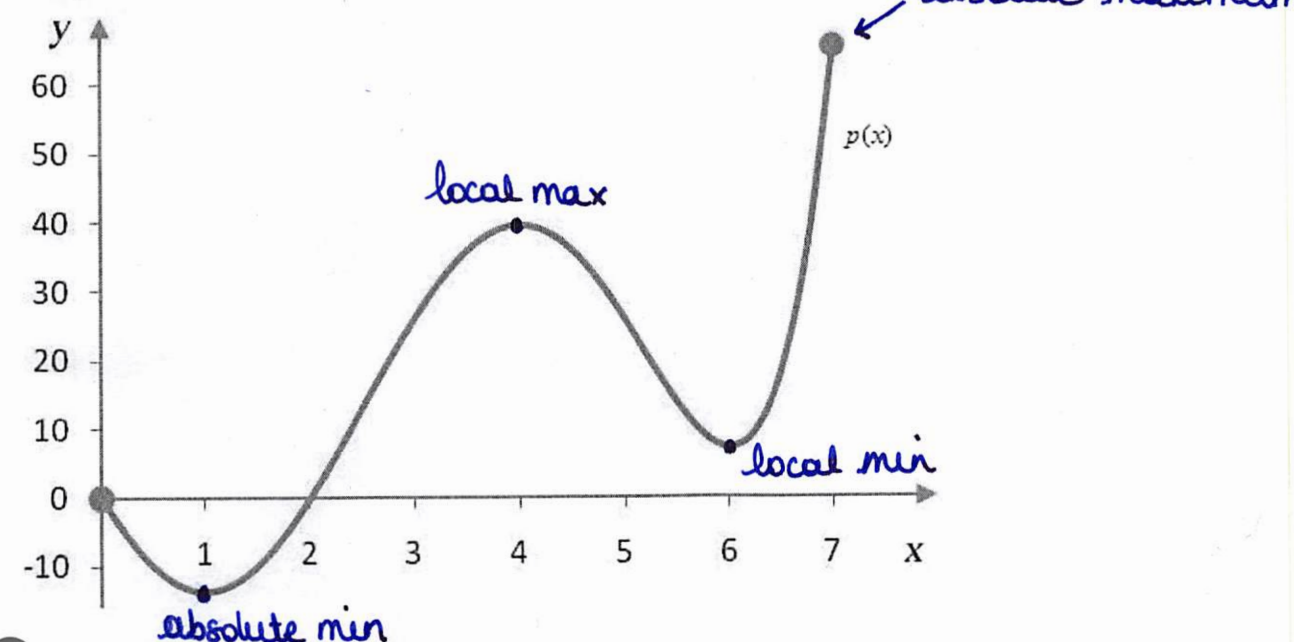
4.1 – Extreme Values of Functions

When describing a graph, we need to distinguish absolute and relative extrema (min and max).

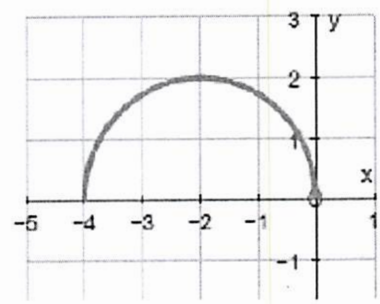
Example:



An **absolute maximum** is a value that is max for the whole domain.
A **relative or local maximum** is a value that is max in an open interval around it.

> *The Extreme Value Theorem*
>
> If f is continuous on a closed interval [a, b], then f has both a minimum and a maximum on the interval

Example 1: The function $f(x)=\sqrt{4-(x+2)^2}$ is shown below. Find the absolute extrema of the function on each given interval. If no maximum or minimum exists, which part of the extreme value theorem hypothesis isn't satisfied?



(a) [–4, 0]
max: 2
min: 0

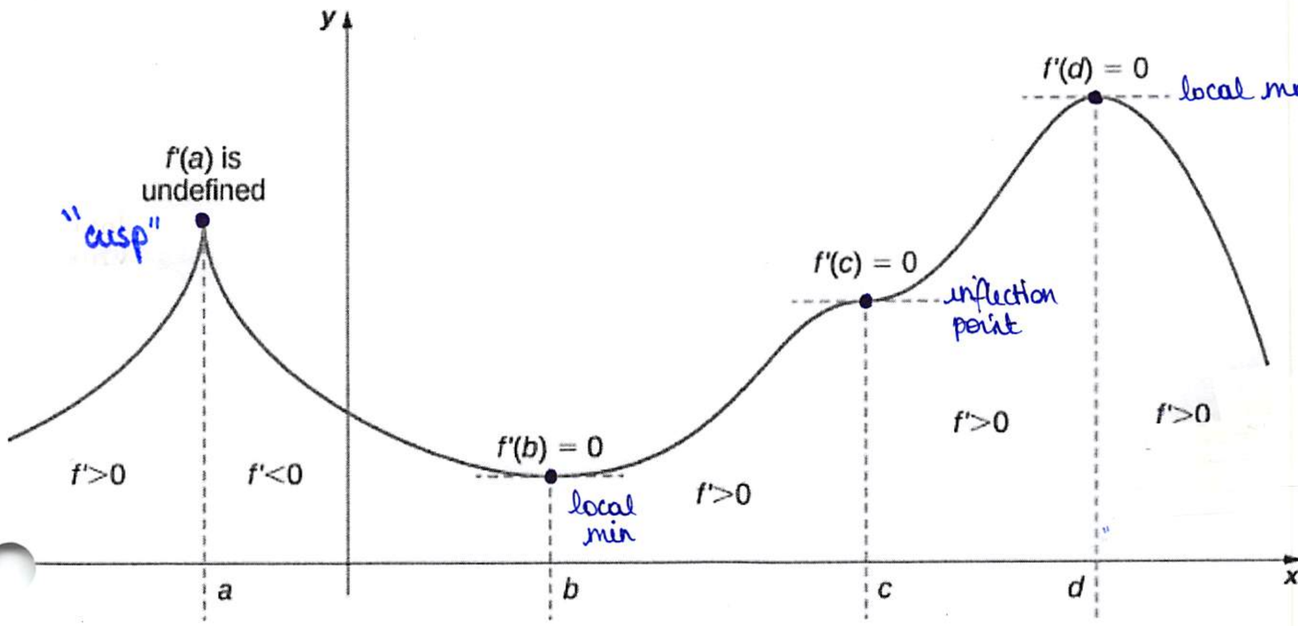
(b) [–2, 0)
max: 2
no min.
(the interval is not closed)

(c) (–4, –2)
no max
No min
the interval is not closed.

Critical Points: We call critical points a point where something happens on the graph (a change in direction for example).

Let f be a function defined at a point c. c is a critical point if $f'(c) = 0$ or if f' is undefined at c.

~~Important: If c is a local extremum, then $f'(c) = 0$. But the converse is not necessarily true!~~



To find absolute/local extrema:

- Determine the critical points.
- Compare the values of the function at all the critical points as well of the end points of the domain (or limits if relevant)

Examples: Find the extrema for the following functions:

1) $f(x) = 3x^4 - 4x^3$

$D = \mathbb{R}$

$f'(x) = 12x^3 - 12x^2$

$= 12x^2(x-1)$

$\Rightarrow \begin{cases} \text{no local max} \\ \text{local min @ 1 and ABS} : (1;-1) \end{cases}$

$f'(x) = 0 \iff x = 0$ or $x = 1$ (critical points)

x	$-\infty$		0		1		$+\infty$
$f(x)$	$+\infty$	$\searrow$	0	$\searrow$	-1	$\nearrow$	$+\infty$

$f(0) = 0$

$f(1) = -1$

$\lim_{x \to -\infty} f(x) = +\infty$

$\lim_{x \to +\infty} f(x) = +\infty$

2) $g(x) = (x+2)^{\frac{2}{3}}$

$D = \mathbb{R}$

$g'(x) = \frac{2}{3}(x+2)^{-1/3} = \dfrac{2}{3(x+2)^{1/3}}$

- g is differentiable on $(-\infty; -2)$ and on $(-2; +\infty)$
- $g'(x) \neq 0$ on both intervals.

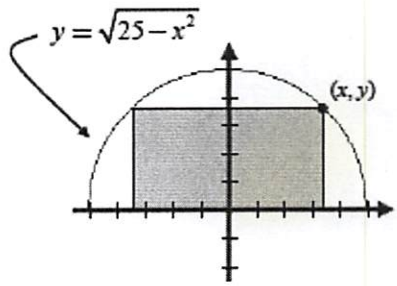
x	$-\infty$		-2		$+\infty$
$g(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$+\infty$

$\lim\limits_{x \to \pm\infty} g(x) = +\infty$

$g(-2) = 0$

$\Rightarrow \begin{cases} \text{no max} \\ \text{min @ } -2 \end{cases}$ $(-2; 0)$

3) A rectangle is bounded by the x – axis and the semicircle $y = \sqrt{25 - x^2}$. What length and width should the rectangle have so that its area is a maximum?



$A = 2x \cdot y$

$A(x) = 2x\sqrt{25 - x^2}$

$D = [0; 5]$

$A'(x) = 2\sqrt{25 - x^2} + 2x \cdot \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x)$

$= 2\sqrt{25 - x^2} - \dfrac{2x^2}{\sqrt{25 - x^2}}$

$= \dfrac{2(25 - x^2) - 2x^2}{\sqrt{25 - x^2}}$

$= \dfrac{-4x^2 + 50}{\sqrt{25 - x^2}}$

$A'(x) = 0 \Leftrightarrow 4x^2 = 50$

$x^2 = \dfrac{25}{2}$

$x = \dfrac{5}{\sqrt{2}}$ $(x \geq 0)$

x	0		$\frac{5\sqrt{2}}{2}$		5
$A(x)$	0	$\nearrow$	25	$\searrow$	0

$A(5) = 0$

$A(0) = 0$

$A\left(\frac{5\sqrt{2}}{2}\right) = 25$

Max Area: when $x = \frac{5\sqrt{2}}{2}$

$\Rightarrow$ length: $5\sqrt{2}$ width: $\frac{5\sqrt{2}}{2}$

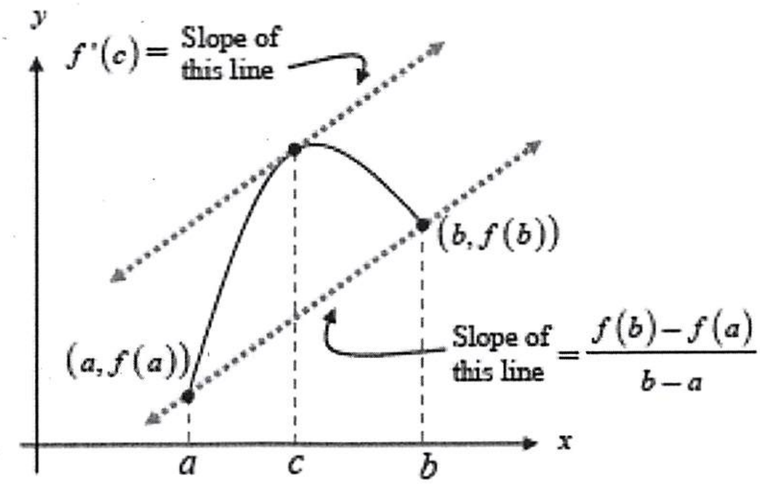
4.2 – Mean Value Theorem

The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then *there exists* a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Basically, the Mean Value Theorem says that the average rate of change over the entire interval is equal to the instantaneous rate of change at some point in the interval. It's an existence theorem. It doesn't say when, it just says that at some point, it happens.



Example 1: A plane begins its takeoff at 2:00 pm on a 2500-mile flight. The plane arrives at its destination at 7:30 pm (ignore time zone changes). Explain why there were at least two times during the flight when the speed of the plane was 400 miles per hour. Average speed: $\frac{2500}{5.5} \approx 454$ miles/hour.

The distance is a continuous function of time between 2pm and 7:30 pm. It is differentiable over $(2; 7.5)$. There exists a time $c \in (2; 7.5)$ when the speed will be 454 miles/hour. Before reaching that speed it was 400 mi/h at some point. Same after...

Example 2: Apply the Mean Value Theorem to the function on the indicated interval. In each case, make sure the hypothesis is true, ~~then find all values of c in the interval that are guaranteed by the MVT~~. x

a) $f(x) = x(x^2 - x - 2)$ on the interval $[-1, 1]$

f is continuous on $[-1; 1]$ and differentiable on $(-1; 1)$

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 0}{2} = -1$$

$\Rightarrow$ There is a $c \in (-1; 1)$ where $f'(c) = -1$.

Test for Increasing and Decreasing Functions:

Let f be a function continuous on an interval $[a; b]$ and differentiable on $(a; b)$

- f is increasing on $[a; b]$ if $f'(x) > 0$ for all x in $(a; b)$ except potentially on a finite number of points (where it could be 0).
  f is decreasing on $[a; b]$ if $f'(x) < 0$ for all x in $(a; b)$ except potentially on a finite number of points(where it could be 0).
  f is constant on $[a; b]$ if $f'(x) = 0$ for all x in $(a; b)$.

The critical points are the points where your graph might change direction. Therefore, to determine the variations of your function, you should:

1- determine its domain.
2- find the critical points.
3- Determine the extrema values as well as the limits.

All this information should be summarized in a variation table. **BUT the table is not an acceptable answer on the exam and on tests. You MUST write a sentence to give your answer.**

Examples: Determine the variations of

1) $f(x) = 4x^3 - 15x^2 - 18x + 7$.

$D = \mathbb{R}$

$\forall x \in \mathbb{R}: f'(x) = 12x^2 - 30x - 18$
$= 6(2x^2 - 5x - 3)$

$\Delta = 25 - 4(2)(-3) = 49$

zeros: $\frac{5 \pm 7}{4}$ ↗ $-1/2$, ↘ 3

$f'(x) = 0 \Leftrightarrow x = 3$ or $x = -\frac{1}{2}$

x	$-\infty$		$-1/2$		3		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	↗	$\frac{47}{4}$	↘	-74	↗	$+\infty$

$\lim_{x \to -\infty} f(x) = -\infty$

$f(-1/2) = \frac{47}{4}$

$f(3) = -74$

$\lim_{x \to +\infty} f(x) = +\infty$

f is increasing on $(-\infty; -1/2]$ and on $[3; +\infty)$ because f' is positive on $(-\infty, -\frac{1}{2})$ and on $(3, +\infty)$

f is decreasing on $[-1/2; 3]$ because f' is negative on $(-1/2, 3)$

2) $g(x) = (x^2 - 9)^{\frac{2}{3}}$

$D = \mathbb{R}$ $\quad g$ is differentiable on $\mathbb{R} \setminus \{\pm 3\}$

$\forall x \in \mathbb{R} \setminus \{\pm 3\}: \quad g'(x) = \frac{2}{3}(x^2-9)^{-1/3} \cdot 2x = \frac{4x}{3(x^2-9)^{1/3}}$

critical points: 0 ; 3 and -3.

x	$-\infty$		-3		0		3		$+\infty$
$g'(x)$		$-$	$\|$	$+$	0	$-$	$\|$	$+$	
$g(x)$	$+\infty$	↘	0	↗	$3\sqrt[3]{3}$	↘	0	↗	$+\infty$

$\lim_{x \to -\infty} g(x) = +\infty$

$g(-3) = 0$

$g(0) = \sqrt[3]{81} = 3\sqrt[3]{3}$

$g(3) = 0$

$\lim_{x \to +\infty} g(x) = +\infty$

g is decreasing on $(-\infty; -3]$ and $[0;3]$ because $g'(x) < 0$ on $(-\infty, -3)$ and on $(0,3)$

g is increasing on $[-3;0]$ and $[3;+\infty)$ because $g'(x) > 0$ on $(-3,0)$ and on $(3,+\infty)$

<u>**Antiderivatives**</u>:

If you are given the derivative of a function, the action to determine the original function is called **antidifferentiation** or finding the antiderivative.

Example: Suppose that $f'(x) = 2x + 3$, what could f be?

$\rightarrow \quad f(x) = x^2 + 3x \quad$ or $\quad f(x) = x^2 + 3x + 5 \quad$ or $\quad f(x) = x^2 + 3x - 2$

...

IMPORTANT: A function never has only 1 antiderivative. They all differ by a constant.
You need to be given one more info about the function in order to determine that constant.

Example: Suppose that $f'(x) = 2x + 3$, and that $f(0) = 2$ what could f be?

$f(x) = x^2 + 3x + c$

$f(0) = 2$

$0^2 + 3(0) + c = 2$

$c = 2$

$\Rightarrow \boxed{f(x) = x^2 + 3x + 2}$

4.3 – Connecting f' and f'' with the graph of f

The First Derivative Test

Let f be a continuous function, and let c be a critical point.

1. If f' changes sign from positive to negative at c, then f has a local maximum value at c.
2. If f' changes sign from negative to positive at c, then f has a local minimum value at c.
3. If f' DOES NOT change signs, then there is no local extreme value at c.

Important : If you are asked to find the absolute maximum (or just a maximum) of a function on a closed interval, you MUST test the endpoints also, and it may be just as simple to plug in the endpoints and the critical points.

<u>Concavity</u>:

Definition of Concavity

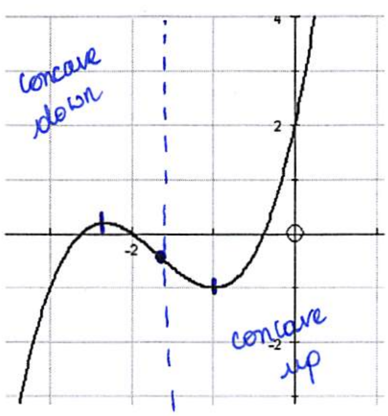
Let $y = f(x)$ be a differentiable function on an interval I. The graph of $f(x)$ is **concave up** on I if f' is increasing on I, and **concave down** on I if f' is decreasing on I.

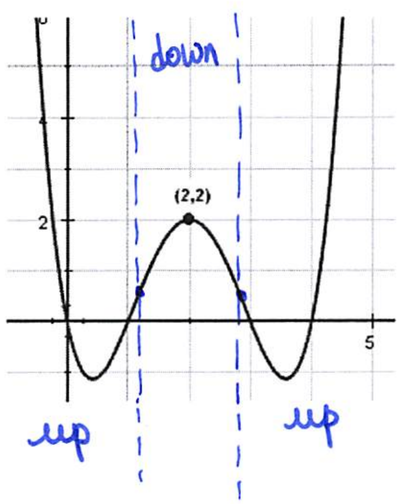
In other words:

oncavity deals with how a graph is curved. A graph that is concave up looks like  , while a graph that is concave down looks like  . We can use the SECOND derivative to determine the concavity of a function.

Concavity TEST

The graph of a twice-differentiable function $y = f(x)$ is **concave up** on any interval where $y'' > 0$, and **concave down** on any interval where $y'' < 0$.





Example: Determine the variations and the concavity of $f(x) = x^4 - 4x^3$.

$D = \mathbb{R}$

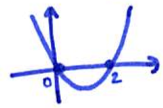
$f'(x) = 4x^3 - 12x^2$ on $\mathbb{R}$

$= 4x^2(x-3)$

critical points : 0 and 3

$f''(x) = 12x^2 - 24x$

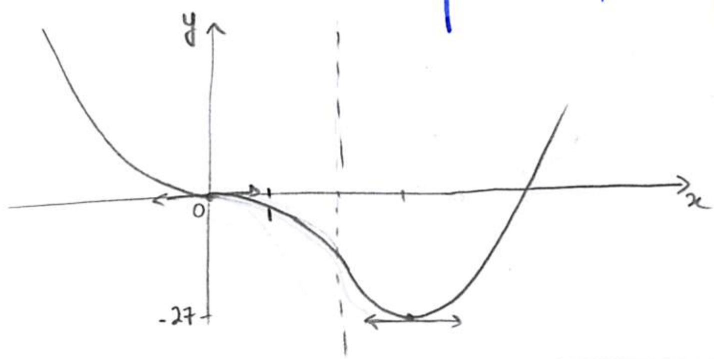
$= 12x(x-2)$



x	$-\infty$		0		3		$+\infty$
$f'(x)$		$-$	0	$-$	0	$+$	
$f(x)$	$+\infty$	↘	0	↘	-27	↗	$+\infty$

no local extremum (the derivative doesn't change sign).

x	$-\infty$		0		2		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
$f(x)$		concave up		concave down		concave up	



Points of Inflection:

> *Definition*
>
> **A point where the graph of a function has a tangent line (even if it's a vertical tangent line) AND where the concavity changes is a point of inflection.**

In other words, a point of inflection is when $f'' = 0$ AND changes signs.

Example: Determine the points of inflection of $f(x) = 3x^5 + 10x^4 + 15x + 7$.

$f'(x) = 15x^4 + 40x^3 + 15$

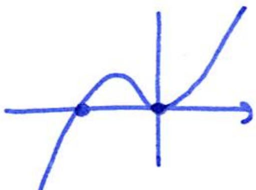
$f''(x) = 60x^3 + 120x^2$

$= 60x^2(x+2)$

f'' changes sign from negative to positive when $x = -2$

$\Rightarrow$ point of inflection when $x = -2$

$\boxed{\text{Point : } (-2 ; 41)}$



Second Derivative Test for Local Extrema

If $f'(c)=0$ (which makes $x=c$ a critical point) AND $f''(c)<0$, then f has a local MAXIMUM at $x=c$.

If $f'(c)=0$ (which makes $x=c$ a critical point) AND $f''(c)>0$, then f has a local MINIMUM at $x=c$.

Example: Use the second derivative test to identify any local extrema of $f(x) = -x^4 + 4x^3 - 4x^2 + 1$.

$f'(x) = -4x^3 + 12x^2 - 8x$

$= -4x(x^2 - 3x + 2)$

$= -4x(x-1)(x-2)$

$f''(x) = -12x^2 + 24x - 8$

$f'(x) = 0$ when $x = 0$ & $x = 1$ & $x = 2$

$f''(0) = -8$ (concave down) $\Rightarrow$ local max @ $(0;1)$

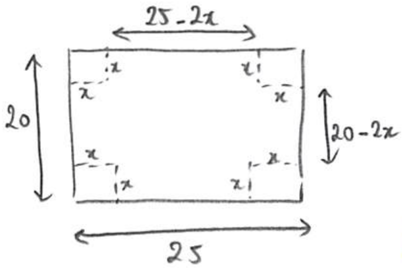
$f''(1) = 4$ (concave up) $\Rightarrow$ local min @ $(1;0)$

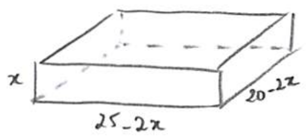
$f''(2) = -8$ (concave down) $\Rightarrow$ local max @ $(2;1)$

4.4 – Modeling And Optimization

In order to optimize a quantity, make a drawing when you can, and you need to express this quantity as a function of only 1 variable in order to determine its extrema values…

Example 1:
An open top box is going to be made by cutting equal squares out of the four corners of a 20x25 cm piece of paper. What size square should be cut out of the paper to make a Maximum volume?





$D = [0; 10]$

$V = x(20-2x)(25-2x)$
$\quad = x(500 - 40x - 50x + 4x^2)$
$\quad = 4x^3 - 90x^2 + 500x$

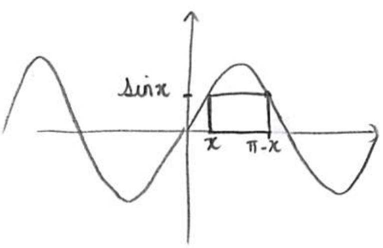
$V'(x) = 12x^2 - 180x + 500$
$\quad = 4(3x^2 - 45x + 125)$

critical points : $\frac{45 \pm \sqrt{525}}{6}$ → ≈ 3.7 ; → $\approx \cancel{11.3}$ not in the domain

x	0		$\frac{45-\sqrt{525}}{6}$		10
$V(x)$	0	↗	820.5	↘	0

size square : $\frac{45 - \sqrt{525}}{6}$ or $\frac{45 - 5\sqrt{21}}{6}$

Example 2: CALCULATOR
A rectangle is going to be inscribed under one arch of the sine curve. One side of the rectangle will be on the *x*-axis. What is the largest area the rectangle can have and what are the dimensions of that area?



$A = \sin x\,(\pi - 2x)$ $\qquad D = [0; \frac{\pi}{2}]$

$A'(x) = \cos x(\pi - 2x) - 2\sin x$

critical point when $x \doteq 0.71$ (calculator)

	0		0.7		$\pi/2$
$A(x)$	0	↗	max	↘	0

$A(0.71) \simeq 1.12$
max area

dimensions : $\pi - 2x \simeq 1.72$
by $\sin x \simeq 0.65$

Example 3:
ppose that $r(x) = 9x$ is the revenue function and $c(x) = x^3 - 6x^2 + 15x$, the cost function for a business, where x represents thousands of units produced and r and c are in thousands of dollars.
Is there a production level that maximizes profit? If so, what is it?

$p(x) = r(x) - c(x)$

$p(x) = -x^3 + 6x^2 - 6x$ $D = [0; +\infty)$

$p'(x) = -3x^2 + 12x - 6$

$= -3(x^2 - 4x + 2)$

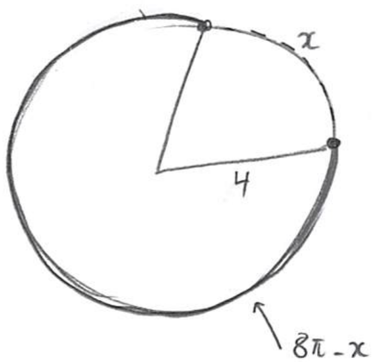
$\Delta = 16 - 8 = 8$

critical points $x = \frac{4 \pm 2\sqrt{2}}{2}$

x	0		$\frac{4-2\sqrt{2}}{2}$		$\frac{4+2\sqrt{2}}{2}$ ≈ 3.414		$+\infty$
$p(x)$	0	$\searrow$	-1.66	$\nearrow$	9.65	$\searrow$	$-\infty$

Maximum profit when 3414 units are produced (profit: \$9656.85)

Example 4: It can get very complicated…
cone is constructed from a flat, circular disk of radius 4cm, by removing a sector of arc length x and then connecting the new edges.
What arc length will produce the cone of maximum volume? And what is that volume?



- $V = \frac{1}{3}\pi r^2 h$

need to express in terms of x...

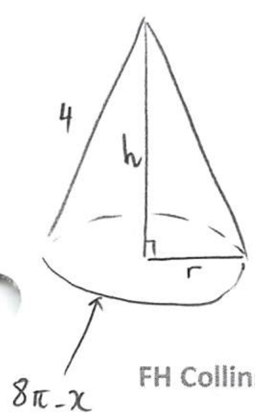
- $2\pi r = 8\pi - x$

$r = \frac{8\pi - x}{2\pi}$

- $h^2 = 4^2 - r^2$

$= 16 - \left(\frac{8\pi - x}{2\pi}\right)^2$

$h = \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}$



$\Rightarrow V = \frac{\pi}{3}\left(\frac{8\pi - x}{2\pi}\right)^2 \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}$

$V' \Rightarrow$ critical points $\Rightarrow$ max on the domain

$D = [0; 8\pi]$

$$r = 4 - \frac{x}{2\pi}$$

$$h^2 = 4^2 - r^2$$
$$= 16 - \left(4 - \frac{x}{2\pi}\right)^2$$
$$= 16 - 16 + \frac{4x}{\pi} - \frac{x^2}{4\pi^2}$$
$$= \frac{x}{4\pi^2}\left(16\pi - x\right)$$
$$h = \frac{1}{2\pi}\sqrt{x(16\pi - x)} = \frac{1}{2\pi}\sqrt{-x^2 + 16\pi x}$$

$$V = \frac{\pi}{3}\left(4 - \frac{x}{2\pi}\right)^2 \cdot \frac{1}{2\pi}\sqrt{x(16\pi - x)}$$
$$V = \frac{1}{6}\left(4 - \frac{x}{2\pi}\right)^2 \sqrt{x(16\pi - x)}$$

$$\rightarrow V' = \frac{1}{6} \times \left[\cancel{2}\left(4 - \frac{x}{2\pi}\right) \times \left(-\frac{1}{\cancel{2}\pi}\right)\sqrt{x(16\pi - x)} + \left(4 - \frac{x}{2\pi}\right)^2 \times \frac{1}{\cancel{2}}\left(x(16\pi - x)\right)^{-1/2} \times \left(-\cancel{2}x + \overset{8}{\cancel{16\pi}}\right)\right]$$

$$= \frac{1}{6}\left(4 - \frac{x}{2\pi}\right)\left[\frac{-x(16\pi - x) + \left(4 - \frac{x}{2\pi}\right)(-x + 8\pi)}{\sqrt{x(16\pi - x)}}\right]$$

$$= \frac{1}{6}\left(4 - \frac{x}{2\pi}\right)\frac{-16\pi x + x^2 - 4x + 32\pi + \frac{x^2}{2\pi} - 4x}{\sqrt{x(16\pi - x)}}$$

$$= \frac{1}{6}\frac{(4 - x/2\pi)}{\sqrt{x(16\pi - x)}} \times \left(x^2\left(1 + \frac{1}{2\pi}\right) + x\left(-8 - 16\pi\right) + 32\pi\right)$$

Critical points: $4 - \frac{x}{2\pi} = 0$

$$x = 8\pi$$

$$\Delta = (8 + 16\pi)^2 - 4\left(1 + \frac{1}{2\pi}\right) \times 32\pi$$
$$= 64 + 256\pi + 256\pi^2 - 128\pi - 64$$
$$= 256\pi^2 + 128\pi$$

$$x = \frac{8 + 16\pi \pm \sqrt{256\pi^2 + 128\pi}}{2 + \frac{1}{\pi}} \nearrow 1.8 \quad \searrow \cancel{48.5} \notin D$$

$V(0) = 0$

$\boxed{V(1.8) \simeq 21.4}$

$V(8\pi) = 0$

4.5 – Linearization and Newton's Method

Linearization:

The goal of linearization is to approximate a curve with a line. Why? Because it's easier to use a line than a curve!
**All you have to do is find the equation of a tangent line and use the tangent line instead of the original function.

Example 1: Consider $f(x)=\sqrt{x}$. We all know that $f(4)=2$, but without a calculator, how can we find $f(4.1)$?

a) Find the equation of the tangent line for $f(x)$ when $x=4$. [Your book refers to this as $L(x)$.]

$L(x) = \frac{1}{4}(x-4)+2$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$ $\quad$ Point : $(4;2)$

$L(x) = \frac{1}{4}x + 1$ $\quad f'(4) = \frac{1}{4}$

b) The tangent line you found is approximately the same as $f(x)$ "centered at $x=4$".
Use your tangent line to approximate $f(4.1)$.

$L(4.1) = \frac{1}{4}(4.1)+1$ $\quad f(4.1) \simeq 2.025$

$= 1.025 + 1$

c) Use a calculator to approximate $f(4.1)$. How close is the approximation?

$f(4.1) \simeq 2.0248$ to the nearest thousandth!

Example 2: Find the linearization of $f(x) = cosx$ at $x = \frac{\pi}{2}$, and use it to approximate $cos1.75$ without a calculator. Then use your calculator to determine the accuracy of the approximation.

$f'(x) = -\sin x$

$f'(\frac{\pi}{2}) = -1$

Point: $(\frac{\pi}{2}; 0)$

$L(x) = -(x - \frac{\pi}{2})$

$L(x) = -x + \frac{\pi}{2}$

$\cos 1.75 \simeq -1.75 + \frac{\pi}{2}$

$\simeq -1.75 + 1.57$

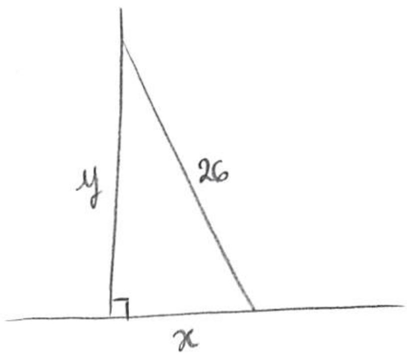
$\simeq -0.18$

calculator : $\cos 1.75 \simeq -0.178$

$|error| < 10^{-2}$.

4.6 – Related Rates

Example 4: Tweety is resting in a bird house 24 feet off the ground. Using a 26 foot ladder which he leaned against the pole holding the bird house, Sylvester tries to steal the small yellow bird. Tweety's bodyguard, Hector the dog, starts pulling the base of the ladder away from the pole at a rate of 2 ft/s. How fast is the ladder falling when it is 10 feet off the ground?



$$\frac{dx}{dt} = 2 \qquad \left.\frac{dy}{dt}\right|_{y=10} = ?$$

need to determine when $y = 10$

2

- $y = \sqrt{26^2 - x^2}$
- $\dfrac{dy}{dt} = -\dfrac{2x}{2\sqrt{676 - x^2}} \cdot \dfrac{dx}{dt}$
- when $y = 10$, $x = \sqrt{26^2 - 10^2}$ $= 24$
- $\left.\dfrac{dy}{dt}\right|_{y=10} = -\dfrac{24}{\sqrt{676-576}} \times 2$

$$= \boxed{-4.8 \text{ ft/s}}$$

Guidelines for Solving Related-Rate Problems

Step 1: Read the problem, really! You'd be amazed how many people skip this step. Then read it again! ☺

Step 2: Draw a diagram showing what's going on. Identify all relevant information and assign variables to what's changing. Use the general case (numbers for values that NEVER change in this situation, and variables for anything that is changing).

♫ : Related Rates usually involve motion … any diagram you draw is like a still picture of what is occurring. Any part of your picture that NEVER changes can be labeled with a constant (or number), but any part of your picture that is in motion or is changing MUST be labeled with a variable!

In other words, if the radius of a circle is increasing and you are asked to find the rate of change in the area at the exact moment when the radius is 5 cm, then your diagram would be a circle, but you would NOT label the radius 5 because it is changing … you would label the radius r.

Step 3: Find the equation that gives the relationship between the variables you just named in step 2. This is sometimes the hardest part, but most problems fall into three categories … a triangle that you can use a trigonometric ratio (involving sides and angles), the Pythagorean theorem (involving all 3 sides of a right triangle), or a known formula like Area, Volume, Distance, etc.

Step 4: Find the particular information (values of variables at the exact moment you drew your diagram) for the problem and write it down. Then, list what you are looking for (normally this would be a derivative).

or

Step 5: Implicitly differentiate the equation with respect to time t. Usually this equation will have at least two derivatives. If it has more than two, be sure you have enough information, *or* you may have to find a relationship between two of the variables, and rewrite the equation in step 3 using this relationship.

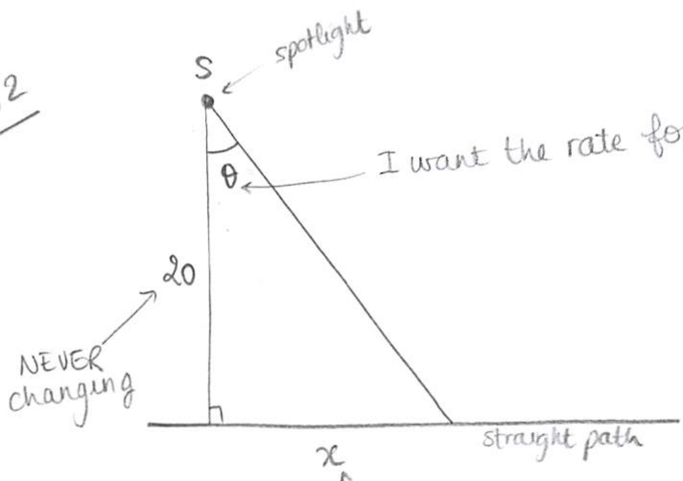
Step 6: Plug in the particular information, and solve for the desired quantity. DO NOT DO THIS UNTIL AFTER YOU HAVE TAKEN THE DERIVATIVE!

Step 7: Write down your answer and circle it with your favorite color. (be sure to use correct units)

Example 5: Bugs and Daffy finished their final act on the *Bugs and Daffy Show* by dancing off the stage with a spotlight covering their every move. If they are moving off the stage along a straight path at a speed of 4 ft/s, and the spotlight is 20 ft away from this path, what rate is the spotlight rotating when they are 15 feet from the point on the path closest to the spotlight?

step 1

step 2



I want the rate for the rotation : $\left.\dfrac{d\theta}{dt}\right|_{x=15}$ Q°

changing at a rate of 4 ft/s

step 3

- $\tan\theta = \dfrac{x}{20}$

step 5

- $\sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{1}{20}\dfrac{dx}{dt}$

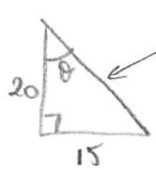
$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{20} \cdot \frac{dx}{dt}$$

need to determine when $x = 15$

4

step 4

- when $x = 15$: $\cos\theta = \dfrac{20}{25} = \dfrac{4}{5}$

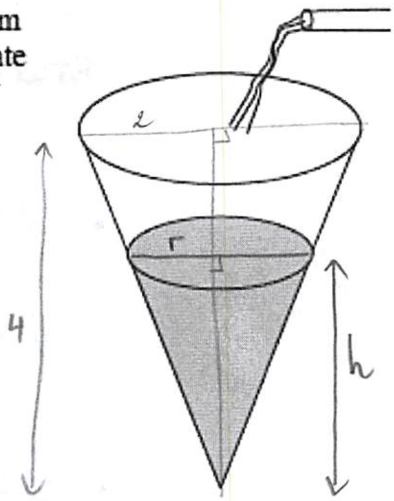


Pyth : 25

step 6

$$\Rightarrow \left.\frac{d\theta}{dt}\right|_{x=15} = \frac{1}{20} \times \frac{16}{25} \times 4 = \boxed{\frac{16}{125}\ \text{rad/s}}$$

Example 6: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m³/min, find the rate at which the water level is rising when the water is 3 m deep. The volume of a circular cone with radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.



$$\frac{dV}{dt} = 2\,m^3/min$$

$$\left.\frac{dh}{dt}\right|_{h=3} = ?$$

- $V = \frac{1}{3}\pi r^2 h$

need to determine

need to determine

- $\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}\cdot h + r^2\frac{dh}{dt}\right)$

3

?

2

$*\ \frac{r}{2} = \frac{h}{4}$

$r = \frac{h}{2}$

$*\ \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$

$\Rightarrow$ when $h=3$: $r = \frac{3}{2}$

$\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(rh\frac{dh}{dt} + r^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{dh}{dt} \times \frac{1}{3}\pi(rh + r^2)$$

$$\frac{dh}{dt} = \frac{3}{\pi(rh + r^2)} \cdot \frac{dV}{dt}$$

- $\left.\frac{dh}{dt}\right|_{h=3} = \frac{3}{\pi\left(\frac{9}{2}+\frac{9}{4}\right)} \cdot 2 = \frac{24}{27\pi} \approx \boxed{0.28\ m/min}$

<u>Note</u>: you could have replaced r by $\frac{h}{2}$ from the start (you always on only 2 variables!)

- $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 . h$

  $V = \frac{1}{12}\pi h^3$

- $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

  $\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$

- $\left.\frac{dh}{dt}\right|_{h=3} = \frac{4}{9\pi} \times 2$

  $= \frac{8}{9\pi}$

  $\simeq 0.28.$

8.2 – L'Hôpital's Rule

We can use derivatives to determine limits when it's an indeterminate form like "$\frac{0}{0}$".

L'Hôpital's Rule:

Suppose that $f(a) = g(a) = 0$ and $g'(a)$ exists, and that $g'(a) \neq 0$.

Then,
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example: Determine $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$ using l'Hôpital's Rule.

$f(x) = \sqrt{1+x} - 1$ $\quad f(0) = 0$ $\quad f'(x) = \frac{1}{2\sqrt{1+x}}$ $\quad f'(0) = \frac{1}{2}$

$g(x) = x$ $\quad g(0) = 0$ $\quad g'(x) = 1$ $\quad g'(0) = 1 \ (\neq 0)$

$\hat{H}$Rule: $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{1}{2}$

L'Hôpital's Rule (stronger form):

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$.

Then,
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example: Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x}-1-x/2}{x^2}$

$f(x) = \sqrt{1+x} - 1 - \frac{x}{2}$ $\quad f(0) = 0$ $\quad f'(x) = \frac{1}{2\sqrt{1+x}} - \frac{1}{2}$

$g(x) = x^2$ $\quad g(0) = 0$ $\quad g'(x) = 2x$ $\quad (g'(0) = 0)$

$$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2}}{2x} = \frac{1 - \sqrt{1+x}}{4x\sqrt{1+x}} = \frac{1-1-x}{4x\sqrt{1+x}(1+\sqrt{1+x})}$$
$$= -\frac{1}{4\sqrt{1+x}(1+\sqrt{1+x})} \xrightarrow{x \to 0} \frac{1}{8}$$

$\therefore \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} = \frac{1}{8}$

L'Hôpital's Rule (for indeterminate forms like $\frac{\infty}{\infty}$):

Suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$.

Then,
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example: Determine

1) $\lim_{x \to \frac{\pi}{2}} \frac{secx}{1+tanx}$

$f(x) = \sec x$ $\quad f'(x) = \sec x \tan x$

$g(x) = 1 + \tan x$ $\quad g'(x) = \sec^2 x$

$\frac{f'(x)}{g'(x)} = \sin x \xrightarrow{x \to \frac{\pi}{2}} \boxed{1}$

2) $\lim_{x \to \infty} \frac{lnx}{2\sqrt{x}}$

$f(x) = \ln x$ $\quad f'(x) = \frac{1}{x}$

$g(x) = 2\sqrt{x}$ $\quad g'(x) = \frac{1}{\sqrt{x}}$

$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \frac{1}{\sqrt{x}} \xrightarrow{x \to +\infty} \boxed{0}$

Hwk: p 450 #