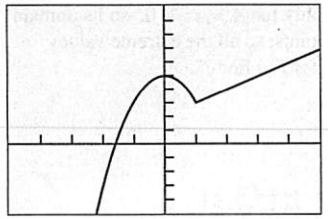


[−5, 5] by [−5, 10]

**Figure 4.8** The function in Example 5.

**SOLUTION**

**Solve Graphically** The graph in Figure 4.8 suggests that $f'(0) = 0$ and that $f'(1)$ does not exist. There appears to be a local maximum value of 5 at $x = 0$ and a local minimum value of 3 at $x = 1$.

**Confirm Analytically** For $x \neq 1$, the derivative is

$$f'(x) = \begin{cases} \dfrac{d}{dx}(5 - 2x^2) = -4x, & x < 1 \\ \dfrac{d}{dx}(x + 2) = 1, & x > 1. \end{cases}$$

The only point where $f' = 0$ is $x = 0$. What happens at $x = 1$?

At $x = 1$, the right- and left-hand derivatives are respectively

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1 + h) + 2 - 3}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1,$$

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{5 - 2(1 + h)^2 - 3}{h}$$

$$= \lim_{h \to 0^-} \frac{-2h(2 + h)}{h} = -4.$$

Since these one-sided derivatives differ, $f$ has no derivative at $x = 1$, and 1 is a secon[d] critical point of $f$.

The domain $(-\infty, \infty)$ has no endpoints, so the only values of $f$ that might be local extrema are those at the critical points:

$$f(0) = 5 \quad \text{and} \quad f(1) = 3.$$

From the formula for $f$, we see that the values of $f$ immediately to either side of $x = 0$ are less than 5, so 5 is a local maximum. Similarly, the values of $f$ immediately to eith[er] side of $x = 1$ are greater than 3, so 3 is a local minimum. *Now try Exercise 4*

Most graphing calculators have built-in methods to find the coordinates of points whe[re] extreme values occur. We must, of course, be sure that we use correct graphs to find the[se] values. The calculus that you learn in this chapter should make you feel more confid[ent] about working with graphs.

### EXAMPLE 6 Using Graphical Methods

Find the extreme values of $f(x) = \ln \left| \dfrac{x}{1 + x^2} \right|$.

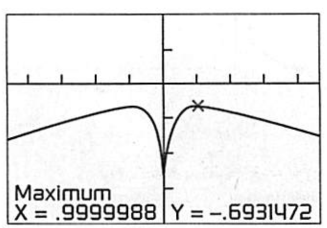
**SOLUTION**

**Solve Graphically** The domain of $f$ is the set of all nonzero real numbers. Figure 4.9 suggests that $f$ is an even function with a maximum value at two points. The coordinat[es] found in this window suggest an extreme value of about $-0.69$ at approximately $x = 1$. Because $f$ is even, there is another extreme of the same value at approximately $x = -1$. The figure also suggests a minimum value at $x = 0$, but $f$ is not defined there.

**Confirm Analytically** The derivative

$$f'(x) = \frac{1 - x^2}{x(1 + x^2)}$$

is defined at every point of the function's domain. The critical points where $f'(x) = 0$ ar[e] $x = 1$ and $x = -1$. The corresponding values of $f$ are both $\ln(1/2) = -\ln 2 \approx -0.69$. *Now try Exercise 3*




[−4.5, 4.5] by [−4, 2]

**Figure 4.9** The function in Example 6.



**EXPLORATION 1 Finding Extreme Values**

Let $f(x) = \left| \dfrac{x}{x^2 + 1} \right|, \ -2 \le x \le 2.$

1. Determine graphically the extreme values of $f$ and where they occur. Find $f'$ at these values of $x$.
2. Graph $f$ and $f'$ (or NDER $(f(x), x, x)$) in the same viewing window. Comment on the relationship between the graphs.
3. Find a formula for $f'(x)$.

## Quick Review 4.1 *(For help, go to Sections 1.2, 2.1, 3.5, and 3.6.)*

In Exercises 1–4, find the first derivative of the function.

1. $f(x) = \sqrt{4 - x}$
2. $f(x) = \dfrac{2}{\sqrt{9 - x^2}}$
3. $g(x) = \cos(\ln x)$
4. $h(x) = e^{2x}$

In Exercises 5–8, match the table with a graph of $f(x)$.

5.

| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | 5 |

6.

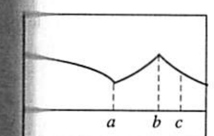
| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | $-5$ |

7.

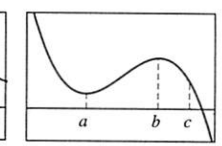
| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | 0 |
| $c$ | $-2$ |

8.

| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | does not exist |
| $c$ | $-1.7$ |



(a)



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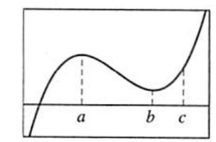
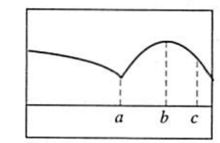


(c)



(d)

In Exercises 9 and 10, find the limit for

$$f(x) = \frac{2}{\sqrt{9 - x^2}}.$$

9. $\lim_{x \to 3^-} f(x)$
10. $\lim_{x \to -3^+} f(x)$

In Exercises 11 and 12, let

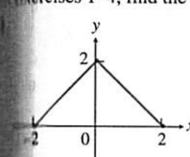
$$f(x) = \begin{cases} x^3 - 2x, & x \le 2 \\ x + 2, & x > 2. \end{cases}$$

11. Find (a) $f'(1)$, (b) $f'(3)$, (c) $f'(2)$.
12. (a) Find the domain of $f'$.
    (b) Write a formula for $f'(x)$.

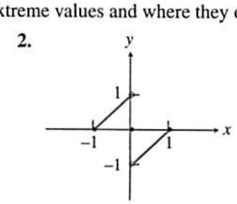
## Section 4.1 Exercises

In Exercises 1–4, find the extreme values and where they occur.

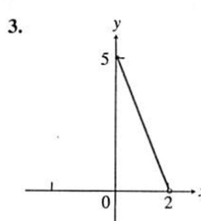
1.



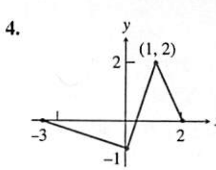
2.



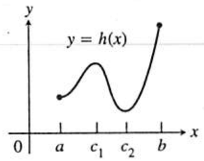
3.

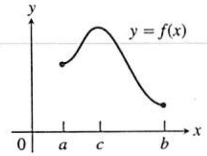


4.



In Exercises 5–10, identify each $x$-value at which any absolute extreme value occurs. Explain how your answer is consistent with the Extreme Value Theorem.

5.



6.
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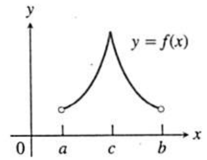


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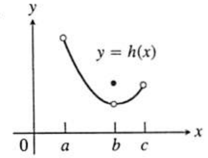
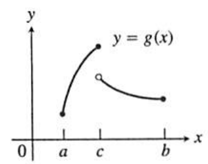
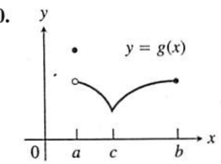



9.



10.



In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur.

11. $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

12. $g(x) = e^{-x}, \quad -1 \le x \le 1$

13. $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$

15. $f(x) = \sin\left(x + \dfrac{\pi}{4}\right), \quad 0 \le x \le \dfrac{7\pi}{4}$

16. $g(x) = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

17. $f(x) = x^{2/5}, \quad -3 \le x < 1$

18. $f(x) = x^{3/5}, \quad -2 < x \le 3$

In Exercises 19–30, find the extreme values of the function and where they occur.

19. $y = 2x^2 - 8x + 9$

20. $y = x^3 - 2x + 4$

21. $y = x^3 + x^2 - 8x + 5$

22. $y = x^3 - 3x^2 + 3x - 2$

23. $y = \sqrt{x^2 - 1}$

24. $y = \dfrac{1}{x^2 - 1}$

25. $y = \dfrac{1}{\sqrt{1 - x^2}}$

26. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

27. $y = \sqrt{3 + 2x - x^2}$

28. $y = \dfrac{3}{2}x^4 + 4x^3 - 9x^2 + 10$

29. $y = \dfrac{x}{x^2 + 1}$

30. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

**Group Activity** In Exercises 31–34, find the extreme values of the function on the interval and where they occur.

31. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

32. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

33. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

34. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

In Exercises 35–42, identify the critical point and determine the local extreme values.

35. $y = x^{2/3}(x + 2)$

36. $y = x^{2/3}(x^2 - 4)$

37. $y = x\sqrt{4 - x^2}$

38. $y = x^2\sqrt{3 - x}$

39. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

40. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

41. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

42. $y = \begin{cases} -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

43. **Writing to Learn** The function

$$V(x) = x(10 - 2x)(16 - 2x), \quad 0 < x < 5,$$

models the volume of a box.

(a) Find the extreme values of $V$.

(b) Interpret any values found in (a) in terms of volume of the box.

44. **Writing to Learn** The function

$$P(x) = 2x + \frac{200}{x}, \quad 0 < x < \infty,$$

models the perimeter of a rectangle of dimensions $x$ by $100/x$.

(a) Find any extreme values of $P$.

(b) Give an interpretation in terms of perimeter of the rectangle for any values found in (a).

## Standardized Test Questions

You should solve the following problems without using a graphing calculator.

45. **True or False** If $f(c)$ is a local maximum of a continuous function $f$ on an open interval $(a, b)$, then $f'(c) = 0$. Justify your answer.

46. **True or False** If $m$ is a local minimum and $M$ is a local maximum of a continuous function $f$ on $(a, b)$, then $m < M$. Justify your answer.

47. **Multiple Choice** Which of the following values is the absolute maximum of the function $f(x) = 4x - x^2 + 6$ on the interval $[0, 4]$?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 10

48. **Multiple Choice** If $f$ is a continuous, decreasing function on $[0, 10]$ with a critical point at $(4, 2)$, which of the following statements *must be false*?

(A) $f(10)$ is an absolute minimum of $f$ on $[0, 10]$.

(B) $f(4)$ is neither a relative maximum nor a relative minimum.

(C) $f'(4)$ does not exist.

(D) $f'(4) = 0$

(E) $f'(4) < 0$

49. **Multiple Choice** Which of the following functions has exactly two local extrema on its domain?

(A) $f(x) = |x - 2|$

(B) $f(x) = x^3 - 6x + 5$

(C) $f(x) = x^3 + 6x - 5$

(D) $f(x) = \tan x$

(E) $f(x) = x + \ln x$

50. **Multiple Choice** If an even function $f$ with domain all real numbers has a local maximum at $x = a$, then $f(-a)$

(A) is a local minimum.

(B) is a local maximum.

(C) is both a local minimum and a local maximum.

(D) could be either a local minimum or a local maximum.

(E) is neither a local minimum nor a local maximum.

## Explorations

In Exercises 51 and 52, give reasons for your answers.

51. **Writing to Learn** Let $f(x) = (x - 2)^{2/3}$.

(a) Does $f'(2)$ exist?

(b) Show that the only local extreme value of $f$ occurs at $x = 2$.

(c) Does the result in (b) contradict the Extreme Value Theorem?

(d) Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by $a$.

52. **Writing to Learn** Let $f(x) = |x^3 - 9x|$.

(a) Does $f'(0)$ exist?

(b) Does $f'(3)$ exist?

(c) Does $f'(-3)$ exist?

(d) Determine all extrema of $f$.



## Extending the Ideas

53. ***Cubic Functions*** Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

(a) Show that $f$ can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

(b) How many local extreme values can $f$ have?

54. ***Proving Theorem 2*** Assume that the function $f$ has a local maximum value at the interior point $c$ of its domain and that $f'(c)$ exists.

(a) Show that there is an open interval containing $c$ such that $f(x) - f(c) \le 0$ for all $x$ in the open interval.

(b) **Writing to Learn** Now explain why we may say

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0.$$

(c) **Writing to Learn** Now explain why we may say

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0.$$

(d) **Writing to Learn** Explain how parts (b) and (c) allow us to conclude $f'(c) = 0$.

(e) **Writing to Learn** Give a similar argument if $f$ has a local minimum value at an interior point.

55. ***Functions with No Extreme Values at Endpoints***

(a) Graph the function

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Explain why $f(0) = 0$ is not a local extreme value of $f$.

(b) **Group Activity** Construct a function of your own that fails to have an extreme value at a domain endpoint.

## Quick Review 4.2 *(For help, go to Sections 1.2, 2.3, and 3.2.)*

In Exercises 1 and 2, find exact solutions to the inequality.

**1.** $2x^2 - 6 < 0$ **2.** $3x^2 - 6 > 0$

In Exercises 3–5, let $f(x) = \sqrt{8 - 2x^2}$.

**3.** Find the domain of $f$.

**4.** Where is $f$ continuous?

**5.** Where is $f$ differentiable?

In Exercises 6–8, let $f(x) = \dfrac{x}{x^2 - 1}$.

**6.** Find the domain of $f$.

**7.** Where is $f$ continuous?

**8.** Where is $f$ differentiable?

In Exercises 9 and 10, find $C$ so that the graph of the function $f$ passes through the specified point.

**9.** $f(x) = -2x + C, \quad (-2, 7)$

**10.** $g(x) = x^2 + 2x + C, \quad (1, -1)$

## Section 4.2 Exercises

In Exercises 1–8, **(a)** state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and **(b)** if it does, find each value of $c$ in the interval $(a, b)$ that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**1.** $f(x) = x^2 + 2x - 1$ on $[0, 1]$

**2.** $f(x) = x^{2/3}$ on $[0, 1]$

**3.** $f(x) = x^{1/3}$ on $[-1, 1]$

**4.** $f(x) = |x - 1|$ on $[0, 4]$

**5.** $f(x) = \sin^{-1} x$ on $[-1, 1]$

**6.** $f(x) = \ln(x - 1)$ on $[2, 4]$

**7.** $f(x) = \begin{cases} \cos x, & 0 \le x < \pi/2 \\ \sin x, & \pi/2 \le x \le \pi \end{cases}$ on $[0, \pi]$

**8.** $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

In Exercises 9 and 10, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

(a) the secant line $AB$.

(b) a tangent line to $f$ in the interval $(a, b)$ that is parallel to $AB$.

**9.** $f(x) = x + \dfrac{1}{x}, \quad 0.5 \le x \le 2$

**10.** $f(x) = \sqrt{x - 1}, \quad 1 \le x \le 3$

**11.** ***Speeding*** A trucker handed in a ticket at a toll booth showing that in 2 h she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

**12.** ***Temperature Change*** It took 20 sec for the temperature to rise from 0°F to 212°F when a thermometer was taken from a freezer and placed in boiling water. Explain why at some moment in that interval the mercury was rising at exactly 10.6°F/sec.

**13.** ***Triremes*** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 h. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

**14.** ***Running a Marathon*** A marathoner ran the 26.2-mi New York City Marathon in 2.2 h. Show that at least twice, the marathoner was running at exactly 11 mph.

In Exercises 15–22, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

**15.** $f(x) = 5x - x^2$ **16.** $g(x) = x^2 - x - 12$

**17.** $h(x) = \dfrac{2}{x}$ **18.** $k(x) = \dfrac{1}{x^2}$

**19.** $f(x) = e^{2x}$ **20.** $f(x) = e^{-0.5x}$

**21.** $y = 4 - \sqrt{x + 2}$ **22.** $y = x^4 - 10x^2 + 9$



In Exercises 23–28, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

**23.** $f(x) = x\sqrt{4 - x}$ **24.** $g(x) = x^{1/3}(x + 8)$

**25.** $h(x) = \dfrac{-x}{x^2 + 4}$ **26.** $k(x) = \dfrac{x}{x^2 - 4}$

**27.** $f(x) = x^3 - 2x - 2\cos x$ **28.** $g(x) = 2x + \cos x$

In Exercises 29–34, find all possible functions $f$ with the given derivative.

**29.** $f'(x) = x$ **30.** $f'(x) = 2$

**31.** $f'(x) = 3x^2 - 2x + 1$ **32.** $f'(x) = \sin x$

**33.** $f'(x) = e^x$ **34.** $f'(x) = \dfrac{1}{x - 1}, \quad x > 1$

In Exercises 35–38, find the function with the given derivative whose graph passes through the point $P$.

**35.** $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$

**36.** $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$

**37.** $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$

**38.** $f'(x) = 2x + 1 - \cos x, \quad P(0, 3)$

**Group Activity** In Exercises 39–42, sketch a graph of a differentiable function $y = f(x)$ that has the given properties.

**39.** (a) local minimum at $(1, 1)$, local maximum at $(3, 3)$

(b) local minima at $(1, 1)$ and $(3, 3)$

(c) local maxima at $(1, 1)$ and $(3, 3)$

**40.** $f(2) = 3, \quad f'(2) = 0$, and

(a) $f'(x) > 0$ for $x < 2$, $f'(x) < 0$ for $x > 2$.

(b) $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$.

(c) $f'(x) < 0$ for $x \ne 2$.

(d) $f'(x) > 0$ for $x \ne 2$.

**41.** $f'(-1) = f'(1) = 0$, $f'(x) > 0$ on $(-1, 1)$, $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.

**42.** A local minimum value that is greater than one of its local maximum values.

**43.** ***Free Fall*** On the moon, the acceleration due to gravity is 1.6 m/sec$^2$.

(a) If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

(b) How far below the point of release is the bottom of the crevasse?

(c) If instead of being released from rest, the rock is thrown into the crevasse from the same point with a downward velocity of 4 m/sec, when will it hit the bottom and how fast will it be going when it does?

**44.** ***Diving*** (a) With what velocity will you hit the water if you step off from a 10-m diving platform?

(b) With what velocity will you hit the water if you dive off the platform with an upward velocity of 2 m/sec?

**45.** **Writing to Learn** The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and at $x = 1$. Its derivative is equal to 1 at every point between 0 and 1, so $f'$ is never zero between 0 and 1, and the graph of $f$ has no tangent parallel to the chord from $(0, 0)$ to $(1, 0)$. Explain why this does not contradict the Mean Value Theorem.

**46.** **Writing to Learn** Explain why there is a zero of $y = \cos x$ between every two zeros of $y = \sin x$.

**47.** ***Unique Solution*** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Also assume that $f(a)$ and $f(b)$ have opposite signs and $f' \ne 0$ between $a$ and $b$. Show that $f(x) = 0$ exactly once between $a$ and $b$.

In Exercises 48 and 49, show that the equation has exactly one solution in the interval. [*Hint:* See Exercise 47.]

**48.** $x^4 + 3x + 1 = 0, \quad -2 \le x \le -1$

**49.** $x + \ln(x + 1) = 0, \quad 0 \le x \le 3$

**50.** ***Parallel Tangents*** Assume that $f$ and $g$ are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between $a$ and $b$ where the tangents to the graphs of $f$ and $g$ are parallel or the same line. Illustrate with a sketch.

### Standardized Test Questions

You may use a graphing calculator to solve the following problems.

**51.** **True or False** If $f$ is differentiable and increasing on $(a, b)$, then $f'(c) > 0$ for every $c$ in $(a, b)$. Justify your answer.

**52.** **True or False** If $f$ is differentiable and $f'(c) > 0$ for every $c$ in $(a, b)$, then $f$ is increasing on $(a, b)$. Justify your answer.

53. **Multiple Choice** If $f(x) = \cos x$, then the Mean Value Theorem guarantees that somewhere between 0 and $\pi/3$, $f'(x) =$

(A) $-\frac{3}{2\pi}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $-\frac{1}{2}$ (D) 0 (E) $\frac{1}{2}$

54. **Multiple Choice** On what interval is the function $g(x) = e^{x^3-6x^2+8}$ decreasing?

(A) $(-\infty, 2]$ (B) $[0, 4]$ (C) $[2, 4]$ (D) $(4, \infty)$ (E) no interval

55. **Multiple Choice** Which of the following functions is an antiderivative of $\frac{1}{\sqrt{x}}$?

(A) $-\frac{1}{\sqrt{2x^3}}$ (B) $-\frac{2}{\sqrt{x}}$ (C) $\frac{\sqrt{x}}{2}$ (D) $\sqrt{x} + 5$ (E) $2\sqrt{x} - 10$

56. **Multiple Choice** All of the following functions satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$ *except*

(A) $\sin x$ (B) $\sin^{-1} x$ (C) $x^{5/3}$ (D) $x^{3/5}$ (E) $\frac{x}{x-2}$

## Explorations

57. ***Analyzing Derivative Data*** Assume that $f$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. The table gives some values of $f'(x)$.

| $x$ | $f'(x)$ | $x$ | $f'(x)$ |
|---|---|---|---|
| −2 | 7 | 0.25 | −4.81 |
| −1.75 | 4.19 | 0.5 | −4.25 |
| −1.5 | 1.75 | 0.75 | −3.31 |
| −1.25 | −0.31 | 1 | −2 |
| −1 | −2 | 1.25 | −0.31 |
| −0.75 | −3.31 | 1.5 | 1.75 |
| −0.5 | −4.25 | 1.75 | 4.19 |
| −0.25 | −4.81 | 2 | 7 |
| 0 | −5 | | |

(a) Estimate where $f$ is increasing, decreasing, and has local extrema.

(b) Find a quadratic regression equation for the data in the table and superimpose its graph on a scatter plot of the data.

(c) Use the model in part (b) for $f'$ and find a formula for $f$ that satisfies $f(0) = 0$.

58. ***Analyzing Motion Data*** Priya's distance $D$ in meters from motion detector is given by the data in Table 4.1.

**Table 4.1 Motion Detector Data**

| $t$ (sec) | $D$ (m) | $t$ (sec) | $D$ (m) |
|---|---|---|---|
| 0.0 | 3.36 | 4.5 | 3.59 |
| 0.5 | 2.61 | 5.0 | 4.15 |
| 1.0 | 1.86 | 5.5 | 3.99 |
| 1.5 | 1.27 | 6.0 | 3.37 |
| 2.0 | 0.91 | 6.5 | 2.58 |
| 2.5 | 1.14 | 7.0 | 1.93 |
| 3.0 | 1.69 | 7.5 | 1.25 |
| 3.5 | 2.37 | 8.0 | 0.67 |
| 4.0 | 3.01 | | |

(a) Estimate when Priya is moving toward the motion detector; away from the motion detector.

(b) **Writing to Learn** Give an interpretation of any local extreme values in terms of this problem situation.

(c) Find a cubic regression equation $D = f(t)$ for the data in Table 4.1 and superimpose its graph on a scatter plot of the data.

(d) Use the model in (c) for $f$ to find a formula for $f'$. Use this formula to estimate the answers to (a).

## Extending the Ideas

59. ***Geometric Mean*** The **geometric mean** of two positive numbers $a$ and $b$ is $\sqrt{ab}$. Show that for $f(x) = 1/x$ on any interval $[a, b]$ of positive numbers, the value of $c$ in the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

60. ***Arithmetic Mean*** The **arithmetic mean** of two numbers $a$ and $b$ is $(a + b)/2$. Show that for $f(x) = x^2$ on any interval $[a, b]$, the value of $c$ in the conclusion of the Mean Value Theorem is $c = (a + b)/2$.

61. ***Upper Bounds*** Show that for any numbers $a$ and $b$, $|\sin b - \sin a| \le |b - a|$.

62. ***Sign of f′*** Assume that $f$ is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that $f'$ is negative at some point between $a$ and $b$.

63. ***Monotonic Functions*** Show that monotonic increasing and decreasing functions are one-to-one.



# 4.3 Connecting $f'$ and $f''$ with the Graph of $f$

**What you'll learn about**

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning about Functions from Derivatives

**. . . and why**

Differential calculus is a powerful problem-solving tool precisely because of its usefulness for analyzing functions.

## First Derivative Test for Local Extrema

As we see once again in Figure 4.18, a function $f$ may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of $f'$ in a critical point's immediate vicinity. As $x$ moves from left to right, the values of $f$ increase where $f' > 0$ and decrease where $f' < 0$.

At the points where $f$ has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where $f$ has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

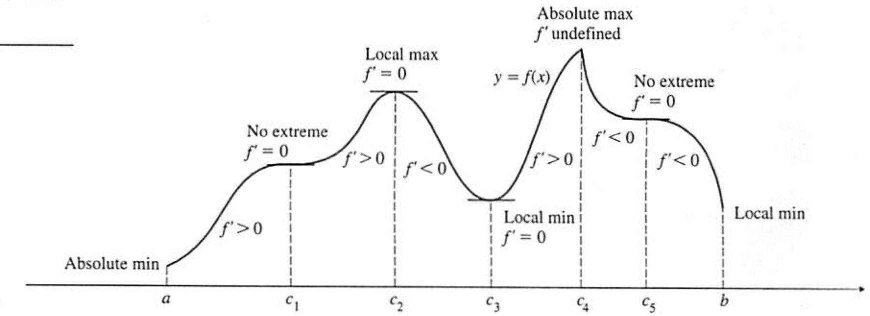


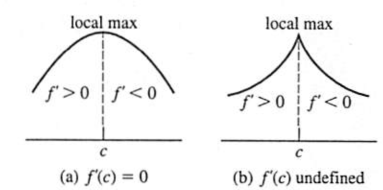

**Figure 4.18** A function's first derivative tells how the graph rises and falls.

**THEOREM 4 First Derivative Test for Local Extrema**

The following test applies to a continuous function $f(x)$.

**At a critical point $c$:**

1. If $f'$ changes sign from positive to negative at $c$ ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then $f$ has a local maximum value at $c$.

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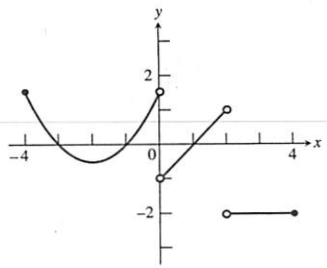


**Figure 4.33** The graph of $f'$, a discontinuous derivative.

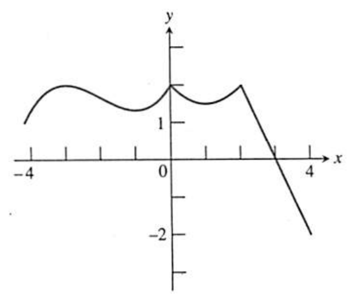


**Figure 4.34** A possible graph of $f$. (Example 9)

Remember also that a function can be continuous and still have points of nondifferentiability (cusps, corners, and points with vertical tangent lines). Thus, a noncontinuous graph of $f'$ could lead to a continuous graph of $f$, as Example 9 shows.

### EXAMPLE 9 Analyzing a Discontinuous Derivative

A function $f$ is continuous on the interval $[-4, 4]$. The discontinuous function $f'$, with domain $[-4, 0) \cup (0, 2) \cup (2, 4]$, is shown in the graph to the right (Figure 4.33).

(a) Find the $x$-coordinates of all local extrema and points of inflection of $f$.

(b) Sketch a possible graph of $f$.

**SOLUTION**

(a) For extrema, we look for places where $f'$ changes sign. There are local maxima at $x = -3$, 0, and 2 (where $f'$ goes from positive to negative) and local minima at $x = -1$ and 1 (where $f'$ goes from negative to positive). There are also local minima at the two endpoints $x = -4$ and 4, because $f'$ starts positive at the left endpoint and ends negative at the right endpoint.

For points of inflection, we look for places where $f''$ changes sign, that is, where the graph of $f'$ changes direction. This occurs only at $x = -2$.

(b) A possible graph of $f$ is shown in Figure 4.34. The derivative information determines the shape of the three components, and the continuity condition determines that the three components must be linked together. ***Now try Exercises 49 and 53.***

### EXPLORATION 2 Finding $f$ from $f'$ and $f''$

A function $f$ is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and $f'$ and $f''$ have the following properties.

| $x$ | $-2 < x < 0$ | $x = 0$ | $0 < x < 2$ | $x = 2$ | $2 < x < 4$ |
|---|---|---|---|---|---|
| $f'$ | + | does not exist | − | 0 | − |
| $f''$ | + | does not exist | + | 0 | − |

1. Find where all absolute extrema of $f$ occur.
2. Find where the points of inflection of $f$ occur.
3. Sketch a possible graph of $f$.

## Quick Review 4.3 *(For help, go to Sections 1.3, 2.2, 3.3, and 3.9.)*

In Exercises 1 and 2, factor the expression and use sign charts to solve the inequality.

**1.** $x^2 - 9 < 0$ **2.** $x^3 - 4x > 0$

In Exercises 3–6, find the domains of $f$ and $f'$.

**3.** $f(x) = xe^x$ **4.** $f(x) = x^{3/5}$

**5.** $f(x) = \dfrac{x}{x-2}$ **6.** $f(x) = x^{2/5}$

In Exercises 7–10, find the horizontal asymptotes of the function's graph.

**7.** $y = (4 - x^2)e^x$ **8.** $y = (x^2 - x)e^{-x}$

**9.** $y = \dfrac{200}{1 + 10e^{-0.5x}}$ **10.** $y = \dfrac{750}{2 + 5e^{-0.1x}}$



## Section 4.3 Exercises

In Exercises 1–6, use the First Derivative Test to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

**1.** $y = x^2 - x - 1$ **2.** $y = -2x^3 + 6x^2 - 3$

**3.** $y = 2x^4 - 4x^2 + 1$ **4.** $y = xe^{1/x}$

**5.** $y = x\sqrt{8 - x^2}$ **6.** $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

In Exercises 7–12, use the Concavity Test to determine the intervals on which the graph of the function is (a) concave up and (b) concave down.

**7.** $y = 4x^3 + 21x^2 + 36x - 20$ **8.** $y = -x^4 + 4x^3 - 4x + 1$

**9.** $y = 2x^{1/5} + 3$ **10.** $y = 5 - x^{1/3}$

**11.** $y = \begin{cases} 2x, & x < 1 \\ 2 - x^2, & x \geq 1 \end{cases}$ **12.** $y = e^x, \quad 0 \leq x \leq 2\pi$

In Exercises 13–20, find all points of inflection of the function.

**13.** $y = xe^x$ **14.** $y = x\sqrt{9 - x^2}$

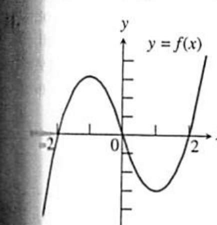
**15.** $y = \tan^{-1} x$ **16.** $y = x^3(4 - x)$

**17.** $y = x^{1/3}(x - 4)$ **18.** $y = x^{1/2}(x + 3)$

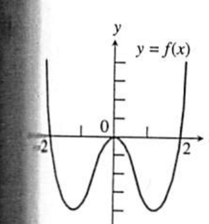
**19.** $y = \dfrac{x^3 - 2x^2 + x - 1}{x - 2}$ **20.** $y = \dfrac{x}{x^2 + 1}$

In Exercises 21 and 22, use the graph of the function $f$ to estimate where (a) $f'$ and (b) $f''$ are 0, positive, and negative.

**21.**

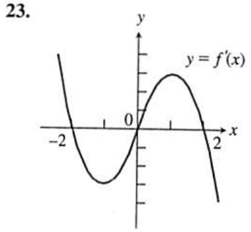



**22.**

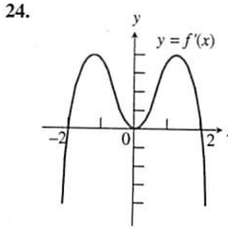



In Exercises 23 and 24, use the graph of the function $f'$ to estimate the intervals on which the function $f$ is (a) increasing or (b) decreasing. Also, (c) estimate the $x$-coordinates of all local extreme values.

**23.**




**24.**



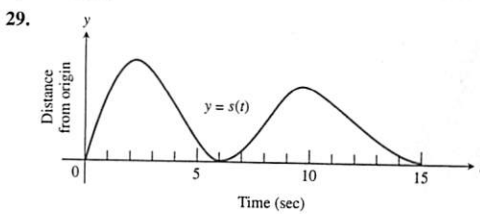

In Exercises 25–28, a particle is moving along the $x$-axis with position function $x(t)$. Find the (a) velocity and (b) acceleration, and (c) describe the motion of the particle for $t \geq 0$.

**25.** $x(t) = t^2 - 4t + 3$ **26.** $x(t) = 6 - 2t - t^2$

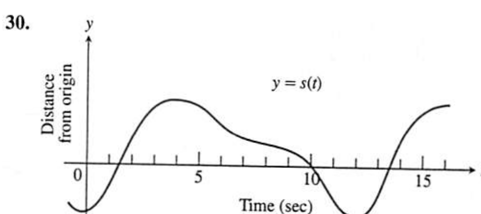
**27.** $x(t) = t^3 - 3t + 3$ **28.** $x(t) = 3t^2 - 2t^3$

In Exercises 29 and 30, the graph of the position function $y = s(t)$ of a particle moving along a line is given. At approximately what times is the particle's (a) velocity equal to zero? (b) acceleration equal to zero?

**29.**




**30.**



**31.** Table 4.3 shows the population of Pennsylvania in each 10-year census between 1830 and 1950.

**Table 4.3 Population of Pennsylvania**

| Years since 1820 | Population in thousands |
|---|---|
| 10 | 1348 |
| 20 | 1724 |
| 30 | 2312 |
| 40 | 2906 |
| 50 | 3522 |
| 60 | 4283 |
| 70 | 5258 |
| 80 | 6302 |
| 90 | 7665 |
| 100 | 8720 |
| 110 | 9631 |
| 120 | 9900 |
| 130 | 10,498 |

*Source:* Bureau of the Census, U.S. Chamber of Commerce.

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve.

(c) Use the regression equation to predict the Pennsylvania population in the 2000 census.

(d) In what year was the Pennsylvania population growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(e) What does the regression equation indicate about the population of Pennsylvania in the long run?

**32.** In 1977, there were 12,168,450 basic cable television subscribers in the U.S. Table 4.4 shows the cumulative number of subscribers added to that baseline number from 1978 to 1985.

**Table 4.4 Growth of Cable Television**

| Years since 1977 | Added Subscribers since 1977 |
|---|---|
| 1 | 1,391,910 |
| 2 | 2,814,380 |
| 3 | 5,671,490 |
| 4 | 11,219,200 |
| 5 | 17,340,570 |
| 6 | 22,113,790 |
| 7 | 25,290,870 |
| 8 | 27,872,520 |

*Source:* Nielsen Media Research, as reported in *The World Almanac and Book of Facts 2004*.

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve. Does it fit the data well?

(c) In what year between 1977 and 1985 were basic cable TV subscriptions growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(d) What does the regression equation indicate about the number of basic cable television subscribers in the long run? (Be sure to add the baseline 1977 number.)

(e) **Writing to Learn** In fact, the long-run number of basic cable subscribers predicted by the regression equation falls short of the actual 2002 number by more than 32 million. What circumstances changed to render the earlier model so ineffective?

In Exercises 33–38, use the Second Derivative Test to find the local extrema for the function.

**33.** $y = 3x - x^3 + 5$

**34.** $y = x^5 - 80x + 100$

**35.** $y = x^3 + 3x^2 - 2$

**36.** $y = 3x^5 - 25x^3 + 60x + 20$

**37.** $y = xe^x$

**38.** $y = xe^{-x}$

In Exercises 39 and 40, use the derivative of the function $y = f(x)$ to find the points at which $f$ has a

(a) local maximum, (b) local minimum, or

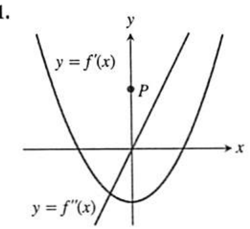
(c) point of inflection.

**39.** $y' = (x - 1)^2(x - 2)$

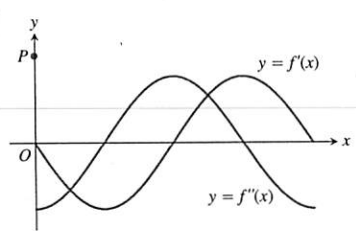
**40.** $y' = (x - 1)^2(x - 2)(x - 4)$

Exercises 41 and 42 show the graphs of the first and second derivatives of a function $y = f(x)$. Copy the figure and add a sketch of a possible graph of $f$ that passes through the point $P$.

**41.**




**42.**






**43. Writing to Learn** If $f(x)$ is a differentiable function and $f'(c) = 0$ at an interior point $c$ of $f$'s domain, must $f$ have a local maximum or minimum at $x = c$? Explain.

**44. Writing to Learn** If $f(x)$ is a twice-differentiable function and $f''(c) = 0$ at an interior point $c$ of $f$'s domain, must $f$ have an inflection point at $x = c$? Explain.

**45. *Connecting f and f′*** Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

**46. *Connecting f and f″*** Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

**47. *Connecting f, f′, and f″*** Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

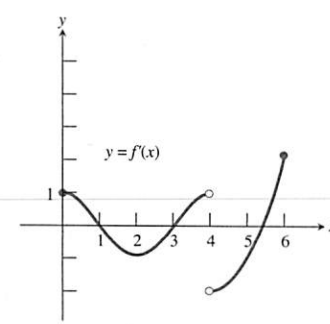
| | |
|---|---|
| $f(-2) = 8$ | $f'(x) > 0$ for $\lvert x\rvert > 2$ |
| $f(0) = 4$ | $f'(x) < 0$ for $\lvert x\rvert < 2$ |
| $f(2) = 0$ | $f''(x) < 0$ for $x < 0$ |
| $f'(2) = f'(-2) = 0$ | $f''(x) > 0$ for $x > 0$ |

**48. *Using Behavior to Sketch*** Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

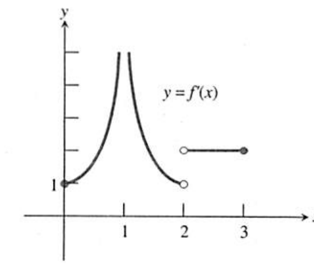
| $x$ | $y$ | Curve |
|---|---|---|
| $x < 2$ | | falling, concave up |
| 2 | 1 | horizontal tangent |
| $2 < x < 4$ | | rising, concave up |
| 4 | 4 | inflection point |
| $4 < x < 6$ | | rising, concave down |
| 6 | 7 | horizontal tangent |
| $x > 6$ | | falling, concave down |

In Exercises 49 and 50, use the graph of $f'$ to estimate the intervals on which the function $f$ is (a) increasing or (b) decreasing. Also, (c) estimate the $x$-coordinates of all local extreme values. (Assume that the function $f$ is continuous, even at the points where $f'$ is undefined.)

**49.** The domain of $f'$ is $[0, 4) \cup (4, 6]$.




**50.** The domain of $f'$ is $[0, 1) \cup (1, 2) \cup (2, 3]$.




**Group Activity** In Exercises 51 and 52, do the following.

(a) Find the absolute extrema of $f$ and where they occur.

(b) Find any points of inflection.

(c) Sketch a possible graph of $f$.

**51.** $f$ is continuous on $[0, 3]$ and satisfies the following.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f$ | 0 | 2 | 0 | −2 |
| $f'$ | 3 | 0 | does not exist | −3 |
| $f''$ | 0 | −1 | does not exist | 0 |

| $x$ | $0 < x < 1$ | $1 < x < 2$ | $2 < x < 3$ |
|---|---|---|---|
| $f$ | + | + | − |
| $f'$ | + | − | − |
| $f''$ | − | − | − |

**52.** $f$ is an even function, continuous on $[-3, 3]$, and satisfies the following.

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f$ | 2 | 0 | −1 |
| $f'$ | does not exist | 0 | does not exist |
| $f''$ | does not exist | 0 | does not exist |

| $x$ | $0 < x < 1$ | $1 < x < 2$ | $2 < x < 3$ |
|---|---|---|---|
| $f$ | + | − | − |
| $f'$ | − | − | + |
| $f''$ | + | − | − |

(d) What can you conclude about $f(3)$ and $f(-3)$?

**Group Activity** In Exercises 53 and 54, sketch a possible graph of a continuous function $f$ that has the given properties.

**53.** Domain $[0, 6]$, graph of $f'$ given in Exercise 49, and $f(0) = 2$.

**54.** Domain $[0, 3]$, graph of $f'$ given in Exercise 50, and $f(0) = -3$.

## Standardized Test Questions

You should solve the following problems without using a graphing calculator.

**55. True or False** If $f''(c) = 0$, then $(c, f(c))$ is a point of inflection. Justify your answer.

**56. True or False** If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum. Justify your answer.

**57. Multiple Choice** If $a < 0$, the graph of $y = ax^3 + 3x^2 + 4x + 5$ is concave up on

(A) $\left(-\infty, -\frac{1}{a}\right)$ (B) $\left(-\infty, \frac{1}{a}\right)$ (C) $\left(-\frac{1}{a}, \infty\right)$

(D) $\left(\frac{1}{a}, \infty\right)$ (E) $(-\infty, -1)$

**58. Multiple Choice** If $f(0) = f'(0) = f''(0) = 0$, which of the following *must be true*?

(A) There is a local maximum of $f$ at the origin.
(B) There is a local minimum of $f$ at the origin.
(C) There is no local extremum of $f$ at the origin.
(D) There is a point of inflection of the graph of $f$ at the origin.
(E) There is a horizontal tangent to the graph of $f$ at the origin.

**59. Multiple Choice** The $x$-coordinates of the points of inflection of the graph of $y = x^5 - 5x^4 + 3x + 7$ are

(A) 0 only (B) 1 only (C) 3 only (D) 0 and 3 (E) 0 and 1

**60. Multiple Choice** Which of the following conditions would enable you to conclude that the graph of $f$ has a point of inflection at $x = c$?

(A) There is a local maximum of $f'$ at $x = c$.
(B) $f''(c) = 0$.
(C) $f''(c)$ does not exist.
(D) The sign of $f'$ changes at $x = c$.
(E) $f$ is a cubic polynomial and $c = 0$.

## Exploration

**61. *Graphs of Cubics*** There is almost no leeway in the location of the inflection point and the extrema of $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, because the one inflection point occurs at $x = -b/(3a)$ and the extrema, if any, must be located symmetrically about this value of $x$. Check this out by examining **(a)** the cubic in Exercise 7 and **(b)** the cubic in Exercise 2. Then **(c)** prove the general case.

## Extending the Ideas

In Exercises 62 and 63, feel free to use a CAS (computer algebra system), if you have one, to solve the problem.

**62. *Logistic Functions*** Let $f(x) = c/(1 + ae^{-bx})$ with $a > 0$, $abc \neq 0$.

**(a)** Show that $f$ is increasing on the interval $(-\infty, \infty)$ if $abc > 0$, and decreasing if $abc < 0$.

**(b)** Show that the point of inflection of $f$ occurs at $x = (\ln|a|)/b$.

**63. *Quartic Polynomial Functions*** Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $a \neq 0$.

**(a)** Show that the graph of $f$ has 0 or 2 points of inflection.

**(b)** Write a condition that must be satisfied by the coefficients if the graph of $f$ has 0 or 2 points of inflection.

## Quick Quiz for AP* Preparation: Sections 4.1–4.3

You should solve these problems without using a graphing calculator.

**1. Multiple Choice** How many critical points does the function $f(x) = (x - 2)^5 (x + 3)^4$ have?

(A) One (B) Two (C) Three (D) Five (E) Nine

**2. Multiple Choice** For what value of $x$ does the function $f(x) = (x - 2)(x - 3)^2$ have a relative maximum?

(A) $-3$ (B) $-\frac{7}{3}$ (C) $-\frac{5}{2}$ (D) $\frac{7}{3}$ (E) $\frac{5}{2}$

**3. Multiple Choice** If $g$ is a differentiable function such that $g(x) < 0$ for all real numbers $x$, and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true?

(A) $f$ has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.
(B) $f$ has a relative minimum at $x = -3$ and a relative maximum at $x = 3$.
(C) $f$ has relative minima at $x = -3$ and at $x = 3$.
(D) $f$ has relative maxima at $x = -3$ and at $x = 3$.
(E) It cannot be determined if $f$ has any relative extrema.

**4. Free Response** Let $f$ be the function given by $f(x) = 3\ln(x^2 + 2) - 2x$ with domain $[-2, 4]$.

**(a)** Find the coordinate of each relative maximum point and each relative minimum point of $f$. Justify your answer.

**(b)** Find the $x$-coordinate of each point of inflection of the graph of $f$.

**(c)** Find the absolute maximum value of $f(x)$.



# 4.4 Modeling and Optimization

**What you'll learn about**

- Examples from Mathematics
- Examples from Business and Industry
- Examples from Economics
- Modeling Discrete Phenomena with Differentiable Functions

**. . . and why**

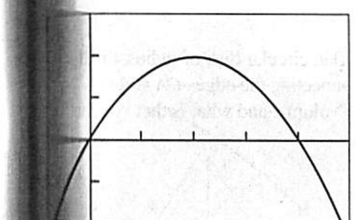
Historically, optimization problems were among the earliest applications of what we now call differential calculus.

## Examples from Mathematics

While today's graphing technology makes it easy to find extrema without calculus, the algebraic methods of differentiation were understandably more practical, and certainly more accurate, when graphs had to be rendered by hand. Indeed, one of the oldest applications of what we now call "differential calculus" (pre-dating Newton and Leibniz) was to find maximum and minimum values of functions by finding where horizontal tangent lines might occur. We will use both algebraic and graphical methods in this section to solve "max-min" problems in a variety of contexts, but the emphasis will be on the *modeling* process that both methods have in common. Here is a strategy you can use:

### Strategy for Solving Max-Min Problems

1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.



[−5, 25] by [−100, 150]

**Figure 4.35** The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 1)

### EXAMPLE 1 Using the Strategy

Find two numbers whose sum is 20 and whose product is as large as possible.

**SOLUTION**

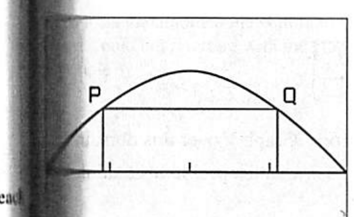
**Model** If one number is $x$, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

**Solve Graphically** We can see from the graph of $f$ in Figure 4.35 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

**Interpret** The two numbers we seek are $x = 10$ and $20 - x = 10$.

*Now try Exercise 1.*

Sometimes we find it helpful to use both analytic and graphical methods together, as in Example 2.




[0, π] by [−0.5, 1.5]

**Figure 4.36** A rectangle inscribed under one arch of $y = \sin x$. (Example 2)

### EXAMPLE 2 Inscribing Rectangles

A rectangle is to be inscribed under one arch of the sine curve (Figure 4.36). What is the largest area the rectangle can have, and what dimensions give that area?

*continued*

## Quick Review 4.4 *(For help, go to Sections 1.6, 4.1, and Appendix A.1.)*

1. Use the first derivative test to identify the local extrema of $y = x^3 - 6x^2 + 12x - 8$.

2. Use the second derivative test to identify the local extrema of $y = 2x^3 + 3x^2 - 12x - 3$.

3. Find the volume of a cone with radius 5 cm and height 8 cm.

4. Find the dimensions of a right circular cylinder with volume 1000 cm$^3$ and surface area 600 cm$^2$.

In Exercises 5–8, rewrite the expression as a trigonometric function of the angle $\alpha$.

5. $\sin(-\alpha)$

6. $\cos(-\alpha)$

7. $\sin(\pi - \alpha)$

8. $\cos(\pi - \alpha)$

In Exercises 9 and 10, use substitution to find the exact solutions of the system of equations.

9. $\begin{cases} x^2 + y^2 = 4 \\ y = \sqrt{3}x \end{cases}$

10. $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \\ y = x + 3 \end{cases}$

## Section 4.4 Exercises

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

1. ***Finding Numbers*** The sum of two nonnegative numbers is 20. Find the numbers if

   (a) the sum of their squares is as large as possible; as small as possible.

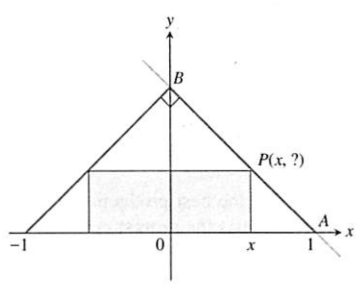
   (b) one number plus the square root of the other is as large as possible; as small as possible.

2. ***Maximizing Area*** What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. ***Maximizing Perimeter*** What is the smallest perimeter possible for a rectangle whose area is 16 in$^2$, and what are its dimensions?

4. ***Finding Area*** Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.

5. ***Inscribing Rectangles*** The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.




   (a) Express the $y$-coordinate of $P$ in terms of $x$. [*Hint:* Write an equation for the line $AB$.]

   (b) Express the area of the rectangle in terms of $x$.

   (c) What is the largest area the rectangle can have, and what are its dimensions?

6. ***Largest Rectangle*** A rectangle has its base on the $x$-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

7. ***Optimal Dimensions*** You are planning to make an open rectangular box from an 8- by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

8. ***Closing Off the First Quadrant*** You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

9. ***The Best Fencing Plan*** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

10. ***The Shortest Fence*** A 216-m$^2$ rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?



11. ***Designing a Tank*** Your iron works has contracted to design and build a 500-ft$^3$, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

    (a) What dimensions do you tell the shop to use?

    (b) **Writing to Learn** Briefly describe how you took weight into account.

12. ***Catching Rainwater*** A 1125-ft$^3$ open-top rectangular tank with a square base $x$ ft on a side and $y$ ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product $xy$.

    (a) If the total cost is

    $$c = 5(x^2 + 4xy) + 10xy,$$

    what values of $x$ and $y$ will minimize it?

    (b) **Writing to Learn** Give a possible scenario for the cost function in (a).

13. ***Designing a Poster*** You are designing a rectangular poster to contain 50 in$^2$ of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

14. ***Vertical Motion*** The height of an object moving vertically is given by

    $$s = -16t^2 + 96t + 112,$$

    with $s$ in ft and $t$ in sec. Find (a) the object's velocity when $t = 0$, (b) its maximum height and when it occurs, and (c) its velocity when $s = 0$.

15. ***Finding an Angle*** Two sides of a triangle have lengths $a$ and $b$, and the angle between them is $\theta$. What value of $\theta$ will maximize the triangle's area? [*Hint:* $A = (1/2)\,ab \sin\theta$.]

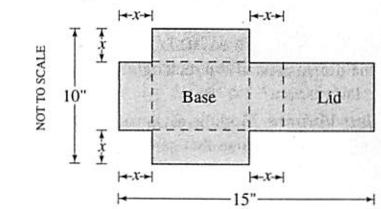
16. ***Designing a Can*** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm$^3$? Compare the result here with the result in Example 4.

17. ***Designing a Can*** You are designing a 1000-cm$^3$ right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius $r$ will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

    $$A = 8r^2 + 2\pi rh$$

    rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 4. In Example 4 the ratio of $h$ to $r$ for the most economical can was 2 to 1. What is the ratio now?

18. ***Designing a Box with Lid*** A piece of cardboard measures 10- by 15-in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



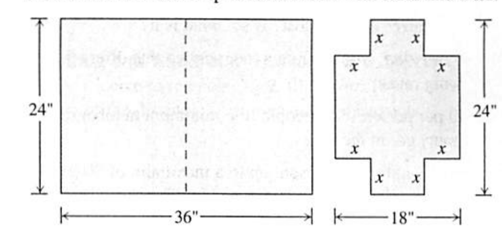

    (a) Write a formula $V(x)$ for the volume of the box.

    (b) Find the domain of $V$ for the problem situation and graph $V$ over this domain.

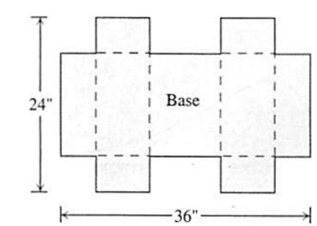
    (c) Use a graphical method to find the maximum volume and the value of $x$ that gives it.

    (d) Confirm your result in part (c) analytically.

19. ***Designing a Suitcase*** A 24- by 36-in. sheet of cardboard is folded in half to form a 24- by 18-in. rectangle as shown in the figure. Then four congruent squares of side length $x$ are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.




The sheet is then unfolded.




    (a) Write a formula $V(x)$ for the volume of the box.

    (b) Find the domain of $V$ for the problem situation and graph $V$ over this domain.

    (c) Use a graphical method to find the maximum volume and the value of $x$ that gives it.

    (d) Confirm your result in part (c) analytically.

    (e) Find a value of $x$ that yields a volume of 1120 in$^3$.

    (f) **Writing to Learn** Write a paragraph describing the issues that arise in part (b).

20. ***Quickest Route*** Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

21. ***Inscribing Rectangles*** A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

22. ***Maximizing Volume*** Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

23. ***Maximizing Profit*** Suppose $r(x) = 8\sqrt{x}$ represents revenue and $c(x) = 2x^2$ represents cost, with $x$ measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

24. ***Maximizing Profit*** Suppose $r(x) = x^2/(x^2 + 1)$ represents revenue and $c(x) = (x - 1)^3/3 - 1/3$ represents cost, with $x$ measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

25. ***Minimizing Average Cost*** Suppose $c(x) = x^3 - 10x^2 - 30x$, where $x$ is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

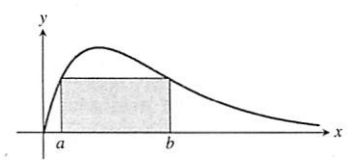
26. ***Minimizing Average Cost*** Suppose $c(x) = xe^x - 2x^2$, where $x$ is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

27. ***Tour Service*** You operate a tour service that offers the following rates:

- \$200 per person if 50 people (the minimum number to book the tour) go on the tour.
- For each additional person, up to a maximum of 80 people total, the rate per person is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

28. **Group Activity** The figure shows the graph of $f(x) = xe^{-x}$, $x \geq 0$.




(a) Find where the absolute maximum of $f$ occurs.

(b) Let $a > 0$ and $b > 0$ be given as shown in the figure. Complete the following table where $A$ is the area of the rectangle in the figure.

| $a$ | $b$ | $A$ |
|---|---|---|
| 0.1 | | |
| 0.2 | | |
| 0.3 | | |
| ⋮ | | |
| 1 | | |

(c) Draw a scatter plot of the data $(a, A)$.

(d) Find the quadratic, cubic, and quartic regression equations for the data in part (b), and superimpose their graphs on a scatter plot of the data.

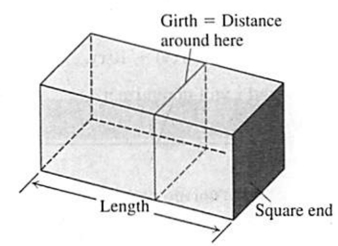
(e) Use each of the regression equations in part (d) to estimate the maximum possible value of the area of the rectangle.

29. ***Cubic Polynomial Functions*** Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$.

(a) Show that $f$ has either 0 or 2 local extrema.

(b) Give an example of each possibility in part (a).

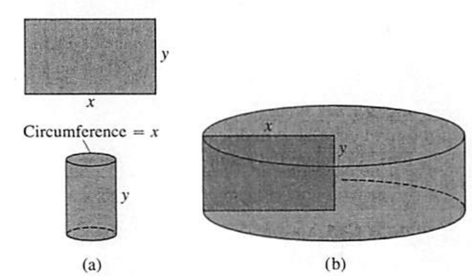
30. ***Shipping Packages*** The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around), as shown in the figure, does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



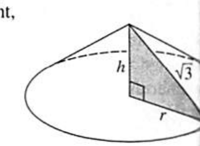

31. ***Constructing Cylinders*** Compare the answers to the following two construction problems.

(a) A rectangular sheet of perimeter 36 cm and dimensions $x$ cm by $y$ cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of $x$ and $y$ give the largest volume?

(b) The same sheet is to be revolved about one of the sides of length $y$ to sweep out the cylinder as shown in part (b) of the figure. What values of $x$ and $y$ give the largest volume?




32. ***Constructing Cones*** A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



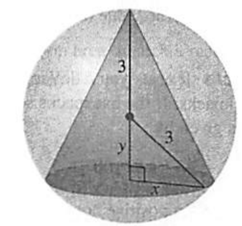



***Finding Parameter Values*** What value of $a$ makes $f(x) = x^2 + (a/x)$ have (a) a local minimum at $x = 2$? (b) a point of inflection at $x = 1$?

***Finding Parameter Values*** Show that $f(x) = x^2 + (a/x)$ cannot have a local maximum for any value of $a$.

***Finding Parameter Values*** What values of $a$ and $b$ make $f(x) = x^3 + ax^2 + bx$ have (a) a local maximum at $x = -1$ and a local minimum at $x = 3$? (b) a local minimum at $x = 4$ and a point of inflection at $x = 1$?

***Inscribing a Cone*** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

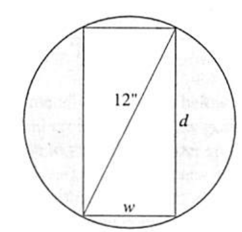



***Strength of a Beam*** The strength $S$ of a rectangular wooden beam is proportional to its width times the square of its depth.

(a) Find the dimensions of the strongest beam that can be cut from a 12-in. diameter cylindrical log.

(b) **Writing to Learn** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) **Writing to Learn** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of $k$? Try it.




***Stiffness of a Beam*** The stiffness $S$ of a rectangular beam is proportional to its width times the cube of its depth.

(a) Find the dimensions of the stiffest beam that can be cut from a 12-in. diameter cylindrical log.

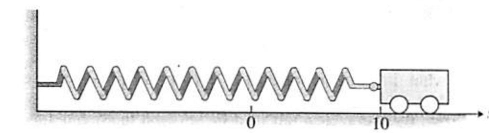
(b) **Writing to Learn** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) **Writing to Learn** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). Wha would be the effect of changing to some other value of $k$? Try it.

39. ***Frictionless Cart*** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time $t$ is $s = 10\cos \pi t$.

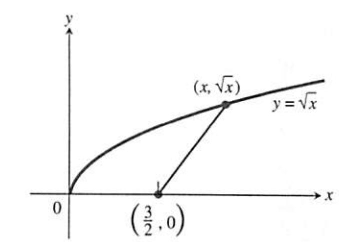
(a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

(b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?




40. ***Electrical Current*** Suppose that at any time $t$ (sec) the current $i$ (amp) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak (largest magnitude) current for this circuit?

41. ***Calculus and Geometry*** How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? [*Hint:* If you minimize the *square* of the distance, you can avoid square roots.]




42. ***Calculus and Geometry*** How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$?

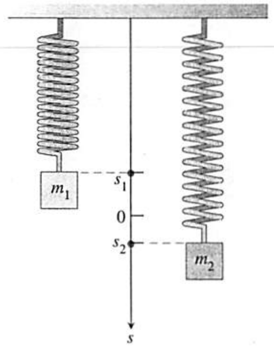
43. **Writing to Learn** Is the function $f(x) = x^2 - x + 1$ ever negative? Explain.

44. **Writing to Learn** You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

(a) Explain why you need to consider values of $x$ only in the interval $[0, 2\pi]$.

(b) Is $f$ ever negative? Explain.

45. ***Vertical Motion*** Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively, with $s_1$ and $s_2$ in meters and $t$ in seconds.




(a) At what times in the interval $t > 0$ do the masses pass each other? [*Hint:* $\sin 2t = 2 \sin t \cos t$.]

(b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2 \cos^2 t - 1$.)

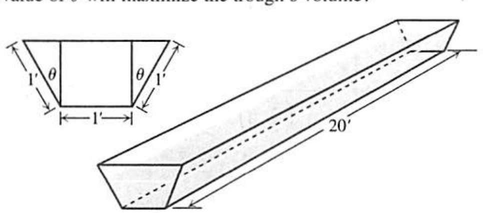
46. ***Motion on a Line*** The positions of two particles on the $s$-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with $s_1$ and $s_2$ in meters and $t$ in seconds.

(a) At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

(b) What is the farthest apart that the particles ever get?

(c) When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

47. ***Finding an Angle*** The trough in the figure is to be made to the dimensions shown. Only the angle $\theta$ can be varied. What value of $\theta$ will maximize the trough's volume?

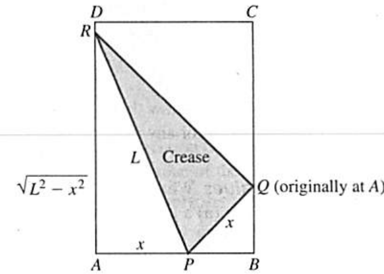



48. **Group Activity** ***Paper Folding*** A rectangular sheet of 8 1/2- by 11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length $L$. Try it with paper.

(a) Show that $L^2 = 2x^3/(2x - 8.5)$.

(b) What value of $x$ minimizes $L^2$?

(c) What is the minimum value of $L$?




49. ***Sensitivity to Medicine*** *(continuation of Exercise 48, Section 3.3)* Find the amount of medicine to which the body is most sensitive by finding the value of $M$ that maximizes the derivative $dR/dM$.

50. ***Selling Backpacks*** It costs you $c$ dollars each to manufacture and distribute backpacks. If the backpacks sell at $x$ dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where $a$ and $b$ are certain positive constants. What selling price will bring a maximum profit?

## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

51. **True or False** A continuous function on a closed interval must attain a maximum value on that interval. Justify your answer.

52. **True or False** If $f'(c) = 0$ and $f(c)$ is not a local maximum, then $f(c)$ is a local minimum. Justify your answer.

53. **Multiple Choice** Two positive numbers have a sum of 60. What is the maximum product of one number times the square of the second number?

(A) 3481

(B) 3600

(C) 27,000

(D) 32,000

(E) 36,000

54. **Multiple Choice** A continuous function $f$ has domain [1, 25] and range [3, 30]. If $f'(x) < 0$ for all $x$ between 1 and 25, what is $f(25)$?

(A) 1

(B) 3

(C) 25

(D) 30

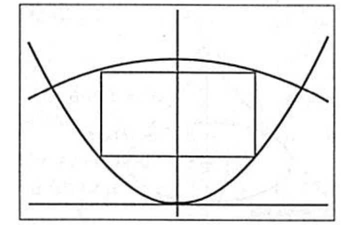
(E) impossible to determine from the information given



55. **Multiple Choice** What is the maximum area of a right triangle with hypotenuse 10?

(A) 24 (B) 25 (C) $25\sqrt{2}$ (D) 48 (E) 50

56. **Multiple Choice** A rectangle is inscribed between the parabolas $y = 4x^2$ and $y = 30 - x^2$ as shown below:



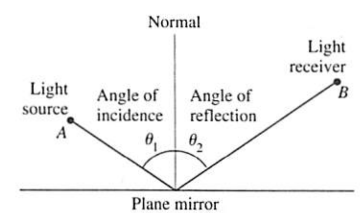
[−3, 3] by [−2, 40]

What is the maximum area of such a rectangle ?

(A) $20\sqrt{2}$ (B) 40 (C) $30\sqrt{2}$ (D) 50 (E) $40\sqrt{2}$

## Explorations

57. ***Fermat's Principle in Optics*** Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Light from a source $A$ is reflected by a plane mirror to a receiver at point $B$, as shown in the figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)




58. ***Tin Pest*** When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious. And indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, $v$ may be considered to be a function of $x$ alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x$ = the amount of product,

$a$ = the amount of substance at the beginning,

$k$ = a positive constant.

At what value of $x$ does the rate $v$ have a maximum? What is the maximum value of $v$?

59. ***How We Cough*** When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity $v$ (in cm/sec) can be modeled by the equation

$$v = c(r_0 - r)r^2, \quad \frac{r_0}{2} \le r \le r_0,$$

where $r_0$ is the rest radius of the trachea in cm and $c$ is a positive constant whose value depends in part on the length of the trachea.

(a) Show that $v$ is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

(b) Take $r_0$ to be 0.5 and $c$ to be 1, and graph $v$ over the interval $0 \le r \le 0.5$. Compare what you see to the claim that $v$ is a maximum when $r = (2/3)r_0$.

60. ***Wilson Lot Size Formula*** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where $q$ is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), $k$ is the cost of placing an order (the same, no matter how often you order), $c$ is the cost of one item (a constant), $m$ is the number of items sold each week (a constant), and $h$ is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

(a) Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula.*)

(b) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace $k$ by $k + bq$, the sum of $k$ and a constant multiple of $q$. What is the most economical quantity to order now?

61. ***Production Level*** Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

62. ***Production Level*** Suppose $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing $x$ items. Find a production level that will minimize the average cost of making $x$ items.

## Extending the Ideas

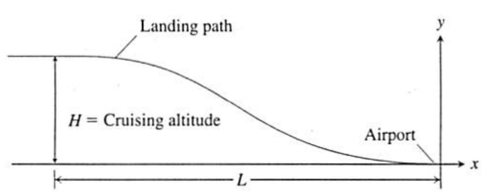
63. ***Airplane Landing Path*** An airplane is flying at altitude $H$ when it begins its descent to an airport runway that is at horizontal ground distance $L$ from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$ where $y(-L) = H$ and $y(0) = 0$.

(a) What is $dy/dx$ at $x = 0$?

(b) What is $dy/dx$ at $x = -L$?

(c) Use the values for $dy/dx$ at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$




In Exercises 64 and 65, you might find it helpful to use a CAS.

64. ***Generalized Cone Problem*** A cone of height $h$ and radius $r$ is constructed from a flat, circular disk of radius $a$ in. as described in Exploration 1.

(a) Find a formula for the volume $V$ of the cone in terms of $x$ and $a$.

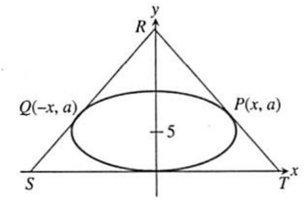
(b) Find $r$ and $h$ in the cone of maximum volume for $a = 4, 5, 6, 8$.

(c) **Writing to Learn** Find a simple relationship between $r$ and $h$ that is independent of $a$ for the cone of maximum volume. Explain how you arrived at your relationship.

65. ***Circumscribing an Ellipse*** Let $P(x, a)$ and $Q(-x, a)$ be two points on the upper half of the ellipse

$$\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$$

centered at $(0, 5)$. A triangle $RST$ is formed by using the tangent lines to the ellipse at $Q$ and $P$ as shown in the figure.




(a) Show that the area of the triangle is

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2,$$

where $y = f(x)$ is the function representing the upper half of the ellipse.

(b) What is the domain of $A$? Draw the graph of $A$. How are the asymptotes of the graph related to the problem situation?

(c) Determine the height of the triangle with minimum area. How is it related to the $y$-coordinate of the center of the ellipse?

(d) Repeat parts (a)–(c) for the ellipse

$$\frac{x^2}{C^2} + \frac{(y-B)^2}{B^2} = 1$$

centered at $(0, B)$. Show that the triangle has minimum area when its height is $3B$.



# 4.5 Linearization and Newton's Method

**What you'll learn about**

- Linear Approximation
- Newton's Method
- Differentials
- Estimating Change with Differentials
- Absolute, Relative, and Percentage Change
- Sensitivity to Change

**. . . and why**

Engineering and science depend on approximations in most practical applications; it is important to understand how approximation techniques work.

## Linear Approximation

In our study of the derivative we have frequently referred to the "tangent line to the curve" at a point. What makes that tangent line so important mathematically is that it provides a *useful representation of the curve itself* if we stay close enough to the point of tangency. We say that differentiable curves are always **locally linear,** a fact that can best be appreciated graphically by zooming in at a point on the curve, as Exploration 1 shows.

**EXPLORATION 1** **Appreciating Local Linearity**

The function $f(x) = (x^2 + 0.0001)^{1/4} + 0.9$ is differentiable at $x = 0$ and hence "locally linear" there. Let us explore the significance of this fact with the help of a graphing calculator.

1. Graph $y = f(x)$ in the "ZoomDecimal" window. What appears to be the behavior of the function at the point $(0, 1)$?
2. Show algebraically that $f$ is differentiable at $x = 0$. What is the equation of the tangent line at $(0, 1)$?
3. Now zoom in repeatedly, keeping the cursor at $(0, 1)$. What is the long-range outcome of repeated zooming?
4. The graph of $y = f(x)$ eventually looks like the graph of a line. What line is it?

We hope that this exploration gives you a new appreciation for the tangent line. As you zoom in on a differentiable function, its graph at that point actually seems to *become* the graph of the tangent line! This observation—that even the most complicated differentiable curve behaves locally like the simplest graph of all, a straight line—is the basis for most of the applications of differential calculus. It is what allows us, for example, to refer to the derivative as the "slope of the curve" or as "the velocity at time $t_0$."

Algebraically, the principle of local linearity means that the *equation* of the tangent line defines a function that can be used to *approximate* a differentiable function near the point of tangency. In recognition of this fact, we give the equation of the tangent line a new name: the *linearization of f at a.* Recall that the tangent line at $(a, f(a))$ has point-slope equation $y - f(a) = f'(x)(x - a)$ (Figure 4.44).

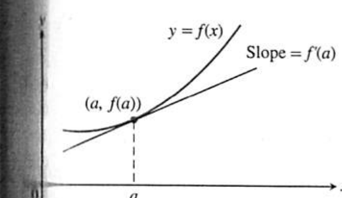



**Figure 4.44** The tangent to the curve $y = f(x)$ at $x = a$ is the line $y = f(a) + f'(a)(x - a)$.

**DEFINITION** **Linearization**

If $f$ is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of $f$ at $a$.** The approximation $f(x) \approx L(x)$ is the **standard linear approximation of $f$ at $a$.** The point $x = a$ is the **center** of the approximation.

## Sensitivity to Change

The equation $df = f'(x)\,dx$ tells how *sensitive* the output of $f$ is to a change in input at dif-ferent values of $x$. The larger the value of $f'$ at $x$, the greater the effect of a given change $dx$

### EXAMPLE 13 Finding Depth of a Well

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1 sec error in measuring the time?

**SOLUTION**

The size of $ds$ in the equation

$$ds = 32t\,dt$$

depends on how big $t$ is. If $t = 2$ sec, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later at $t = 5$ sec, the error caused by the same $dt$ is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

***Now try Exercise 53***

## Quick Review 4.5 *(For help, go to Sections 3.3, 3.6, and 3.9.)*

In Exercises 1 and 2, find $dy/dx$.

1. $y = \sin(x^2 + 1)$
2. $y = \dfrac{x + \cos x}{x + 1}$

In Exercises 3 and 4, solve the equation graphically.

3. $xe^{-x} + 1 = 0$
4. $x^3 + 3x + 1 = 0$

In Exercises 5 and 6, let $f(x) = xe^{-x} + 1$. Write an equation for the line tangent to $f$ at $x = c$.

5. $c = 0$
6. $c = -1$
7. Find where the tangent line in **(a)** Exercise 5 and **(b)** Exercise 6 crosses the $x$-axis.
8. Let $g(x)$ be the function whose graph is the tangent line to the graph of $f(x) = x^3 - 4x + 1$ at $x = 1$. Complete the table.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 0.7 | | |
| 0.8 | | |
| 0.9 | | |
| 1 | | |
| 1.1 | | |
| 1.2 | | |
| 1.3 | | |

In Exercises 9 and 10, graph $y = f(x)$ and its tangent line at $x = c$.

9. $c = 1.5$, $f(x) = \sin x$
10. $c = 4$, $f(x) = \begin{cases} -\sqrt{3 - x}, & x < 3 \\ \sqrt{x - 3}, & x \ge 3 \end{cases}$

## Section 4.5 Exercises

In Exercises 1–6, **(a)** find the linearization $L(x)$ of $f(x)$ at $x = a$. **(b)** How accurate is the approximation $L(a + 0.1) \approx f(a + 0.1)$? See the comparisons following Example 1.

1. $f(x) = x^3 - 2x + 3$, $a = 2$
2. $f(x) = \sqrt{x^2 + 9}$, $a = -4$
3. $f(x) = x + \dfrac{1}{x}$, $a = 1$
4. $f(x) = \ln(x + 1)$, $a = 0$
5. $f(x) = \tan x$, $a = \pi$
6. $f(x) = \cos^{-1} x$, $a = 0$
7. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.
8. Use the linearization $(1 + x)^k \approx 1 + kx$ to approximate the following. State how accurate your approximation is.
   **(a)** $(1.002)^{100}$ **(b)** $\sqrt[3]{1.009}$

In Exercises 9 and 10, use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of $x$ near zero.

9. **(a)** $f(x) = (1 - x)^6$ **(b)** $f(x) = \dfrac{2}{1 - x}$ **(c)** $f(x) = \dfrac{1}{\sqrt{1 + x}}$
10. **(a)** $f(x) = (4 + 3x)^{1/3}$ **(b)** $f(x) = \sqrt{2 + x^2}$
    **(c)** $f(x) = \sqrt[3]{\left(1 - \dfrac{1}{2 + x}\right)^2}$

In Exercises 11–14, approximate the root by using a linearization centered at an appropriate nearby number.

11. $\sqrt{101}$
12. $\sqrt[3]{26}$
13. $\sqrt[3]{998}$
14. $\sqrt{80}$

In Exercises 15–18, use Newton's method to estimate all real solutions of the equation. Make your answers accurate to 6 decimal places.

15. $x^3 + x - 1 = 0$
16. $x^4 + x - 3 = 0$
17. $x^2 - 2x + 1 = \sin x$
18. $x^4 - 2 = 0$

In Exercises 19–26, **(a)** find $dy$, and **(b)** evaluate $dy$ for the given value of $x$ and $dx$.

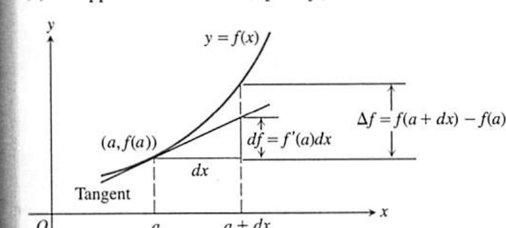
19. $y = x^3 - 3x$, $x = 2$, $dx = 0.05$
20. $y = \dfrac{2x}{1 + x^2}$, $x = -2$, $dx = 0.1$
21. $y = x^2 \ln x$, $x = 1$, $dx = 0.01$
22. $y = x\sqrt{1 - x^2}$, $x = 0$, $dx = -0.2$
23. $y = e^{\sin x}$, $x = \pi$, $dx = -0.1$
24. $y = 3\csc\left(1 - \dfrac{x}{3}\right)$, $x = 1$, $dx = 0.1$
25. $y + xy - x = 0$, $x = 0$, $dx = 0.01$
26. $2y = x^2 - xy$, $x = 2$, $dx = -0.05$

In Exercises 27–30, find the differential.

27. $d(\sqrt{1 - x^2})$
28. $d(e^{5x} + x^5)$
29. $d(\arctan 4x)$
30. $d(8^x + x^8)$

In Exercises 31–34, the function $f$ changes value when $x$ changes from $a$ to $a + dx$. Find

**(a)** the true change $\Delta f = f(a + dx) - f(a)$.
**(b)** the estimated change $df = f'(a)\,dx$.
**(c)** the approximation error $|\Delta f - df|$.



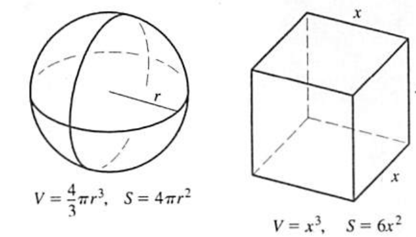

31. $f(x) = x^2 + 2x$, $a = 0$, $dx = 0.1$
32. $f(x) = x^3 - x$, $a = 1$, $dx = 0.1$



33. $f(x) = x^{-1}$, $a = 0.5$, $dx = 0.05$
34. $f(x) = x^4$, $a = 1$, $dx = 0.01$

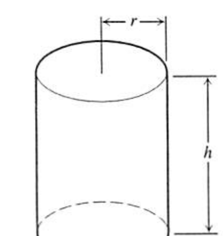
In Exercises 35–40, write a differential formula that estimates the given change in volume or surface area. Then use the formula to estimate the change when the dependent variable changes from 10 cm to 10.05 cm.

35. ***Volume*** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from $a$ to $a + dr$
36. ***Surface Area*** The change in the surface area $S = 4\pi r^2$ of a sphere when the radius changes from $a$ to $a + dr$




$V = \frac{4}{3}\pi r^3,\ S = 4\pi r^2$ $\qquad$ $V = x^3,\ S = 6x^2$

37. ***Volume*** The change in the volume $V = x^3$ of a cube when the edge lengths change from $a$ to $a + dx$
38. ***Surface Area*** The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from $a$ to $a + dx$
39. ***Volume*** The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from $a$ to $a + dr$ and the height does not change
40. ***Surface Area*** The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from $a$ to $a + dh$ and the radius does not change




$V = \pi r^2 h,\ S = 2\pi rh$

In Exercises 41–44, use differentials to estimate the maximum error in measurement resulting from the tolerance of error in the dependent variable. Express answers to the nearest tenth, since that is the precision used to express the tolerance.

41. The area of a circle with radius $10 \pm 0.1$ in.
42. The volume of a sphere with radius $8 \pm 0.3$ in.
43. The volume of a cube with side $15 \pm 0.2$ cm.

44. The area of an equilateral triangle with side $20 \pm 0.5$ cm.

45. ***Linear Approximation*** Let $f$ be a function with $f(0) = 1$ and $f'(x) = \cos(x^2)$.

(a) Find the linearization of $f$ at $x = 0$.

(b) Estimate the value of $f$ at $x = 0.1$.

(c) **Writing to Learn** Do you think the actual value of $f$ at $x = 0.1$ is greater than or less than the estimate in part (b)? Explain.

46. ***Expanding Circle*** The radius of a circle is increased from 2.00 to 2.02 m.

(a) Estimate the resulting change in area.

(b) Estimate as a percentage of the circle's original area.

47. ***Growing Tree*** The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? the tree's cross section area?

48. ***Percentage Error*** The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

49. ***Tolerance*** About how accurately should you measure the side of a square to be sure of calculating the area to within 2% of its true value?

50. ***Tolerance*** (a) About how accurately must the interior diameter of a 10-m high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

(b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

51. ***Minting Coins*** A manufacturer contracts to mint coins for the federal government. The coins must weigh within 0.1% of their ideal weight, so the volume must be within 0.1% of the ideal volume. Assuming the thickness of the coins does not change, what is the percentage change in the volume of the coin that would result from a 0.1% increase in the radius?

52. ***Tolerance*** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of $h$, expressed as a percentage of $h$.

53. ***Estimating Volume*** You can estimate the volume of a sphere by measuring its circumference with a tape measure, dividing by $2\pi$ to get the radius, then using the radius in the volume formula. Find how sensitive your volume estimate is to a 1/8 in. error in the circumference measurement by filling in the table below for spheres of the given sizes. Use differentials when filling in the last column.

| Sphere Type | True Radius | Tape Error | Radius Error | Volume Error |
|---|---|---|---|---|
| Orange | 2 in. | 1/8 in. | | |
| Melon | 4 in. | 1/8 in. | | |
| Beach Ball | 7 in. | 1/8 in. | | |

54. ***Estimating Surface Area*** Change the heading in the last column of the table in Exercise 53 to "Surface Area Error" and find how sensitive the measure of surface area is to a 1/8 in. error in estimating the circumference of the sphere.

55. ***The Effect of Flight Maneuvers on the Heart*** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where $W$ is the work per unit time, $P$ is the average blood pressure, $V$ is the volume of blood pumped out during the unit of time, $\delta$ ("delta") is the density of the blood, $v$ is the average velocity of the exiting blood, and $g$ is the acceleration of gravity.

When $P$, $V$, $\delta$, and $v$ remain constant, $W$ becomes a function of $g$, and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive $W$ is to apparent changes in $g$ caused by flight maneuvers, and this depends on the initial value of $g$. As part of your investigation, you decide to compare the effect on $W$ of a given change $dg$ on the moon, where $g = 5.2$ ft/sec$^2$, with the effect the same change $dg$ would have on Earth, where $g = 32$ ft/sec$^2$. Use the simplified equation above to find the ratio of $dW_{\text{moon}}$ to $dW_{\text{Earth}}$.

56. ***Measuring Acceleration of Gravity*** When the length $L$ of a clock pendulum is held constant by controlling its temperature, the pendulum's period $T$ depends on the acceleration of gravity $g$. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in $g$. By keeping track of $\Delta T$, we can estimate the variation in $g$ from the equation $T = 2\pi(L/g)^{1/2}$ that relates $T$, $g$, and $L$.

(a) With $L$ held constant and $g$ as the independent variable, calculate $dT$ and use it to answer parts (b) and (c).

(b) **Writing to Learn** If $g$ increases, will $T$ increase or decrease? Will a pendulum clock speed up or slow down? Explain.

(c) A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec$^2$ to a new location. This increases the period by $dT = 0.001$ sec. Find $dg$ and estimate the value of $g$ at the new location.



## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

57. **True or False** Newton's method will not find the zero of $f(x) = x/(x^2 + 1)$ if the first guess is greater than 1. Justify your answer.

58. **True or False** If $u$ and $v$ are differentiable functions, then $d(uv) = du\,dv$. Justify your answer.

59. **Multiple Choice** What is the linearization of $f(x) = e^x$ at $x = 1$?

(A) $y = e$ (B) $y = ex$ (C) $y = e^x$
(D) $y = x - e$ (E) $y = e(x - 1)$

60. **Multiple Choice** If $y = \tan x$, $x = \pi$, and $dx = 0.5$, what does $dy$ equal?

(A) $-0.25$ (B) $-0.5$ (C) 0 (D) 0.5 (E) 0.25

61. **Multiple Choice** If Newton's method is used to find the zero of $f(x) = x - x^3 + 2$, what is the third estimate if the first estimate is 1?

(A) $-\frac{3}{4}$ (B) $\frac{3}{2}$ (C) $\frac{8}{5}$ (D) $\frac{18}{11}$ (E) 3

62. **Multiple Choice** If the linearization of $y = \sqrt[3]{x}$ at $x = 64$ is used to approximate $\sqrt[3]{66}$, what is the percentage error?

(A) 0.01% (B) 0.04% (C) 0.4% (D) 1% (E) 4%

## Explorations

63. ***Newton's Method*** Suppose your first guess in using Newton's method is lucky in the sense that $x_1$ is a root of $f(x) = 0$. What happens to $x_2$ and later approximations?

64. ***Oscillation*** Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_2 = -h$ if $x_1 = h$, and to $x_2 = h$ if $x_1 = -h$. Draw a picture that shows what is going on.

65. ***Approximations that Get Worse and Worse*** Apply Newton's method to $f(x) = x^{1/3}$ with $x_1 = 1$, and calculate $x_2$, $x_3$, $x_4$, and $x_5$. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

66. ***Quadratic Approximations***

(a) Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i. $Q(a) = f(a)$,

ii. $Q'(a) = f'(a)$,

iii. $Q''(a) = f''(a)$.

Determine the coefficients $b_0$, $b_1$, and $b_2$.

(b) Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

(c) Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point (0, 1). Comment on what you see.

(d) Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph $g$ and its quadratic approximation together. Comment on what you see.

(e) Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph $h$ and its quadratic approximation together. Comment on what you see.

(f) What are the linearizations of $f$, $g$, and $h$ at the respective points in parts (b), (d), and (e)?

67. ***Multiples of Pi*** Store any number as X in your calculator. Then enter the command X−tan(X)→X and press the ENTER key repeatedly until the displayed value stops changing. The result is always an integral multiple of $\pi$. Why is this so? [*Hint:* These are zeros of the sine function.]

## Extending the Ideas

68. ***Formulas for Differentials*** Verify the following formulas.

(a) $d(c) = 0$ ($c$ a constant)

(b) $d(cu) = c\,du$ ($c$ a constant)

(c) $d(u + v) = du + dv$

(d) $d(u \cdot v) = u\,dv + v\,du$

(e) $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

(f) $d(u^n) = nu^{n-1}\,du$

69. ***Linearization*** Show that the approximation of $\tan x$ by its linearization at the origin must improve as $x \to 0$ by showing that

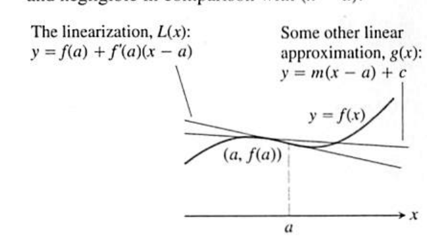
$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

70. ***The Linearization is the Best Linear Approximation*** Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ ($m$ and $c$ constants). If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using $g$ as a linear approximation of $f$ instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on $g$ the conditions

i. $E(a) = 0$, The error is zero at $x = a$.

ii. $\lim_{x \to a} \frac{E(x)}{x - a} = 0$, The error is negligible when compared with $(x - a)$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $(x - a)$.




71. **Writing to Learn** Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x + 1}$ and $\sin x$?

X1T = 2T
Y1T = 0
X2T = 0
$Y2T = \sqrt{10^2 - (2T)^2}$

2. What minimum and maximum values of T make sense in this problem?
3. Put your grapher in parametric and simultaneous modes. Enter the parametric equations and change the graphing style to "0" (the little ball) if your grapher has this feature. Set Tmin=0, Tmax=5, Tstep=5/20, Xmin=−1, Xmax=17, Xscl=0, Ymin=−1, Ymax=11, and Yscl=0. You can speed up the action by making the denominator in the Tstep smaller or slow it down by making it larger.
4. Press GRAPH and watch the two ends of the ladder move as time changes. Do both ends seem to move at a constant rate?
5. To see the simulation again, enter "ClrDraw" from the DRAW menu.
6. If $y$ represents the vertical height of the top of the ladder and $x$ the distance of the bottom from the wall, relate $y$ and $x$ and find $dy/dt$ in terms of $x$ and $y$. (Remember that $dx/dt = 2$.)
7. Find $dy/dt$ when $t = 3$ and interpret its meaning. Why is it negative?
8. In theory, how fast is the top of the ladder moving as it hits the ground?

Figure 4.58 shows you how to write a calculator program that animates the falling lad der as a line segment.

```
PROGRAM:LADDER
:For (A, 0, 5, .25)
:ClrDraw
:Line(2,2+√ (100−
(2A)²), 2+2A, 2)
:If A=0:Pause
:End
```

```
WINDOW
 Xmin=2
 Xmax=20
 Xscl=0
 Ymin=1
 Ymax=13
 Yscl=0
 Xres=1
```

**Figure 4.58** This 5-step program (with the viewing window set as shown) will animate the ladde in Exploration 1. Be sure any functions in the "Y=" register are turned off. Run the program and th ladder appears against the wall; push ENTER to start the bottom moving away from the wall.
For an enhanced picture, you can insert the commands ":Pt-On(2,2+√(100−(2A)²),2)" and ":Pt-On(2+2A,2,2)" on either side of the middle line of the program.

## eview 4.6 (For help, go to Sections 1.1, 1.4, and 3.7.)

d 2, find the distance between the points

, 0) 2. $A(0, a)$, $B(b, 0)$

find $dy/dx$.

: + y

· xy

6. $\ln(x + y) = 2x$

In Exercises 7 and 8, find a parametrization for the line segment wit endpoints $A$ and $B$.

7. $A(-2, 1)$, $B(4, -3)$ 8. $A(0, -4)$, $B(5, 0)$

In Exercises 9 and 10, let $x = 2\cos t$, $y = 2\sin t$. Find a paramete interval that produces the indicated portion of the graph.

9. The portion in the second and third quadrants, including the points on the axes.
10. The portion in the fourth quadrant, including the points on the axe



## Section 4.6 Exercises

In Exercises 1–41, assume all variables are differentiable functions of $t$.

1. ***Area*** The radius $r$ and area $A$ of a circle are related by the equation $A = \pi r^2$. Write an equation that relates $dA/dt$ to $dr/dt$.

2. ***Surface Area*** The radius $r$ and surface area $S$ of a sphere are related by the equation $S = 4\pi r^2$. Write an equation that relates $dS/dt$ to $dr/dt$.

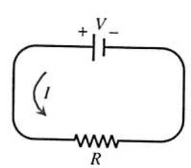
3. ***Volume*** The radius $r$, height $h$, and volume $V$ of a right circular cylinder are related by the equation $V = \pi r^2 h$.
(a) How is $dV/dt$ related to $dh/dt$ if $r$ is constant?
(b) How is $dV/dt$ related to $dr/dt$ if $h$ is constant?
(c) How is $dV/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

4. ***Electrical Power*** The power $P$ (watts) of an electric circuit is related to the circuit's resistance $R$ (ohms) and current $I$ (amperes) by the equation $P = RI^2$.
(a) How is $dP/dt$ related to $dR/dt$ and $dI/dt$?
(b) How is $dR/dt$ related to $dI/dt$ if $P$ is constant?

5. ***Diagonals*** If $x$, $y$, and $z$ are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. How is $ds/dt$ related to $dx/dt$, $dy/dt$, and $dz/dt$?

6. ***Area*** If $a$ and $b$ are the lengths of two sides of a triangle, and $\theta$ the measure of the included angle, the area $A$ of the triangle is $A = (1/2)\,ab\sin\theta$. How is $dA/dt$ related to $da/dt$, $db/dt$, and $d\theta/dt$?

7. ***Changing Voltage*** The voltage $V$ (volts), current $I$ (amperes), and resistance $R$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that $V$ is increasing at the rate of 1 volt/sec while $I$ is decreasing at the rate of 1/3 amp/sec. Let $t$ denote time in sec.




(a) What is the value of $dV/dt$?
(b) What is the value of $dI/dt$?
(c) Write an equation that relates $dR/dt$ to $dV/dt$ and $dI/dt$.
(d) **Writing to Learn** Find the rate at which $R$ is changing when $V = 12$ volts and $I = 2$ amp. Is $R$ increasing, or decreasing? Explain.

8. ***Heating a Plate*** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/sec. At what rate is the plate's area increasing when the radius is 50 cm?

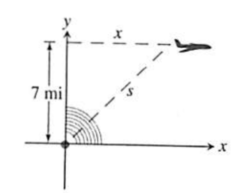
9. ***Changing Dimensions in a Rectangle*** The length $\ell$ of a rectangle is decreasing at the rate of 2 cm/sec while the width $w$ is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of
(a) the area, (b) the perimeter, and
(c) the length of a diagonal of the rectangle.
(d) **Writing to Learn** Which of these quantities are decreasing, and which are increasing? Explain.

10. ***Changing Dimensions in a Rectangular Box*** Suppose that the edge lengths $x$, $y$, and $z$ of a closed rectangular box are changing at the following rates:
$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$
Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

11. ***Inflating Balloon*** A spherical balloon is inflated with helium at the rate of $100\pi$ ft³/min.
(a) How fast is the balloon's radius increasing at the instant the radius is 5 ft?
(b) How fast is the surface area increasing at that instant?

12. ***Growing Raindrop*** Suppose that a droplet of mist is a perfect sphere and that, through condensation, the droplet picks up moisture at a rate proportional to its surface area. Show that under these circumstances the droplet's radius increases at a constant rate.

13. ***Air Traffic Control*** An airplane is flying at an altitude of 7 mi and passes directly over a radar antenna as shown in the figure. When the plane is 10 mi from the antenna ($s = 10$), the radar detects that the distance $s$ is changing at the rate of 300 mph. What is the speed of the airplane at that moment?



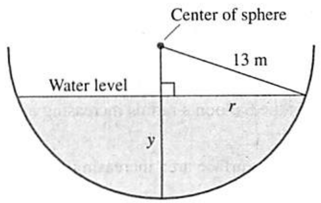

14. ***Flying a Kite*** Inge flies a kite at a height of 300 ft, the wind carrying the kite horizontally away at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

15. ***Boring a Cylinder*** The mechanics at Lincoln Automotive are reboring a 6-in. deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

**16.** ***Growing Sand Pile*** Sand falls from a conveyor belt at the rate of 10 $m^3$/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Give your answer in cm/min.

**17.** ***Draining Conical Reservoir*** Water is flowing at the rate of 50 $m^3$/min from a concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. **(a)** How fast is the water level falling when the water is 5 m deep? **(b)** How fast is the radius of the water's surface changing at that moment? Give your answer in cm/min.

**18.** ***Draining Hemispherical Reservoir*** Water is flowing at the rate of 6 $m^3$/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions given that the volume of water in a hemispherical bowl of radius $R$ is $V = (\pi/3)y^2(3R - y)$ when the water is $y$ units deep.

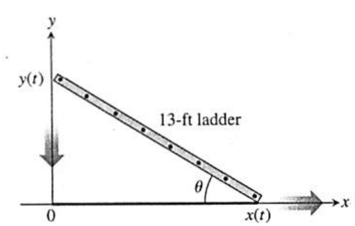



**(a)** At what rate is the water level changing when the water is 8 m deep?

**(b)** What is the radius $r$ of the water's surface when the water is $y$ m deep?

**(c)** At what rate is the radius $r$ changing when the water is 8 m deep?

**19.** ***Sliding Ladder*** A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

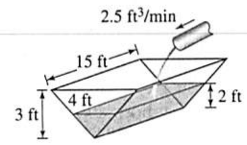



**(a)** How fast is the top of the ladder sliding down the wall at that moment?

**(b)** At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment?

**(c)** At what rate is the angle $\theta$ between the ladder and the ground changing at that moment?

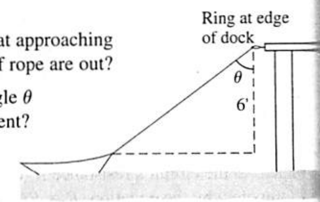
**20.** ***Filling a Trough*** A trough is 15 ft long and 4 ft across the top as shown in the figure. Its ends are isosceles triangles with height 3 ft. Water runs into the trough at the rate of 2.5 $ft^3$/min. How fast is the water level rising when it is 2 ft deep?



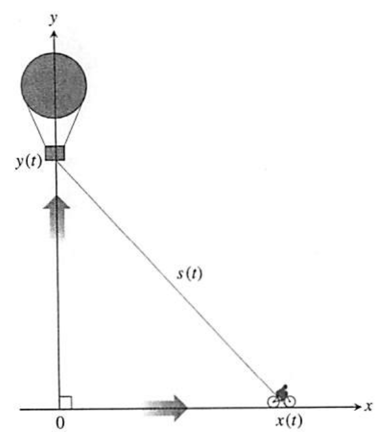

**21.** ***Hauling in a Dinghy*** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow as shown in the figure. The rope is hauled in at the rate of 2 ft/sec.

**(a)** How fast is the boat approaching the dock when 10 ft of rope are out?

**(b)** At what rate is angle $\theta$ changing at that moment?




**22.** ***Rising Balloon*** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later (see figure)?




In Exercises 23 and 24, a particle is moving along the curve $y = f(x)$.

**23.** Let $y = f(x) = \dfrac{10}{1 + x^2}$.

If $dx/dt = 3$ cm/sec, find $dy/dt$ at the point where

**(a)** $x = -2$. **(b)** $x = 0$. **(c)** $x = 20$.

**24.** Let $y = f(x) = x^3 - 4x$.

If $dx/dt = -2$ cm/sec, find $dy/dt$ at the point where

**(a)** $x = -3$. **(b)** $x = 1$. **(c)** $x = 4$.



**25.** ***Particle Motion*** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its $x$-coordinate (in meters) increases at a constant rate of 10 m/sec. How fast is the angle of inclination $\theta$ of theline joining the particle to the origin changing when $x = 3$?

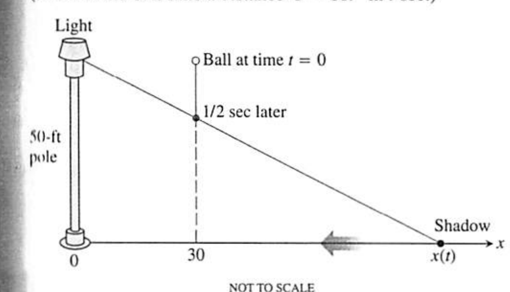
**26.** ***Particle Motion*** A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its $x$-coordinate (in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination $\theta$ of the line joining the particle to the origin changing when $x = -4$?

**27.** ***Melting Ice*** A spherical iron ball is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 8 mL/min, how fast is the outer surface area of ice decreasing when the outer diameter (ball plus ice) is 20 cm?

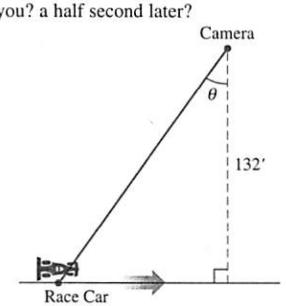
**28.** ***Particle Motion*** A particle $P(x, y)$ is moving in the coordinate plane in such a way that $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

**29.** ***Moving Shadow*** A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

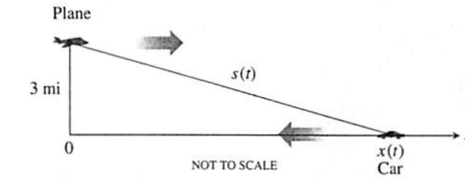
**30.** ***Moving Shadow*** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light as shown below. How fast is the ball's shadow moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ in $t$ sec.)



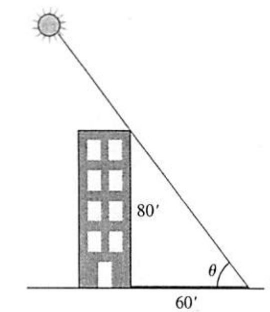

**31.** ***Moving Race Car*** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec) as shown in the figure. About how fast will your camera angle $\theta$ be changing when the car is right in front of you? a half second later?



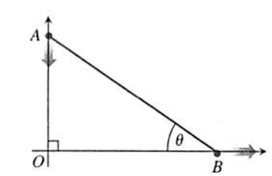

**32.** ***Speed Trap*** A highway patrol airplane flies 3 mi above a level, straight road at a constant rate of 120 mph. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mph. Find the car's speed along the highway.




**33.** ***Building's Shadow*** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long as shown in the figure. At the moment in question, the angle $\theta$ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow length decreasing? Express your answer in in./min, to the nearest tenth. (Remember to use radians.)




**34.** ***Walkers*** $A$ and $B$ are walking on straight streets that meet at right angles. $A$ approaches the intersection at 2 m/sec and $B$ moves away from the intersection at 1 m/sec as shown in the figure. At what rate is the angle $\theta$ changing when $A$ is 10 m from the intersection and $B$ is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.




**35.** ***Moving Ships*** Two ships are steaming away from a point $O$ along routes that make a 120° angle. Ship $A$ moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yards). Ship $B$ moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

**36. True or False** If the radius of a circle is expanding at a constant rate, then its circumference is increasing at a constant rate. Justify your answer.

**37. True or False** If the radius of a circle is expanding at a constant rate, then its area is increasing at a constant rate. Justify your answer.

**38. Multiple Choice** If the volume of a cube is increasing at 24 $in^3$/ min and each edge of the cube is increasing at 2 in./min, what is the length of each edge of the cube?

(A) 2 in. (B) $2\sqrt{2}$ in. (C) $\sqrt[3]{12}$ in. (D) 4 in. (E) 8 in.

**39. Multiple Choice** If the volume of a cube is increasing at 24 $in^3$/ min and the surface area of the cube is increasing at 12 $in^2$/ min, what is the length of each edge of the cube?

(A) 2 in. (B) $2\sqrt{2}$ in. (C) $\sqrt[3]{12}$ in. (D) 4 in. (E) 8 in.

**40. Multiple Choice** A particle is moving around the unit circle (the circle of radius 1 centered at the origin). At the point (0.6, 0.8) the particle has horizontal velocity $dx/dt = 3$. What is its vertical velocity $dy/dt$ at that point?

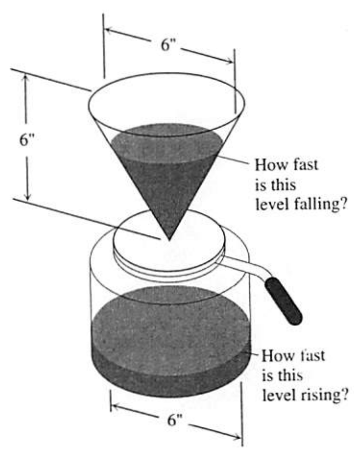
(A) −3.875 (B) −3.75 (C) −2.25 (D) 3.75 (E) 3.875

**41. Multiple Choice** A cylindrical rubber cord is stretched at a constant rate of 2 cm per second. Assuming its volume does not change, how fast is its radius shrinking when its length is 100 cm and its radius is 1 cm?

(A) 0 cm/sec (B) 0.01 cm/sec (C) 0.02 cm/sec
(D) 2 cm/sec (E) 3.979 cm/sec

## Explorations

**42. *Making Coffee*** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 $in^3$/min.




(a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

(b) How fast is the level in the cone falling at that moment?

**43. *Cost, Revenue, and Profit*** A company can manufacture $x$ items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (all amounts in thousands). Find $dc/dt$, $dr/dt$, and $dp/dt$ for the following values of $x$ and $dx/dt$.

(a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$.

(b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$.

**44. Group Activity *Cardiac Output*** In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where $Q$ is the number of milliliters of $CO_2$ you exhale in a minute and $D$ is the difference between the $CO_2$ concentration (mL/L) in the blood pumped to the lungs and the $CO_2$ concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233 \text{ mL/min}}{41 \text{ mL/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that $D$ is decreasing at the rate of 2 units a minute but that $Q$ remains unchanged. What is happening to the cardiac output?

## Extending the Ideas

**45. *Motion along a Circle*** A wheel of radius 2 ft makes 8 revolutions about its center every second.

(a) Explain how the parametric equations

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

can be used to represent the motion of the wheel.

(b) Express $\theta$ as a function of time $t$.

(c) Find the rate of horizontal movement and the rate of vertical movement of a point on the edge of the wheel when it is at the position given by $\theta = \pi/4$, $\pi/2$, and $\pi$.

**46. *Ferris Wheel*** A Ferris wheel with radius 30 ft makes one revolution every 10 sec.

(a) Assume that the center of the Ferris wheel is located at the point (0, 40), and write parametric equations to model its motion. [*Hint:* See Exercise 45.]

(b) At $t = 0$ the point $P$ on the Ferris wheel is located at (30, 40). Find the rate of horizontal movement, and the rate of vertical movement of the point $P$ when $t = 5$ sec and $t = 8$ sec.



**7. *Industrial Production*** (a) Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time $t$ in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time $t$. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, $y$.

(b) Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

## uick Quiz for AP* Preparation: Sections 4.4–4.6

You may use a graphing calculator to solve the following problems.

**1. Multiple Choice** If Newton's method is used to approximate the real root of $x^3 + 2x - 1 = 0$, what would the third approximation, $x_3$, be if the first approximation is $x_1 = 1$?

(A) 0.453 (B) 0.465 (C) 0.495 (D) 0.600 (E) 1.977

**. Multiple Choice** The sides of a right triangle with legs $x$ and $y$ and hypotenuse $z$ increase in such a way that $dz/dt = 1$ and $dx/dt = 3\,dy/dt$. At the instant when $x = 4$ and $y = 3$, what is $dx/dt$ ?

(A) $\frac{1}{3}$ (B) 1 (C) 2 (D) $\sqrt{5}$ (E) 5

**. Multiple Choice** An observer 70 meters south of a railroad crossing watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

(A) 57.60 (B) 57.88 (C) 59.20 (D) 60.00 (E) 67.40

**4. Free Response** (a) Approximate $\sqrt{26}$ by using the linearization of $y = \sqrt{x}$ at the point (25, 5). Show the computation that leads to your conclusion.

(b) Approximate $\sqrt{26}$ by using a first guess of 5 and one iteration of Newton's method to approximate the zero of $x^2 - 26$. Show the computation that leads to your conclusion.

(c) Approximate $\sqrt[3]{26}$ by using an appropriate linearization. Show the computation that leads to your conclusion.

## hapter 4 Key Terms

olute change (p. 240)
olute maximum value (p. 187)
olute minimum value (p. 187)
iderivative (p. 200)
idifferentiation (p. 200)
ithmetic mean (p. 204)
rage cost (p. 224)
ter of linear approximation (p. 233)
ncave down (p. 207)
ncave up (p. 207)
ncavity test (p. 208)
tical point (p. 190)
reasing function (p. 198)
ferential (p. 237)
ferential estimate of change (p. 239)
ferential of a function (p. 239)
extrema (p. 187)
Extreme Value Theorem (p. 188)
first derivative test (p. 205)
first derivative test for local extrema (p. 205)
geometric mean (p. 204)
global maximum value (p. 177)
global minimum value (p. 177)
increasing function (p. 198)
linear approximation (p. 233)
linearization (p. 233)
local linearity (p. 233)
local maximum value (p. 189)
local minimum value (p. 189)
logistic curve (p. 210)
logistic regression (p. 211)
marginal analysis (p. 223)
marginal cost and revenue (p. 223)
Mean Value Theorem (p. 196)
monotonic function (p. 198)
Newton's method (p. 235)
optimization (p. 219)
percentage change (p. 240)
point of inflection (p. 208)
profit (p. 223)
quadratic approximation (p. 245)
related rates (p. 246)
relative change (p. 240)
relative extrema (p. 189)
Rolle's Theorem (p. 196)
second derivative test
for local extrema (p. 211)
standard linear approximation (p. 233)

# Chapter 4 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1 and 2, use analytic methods to find the global extreme values of the function on the interval and state where they occur.

**1.** $y = x\sqrt{2-x}, \quad -2 \le x \le 2$

**2.** $y = x^3 - 9x^2 - 21x - 11, \quad -\infty < x < \infty$

In Exercises 3 and 4, use analytic methods. Find the intervals on which the function is

(a) increasing, (b) decreasing,

(c) concave up, (d) concave down.

Then find any

(e) local extreme values, (f) inflection points.

**3.** $y = x^2 e^{1/x^2}$ **4.** $y = x\sqrt{4-x^2}$

In Exercises 5–16, find the intervals on which the function is

(a) increasing, (b) decreasing,

(c) concave up, (d) concave down.

Then find any

(e) local extreme values, (f) inflection points.

**5.** $y = 1 + x - x^2 - x^4$ **6.** $y = e^{x-1} - x$

**7.** $y = \dfrac{1}{\sqrt[4]{1-x^2}}$ **8.** $y = \dfrac{x}{x^3 - 1}$

**9.** $y = \cos^{-1} x$ **10.** $y = \dfrac{x}{x^2 + 2x + 3}$

**11.** $y = \ln|x|, \quad -2 \le x \le 2, \quad x \ne 0$

**12.** $y = \sin 3x + \cos 4x, \quad 0 \le x \le 2\pi$

**13.** $y = \begin{cases} e^{-x}, & x \le 0 \\ 4x - x^3, & x > 0 \end{cases}$

**14.** $y = -x^5 + \dfrac{7}{3}x^3 + 5x^2 + 4x + 2$

**15.** $y = x^{4/5}(2 - x)$

**16.** $y = \dfrac{5 - 4x + 4x^2 - x^3}{x - 2}$

In Exercises 17 and 18, use the derivative of the function $y = f(x)$ to find the points at which $f$ has a

(a) local maximum, (b) local minimum, or

(c) point of inflection.

**17.** $y' = 6(x+1)(x-2)^2$ **18.** $y' = 6(x+1)(x-2)$

In Exercises 19–22, find all possible functions with the given derivative.

**19.** $f'(x) = x^{-5} + e^{-x}$ **20.** $f'(x) = \sec x \tan x$

**21.** $f'(x) = \dfrac{2}{x} + x^2 + 1, \ x > 0$ **22.** $f'(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

In Exercises 23 and 24, find the function with the given derivative whose graph passes through the point $P$.

**23.** $f'(x) = \sin x + \cos x, \quad P(\pi, 3)$

**24.** $f'(x) = x^{1/3} + x^2 + x + 1, \quad P(1, 0)$

In Exercises 25 and 26, the velocity $v$ or acceleration $a$ of a particle is given. Find the particle's position $s$ at time $t$.

**25.** $v = 9.8t + 5, \quad s = 10$ when $t = 0$

**26.** $a = 32, \quad v = 20$ and $s = 5$ when $t = 0$

In Exercises 27–30, find the linearization $L(x)$ of $f(x)$ at $x = a$.

**27.** $f(x) = \tan x, \quad a = -\pi/4$ **28.** $f(x) = \sec x, \quad a = \pi/4$

**29.** $f(x) = \dfrac{1}{1 + \tan x}, \quad a = 0$ **30.** $f(x) = e^x + \sin x, \quad a = 0$

In Exercises 31–34, use the graph to answer the questions.

**31.** Identify any global extreme values of $f$ and the values of $x$ at which they occur.

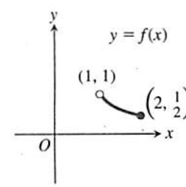



Figure for Exercise 31

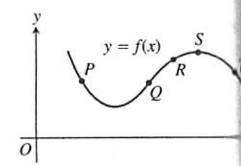


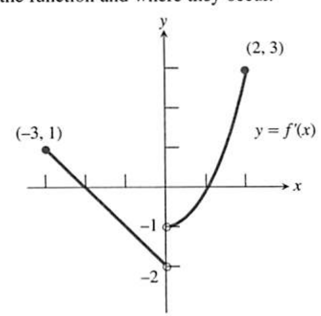

Figure for Exercise 32

**32.** At which of the five points on the graph of $y = f(x)$ shown here

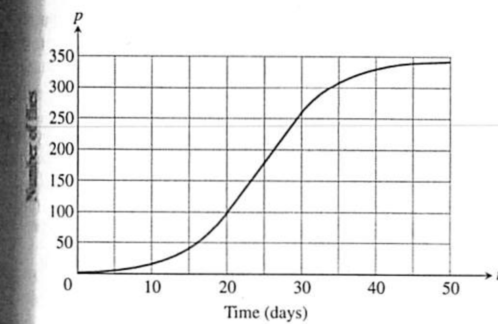
(a) are $y'$ and $y''$ both negative?

(b) is $y'$ negative and $y''$ positive?

**33.** Estimate the intervals on which the function $y = f(x)$ is (a) increasing; (b) decreasing. (c) Estimate any local extreme values of the function and where they occur.




**34.** Here is the graph of the fruit fly population from Section 2.4, Example 2. On approximately what day did the population's growth rate change from increasing to decreasing?






**35.** ***Connecting f and f′*** The graph of $f'$ is shown in Exercise 33. Sketch a possible graph of $f$ given that it is continuous with domain $[-3, 2]$ and $f(-3) = 0$.

**36.** ***Connecting f, f′, and f″*** The function $f$ is continuous on $[0, 3]$ and satisfies the following.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f$ | 0 | −2 | 0 | 3 |
| $f'$ | −3 | 0 | does not exist | 4 |
| $f''$ | 0 | 1 | does not exist | 0 |

| $x$ | $0 < x < 1$ | $1 < x < 2$ | $2 < x < 3$ |
|---|---|---|---|
| $f$ | − | − | + |
| $f'$ | − | + | + |
| $f''$ | + | + | + |

(a) Find the absolute extrema of $f$ and where they occur.

(b) Find any points of inflection.

(c) Sketch a possible graph of $f$.

**37.** ***Mean Value Theorem*** Let $f(x) = x \ln x$.

(a) **Writing to Learn** Show that $f$ satisfies the hypotheses of the Mean Value Theorem on the interval $[a, b] = [0.5, 3]$.

(b) Find the value(s) of $c$ in $(a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(c) Write an equation for the secant line $AB$ where $A = (a, f(a))$ and $B = (b, f(b))$.

(d) Write an equation for the tangent line that is parallel to the secant line $AB$.

**38.** ***Motion along a Line*** A particle is moving along a line with position function $s(t) = 3 + 4t - 3t^2 - t^3$. Find the (a) velocity and (b) acceleration, and (c) describe the motion of the particle for $t \ge 0$.

**39.** ***Approximating Functions*** Let $f$ be a function with $f'(x) = \sin x^2$ and $f(0) = -1$.

(a) Find the linearization of $f$ at $x = 0$.

(b) Approximate the value of $f$ at $x = 0.1$.

(c) **Writing to Learn** Is the actual value of $f$ at $x = 0.1$ greater than or less than the approximation in (b)?

**40.** ***Differentials*** Let $y = x^2 e^{-x}$. Find (a) $dy$ and (b) evaluate $dy$ for $x = 1$ and $dx = 0.01$.

**41.** Table 4.5 shows the growth of the population of Tennessee from the 1850 census to the 1910 census. The table gives the population growth beyond the baseline number from the 1840 census, which was 829,210.

**Table 4.5 Population Growth of Tennessee**

| Years since 1840 | Growth Beyond 1840 Population |
|---|---|
| 10 | 173,507 |
| 20 | 280,591 |
| 30 | 429,310 |
| 40 | 713,149 |
| 50 | 938,308 |
| 60 | 1,191,406 |
| 70 | 1,355,579 |

*Source:* Bureau of the Census, U.S. Chamber of Commerce

(a) Find the logistic regression for the data.

(b) Graph the data in a scatter plot and superimpose the regression curve.

(c) Use the regression equation to predict the Tennessee population in the 1920 census. Be sure to add the baseline 1840 number. (The actual 1920 census value was 2,337,885.)

(d) In what year during the period was the Tennessee population growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

(e) What does the regression equation indicate about the population of Tennessee in the long run?

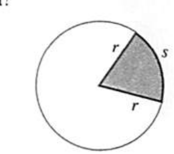
(f) **Writing to Learn** In fact, the population of Tennessee had already passed the long-run value predicted by this regression curve by 1930. By 2000 it had surpassed the prediction by more than 3 million people! What historical circumstances could have made the early regression so unreliable?

**42.** ***Newton's Method*** Use Newton's method to estimate all real solutions to $2\cos x - \sqrt{1 + x} = 0$. State your answers accurate to 6 decimal places.

**43.** ***Rocket Launch*** A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec². How fast will the rocket be going 1 min later?

**44.** ***Launching on Mars*** The acceleration of gravity near the surface of Mars is 3.72 m/sec². If a rock is blasted straight up from the surface with an initial velocity of 93 m/sec (about 208 mph), how high does it go?

**45.** ***Area of Sector*** If the perimeter of the circular sector shown here is fixed at 100 ft, what values of $r$ and $s$ will give the sector the greatest area?

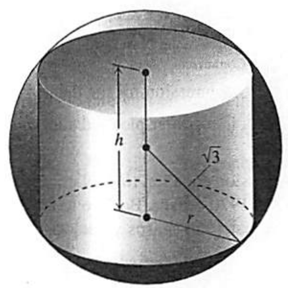


**46. *Area of Triangle*** An isosceles triangle has its vertex at the origin and its base parallel to the $x$-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

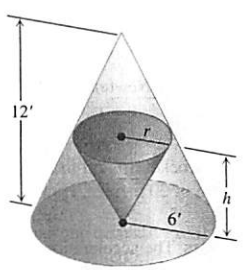
**47. *Storage Bin*** Find the dimensions of the largest open-top storage bin with a square base and vertical sides that can be made from 108 $ft^2$ of sheet steel. (Neglect the thickness of the steel and assume that there is no waste.)

**48. *Designing a Vat*** You are to design an open-top rectangular stainless-steel vat. It is to have a square base and a volume of 32 $ft^3$; to be welded from quarter-inch plate, and weigh no more than necessary. What dimensions do you recommend?

**49. *Inscribing a Cylinder*** Find the height and radius of the largest right circular cylinder that can be put into a sphere of radius $\sqrt{3}$ as described in the figure.



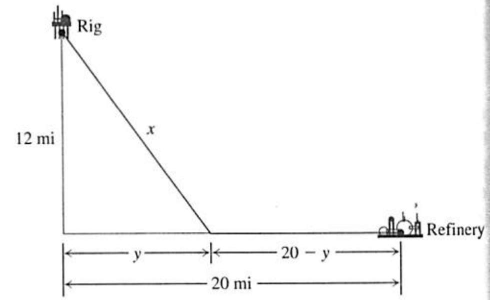

**50. *Cone in a Cone*** The figure shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of $r$ and $h$ will give the smaller cone the largest possible volume?




**51. *Box with Lid*** Repeat Exercise 18 of Section 4.4 but this time remove the two equal squares from the corners of a 15-in. side.

**52. *Inscribing a Rectangle*** A rectangle is inscribed under one arch of $y = 8 \cos (0.3x)$ with its base on the $x$-axis and its upper two vertices on the curve symmetric about the $y$-axis. What is the largest area the rectangle can have?

**53. *Oil Refinery*** A drilling rig 12 mi offshore is to be connected by a pipe to a refinery onshore, 20 mi down the coast from the rig as shown in the figure. If underwater pipe costs \$40,000 per mile and land-based pipe costs \$30,000 per mile, what values of $x$ and $y$ give the least expensive connection?




**54. *Designing an Athletic Field*** An athletic field is to be built in the shape of a rectangle $x$ units long capped by semicircular regions of radius $r$ at the two ends. The field is to be bounded by a 400-m running track. What values of $x$ and $r$ will give the rectangle the largest possible area?

**55. *Manufacturing Tires*** Your company can manufacture $x$ hundred grade A tires and $y$ hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

**56. *Particle Motion*** The positions of two particles on the $s$-axis are $s_1 = \cos t$ and $s_2 = \cos (t + \pi/4)$.

**(a)** What is the farthest apart the particles ever get?

**(b)** When do the particles collide?

**57. *Open-top Box*** An open-top rectangular box is constructed from a 10- by 16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

**58. *Changing Area*** The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?



***Particle Motion*** The coordinates of a particle moving in the plane are differentiable functions of time $t$ with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle approaching the origin as it passes through the point (5, 12)?

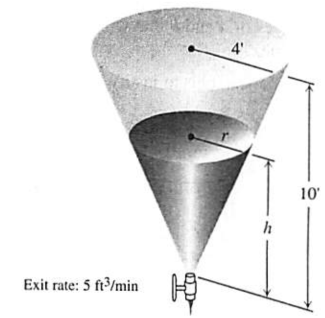
***Changing Cube*** The volume of a cube is increasing at the rate of 1200 $cm^3$/min at the instant its edges are 20 cm long. At what rate are the edges changing at that instant?

***Particle Motion*** A point moves smoothly along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the constant rate of 11 units per second. Find $dx/dt$ when $x = 3$.

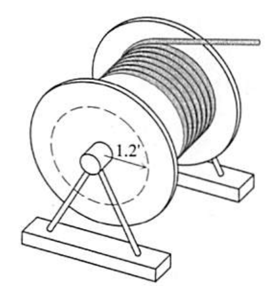
***Draining Water*** Water drains from the conical tank shown in the figure at the rate of 5 $ft^3$/min.

**(a)** What is the relation between the variables $h$ and $r$?

**(b)** How fast is the water level dropping when $h = 6$ ft?

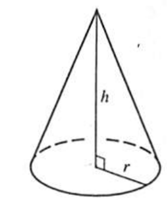



***Stringing Telephone Cable*** As telephone cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius as suggested in the figure. If the truck pulling the cable moves at a constant rate of 6 ft/sec, use the equation $s = r\theta$ to find how fast (in rad/sec) the spool is turning when the layer of radius 1.2 ft is being unwound.




**64. *Throwing Dirt*** You sling a shovelful of dirt up from the bottom of a 17-ft hole with an initial velocity of 32 ft/sec. Is that enough speed to get the dirt out of the hole, or had you better duck?

**65. *Estimating Change*** Write a formula that estimates the change that occurs in the volume of a right circular cone (see figure) when the radius changes from $a$ to $a + dr$ and the height does not change.




$V = \frac{1}{3}\pi r^2 h$

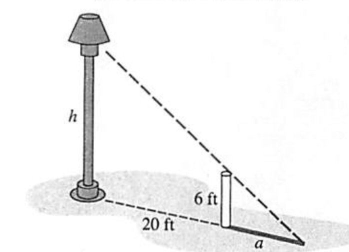
**66. *Controlling Error***

**(a)** How accurately should you measure the edge of a cube to reasonably sure of calculating the cube's surface area with an error of no more than 2%?

**(b)** Suppose the edge is measured with the accuracy required i part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate t percentage error in the volume calculation that might result fr using the edge measurement.

**67. *Compounding Error*** The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volum of the sphere. Estimate the percentage errors in the calculated values of **(a)** the radius, **(b)** the surface area, and **(c)** the volum

**68. *Finding Height*** To find the height of a lamppost (see figure) you stand a 6-ft pole 20 ft from the lamp and measure the leng $a$ of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$, and estimate the possible error in the result.



**69.** ***Decreasing Function*** Show that the function $y = \sin^2 x - 3x$ decreases on every interval in its domain.

## AP Examination Preparation

You should solve the following problems without using a calculator.

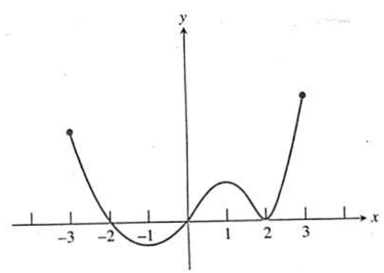
**70.** The accompanying figure shows the graph of the derivative of a function $f$. The domain of $f$ is the closed interval $[-3, 3]$.

**(a)** For what values of $x$ in the open interval $(-3, 3)$ does $f$ have a relative maximum? Justify your answer.

**(b)** For what values of $x$ in the open interval $(-3, 3)$ does $f$ have a relative minimum? Justify your answer.

**(c)** For what values of $x$ is the graph of $f$ concave up? Justify your answer.

**(d)** Suppose $f(-3) = 0$. Sketch a possible graph of $f$ on the domain $[-3, 3]$.




**71.** The volume $V$ of a cone $(V = \frac{1}{3}\pi r^2 h)$ is increasing at the rate of $4\pi$ cubic inches per second. At the instant when the radius of the cone is 2 inches, its volume is $8\pi$ cubic inches and the radius is increasing at 1/3 inch per second.

**(a)** At the instant when the radius of the cone is 2 inches, what is the rate of change of the area of its base?

**(b)** At the instant when the radius of the cone is 2 inches, what is the rate of change of its height $h$?

**(c)** At the instant when the radius of the cone is 2 inches, what is the instantaneous rate of change of the area of its base with respect to its height $h$?

**72.** A piece of wire 60 inches long is cut into six sections, two of length $a$ and four of length $b$. Each of the two sections of length $a$ is bent into the form of a circle and the circles are then joined by the four sections of length $b$ to make a frame for a model of a right circular cylinder, as shown in the accompanying figure.

**(a)** Find the values of $a$ and $b$ that will make the cylinder of maximum volume.

**(b)** Use differential calculus to justify your answer in part (a).