Extra Practice Chapter 4

Let f be a twice-differentiable function with $f(0) = 4$. The derivative of f is given by $f'(x) = \sin(x^2 - 2x + 1)$ for $-2 \le x \le 2$.

(a) Find all values of x in the interval $-2 < x < 2$ at which f has a critical point. Classify each as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(b) Use the line tangent to the graph of f at $x = 0$ to approximate $f(0.25)$.

(c) On the interval $0 \le x \le 0.25$, $f'(x) > 0$ and $f''(x) < 0$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.25)$? Give a reason for your answer.

(d) Using the Mean Value Theorem, explain why the average rate of change of f over the interval $-2 \le x \le 2$ cannot equal 1.25.

a) $f'(x) = 0 \Rightarrow x = -1.507$, $x \simeq -0.772$, $x = 1$
f has a relative max at -1.507 because f' changes from + to −
relative min at $x = -0.772$ because...
neither a relative min nor max at $x = 1$ because f' does not change sign

b) $f'(0) = \sin 1 = 0.841$
$f(0.25) \simeq f(0) + f'(0)(0.25) = 4.210$

c) $f''(x) < 0$ for $x \in [0, 0.25] \Rightarrow f$ is concave down for $x \in [0, 0.25]$
$\Rightarrow$ tangent line approx is an overestimate for $f(0.25)$

d) f' is differentiable for $-2 \le x \le 2 \Rightarrow$ continuous
$\Rightarrow$ MVT can be applied on $[-2, 2]$. There should be a number $c \in (-2, 2)$ such that $f'(c) =$ average rate of change over $[-2, 2]$.
However, since $-1 \le f'(x) \le 1$, there can be no number c such that $f'(c) = 1.25$
Therefore the average rate of change cannot equal 1.25

$\lim_{x \to 0} \frac{x^2}{1 - \cos x}$ is

(A) -2

(B) 0

(C) 1

(D) 2

(E) nonexistent

Let f be a twice-differentiable function with $f(1.5) = 3$. The derivative of f is given by $f'(x) = (7x - 19)\sin(x^2 - 4x + 4)$ for $1 \leq x \leq 4$.

(a) Find all values of x in the interval $1 < x < 4$ at which f has a critical point. Classify each as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(b) Use the line tangent to the graph of f at $x = 1.5$ to approximate $f(1.8)$.

On the interval $1.5 \leq x \leq 1.8$, $f'(x) < 0$ and $f''(x) > 0$. Is the approximation found in part (b) an overestimate or an underestimate for $f(1.8)$? Give a reason for your answer.

Using the Mean Value Theorem, explain why the average rate of change of f over the interval $1 \leq x \leq 4$ cannot equal 6.5.

a) $f'(x) = 0 \Rightarrow x = 2,\ x = 2.714,\ x = 3.772$
No relative min or max at 2 because f' does not change sign.
Relative min at 2.714 (f' − to +) Relative max at 3.772 (f' + to −)

b) $f'(1.5) = -2.102934$ $f(1.8) \simeq f(1.5) + f'(1.5)(0.3) = 2.369$

c) $f''(x) > 0$ for $x \in [1.5, 1.8] \Rightarrow f$ is concave up $\Rightarrow$ underestimate for $f(1.8)$

d) f differentiable on $[1,4] \Rightarrow$ continuous
MVT applies: there should exist $c \in (1,4)$ such that $f'(c) =$ av. rate change over $[1,4]$
However: $f'(x) = 6.5$ has no sol. on $[1,4]$.
So there is no c such that $f'(c) = 6.5 \Rightarrow$ av rate change $\neq 6.5$

$\lim\limits_{x \to \infty} \frac{x^3}{e^{3x}}$ is

(A) 0

(B) $\frac{2}{9}$

(C) $\frac{2}{3}$

(D) 1

(E) infinite

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x-2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

a) Find $h'(2)$.

b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

c) The function h satisfies $h(x) = \frac{x^2-4}{1-(f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

a) $h'(2) = \frac{2}{3}$

b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$ $\quad a'(2) = 160$

c) h differentiable ⇒ h continuous so $\lim_{x\to 2} h(x) = h(2) = 4$

i.e $\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{(f(x))^3} = 4$

since $\lim_{x\to 2}(x^2-4) = 0$, then we must have $\lim_{x\to 2}(1-(f(x))^3) = 0$

Thus $\lim_{x\to 2} f(x) = 1$

Since f is differentiable, then f is continuous so $f(2) = \lim_{x\to 2} f(x) = 1$

since f is twice differentiable, then f' is continuous so $\lim_{x\to 2} f'(x) = f'(2)$ exists

Using l'Hopital's Rule:

$$\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4$$

$\therefore f'(2) = -\frac{1}{3}$

d) g and h differentiable ⇒ g and h continuous.

so $\lim_{x\to 2} g(x) = g(2) = 4$ and $\lim_{x\to 2} h(x) = h(2) = 4$

Since $g(x) \leq k(x) \leq h(x)$ for $x \in (1,3)$, then squeezing theorem ⇒ $\lim_{x\to 2} k(x) = 4$

Also: $4 = g(2) \leq k(2) \leq h(2) = 4$ so $k(2) = 4$

Thus k is continuous at $x = 2$.

$\lim_{h\to 0} \frac{e^{(2+h)} - e^2}{h} =$

- A) 0
- B) 1
- C) $2e$
- D) e^2
- E) $2e^2$

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
2	0	0	5	7

The third derivative of the function f is continuous on the interval(0,4) . Values for f and its first three derivatives at$x$=2 are given in the table above. What is $\lim_{x\to 2}\frac{f(x)}{(x-2)^2}$?

- (A) 0
- (B) $\frac{5}{2}$
- (C) 5
- (D) 7
- (E) The limit does not exist.

x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

Suppose $f'(5)=3$ and $f''(x)<0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x=5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7)\ge 4/3$.

Eq of tangent line: $y=-2+3(x-5)$

Since $f''(x)<0$ for $x\in[5,8]$, the line tangent to the graph of $y=f(x)$ at $x=5$ lies above the graph for all $x \in (5,8]$.

Therefore $f(7)\le -2+3\times 2=4$

An equation for the secant line is: $y=-2+\frac{5}{3}(x-5)$

Since $f''(x)<0$ for $x\in[5,8]$, the secant line connecting $(5,f(5))$ and $(8,f(8))$ lies below the graph of $y=f(x)$ for $x\in(5,8)$

Therefore: $f(7)\ge -2+\frac{5}{3}\cdot 2=\frac{4}{3}$

A differentiable function f has the property that $f(5) = 3$ and $f'(5) = 4$. What is the estimate for $f(4.8)$ using the local linear approximation for f at x= 5 ?

A) 2.2

B) 2.8

C) 3.4

D) 3.8

E) 4.6

x	−1.5	−1.0	−0.5	0	0.5	1.0	1.5
$f(x)$	−1	−4	−6	−7	−6	−4	−1
$f'(x)$	−7	−5	−3	0	3	5	7

Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \leq x \leq 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \leq x \leq 1.5$.

Write an equation of the line tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

$$\left.\begin{array}{l}\text{Point}: (1, f(1)) \text{ i.e } (1,-4) \\ \text{slope}: f'(1) = 5\end{array}\right\} \quad y+4 = 5(x-1) \quad \boxed{y = 5x-9}$$

$f(1.2) \simeq 5(1.2) - 9 = -3$

$f''(x) > 0$ for $x \in [-1.5, 1.5]$ $\Rightarrow$ underapprox

$f(1.2) \geq -3$

The function f is continuous for $-2 \leq x \leq 2$ and $f(-2) = f(2) = 0$. If there is no c, where $-2 < c < 2$ for which $f'(c) = 0$, which of the following statements must be true?

A) For $-2 < k < 2$, $f'(k) > 0$

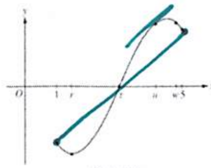
B) For $-2 < k < 2$, $f(k) < 0$

C) For $-2 < k < 2$, $f(k)$ exists.

D) For $-2 < k < 2$, $f(k)$ exists, but f is not continuous.

E) For some k, where $-2 < k < 2$, $f(k)$ does not exist.

$f'(k)$



Graph of f

The figure above shows the graph of the differentiable function f for $1 \leq x \leq 5$. Which of the following could be the x-coordinate of a point at which the line tangent to the graph of f is parallel to the secant line through the points $(1,\ f(1))$ and $(5,\ f(5))$?

(A) r

(B) t

(C) u ✓

(D) w

(E) There is no such point.

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	−20	−30	−20	−14	−10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

A car travels on a straight track. During the time interval $0 \leq t \leq 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

For $0<t<60$, must there be a time t when $a(t)=0$? Justify your answer.

$v(0) = v(25)$ V is continuous on $[0, 25]$

MVT guarantees $t \in (0, 25)$ such that $a(t) = v'(t) = 0$.

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

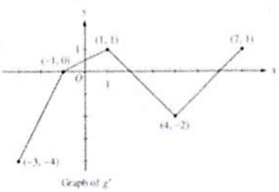
Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

Is there a time $t, 2 \le t \le 4$, at which $C'(t)=2$? Justify your answer.

C is differentiable => C is continuous (on closed interval)

$$\frac{C(4)-C(2)}{4-2} = \frac{12.8-8.8}{2} = 2$$

Therefore, by MVT, there is at least one time $t \in (0,4)$ for which $C(t) = 2$

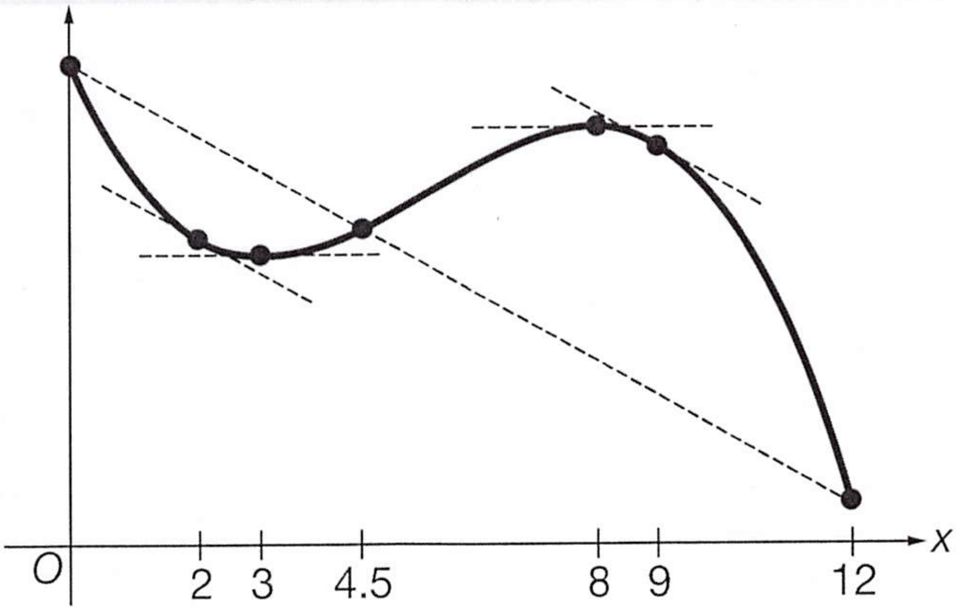


Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

$$\frac{g'(7)-g'(3)}{7-(-3)} = \frac{1-(-4)}{10} = \frac{1}{2}$$

No, the MVT does not guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$



The function f shown in the figure above is continuous on the closed interval $[0, 12]$ and differentiable on the open interval $(0, 12)$. Based on the graph, what are all values of x that satisfy the conclusion of the Mean Value Theorem applied to f on the closed interval $[0, 12]$?

A 4.5 only because this is the value where $f(x)$ equals the average rate of change of f on $[0, 12]$.

B 3 and 8 because these are the values where $f'(x) = 0$ on $[0, 12]$.

C 2 and 9 only because these are the values where the instantaneous rate of change of f at those values is equal to the average rate of change of f on $[0, 12]$.

D 2, 4.5, and 9 because these are the values where either the instantaneous rate of change of f at the value is equal to the average rate of change of f on $[0, 12]$ or the value of $f(x)$ is equal to the average rate of change of f on $[0, 12]$.