(f(x)+g(x))^2}$$

d) Determine $f''(x)$ for $f(x) = \sin(x^2)$ [1.5]

$f'(x) = \cos(x^2) \cdot 2x$

$f''(x) = -\sin(x^2) \cdot (2x)^2 + 2\cos(x^2)$

$= -4x^2 \sin x^2 + 2\cos x^2$

e) Find an equation of the normal line to the curve $x^2 + xy = y^3$ at point $(2,2)$. [2.5]

implicit diff: $2x + y + xy' = 3y^2 \cdot y'$

$y'(3y^2 - x) = 2x + y$

$y' = \frac{2x+y}{3y^2 - x}$

$\left.\frac{dy}{dx}\right|_{\substack{x=2 \\ y=2}} = \frac{4+2}{12-2} = \frac{6}{10} = \frac{3}{5}$

normal line: slope $= -\frac{5}{3}$ $\Rightarrow$ $y - 2 = -\frac{5}{3}(x-2)$

or $y = -\frac{5}{3}x + \frac{16}{3}$

g) Find $f^{-1}(2)$ if $f(x) = x^3 + x$. [2]

$f'(x) = 3x^2 + 1$

Point: $x^3 + x = 2$

$x^3 + x - 2 = 0$

$(x-1)(x^2 + x + 2) = 0$

$\Delta = -7 < 0$

$\Rightarrow x = 1$

$(f^{-1})'(2) = \dfrac{1}{f'(1)} = \dfrac{1}{4}$

7. A point moves along the x-axis so that its position x at time t is given by $x = t^3 - 6t^2 + 9t + 1$ [6]
Determine:

a) the position of the particle at the start.

$x(0) = 1$

b) the initial velocity of the particle.

$x'(t) = 3t^2 - 12t + 9 \quad x'(0) = 9$

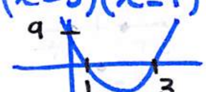
c) the time intervals on which the point is moving to the right.

The point is moving to the right when $x'(t) > 0$

$3t^2 - 12t + 9 > 0$

$3(x-3)(x-1) > 0$

intervals: $(0,1)$ and $(3,+\infty)$



d) the time intervals on which the point is speeding up. Justify.

The point is speeding up when x' and x'' have the same sign

$x''(t) = 6t - 12$

$= 6(t-2)$

$t > 2$

x	0		1		2		3		$+\infty$
$x'(t)$		+	0	−		−	0	+	
$x''(t)$		−		−	0	+		+	

$\Rightarrow \boxed{(1,2) \text{ and } (3,+\infty)}$

e) the average velocity over the time interval [0,4].

$\dfrac{x(4) - x(0)}{4} = \dfrac{5-1}{4} = \boxed{1}$

f) At what instant is the velocity equal to the average velocity over [0,4].

$x'(t) = 1$

$3t^2 - 12t + 9 = 1$

$3t^2 - 12t + 8 = 0$

$\Delta = 144 - 4(3)(8)$

$= 48$

$t = \dfrac{12 \pm \sqrt{48}}{6}$ → $\boxed{\dfrac{12 - \sqrt{48}}{6}}$, $\boxed{\dfrac{12 + \sqrt{48}}{6}}$

Both on the interval [0,4]