Keeping Track of Units

Notice in Example 3 that we are summing products of the form $\pi(16 - x^2)$ (a cross section area, measured in square units) times Δx (a length, measured in units). The products are therefore measured in cubic units, which are the correct units for volume.

The value for $n = 1000$ compares *very* favorably with the true volume,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4)^3 = \frac{256\pi}{3} \approx 268.0825731.$$

Even for $n = 10$ the difference between the MRAM approximation and the true vol-ume is a small percentage of V:

$$\frac{|\text{MRAM}_{10} - V|}{V} = \frac{\text{MRAM}_{10} - 256\pi/3}{256\pi/3} \le 0.005.$$

That is, the error percentage is about one half of one percent! ***Now try Exercise***

Cardiac Output

So far we have seen applications of the RAM process to finding distance traveled and vol-ume. These applications hint at the usefulness of this technique. To suggest its versatil-ity we will present an application from human physiology.

The number of liters of blood your heart pumps in a fixed time interval is called your *cardiac output*. For a person at rest, the rate might be 5 or 6 liters per minute. During stren-uous exercise the rate might be as high as 30 liters per minute. It might also be altered sig-nificantly by disease. How can a physician measure a patient's cardiac output without interrupting the flow of blood?

One technique is to inject a dye into a main vein near the heart. The dye is drawn into the right side of the heart and pumped through the lungs and out the left side of the heart into the aorta, where its concentration can be measured every few seconds as the blood flows past. The data in Table 5.2 and the plot in Figure 5.10 (obtained from the data) show the response of a healthy, resting patient to an injection of 5.6 mg of dye.

Table 5.2 **Dye Concentration Data**

Seconds after Injection t	Dye Concentration (adjusted for recirculation) c
5	0
7	3.8
9	8.0
11	6.1
13	3.6
15	2.3
17	1.45
19	0.91
21	0.57
23	0.36
25	0.23
27	0.14
29	0.09
31	0

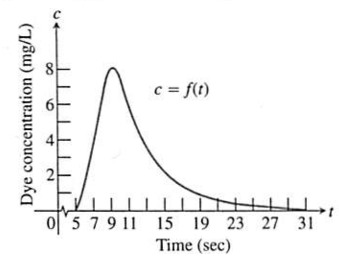


Figure 5.10 The dye concentration data from Table 5.2, plotted and fitted with a smooth curve. Time is measured with $t = 0$ at the time of injection. The dye concentration is zero at the beginning while the dye passes through the lungs. It then rises to a maximum at about $t = 9$ sec and tapers to zero by $t = 31$ sec.

The graph shows dye concentration (measured in milligrams of dye per liter of blood) as a function of time (in seconds). How can we use this graph to obtain the cardiac output (measured in liters of blood per second)? The trick is to divide the *number of mg of dye* by the *area under the dye concentration curve.* You can see why this works if you consider what happens to the units:

$$\frac{\text{mg of dye}}{\text{units of area under curve}} = \frac{\text{mg of dye}}{\frac{\text{mg of dye}}{\text{L of blood}} \cdot \text{sec}} = \frac{\text{mg of dye}}{\text{sec}} \cdot \frac{\text{L of blood}}{\text{mg of dye}} = \frac{\text{L of blood}}{\text{sec}}.$$

So you are now ready to compute like a cardiologist.

Charles Richard Drew (1904–1950)

Millions of people are alive today because of Charles Drew's pioneering work on blood plasma and the preservation of human blood for transfusion. After directing the Red Cross program that collected plasma for the Armed Forces in World War II, Drew went on to become Head of Surgery at Howard University and Chief of Staff at Freedmen's Hospital in Washington, D.C.

EXAMPLE 4 Computing Cardiac Output from Dye Concentration

Estimate the cardiac output of the patient whose data appear in Table 5.2 and Figure 5.10. Give the estimate in liters per minute.

SOLUTION

We have seen that we can obtain the cardiac output by dividing the amount of dye (5.6 mg for our patient) by the area under the curve in Figure 5.10. Now we need to find the area. Our geometry formulas do not apply to this irregularly shaped region, and the RAM program is useless without a formula for the function. Nonetheless, we can draw the MRAM rectangles ourselves and estimate their heights from the graph. In Figure 5.11 each rectangle has a base 2 units long and a height $f(m_i)$ equal to the height of the curve above the midpoint of the base.

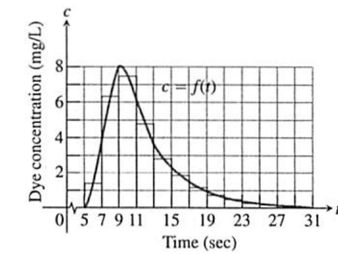


Figure 5.11 The region under the concentration curve of Figure 5.10 is approximated with rectangles. We ignore the portion from $t = 29$ to $t = 31$; its concentration is negligible. (Example 4)

The area of each rectangle, then, is $f(m_i)$ times 2, and the sum of the rectangular areas is the MRAM estimate for the area under the curve:

$$\begin{aligned}\text{Area} &\approx f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + \cdots + f(28)\cdot 2\\ &\approx 2\cdot(1.4 + 6.3 + 7.5 + 4.8 + 2.8 + 1.9 + 1.1\\ &\qquad + 0.7 + 0.5 + 0.3 + 0.2 + 0.1)\\ &= 2\cdot(27.6) = 55.2\ (\text{mg/L})\cdot\text{sec}.\end{aligned}$$

Dividing 5.6 mg by this figure gives an estimate for cardiac output in liters per second. Multiplying by 60 converts the estimate to liters per minute:

$$\frac{5.6\text{ mg}}{55.2\text{ mg}\cdot\text{sec/L}} \cdot \frac{60\text{ sec}}{1\text{ min}} \approx 6.09\text{ L/min}.$$

Now try Exercise 15.

Quick Review 5.1

As you answer the questions in Exercises 1–10, try to associate the answers with area, as in Figure 5.1.

1. A train travels at 80 mph for 5 hours. How far does it travel?
2. A truck travels at an average speed of 48 mph for 3 hours. How far does it travel?
3. Beginning at a standstill, a car maintains a constant acceleration of 10 ft/sec^2 for 10 seconds. What is its velocity after 10 seconds? Give your answer in ft/sec and then convert it to mi/h.
4. In a vacuum, light travels at a speed of 300,000 kilometers per second. How many kilometers does it travel in a year? (This distance equals one *light-year.*)
5. A long distance runner ran a race in 5 hours, averaging 6 mph for the first 3 hours and 5 mph for the last 2 hours. How far did she run?
6. A pump working at 20 gallons/minute pumps for an hour. How many gallons are pumped?

7. At 8:00 P.M. the temperature began dropping at a rate of 1 degree Celsius per hour. Twelve hours later it began rising at a rate of 1.5 degrees per hour for six hours. What was the net change in temperature over the 18-hour period?

8. Water flows over a spillway at a steady rate of 300 cubic feet per second. How many cubic feet of water pass over the spillway in one day?

9. A city has a population density of 350 people per square mile in an area of 50 square miles. What is the population of the city?

10. A hummingbird in flight beats its wings at a rate of 70 times per second. How many times does it beat its wings in an hour if it is in flight 70% of the time?

Section 5.1 Exercises

1. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = 5$ for time $t \geq 0$. Where is the particle at $t = 4$?

2. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = 2t + 1$ for time $t \geq 0$. Where is the particle at $t = 4$?

3. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2 + 1$ for time $t \geq 0$. Where is the particle at $t = 4$? Approximate the area under the curve using four rectangles of equal width and heights determined by the midpoints of the intervals, as in Example 1.

4. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2 + 1$ for time $t \geq 0$. Where is the particle at $t = 5$? Approximate the area under the curve using five rectangles of equal width and heights determined by the midpoints of the intervals, as in Example 1.

Exercises 5–8 refer to the region R enclosed between the graph of the function $y = 2x - x^2$ and the x-axis for $0 \leq x \leq 2$.

5. **(a)** Sketch the region R.

(b) Partition $[0, 2]$ into 4 subintervals and show the four rectangles that LRAM uses to approximate the area of R. Compute the LRAM sum without a calculator.

6. Repeat Exercise 1(b) for RRAM and MRAM.

7. Using a calculator program, find the RAM sums that complete the following table.

n	LRAM_n	MRAM_n	RRAM_n
10			
50			
100			
500			

8. Make a conjecture about the area of the region R.

In Exercises 9–12, use RAM to estimate the area of the region enclosed between the graph of f and the x-axis for $a \leq x \leq b$.

9. $f(x) = x^2 - x + 3, \quad a = 0, \quad b = 3$

10. $f(x) = \dfrac{1}{x}, \quad a = 1, \quad b = 3$

11. $f(x) = e^{-x^2}, \quad a = 0, \quad b = 2$

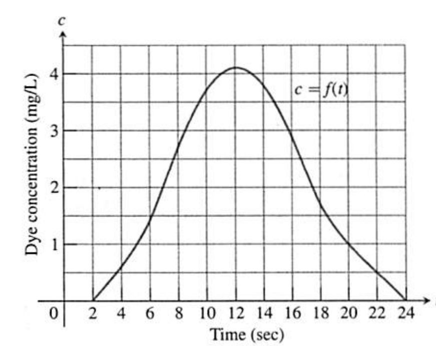
12. $f(x) = \sin x, \quad a = 0, \quad b = \pi$

13. *(Continuation of Example 3)* Use the slicing technique of Example 3 to find the MRAM sums that approximate the volume of a sphere of radius 5. Use $n = 10, 20, 40, 80,$ and 160.

14. *(Continuation of Exercise 13)* Use a geometry formula to find the volume V of the sphere in Exercise 13 and find **(a)** the error and **(b)** the percentage error in the MRAM approximation for each value of n given.

15. ***Cardiac Output*** The following table gives dye concentrations for a dye-concentration cardiac-output determination like the one in Example 4. The amount of dye injected in this patient was 5 mg instead of 5.6 mg. Use rectangles to estimate the area under the dye concentration curve and then go on to estimate the patient's cardiac output.

Seconds after Injection t	Dye Concentration (adjusted for recirculation) c
2	0
4	0.6
6	1.4
8	2.7
10	3.7
12	4.1
14	3.8
16	2.9
18	1.7
20	1.0
22	0.5
24	0



Distance Traveled The table below shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine, using 10 subintervals of length 1 with **(a)** left-endpoint values (LRAM) and **(b)** right-endpoint values (RRAM).

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

Distance Traveled Upstream You are walking along the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the table below. About how far upstream does the bottle travel during that hour? Find the **(a)** LRAM and **(b)** RRAM estimates using 12 subintervals of length 5.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

Length of a Road You and a companion are driving along a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the table below. (The velocity was converted from mi/h to ft/sec using 30 mi/h = 44 ft/sec.) Estimate the length of the road by averaging the LRAM and RRAM sums.

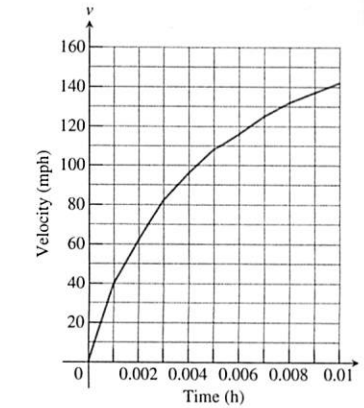
Time (sec)	Velocity (ft/sec)	Time (sec)	Velocity (ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

Distance from Velocity Data The table below gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour.)

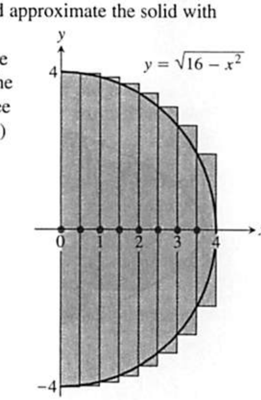
Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		

(a) Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

(b) Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?



20. ***Volume of a Solid Hemisphere*** To estimate the volume of a solid hemisphere of radius 4, imagine its axis of symmetry to be the interval $[0, 4]$ on the x-axis. Partition $[0, 4]$ into eight subintervals of equal length and approximate the solid with cylinders based on the circular cross sections of the hemisphere perpendicular to the x-axis at the subintervals' left endpoints. (See the accompanying profile view.)



(a) **Writing to Learn** Find the sum S_8 of the volumes of the cylinders. Do you expect S_8 to overestimate V? Give reasons for your answer.

(b) Express $|V - S_8|$ as a percentage of V to the nearest percent.

21. Repeat Exercise 20 using cylinders based on cross sections at the right endpoints of the subintervals.

22. ***Volume of Water in a Reservoir*** A reservoir shaped like a hemispherical bowl of radius 8 m is filled with water to a depth of 4 m.

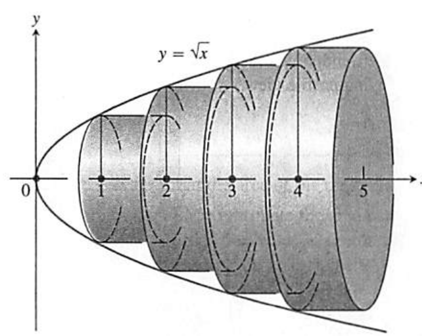
(a) Find an estimate S of the water's volume by approximating the water with eight circumscribed solid cylinders.

(b) It can be shown that the water's volume is $V = (320\pi)/3$ m^3. Find the error $|V - S|$ as a percentage of V to the nearest percent.

23. ***Volume of Water in a Swimming Pool*** A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using **(a)** left-endpoint values and **(b)** right-endpoint values.

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

24. ***Volume of a Nose Cone*** The nose "cone" of a rocket is a *paraboloid* obtained by revolving the curve $y = \sqrt{x}, 0 \le x \le 5$ about the x-axis, where x is measured in feet. Estimate the volume V of the nose cone by partitioning [0, 5] into five subintervals of equal length, slicing the cone with planes perpendicular to the x-axis at the subintervals' left endpoints, constructing cylinders of height 1 based on cross sections at these points, and finding the volumes of these cylinders. (See the accompanying figure.)



25. ***Volume of a Nose Cone*** Repeat Exercise 24 using cylinders based on cross sections at the *midpoints* of the subintervals.

26. ***Free Fall with Air Resistance*** An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown in the table below.

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

(a) Use LRAM_5 to find an upper estimate for the speed when $t = 5$.

(b) Use RRAM_5 to find a lower estimate for the speed when $t = 5$.

(c) Use upper estimates for the speed during the first second, second second, and third second to find an upper estimate for the distance fallen when $t = 3$.

27. ***Distance Traveled by a Projectile*** An object is shot straight upward from sea level with an initial velocity of 400 ft/sec.

(a) Assuming gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32$ ft/sec^2 for the gravitational constant.

(b) Find a lower estimate for the height attained after 5 sec.

28. ***Water Pollution*** Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the table below.

Time (h)	0	1	2	3	4
Leakage (gal/h)	50	70	97	136	190

Time (h)	5	6	7	8
Leakage (gal/h)	265	369	516	720

(a) Give an upper and lower estimate of the total quantity of oil that has escaped after 5 hours.

(b) Repeat part (a) for the quantity of oil that has escaped after 8 hours.

(c) The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all of the oil has leaked? In the best case?

29. ***Air Pollution*** A power plant generates electricity by burning oil. Pollutants produced by the burning process are removed by scrubbers in the smokestacks. Over time the scrubbers become less efficient and eventually must be replaced when the amount of pollutants released exceeds government standards. Measurements taken at the end of each month determine the rate at which pollutants are released into the atmosphere as recorded in the table below.

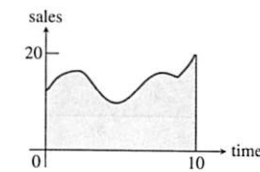
Month	Jan	Feb	Mar	Apr	May	Jun
Pollutant Release Rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant Release Rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

(a) Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

(b) In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

Writing to Learn The graph shows the sales record for a company over a 10-year period. If sales are measured in millions of units per year, explain what information can be obtained from the area of the region, and why.



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

True or False If f is a positive, continuous, increasing function on $[a, b]$, then LRAM gives an area estimate that is less than the true area under the curve. Justify your answer.

True or False For a given number of rectangles, MRAM always gives a more accurate approximation to the true area under the curve than RRAM or LRAM. Justify your answer.

Multiple Choice If an MRAM sum with four rectangles of equal width is used to approximate the area enclosed between the x-axis and the graph of $y = 4x - x^2$, the approximation is

(A) 10 (B) 10.5 (C) $10.\overline{6}$ (D) 10.75 (E) 11

Multiple Choice If f is a positive, continuous function on an interval $[a, b]$, which of the following rectangular approximation methods has a limit equal to the actual area under the curve from a to b as the number of rectangles approaches infinity?

I. LRAM

II. RRAM

III. MRAM

(A) I and II only

(B) III only

(C) I and III only

(D) I, II, and III

(E) None of these

35. **Multiple Choice** An LRAM sum with 4 equal subdivisions is used to approximate the area under the sine curve from $x = 0$ to $x = \pi$. What is the approximation?

(A) $\frac{\pi}{4}\left(0 + \frac{\pi}{4} + \frac{\pi}{2} + \frac{3\pi}{4}\right)$ (B) $\frac{\pi}{4}\left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1\right)$

(C) $\frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right)$ (D) $\frac{\pi}{4}\left(0 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right)$

(E) $\frac{\pi}{4}\left(\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1\right)$

36. **Multiple Choice** A truck moves with positive velocity $v(t)$ from time $t = 3$ to time $t = 15$. The area under the graph of $y = v(t)$ between 3 and 15 gives

(A) the velocity of the truck at $t = 15$.

(B) the acceleration of the truck at $t = 15$.

(C) the position of the truck at $t = 15$.

(D) the distance traveled by the truck from $t = 3$ to $t = 15$.

(E) the average position of the truck in the interval from $t = 3$ to $t = 15$.

Exploration

37. **Group Activity** ***Area of a Circle*** Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n.

(a) 4 (square) (b) 8 (octagon) (c) 16

(d) Compare the areas in parts (a), (b), and (c) with the area of the circle.

Extending the Ideas

38. ***Rectangular Approximation Methods*** Prove or disprove the following statement: MRAM_n is always the average of LRAM_n and RRAM_n.

39. ***Rectangular Approximation Methods*** Show that if f is a nonnegative function on the interval $[a, b]$ and the line $x = (a + b)/2$ is a line of symmetry of the graph of $y = f(x)$, then $\text{LRAM}_n f = \text{RRAM}_n f$ for every positive integer n.

40. *(Continuation of Exercise 37)*

(a) Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

(b) Compute the limit of the area of the inscribed polygon as $n \to \infty$.

(c) Repeat the computations in parts (a) and (b) for a circle of radius r.

A Nonintegrable Function

How "bad" does a function have to be before it is *not* integrable? One way to defeat integrability is to be unbounded (like $y = 1/x$ near $x = 0$), which can prevent the Riemann sums from tending to a finite limit. Another, more subtle, way is to be bounded but badly discontinuous, like the *characteristic function of the rationals:*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

No matter what partition we take of the closed interval [0, 1], every subinterval contains both rational and irrational numbers. That means that we can always form a Riemann sum with all rational c_k's (a Riemann sum of 1) or all irrational c_k's (a Riemann sum of 0). The sums can therefore never tend toward a unique limit.

EXPLORATION 2 More Discontinuous Integrands

1. Explain why the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

is not continuous on [0, 3]. What kind of discontinuity occurs?

2. Use areas to show that

$$\int_0^3 \frac{x^2 - 4}{x - 2}\,dx = 10.5.$$

3. Use areas to show that

$$\int_0^5 \text{int}(x)\,dx = 10.$$

Quick Review 5.2

In Exercises 1–3, evaluate the sum.

1. $\sum_{n=1}^{5} n^2$
2. $\sum_{k=0}^{4} (3k - 2)$
3. $\sum_{j=0}^{4} 100(j + 1)^2$

In Exercises 4–6, write the sum in sigma notation.

4. $1 + 2 + 3 + \cdots + 98 + 99$
5. $0 + 2 + 4 + \cdots + 48 + 50$
6. $3(1)^2 + 3(2)^2 + \cdots + 3(500)^2$

In Exercises 7 and 8, write the expression as a single sum in sigma notation.

7. $2\sum_{x=1}^{50} x^2 + 3\sum_{x=1}^{50} x$
8. $\sum_{k=0}^{8} x^k + \sum_{k=9}^{20} x^k$
9. Find $\sum_{k=0}^{n} (-1)^k$ if n is odd.
10. Find $\sum_{k=0}^{n} (-1)^k$ if n is even.

Section 5.2 Exercises

In Exercises 1–6, each c_k is chosen from the kth subinterval of a regular partition of the indicated interval into n subintervals of length Δx. Express the limit as a definite integral.

1. $\lim_{n\to\infty} \sum_{k=1}^{n} c_k^2 \Delta x$, [0, 2]
2. $\lim_{n\to\infty} \sum_{k=1}^{n} (c_k^2 - 3c_k)\Delta x$, [−7, 5]
3. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{c_k}\Delta x$, [1, 4]
4. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1 - c_k}\Delta x$, [2, 3]
5. $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\,\Delta x$, [0, 1]
6. $\lim_{n\to\infty} \sum_{k=1}^{n} (\sin^3 c_k)\,\Delta x$, $[-\pi, \pi]$

In Exercises 7–12, evaluate the integral.

7. $\int_{-2}^{1} 5\,dx$
8. $\int_{3}^{7} (-20)\,dx$
9. $\int_{0}^{3} (-160)\,dt$
10. $\int_{-4}^{-1} \frac{\pi}{2}\,d\theta$
11. $\int_{-2.1}^{3.4} 0.5\,ds$
12. $\int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\,dr$

In Exercises 13–22, use the graph of the integrand and areas to evaluate the integral.

13. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$
14. $\int_{1/2}^{3/2} (-2x + 4)\,dx$
15. $\int_{-3}^{3} \sqrt{9 - x^2}\,dx$
16. $\int_{-4}^{0} \sqrt{16 - x^2}\,dx$
17. $\int_{-2}^{1} |x|\,dx$
18. $\int_{-1}^{1} (1 - |x|)\,dx$
19. $\int_{-1}^{1} (2 - |x|)\,dx$
20. $\int_{-1}^{1} (1 + \sqrt{1 - x^2})\,dx$
21. $\int_{\pi}^{2\pi} \theta\,d\theta$
22. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$

In Exercises 23–28, use areas to evaluate the integral.

23. $\int_{0}^{b} x\,dx$, $b > 0$
24. $\int_{0}^{b} 4x\,dx$, $b > 0$
25. $\int_{a}^{b} 2s\,ds$, $0 < a < b$
26. $\int_{a}^{b} 3t\,dt$, $0 < a < b$
27. $\int_{a}^{2a} x\,dx$, $a > 0$
28. $\int_{a}^{\sqrt{3}a} x\,dx$, $a > 0$

In Exercises 29–32, express the desired quantity as a definite integral and evaluate the integral using Theorem 2.

29. Find the distance traveled by a train moving at 87 mph from 8:00 A.M. to 11:00 A.M.
30. Find the output from a pump producing 25 gallons per minute during the first hour of its operation.
31. Find the calories burned by a walker burning 300 calories per hour between 6:00 P.M. and 7:30 P.M.
32. Find the amount of water lost from a bucket leaking 0.4 liters per hour between 8:30 A.M. and 11:00 A.M.

In Exercises 33–36, use NINT to evaluate the expression.

33. $\int_{0}^{5} \frac{x}{x^2 + 4}\,dx$
34. $3 + 2\int_{0}^{\pi/3} \tan x\,dx$
35. Find the area enclosed between the x-axis and the graph of $y = 4 - x^2$ from $x = -2$ to $x = 2$.
36. Find the area enclosed between the x-axis and the graph of $y = x^2e^{-x}$ from $x = -1$ to $x = 3$.

In Exercises 37–40, **(a)** find the points of discontinuity of the integrand on the interval of integration, and **(b)** use area to evaluate the integral.

37. $\int_{-2}^{3} \frac{x}{|x|}\,dx$
38. $\int_{-6}^{5} 2\,\text{int}(x - 3)\,dx$
39. $\int_{-3}^{4} \frac{x^2 - 1}{x + 1}\,dx$
40. $\int_{-5}^{6} \frac{9 - x^2}{x - 3}\,dx$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

41. **True or False** If $\int_a^b f(x)\,dx > 0$, then $f(x)$ is positive for all x in $[a, b]$. Justify your answer.

42. **True or False** If $f(x)$ is positive for all x in $[a, b]$, then $\int_a^b f(x)\,dx > 0$. Justify your answer.

43. **Multiple Choice** If $\int_2^5 f(x)\,dx = 18$, then $\int_2^5 (f(x) + 4)\,dx =$
(A) 20 (B) 22 (C) 23 (D) 25 (E) 30

44. **Multiple Choice** $\int_{-4}^{4} (4 - |x|)\,dx =$
(A) 0 (B) 4 (C) 8 (D) 16 (E) 32

45. **Multiple Choice** If the interval $[0, \pi]$ is divided into n subintervals of length π/n and c_k is chosen from the kth subinterval, which of the following is a Riemann sum?
(A) $\sum_{k=1}^{n} \sin(c_k)$ (B) $\sum_{k=1}^{\infty} \sin(c_k)$ (C) $\sum_{k=1}^{n} \sin(c_k)\left(\frac{\pi}{n}\right)$
(D) $\sum_{k=1}^{n} \sin\left(\frac{\pi}{n}\right)(c_k)$ (E) $\sum_{k=1}^{n} \sin(c_k)\left(\frac{\pi}{k}\right)$

46. **Multiple Choice** Which of the following quantities would n be represented by the definite integral $\int_0^8 70\,dt$?
(A) The distance traveled by a train moving at 70 mph for 8 minutes.
(B) The volume of ice cream produced by a machine making 7 gallons per hour for 8 hours.
(C) The length of a track left by a snail traveling at 70 cm per hour for 8 hours.
(D) The total sales of a company selling $70 of merchandise pe hour for 8 hours.
(E) The amount the tide has risen 8 minutes after low tide if it rises at a rate of 70 mm per minute during that period.

Explorations

In Exercises 47–56, use graphs, your knowledge of area, and the fac that

$$\int_0^1 x^3\,dx = \frac{1}{4}$$

to evaluate the integral.

47. $\int_{-1}^{1} x^3\,dx$
48. $\int_{0}^{1} (x^3 + 3)\,dx$
49. $\int_{2}^{3} (x - 2)^3\,dx$
50. $\int_{-1}^{1} |x|^3\,dx$
51. $\int_{0}^{1} (1 - x^3)\,dx$
52. $\int_{-1}^{2} (|x| - 1)^3\,dx$
53. $\int_{0}^{2} \left(\frac{x}{2}\right)^3 dx$
54. $\int_{-8}^{8} x^3\,dx$
55. $\int_{0}^{1} (x^3 - 1)\,dx$
56. $\int_{0}^{1} \sqrt[3]{x}\,dx$

Extending the Ideas

57. Writing to Learn The function

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

is defined on [0, 1] and has a single point of discontinuity at $x = 0$.

(a) What happens to the graph of f as x approaches 0 from the right?

(b) The function f is not integrable on [0, 1]. Give a convincing argument based on Riemann sums to explain why it is not.

58. It can be shown by mathematical induction (see Appendix 2) that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use this fact to give a formal proof that

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

by following the steps given in the next column.

(a) Partition [0, 1] into n subintervals of length $1/n$. Show that the RRAM Riemann sum for the integral is

$$\sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}\right).$$

(b) Show that this sum can be written as

$$\frac{1}{n^3} \cdot \sum_{k=1}^{n} k^2.$$

(c) Show that the sum can therefore be written as

$$\frac{(n+1)(2n+1)}{6n^2}.$$

(d) Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}\right) = \frac{1}{3}.$$

(e) Explain why the equation in part (d) proves that

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

5.3 Definite Integrals and Antiderivatives

What you'll learn about

- Properties of Definite Integrals
- Average Value of a Function
- Mean Value Theorem for Definite Integrals
- Connecting Differential and Integral Calculus

. . . and why

Working with the properties of definite integrals helps us to understand better the definite integral. Connecting derivatives and definite integrals sets the stage for the Fundamental Theorem of Calculus.

Properties of Definite Integrals

In defining $\int_a^b f(x)$ as a limit of sums $\sum c_k\,\Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we integrated in the *opposite direction*? The integral would become $\int_b^a f(x)\,dx$—again a limit of sums of the form $\sum f(c_k)\Delta x_k$—but this time each of the Δx_k's would be negative as the x-values *decreased* from b to a. This would change the signs of all the terms in each Riemann sum, and ultimately the sign of the definite integral. This suggests the rule

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Since the original definition did not apply to integrating backwards over an interval, we can treat this rule as a logical extension of the definition.

Although $[a, a]$ is technically not an interval, another logical extension of the definition is that $\int_a^a f(x)\,dx = 0$.

These are the first two rules in Table 5.3. The others are inherited from rules that hold for Riemann sums. However, the limit step required to *prove* that these rules hold in the limit (as the norms of the partitions tend to zero) places their mathematical verification beyond the scope of this course. They should make good sense nonetheless.

Table 5.3 Rules for Definite Integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero:*	$\int_a^a f(x)\,dx = 0$	Also a definition
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any number k
	$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	

6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ $\quad g = 0$

The implications of the previous last equation were enormous for the discoverers of calculus. It meant that they could evaluate the definite integral of f from a to any number x simply by computing $F(x) - F(a)$, where *F is any antiderivative of f.*

EXAMPLE 4 Finding an Integral Using Antiderivatives

Find $\int_0^{\pi} \sin x\, dx$ using the formula $\int_a^x f(t)\, dt = F(x) - F(a)$.

SOLUTION

Since $\sin x$ is the rate of change of the quantity $F(x) = -\cos x$, that is, $F'(x) = \sin x$

$$\int_0^{\pi} \sin x\, dx = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

This explains how we obtained the value for Exploration 1 of the previous section.

Now try Exercise 21.

Quick Review 5.3 *(For help, go to Sections 3.6, 3.8, and 3.9.)*

In Exercises 1–10, find dy/dx.

1. $y = -\cos x$ **2.** $y = \sin x$

3. $y = \ln(\sec x)$ **4.** $y = \ln(\sin x)$

5. $y = \ln(\sec x + \tan x)$ **6.** $y = x \ln x - x$

7. $y = \dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$ **8.** $y = \dfrac{1}{2^x + 1}$

9. $y = xe^x$ **10.** $y = \tan^{-1} x$

Section 5.3 Exercises

The exercises in this section are designed to reinforce your understanding of the definite integral from the algebraic and geometric points of view. For this reason, you should not use the numerical integration capability of your calculator (NINT) except perhaps to support an answer.

1. Suppose that f and g are continuous functions and that

$$\int_1^2 f(x)\, dx = -4, \quad \int_1^5 f(x)\, dx = 6, \quad \int_1^5 g(x)\, dx = 8.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_2^2 g(x)\, dx$ (b) $\int_5^1 g(x)\, dx$

(c) $\int_1^2 3f(x)\, dx$ (d) $\int_2^5 f(x)\, dx$

(e) $\int_1^5 [f(x) - g(x)]\, dx$ (f) $\int_1^5 [4f(x) - g(x)]\, dx$

2. Suppose that f and h are continuous functions and that

$$\int_1^9 f(x)\, dx = -1, \quad \int_7^9 f(x)\, dx = 5, \quad \int_7^9 h(x)\, dx = 4.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_1^9 -2f(x)\, dx$ (b) $\int_7^9 [f(x) + h(x)]\, dx$

(c) $\int_7^9 [2f(x) - 3h(x)]\, dx$ (d) $\int_9^1 f(x)\, dx$

(e) $\int_1^7 f(x)\, dx$ (f) $\int_9^7 [h(x) - f(x)]\, dx$

3. Suppose that $\int_1^2 f(x)\, dx = 5$. Find each integral.

(a) $\int_1^2 f(u)\, du$ (b) $\int_1^2 \sqrt{3} f(z)\, dz$

(c) $\int_2^1 f(t)\, dt$ (d) $\int_1^2 [-f(x)]\, dx$

4. Suppose that $\int_{-3}^0 g(t)\, dt = \sqrt{2}$. Find each integral.

(a) $\int_0^{-3} g(t)\, dt$ (b) $\int_{-3}^0 g(u)\, du$

(c) $\int_{-3}^0 [-g(x)]\, dx$ (d) $\int_{-3}^0 \dfrac{g(r)}{\sqrt{2}}\, dr$

5. Suppose that f is continuous and that

$$\int_0^3 f(z)\, dz = 3 \quad \text{and} \quad \int_0^4 f(z)\, dz = 7.$$

Find each integral.

(a) $\int_3^4 f(z)\, dz$ (b) $\int_4^3 f(t)\, dt$

6. Suppose that h is continuous and that

$$\int_{-1}^1 h(r)\, dr = 0 \quad \text{and} \quad \int_{-1}^3 h(r)\, dr = 6.$$

Find each integral.

(a) $\int_1^3 h(r)\, dr$ (b) $-\int_3^1 h(u)\, du$

7. Show that the value of $\int_0^1 \sin(x^2)\, dx$ cannot possibly be 2.

8. Show that the value of $\int_0^1 \sqrt{x+8}\, dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

9. ***Integrals of Nonnegative Functions*** Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \geq 0 \text{ on } [a, b] \;\Rightarrow\; \int_a^b f(x)\, dx \geq 0.$$

10. ***Integrals of Nonpositive Functions*** Show that if f is integrable then

$$f(x) \leq 0 \text{ on } [a, b] \;\Rightarrow\; \int_a^b f(x)\, dx \leq 0.$$

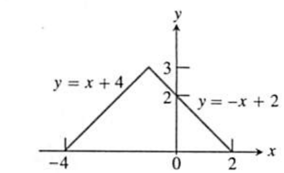
In Exercises 11–14, use NINT to find the average value of the function on the interval. At what point(s) in the interval does the function assume its average value?

11. $y = x^2 - 1, \quad [0, \sqrt{3}]$ **12.** $y = -\dfrac{x^2}{2}, \quad [0, 3]$

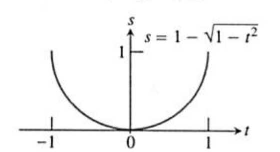
13. $y = -3x^2 - 1, \quad [0, 1]$ **14.** $y = (x-1)^2, \quad [0, 3]$

In Exercises 15–18, find the average value of the function on the interval without integrating, by appealing to the geometry of the region between the graph and the x-axis.

15. $f(x) = \begin{cases} x + 4, & -4 \leq x \leq -1, \\ -x + 2, & -1 < x \leq 2, \end{cases}$ on $[-4, 2]$



16. $f(t) = 1 - \sqrt{1 - t^2}, \quad [-1, 1]$



17. $f(t) = \sin t, \quad [0, 2\pi]$

18. $f(\theta) = \tan\theta, \quad \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

In Exercises 19–30, interpret the integrand as the rate of change of a quantity and evaluate the integral using the antiderivative of the quantity, as in Example 4.

19. $\int_{\pi}^{2\pi} \sin x\, dx$ **20.** $\int_0^{\pi/2} \cos x\, dx$

21. $\int_0^1 e^x\, dx$ **22.** $\int_0^{\pi/4} \sec^2 x\, dx$

23. $\int_1^4 2x\, dx$ **24.** $\int_{-1}^2 3x^2\, dx$

25. $\int_{-2}^6 5\, dx$ **26.** $\int_3^7 8\, dx$

27. $\int_{-1}^1 \dfrac{1}{1+x^2}\, dx$ **28.** $\int_0^{1/2} \dfrac{1}{\sqrt{1-x^2}}\, dx$

29. $\int_1^e \dfrac{1}{x}\, dx$ **30.** $\int_1^4 -x^{-2}\, dx$

In Exercises 31–36, find the average value of the function on the interval, using antiderivatives to compute the integral.

31. $y = \sin x, \quad [0, \pi]$ **32.** $y = \dfrac{1}{x}, \quad [e, 2e]$

33. $y = \sec^2 x, \quad \left[0, \dfrac{\pi}{4}\right]$ **34.** $y = \dfrac{1}{1+x^2}, \quad [0, 1]$

35. $y = 3x^2 + 2x, \quad [-1, 2]$ **36.** $y = \sec x \tan x, \quad \left[0, \dfrac{\pi}{3}\right]$

37. Group Activity Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^4}\, dx.$$

38. Group Activity *(Continuation of Exercise 37)* Use the Max-Min Inequality to find upper and lower bounds for the values of

$$\int_0^{0.5} \frac{1}{1+x^4}\, dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1+x^4}\, dx.$$

Add these to arrive at an improved estimate for

$$\int_0^1 \frac{1}{1+x^4}\, dx.$$

39. Writing to Learn If $av(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $av(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b av(f)\, dx = \int_a^b f(x)\, dx?$$

Give reasons for your answer.

40. Writing to Learn A driver averaged 30 mph on a 150-mile trip and then returned over the same 150 miles at the rate of 50 mph. He figured that his average speed was 40 mph for the entire trip.

(a) What was his total distance traveled?

(b) What was his total time spent for the trip?

(c) What was his average speed for the trip?

(d) Explain the error in the driver's reasoning.

41. Writing to Learn A dam released 1000 m^3 of water at 10 m^3/min and then released another 1000 m^3 at 20 m^3/min. What was the average rate at which the water was released? Give reasons for your answer.

42. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\, dx$.

43. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\, dx$.

44. Show that the average value of a linear function $L(x)$ on $[a, b]$ is

$$\frac{L(a) + L(b)}{2}.$$

[*Caution*: This simple formula for average value does *not* work for functions in general!]

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

45. True or False The average value of a function f on $[a, b]$ always lies between $f(a)$ and $f(b)$. Justify your answer.

46. True or False If $\int_a^b f(x)\, dx = 0$, then $f(a) = f(b)$. Justify your answer.

47. Multiple Choice If $\int_3^7 f(x)\, dx = 5$ and $\int_3^7 g(x)\, dx = 3$, then all of the following must be true *except*

(A) $\int_3^7 f(x)g(x)dx = 15$

(B) $\int_3^7 [f(x) + g(x)]dx = 8$

(C) $\int_3^7 2f(x)\, dx = 10$

(D) $\int_3^7 [f(x) - g(x)]dx = 2$

(E) $\int_7^3 [g(x) - f(x)]dx = 2$

48. Multiple Choice If $\int_2^5 f(x)\, dx = 12$ and $\int_5^8 f(x)\, dx = 4$, then all of the following must be true *except*

(A) $\int_2^8 f(x)\, dx = 16$

(B) $\int_2^5 f(x)\, dx - \int_5^8 3f(x)\, dx = 0$

(C) $\int_5^2 f(x)\, dx = -12$

(D) $\int_{-5}^{-8} f(x)\, dx = -4$

(E) $\int_2^6 f(x)\, dx + \int_6^8 f(x)\, dx = 16$

49. Multiple Choice What is the average value of the cosine function on the interval [1, 5]?

(A) −0.990 (B) −0.450 (C) −0.128 (D) 0.412 (E) 0.998

50. Multiple Choice If the average value of the function f on the interval $[a, b]$ is 10, then $\int_a^b f(x)\, dx =$

(A) $\frac{10}{b-a}$ (B) $\frac{f(a)+f(b)}{10}$ (C) $10b - 10a$ (D) $\frac{b-a}{10}$ (E) $\frac{f(b)+f(a)}{20}$

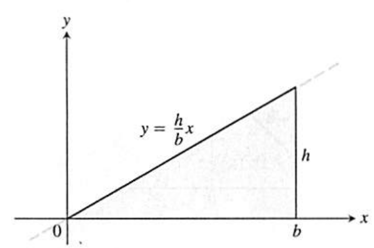
Exploration

51. *Comparing Area Formulas* Consider the region in the first quadrant under the curve $y = (h/b)\,x$ from $x = 0$ to $x = b$ (see figure).

(a) Use a geometry formula to calculate the area of the region.

(b) Find all antiderivatives of y.

(c) Use an antiderivative of y to evaluate $\int_0^b y(x)\, dx$.



xtending the Ideas

52. *Graphing Calculator Challenge* If $k > 1$, and if the average value of x^k on $[0, k]$ is k, what is k? Check your result with a CAS if you have one available.

53. Show that if $F'(x) = G'(x)$ on $[a, b]$, then $F(b) - F(a) = G(b) - G(a)$.

uick Quiz for AP* Preparation: Sections 5.1–5.3

You should solve the following problems without using a calculator.

1. Multiple Choice If $\int_a^b f(x)\, dx = a + 2b$, then $\int_a^b (f(x) + 3)\, dx =$

(A) $a + 2b + 3$ (B) $3b - 3a$ (C) $4a - b$ (D) $5b - 2a$ (E) $5b - 3a$

2. Multiple Choice The expression

$$\frac{1}{20}\left(\sqrt{\frac{1}{20}} + \sqrt{\frac{2}{20}} + \sqrt{\frac{3}{20}} + \cdots + \sqrt{\frac{20}{20}}\right)$$

is a Riemann sum approximation for

(A) $\int_0^1 \sqrt{\frac{x}{20}}\, dx$ (B) $\int_0^1 \sqrt{x}\, dx$ (C) $\frac{1}{20}\int_0^1 \sqrt{\frac{x}{20}}\, dx$ (D) $\frac{1}{20}\int_0^1 \sqrt{x}\, dx$ (E) $\frac{1}{20}\int_0^{20} \sqrt{x}\, dx$

3. Multiple Choice What are all values of k for which $\int_2^k x^2\, dx = 0$?

(A) −2 (B) 0 (C) 2 (D) −2 and 2 (E) −2, 0, and 2

4. Free Response Let f be a function such that $f''(x) = 6x + 12$.

(a) Find $f(x)$ if the graph of f is tangent to the line $4x - y = 5$ at the point $(0, -5)$.

(b) Find the average value of $f(x)$ on the closed interval $[-1, 1]$.

SOLUTION

First, we note that $h'(x) = f(x)$, so the graph of f is also the graph of the derivative of
Also, h is continuous because it is differentiable.

(a) $h(1) = \int_1^1 f(t)\,dt = 0$.

(b) $h(0) = \int_1^0 f(t)\,dt < 0$, because we are integrating from right to left under a positive function.

(c) The derivative of h is positive on (0,1), positive on (1, 4), and negative on (4, 8), so the continuous function h is increasing on [0, 4] and decreasing on [4, 8]. Thus $f(4)$ is a maximum.

(d) The sign analysis of the derivative above shows that the minimum value occurs at an endpoint of the interval [0, 8]. We see by comparing areas that $h(0) = \int_1^0 f(t)\,dt \approx -0$ while $h(8) = \int_1^8 f(t)\,dt$ is a negative number considerably less than -1. Thus $f(8)$ is a minimum.

(e) The points of inflection occur where $h' = f$ changes direction, that is, at $x = 1$, $x = 3$, and $x = 6$.

Now try Exercise

Quick Review 5.4 *(For help, go to Sections 3.6, 3.7, and 3.9.)*

In Exercises 1–10, find dy/dx.

1. $y = \sin(x^2)$

2. $y = (\sin x)^2$

3. $y = \sec^2 x - \tan^2 x$

4. $y = \ln(3x) - \ln(7x)$

5. $y = 2^x$

6. $y = \sqrt{x}$

7. $y = \dfrac{\cos x}{x}$

8. $y = \sin t$ and $x = \cos t$

9. $xy + x = y^2$

10. $dx/dy = 3x$

Section 5.4 Exercises

In Exercises 1–20, find dy/dx.

1. $y = \int_0^x (\sin^2 t)\,dt$

2. $y = \int_2^x (3t + \cos t^2)\,dt$

3. $y = \int_0^x (t^3 - t)^5\,dt$

4. $y = \int_{-2}^x \sqrt{1 + e^{5t}}\,dt$

5. $y = \int_2^x (\tan^3 u)du$

6. $y = \int_4^x e^u \sec u\,du$

7. $y = \int_7^x \dfrac{1 + t}{1 + t^2}\,dt$

8. $y = \int_{-\pi}^x \dfrac{2 - \sin t}{3 + \cos t}\,dt$

9. $y = \int_0^{x^2} e^{t^2}\,dt$

10. $y = \int_6^{x^2} \cot 3t\,dt$

11. $y = \int_2^{5x} \dfrac{\sqrt{1 + u^2}}{u}\,du$

12. $y = \int_\pi^{\pi - x} \dfrac{1 + \sin^2 u}{1 + \cos^2 u}\,du$

13. $y = \int_x^6 \ln(1 + t^2)\,dt$

14. $y = \int_x^7 \sqrt{2t^4 + t + 1}\,dt$

15. $y = \int_{x^3}^5 \dfrac{\cos t}{t^2 + 2}\,dt$

16. $y = \int_{5x^2}^{25} \dfrac{t^2 - 2t + 9}{t^3 + 6}\,dt$

17. $y = \int_{\sqrt{x}}^0 \sin(r^2)\,dr$

18. $y = \int_{3x^2}^{10} \ln(2 + p^2)\,dp$

19. $\int_{x^2}^{x^3} \cos(2t)\,dt$

20. $y = \int_{\sin x}^{\cos x} t^2\,dt$

In Exercises 21–26, construct a function of the form $y = \int_a^x f(t)\,dt + C$ that satisfies the given conditions.

21. $\dfrac{dy}{dx} = \sin^3 x$, and $y = 0$ when $x = 5$.

22. $\dfrac{dy}{dx} = e^x \tan x$, and $y = 0$ when $x = 8$.

23. $\dfrac{dy}{dx} = \ln(\sin x + 5)$, and $y = 3$ when $x = 2$.

24. $\dfrac{dy}{dx} = \sqrt{3 - \cos x}$, and $y = 4$ when $x = -3$.

25. $\dfrac{dy}{dx} = \cos^2 5x$, and $y = -2$ when $x = 7$.

26. $\dfrac{dy}{dx} = e^{\sqrt{x}}$, and $y = 1$ when $x = 0$.

In Exercises 27–40, evaluate each integral using Part 2 of the Fundamental Theorem. Support your answer with NINT if you are unsure.

27. $\int_{1/2}^3 \left(2 - \dfrac{1}{x}\right) dx$

28. $\int_2^{-1} 3^x\,dx$

29. $\int_0^1 (x^2 + \sqrt{x})\,dx$

30. $\int_0^5 x^{3/2}\,dx$

31. $\int_1^{32} x^{-6/5}\,dx$

32. $\int_{-2}^{-1} \dfrac{2}{x^2}\,dx$

33. $\int_0^\pi \sin x\,dx$

34. $\int_0^\pi (1 + \cos x)\,dx$

35. $\int_0^{\pi/3} 2\sec^2\theta\,d\theta$

36. $\int_{\pi/6}^{5\pi/6} \csc^2\theta\,d\theta$

37. $\int_{\pi/4}^{3\pi/4} \csc x \cot x\,dx$

38. $\int_0^{\pi/3} 4\sec x \tan x\,dx$

39. $\int_{-1}^1 (r + 1)^2\,dr$

40. $\int_0^4 \dfrac{1 - \sqrt{u}}{\sqrt{u}}\,du$

In Exercises 41–44, find the total area of the region between the curve and the x-axis.

41. $y = 2 - x, \quad 0 \le x \le 3$

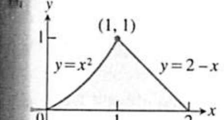
42. $y = 3x^2 - 3, \quad -2 \le x \le 2$

43. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

44. $y = x^3 - 4x, \quad -2 \le x \le 2$

In Exercises 45–48, find the area of the shaded region.

45.



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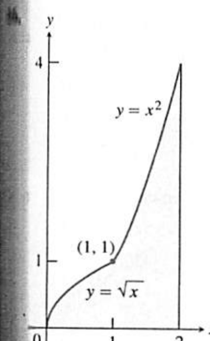

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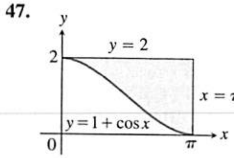

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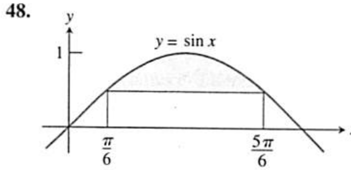


In Exercises 49–54, use NINT to solve the problem.

49. Evaluate $\int_0^{10} \dfrac{1}{3 + 2\sin x}\,dx$.

50. Evaluate $\int_{-0.8}^{0.8} \dfrac{2x^4 - 1}{x^4 - 1}\,dx$.

51. Find the area of the semielliptical region between the x-axis and the graph of $y = \sqrt{8 - 2x^2}$.

52. Find the average value of $\sqrt{\cos x}$ on the interval $[-1, 1]$.

53. For what value of x does $\int_0^x e^{-t^2}\,dt = 0.6$?

54. Find the area of the region in the first quadrant enclosed by the coordinate axes and the graph of $x^3 + y^3 = 1$.

In Exercises 55 and 56, find K so that

$$\int_a^x f(t)\,dt + K = \int_b^x f(t)\,dt.$$

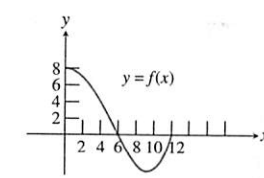
55. $f(x) = x^2 - 3x + 1; \quad a = -1; \quad b = 2$

56. $f(x) = \sin^2 x; \quad a = 0; \quad b = 2$

57. Let

$$H(x) = \int_0^x f(t)\,dt,$$

where f is the continuous function with domain [0, 12] graphed here.



(a) Find $H(0)$.

(b) On what interval is H increasing? Explain.

(c) On what interval is the graph of H concave up? Explain.

(d) Is $H(12)$ positive or negative? Explain.

(e) Where does H achieve its maximum value? Explain.

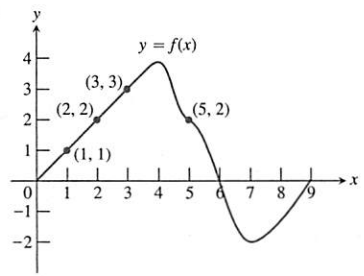
(f) Where does H achieve its minimum value? Explain.

In Exercises 58 and 59, f is the differentiable function whose graph is shown in the given figure. The position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the questions. Give reasons for your answers.

58.



(a) What is the particle's velocity at time $t = 5$?

(b) Is the acceleration of the particle at time $t = 5$ positive or negative?

(c) What is the particle's position at time $t = 3$?

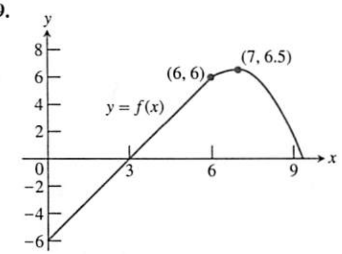
(d) At what time during the first 9 sec does s have its largest value?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? away from the origin?

(g) On which side of the origin does the particle lie at time $t = 9$?

59.



(a) What is the particle's velocity at time $t = 3$?

(b) Is the acceleration of the particle at time $t = 3$ positive or negative?

(c) What is the particle's position at time $t = 3$?

(d) When does the particle pass through the origin?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? away from the origin?

(g) On which side of the origin does the particle lie at time $t = 9$?

60. Suppose $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

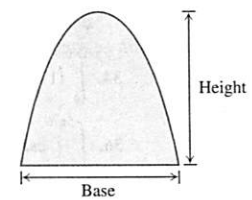
61. ***Linearization*** Find the linearization of

$$f(x) = 2 + \int_0^x \frac{10}{1+t}\,dt \quad \text{at} \quad x = 0.$$

62. Find $f(4)$ if $\int_0^x f(t)\,dt = x \cos \pi x$.

63. ***Finding Area*** Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is always $2/k$.

64. ***Archimedes' Area Formula for Parabolas*** Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times, discovered that the area under a parabolic arch like the one shown here is always two-thirds the base times the height.



(a) Find the area under the parabolic arch

$$y = 6 - x - x^2, \quad -3 \le x \le 2.$$

(b) Find the height of the arch.

(c) Show that the area is two-thirds the base times the height.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

65. **True or False** If f is continuous on an open interval I containing a, then F defined by $F(x) = \int_a^x f(t)\,dt$ is continuous on I. Justify your answer.

66. **True or False** If $b > a$, then $\frac{d}{dx}\int_a^b e^{x^2}\,dx$ is positive. Justify your answer.

67. **Multiple Choice** Let $f(x) = \int_a^x \ln(2 + \sin t)dt$. If $f(3) = 4$, then $f(5) =$

(A) 0.040 (B) 0.272 (C) 0.961 (D) 4.555 (E) 6.667

68. **Multiple Choice** What is $\lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$?

(A) 0 (B) 1 (C) $f'(x)$ (D) $f(x)$ (E) nonexistent

69. **Multiple Choice** At $x = \pi$, the linearization of $f(x) = \int_\pi^x \cos^3 t\,dt$ is

(A) $y = -1$ (B) $y = -x$ (C) $y = \pi$
(D) $y = x - \pi$ (E) $y = \pi - x$

70. **Multiple Choice** The area of the region enclosed between the graph of $y = \sqrt{1 - x^4}$ and the x-axis is

(A) 0.886 (B) 1.253 (C) 1.414
(D) 1.571 (E) 1.748

Explorations

71. ***The Sine Integral Function*** The sine integral function

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is one of the many useful functions in engineering that are defined as integrals. Although the notation does not show it, the function being integrated is

$$f(t) = \begin{cases} \frac{\sin t}{t}, & t \ne 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the origin.

(a) Show that $\text{Si}(x)$ is an odd function of x.

(b) What is the value of $\text{Si}(0)$?

(c) Find the values of x at which $\text{Si}(x)$ has a local extreme value.

(d) Use NINT to graph $\text{Si}(x)$.

72. ***Cost from Marginal Cost*** The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find

(a) $c(100) - c(1)$, the cost of printing posters 2 to 100.

(b) $c(400) - c(100)$, the cost of printing posters 101 to 400.

73. ***Revenue from Marginal Revenue*** Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2},$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

74. ***Average Daily Holding Cost*** Solon Container receives 450 drums of plastic pellets every 30 days. The inventory function (drums on hand as a function of days) is $I(x) = 450 - x^2/2$.

(a) Find the average daily inventory (that is, the average value of $I(x)$ for the 30-day period).

(b) If the holding cost for one drum is \$0.02 per day, find the average daily holding cost (that is, the per-drum holding cost times the average daily inventory).

75. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

(a) h is a twice-differentiable function of x.

(b) h and dh/dx are both continuous.

(c) The graph of h has a horizontal tangent at $x = 1$.

(d) h has a local maximum at $x = 1$.

(e) h has a local minimum at $x = 1$.

(f) The graph of h has an inflection point at $x = 1$.

(g) The graph of dh/dx crosses the x-axis at $x = 1$.

Extending the Ideas

76. **Writing to Learn** If f is an odd continuous function, give a graphical argument to explain why $\int_0^x f(t)\,dt$ is even.

77. **Writing to Learn** If f is an even continuous function, give a graphical argument to explain why $\int_0^x f(t)\,dt$ is odd.

78. **Writing to Learn** Explain why we can conclude from Exercises 76 and 77 that every even continuous function is the derivative of an odd continuous function and vice versa.

79. Give a convincing argument that the equation

$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

has exactly one solution. Give its approximate value.

Table 5.5

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Then apply Simpson's Rule with $n = 4$ and $h = 1/2$:

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2\left(5\right) + 4\left(\frac{405}{16}\right) + 80\right)$$

$$= \frac{385}{12}.$$

This estimate differs from the exact value (32) by only 1/12, a percentage error of le than three-tenths of one percent—and this was with just 4 subintervals.

Now try Exercis

There are still other algorithms for approximating definite integrals, most of th involving fancy numerical analysis designed to make the calculations more efficient high-speed computers. Some are kept secret by the companies that design the machines any case, we will not deal with them here.

Error Analysis

After finding that the trapezoidal approximation in Example 1 overestimated the integ we pointed out that this could have been predicted from the concavity of the curve we w approximating.

Knowing something about the error in an approximation is more than just an interes sidelight. Despite what your years of classroom experience might have suggested, ex answers are not always easy to find in mathematics. It is fortunate that for all *practical* p poses exact answers are also rarely necessary. (For example, a carpenter who computes need for a board of length $\sqrt{34}$ feet will happily settle for an approximation when cut the board.)

Suppose that an exact answer really can *not* be found, but that we know that an appr imation within 0.001 unit is good enough. How can we tell that our approximation is wi 0.001 if we do not know the exact answer? This is where knowing something about the er is critical.

Since the Trapezoidal Rule approximates curves with straight lines, it seems reason that the error depends on how "curvy" the graph is. This suggests that the error depend the second derivative. It is also apparent that the error depends on the length h of the sub tervals. It can be shown that if f'' is continuous the error in the trapezoidal approximat denoted E_T, satisfies the inequality

$$|E_T| \le \frac{b-a}{12} h^2 M_{f''},$$

where $[a, b]$ is the interval of integration, h is the length of each subinterval, and $M_{f''}$ is maximum value of $|f''|$ on $[a, b]$.

It can also be shown that the error E_S in Simpson's Rule depends on h and the fo derivative. It satisfies the inequality

$$|E_S| \le \frac{b-a}{180} h^4 M_{f^{(4)}},$$

where $[a, b]$ is the interval of integration, h is the length of each subinterval, and $M_{f^{(4)}}$ is maximum value of $|f^{(4)}|$ on $[a, b]$, provided that $f^{(4)}$ is continuous.

For comparison's sake, if all the assumptions hold, we have the following er *bounds.*

Error Bounds

If T and S represent the approximations to $\int_a^b f(x)\,dx$ given by the Trapezoidal Rule and Simpson's Rule, respectively, then the errors E_T and E_S satisfy

$$|E_T| \le \frac{b-a}{12} h^2 M_{f''} \quad \text{and} \quad |E_S| \le \frac{b-a}{180} h^4 M_{f^{(4)}}.$$

If we disregard possible differences in magnitude between $M_{f''}$ and $M_{f^{(4)}}$, we notice immediately that $(b - a)/180$ is one-fifteenth the size of $(b - a)/12$, giving S an obvious advantage over T as an approximation. That, however, is almost insignificant when compared to the fact that the trapezoid error varies as the *square* of h, while Simpson's error varies as the *fourth power* of h. (Remember that h is already a small number in most partitions.)

Table 5.6 shows T and S values for approximations of $\int_1^2 1/x\,dx$ using various values of n. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule. In particular, notice that when we double the value of n (thereby halving the value of h), the T error is divided by 2 *squared,* while the S error is divided by 2 *to the fourth.*

Table 5.6 Trapezoidal Rule Approximations (T_n) and Simpson's Rule Approximations (S_n) of ln 2 = $\int_1^2 (1/x)\,dx$

n	T_n	|Error| less than ...	S_n	|Error| less than ...
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

ble 5.7 Approximations of $\int_1^5 (\sin x)/x\,dx$

Method	Subintervals	Value
LRAM	50	0.6453898
RRAM	50	0.5627293
MRAM	50	0.6037425
TRAP	50	0.6040595
SIMP	50	0.6038481
NINT	Tol = 0.00001	0.6038482

This has a dramatic effect as h gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places, and for $n = 100$ agrees to nine decimal places (billionths)!

We close by showing you the values (Table 5.7) we found for $\int_1^5 (\sin x)/x\,dx$ by six different calculator methods. The exact value of this integral to six decimal places is 0.603848, so both Simpson's method with 50 subintervals and NINT give results accurate to at least six places (millionths).

uick Review 5.5 *(For help, go to Sections 3.9 and 4.3.)*

xercises 1–10, tell whether the curve is concave up or concave n on the given interval.

$y = \cos x$ on $[-1, 1]$

$y = x^4 - 12x - 5$ on $[8, 17]$

$y = 4x^3 - 3x^2 + 6$ on $[-8, 0]$

$y = \sin(x/2)$ on $[48\pi, 50\pi]$

$y = e^{2x}$ on $[-5, 5]$

6. $y = \ln x$ on $[100, 200]$

7. $y = \frac{1}{x}$ on $[3, 6]$

8. $y = \csc x$ on $[0, \pi]$

9. $y = 10^{10} - 10x^{10}$ on $[10, 10^{10}]$

10. $y = \sin x - \cos x$ on $[1, 2]$

Section 5.5 Exercises

In Exercises 1–6, **(a)** use the Trapezoidal Rule with $n = 4$ to approximate the value of the integral. **(b)** Use the concavity of the function to predict whether the approximation is an overestimate or an underestimate. Finally, **(c)** find the integral's exact value to check your answer.

1. $\int_0^2 x\,dx$ **2.** $\int_0^2 x^2\,dx$

3. $\int_0^2 x^3\,dx$ **4.** $\int_1^2 \frac{1}{x}\,dx$

5. $\int_0^4 \sqrt{x}\,dx$ **6.** $\int_0^{\pi} \sin x\,dx$

7. Use the function values in the following table and the Trapezoidal Rule with $n = 6$ to approximate $\int_0^6 f(x)\,dx$.

x	0	1	2	3	4	5	6
$f(x)$	12	10	9	11	13	16	18

8. Use the function values in the following table and the Trapezoidal Rule with $n = 6$ to approximate $\int_2^8 f(x)\,dx$.

x	2	3	4	5	6	7	8
$f(x)$	16	19	17	14	13	16	20

9. ***Volume of Water in a Swimming Pool*** A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

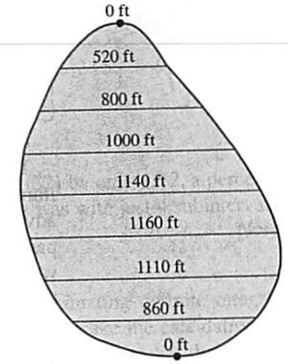
$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

10. ***Stocking a Fish Pond*** As the fish and game warden of your township, you are responsible for stocking the town pond with fish before the fishing season. The average depth of the pond is 20 feet. Using a scaled map, you measure distances across the pond at 200-foot intervals, as shown in the diagram.

(a) Use the Trapezoidal Rule to estimate the volume of the pond.

(b) You plan to start the season with one fish per 1000 cubic feet. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sell if the average seasonal catch is 20 fish per license?



Vertical spacing = 200 ft

11. ***Audi S4 Quattro Cabriolet*** The accompanying table show[s] time-to-speed data for a 2004 Audi S4 Quattro Cabriolet accelerating from rest to 130 mph. How far had the Cabriolet traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: the time intervals vary in length.)

Speed Change: Zero to	Time (sec)
30 mph	2.0
40 mph	3.2
50 mph	4.5
60 mph	5.8
70 mph	7.7
80 mph	9.5
90 mph	11.6
100 mph	14.9
110 mph	17.8
120 mph	21.7
130 mph	26.3

Source: Car and Driver, July 2004.

12. The table below records the velocity of a bobsled at 1-second intervals for the first eight seconds of its run. Use the Trapezoidal Rule to approximate the distance the bobsled tra[veled] during that 8-second interval. (Give your final answer in feet.)

Time (Seconds)	Speed (Miles/hr)
0	0
1	3
2	7
3	12
4	17
5	25
6	33
7	41
8	48

[In E]xercises 13–18, **(a)** use Simpson's Rule with $n = 4$ to [appr]oximate the value of the integral and **(b)** find the exact [val]ue of the integral to check your answer. (Note that these are [the s]ame integrals as Exercises 1–6, so you can also compare it [w]ith the Trapezoidal Rule approximation.)

$\int_0^2 x\,dx$ **14.** $\int_0^2 x^2\,dx$

$\int_0^2 x^3\,dx$ **16.** $\int_1^2 \frac{1}{x}\,dx$

$\int_0^4 \sqrt{x}\,dx$ **18.** $\int_0^{\pi} \sin x\,dx$

Consider the integral $\int_{-1}^{3} (x^3 - 2x)\,dx$.

(a) Use Simpson's Rule with $n = 4$ to approximate its value.

(b) Find the exact value of the integral. What is the error, $|E_S|$?

(c) Explain how you could have predicted what you found in (b) from knowing the error-bound formula.

(d) Writing to Learn Is it possible to make a general statement about using Simpson's Rule to approximate integrals of cubic polynomials? Explain.

Writing to Learn In Example 2 (before rounding) we found the average temperature to be 65.17 degrees when we used the integral approximation, yet the average of the 13 discrete temperatures is only 64.69 degrees. Considering the shape of the temperature curve, explain why you would expect the average of the 13 discrete temperatures to be less than the average value of the temperature function on the entire interval.

(Continuation of Exercise 20)

(a) In the Trapezoidal Rule, every function value in the sum is doubled except for the two endpoint values. Show that if you double the endpoint values, you get 70.08 for the average temperature.

(b) Explain why it makes more sense to not double the endpoint values if we are interested in the average temperature over the entire 12-hour period.

Group Activity For most functions, Simpson's Rule gives a better approximation to an integral than the Trapezoidal Rule for a given value of n. Sketch the graph of a function on a closed interval for which the Trapezoidal Rule obviously gives a better approximation than Simpson's Rule for $n = 4$.

[In E]xercises 23–26, use a calculator program to find the Simpson's [Rule] approximations with $n = 50$ and $n = 100$.

$\int_{-1}^{1} 2\sqrt{1 - x^2}\,dx$ The exact value is π.

$\int_0^1 \sqrt{1 + x^4}\,dx$ An integral that came up in Newton's research

$\int_0^{\pi/2} \frac{\sin x}{x}\,dx$

$\int_0^{\pi/2} \sin(x^2)\,dx$ An integral associated with the diffraction of light

27. Consider the integral $\int_0^{\pi} \sin x\,dx$.

(a) Use a calculator program to find the Trapezoidal Rule approximations for $n = 10$, 100, and 1000.

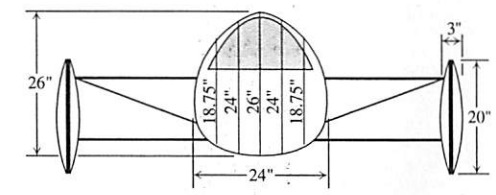
(b) Record the errors with as many decimal places of accuracy as you can.

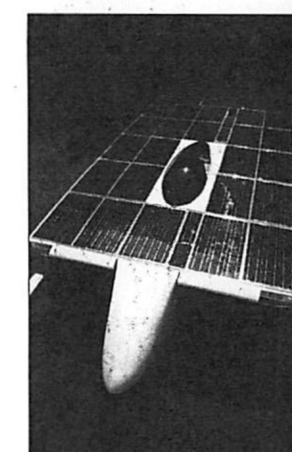
(c) What pattern do you see?

(d) Writing to Learn Explain how the error bound for E_T accounts for the pattern.

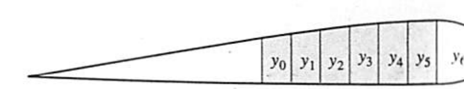
28. *(Continuation of Exercise 27)* Repeat Exercise 27 with Simpson's Rule and E_S.

29. ***Aerodynamic Drag*** A vehicle's aerodynamic drag is determined in part by its cross section area, so, all other things being equal, engineers try to make this area as small as possible. Use Simpson's Rule to estimate the cross section area of the body of James Worden's solar-powered Solectria® automobile at M.I.T. from the diagram below.





30. ***Wing Design*** The design of a new airplane requires a gasoline tank of constant cross section area in each wing. A scale drawing of a cross section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft³. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

31. True or False The Trapezoidal Rule will underestimate $\int_a^b f(x)\,dx$ if the graph of f is concave up on $[a, b]$. Justify your answer.

32. True or False For a given value of n, the Trapezoidal Rule with n subdivisions will always give a more accurate estimate of $\int_a^b f(x)\,dx$ than a right Riemann sum with n subdivisions. Justify your answer.

33. Multiple Choice Using 8 equal subdivisions of the interval [2, 12], the LRAM approximation of $\int_2^{12} f(x)\,dx$ is 16.6 and the trapezoidal approximation is 16.4. What is the RRAM approximation?

(A) 16.2 (B) 16.5

(C) 16.6 (D) 16.8

(E) It cannot be determined from the given information.

34. Multiple Choice If three equal subdivisions of $[-2, 4]$ are used, what is the trapezoidal approximation of $\int_{-2}^{4} \frac{e^x}{2}\,dx$?

(A) $e^4 + e^2 + e^0 + e^{-2}$

(B) $e^4 + 2e^2 + 2e^0 + e^{-2}$

(C) $\frac{1}{2}(e^4 + e^2 + e^0 + e^{-2})$

(D) $\frac{1}{2}(e^4 + 2e^2 + 2e^0 + e^{-2})$

(E) $\frac{1}{4}(e^4 + 2e^2 + 2e^0 + e^{-2})$

35. Multiple Choice The trapezoidal approximation of $\int_0^{\pi} \sin x\,dx$ using 4 equal subdivisions of the interval of integration is

(A) $\frac{\pi}{2}$

(B) π

(C) $\frac{\pi}{4}(1 + \sqrt{2})$

(D) $\frac{\pi}{2}(1 + \sqrt{2})$

(E) $\frac{\pi}{4}(2 + \sqrt{2})$

36. Multiple Choice Suppose f, f', and f'' are all positive on the interval $[a, b]$, and suppose we compute LRAM, RRAM, and trapezoidal approximations of $I = \int_a^b f(x)\,dx$ using the same number of equal subdivisions of $[a, b]$. If we denote the three approximations of I as L, R, and T respectively, which of the following is true?

(A) $R < T < I < L$ (B) $R < I < T < L$ (C) $L < I < T < R$

(D) $L < T < I < R$ (E) $L < I < R < T$

Explorations

37. Consider the integral $\int_{-1}^{1} \sin(x^2)\,dx$.

(a) Find f'' for $f(x) = \sin(x^2)$.

(b) Graph $y = f''(x)$ in the viewing window $[-1, 1]$ by $[-3,$

(c) Explain why the graph in part (b) suggests that $|f''(x)| \le$ for $-1 \le x \le 1$.

(d) Show that the error estimate for the Trapezoidal Rule in this case becomes

$$|E_T| \le \frac{h^2}{2}.$$

(e) Show that the Trapezoidal Rule error will be less than or equal to 0.01 if $h \le 0.1$.

(f) How large must n be for $h \le 0.1$?

38. Consider the integral $\int_{-1}^{1} \sin(x^2)\,dx$.

(a) Find $f^{(4)}$ for $f(x) = \sin(x^2)$. (You may want to check your work with a CAS if you have one available.)

(b) Graph $y = f^{(4)}(x)$ in the viewing window $[-1, 1]$ by $[-30, 10]$.

(c) Explain why the graph in part (b) suggests that $|f^{(4)}(x)| \le$ for $-1 \le x \le 1$.

(d) Show that the error estimate for Simpson's Rule in this ca becomes

$$|E_S| \le \frac{h^4}{3}.$$

(e) Show that the Simpson's Rule error will be less than or eq to 0.01 if $h \le 0.4$.

(f) How large must n be for $h \le 0.4$?

Extending the Ideas

39. Using the definitions, prove that, in general,

$$T_n = \frac{\text{LRAM}_n + \text{RRAM}_n}{2}.$$

40. Using the definitions, prove that, in general,

$$S_{2n} = \frac{\text{MRAM}_n + 2T_{2n}}{3}.$$

uick Quiz for AP* Preparation: Sections 5.4 and 5.5

You may use a graphing calculator to solve the following problems.

Multiple Choice The function f is continuous on the closed interval [1, 7] and has values that are given in the table below.

x	1	4	6	7
$f(x)$	10	30	40	20

Using the subintervals [1, 4], [4, 6], and [6, 7], what is the trapezoidal approximation of $\int_1^7 f(x)\,dx$?

(A) 110 (B) 130 (C) 160 (D) 190 (E) 210

Multiple Choice Let $F(x)$ be an antiderivative of $\sin^3 x$. If $F(1) = 0$, then $F(8) =$

(A) 0.00 (B) 0.021 (C) 0.373 (D) 0.632 (E) 0.968

3. Multiple Choice Let $f(x) = \int_{-2}^{x^2-3x} e^{t^2}\,dt$. At what value of x is $f(x)$ a minimum?

(A) For no value of x (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

4. Free Response Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.

(a) Use the Trapezoidal Rule with four equal subdivisions of the closed interval [0, 2] to approximate $F(2)$.

(b) On what interval or intervals is F increasing? Justify your answer.

(c) If the average rate of change of F on the closed interval [0, 3] is k, find $\int_0^3 \sin(t^2)\,dt$ in terms of k.

hapter 5 Key Terms

hapter 5 Review Exercises

collection of exercises marked in red could be used as a chapter test.

rcises 1–6 refer to the region R in the first quadrant enclosed by axis and the graph of the function $y = 4x - x^3$.

Sketch R and partition it into four subregions, each with a base of length $\Delta x = 1/2$.

Sketch the rectangles and compute (by hand) the area for the LRAM_4 approximation.

Sketch the rectangles and compute (by hand) the area for the MRAM_4 approximation.

Sketch the rectangles and compute (by hand) the area for the RRAM_4 approximation.

Sketch the trapezoids and compute (by hand) the area for the T_4 approximation.

6. Find the exact area of R by using the Fundamental Theorem of Calculus.

7. Use a calculator program to compute the RAM approximations in the following table for the area under the graph of $y = 1/x$ from $x = 1$ to $x = 5$.

n	LRAM_n	MRAM_n	RRAM_n
10			
20			
30			
50			
100			
1000			

8. *(Continuation of Exercise 7)* Use the Fundamental Theorem of Calculus to determine the value to which the sums in the table are converging.

9. Suppose

$$\int_{-2}^{2} f(x)\,dx = 4, \quad \int_{2}^{5} f(x)\,dx = 3, \quad \int_{-2}^{5} g(x)\,dx = 2.$$

Which of the following statements are true, and which, if any, are false?

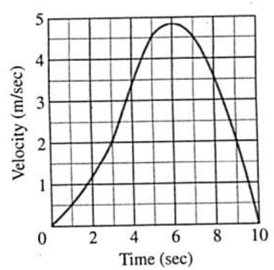
(a) $\int_{5}^{2} f(x)\,dx = -3$

(b) $\int_{-2}^{5} [f(x) + g(x)]\,dx = 9$

(c) $f(x) \le g(x)$ on the interval $-2 \le x \le 5$

10. The region under one arch of the curve $y = \sin x$ is revolved around the x-axis to form a solid. **(a)** Use the method of Example 3, Section 5.1, to set up a Riemann sum that approximates the volume of the solid. **(b)** Find the volume using NINT.

11. The accompanying graph shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. **(a)** About how far did the body travel during those 10 seconds?



(b) Sketch a graph of position (s) as a function of time (t) for $0 \le t \le 10$, assuming $s(0) = 0$.

12. The interval [0, 10] is partitioned into n subintervals of length $\Delta x = 10/n$. We form the following Riemann sums, choosing each c_k in the k^{th} subinterval. Write the limit as $n \to \infty$ of each Riemann sum as a definite integral.

(a) $\sum_{k=1}^{n} (c_k)^3 \Delta x$

(b) $\sum_{k=1}^{n} c_k(\sin c_k)\Delta x$

(c) $\sum_{k=1}^{n} c_k(3c_k - 2)^2 \Delta x$

(d) $\sum_{k=1}^{n} (1 + c_k^2)^{-1} \Delta x$

(e) $\sum_{k=1}^{n} \pi(9 - \sin^2(\pi c_k/10))\Delta x$

In Exercises 13 and 14, find the total area between the curve and the x-axis.

13. $y = 4 - x, \quad 0 \le x \le 6$

14. $y = \cos x, \quad 0 \le x \le \pi$

In Exercises 15–24, evaluate the integral analytically by using the Integral Evaluation Theorem (Part 2 of the Fundamental Theorem, Theorem 4).

15. $\int_{-2}^{2} 5\,dx$

16. $\int_{2}^{5} 4x\,dx$

17. $\int_{0}^{\pi/4} \cos x\,dx$

18. $\int_{-1}^{1} (3x^2 - 4x + 7)\,dx$

19. $\int_{0}^{1} (8s^3 - 12s^2 + 5)\,ds$

20. $\int_{1}^{2} \frac{4}{x^2}\,dx$

21. $\int_{1}^{27} y^{-4/3}\,dy$

22. $\int_{1}^{4} \frac{dt}{t\sqrt{t}}$

23. $\int_{0}^{\pi/3} \sec^2\theta\,d\theta$

24. $\int_{1}^{e} (1/x)\,dx$

In Exercises 25–29, evaluate the integral.

25. $\int_{0}^{1} \frac{36}{(2x+1)^3}\,dx$

26. $\int_{1}^{2} \left(x + \frac{1}{x^2}\right) dx$

27. $\int_{-\pi/3}^{0} \sec x \tan x\,dx$

28. $\int_{-1}^{1} 2x \sin(1 - x^2)\,dx$

29. $\int_{0}^{2} \frac{2}{y+1}\,dy$

In Exercises 30–32, evaluate the integral by interpreting it as areas and using formulas from geometry.

30. $\int_{0}^{2} \sqrt{4 - x^2}\,dx$

31. $\int_{-4}^{8} |x|\,dx$

32. $\int_{-8}^{8} 2\sqrt{64 - x^2}\,dx$

33. ***Oil Consumption on Pathfinder Island*** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced.

Day	Oil Consumption Rate (liters/hour)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

(a) Give an upper estimate and a lower estimate for the amount of oil consumed by the generator during that week.

(b) Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Rubber-Band-Powered Sled A sled powered by a wound rubber band moves along a track until friction and the unwinding of the rubber band gradually slow it to a stop. A speedometer in the sled monitors its speed, which is recorded at 3-second intervals during the 27-second run.

Time (sec)	Speed (ft/sec)
0	5.30
3	5.25
6	5.04
9	4.71
12	4.25
15	3.66
18	2.94
21	2.09
24	1.11
27	0

(a) Give an upper estimate and a lower estimate for the distance traveled by the sled.

(b) Use the Trapezoidal Rule to estimate the distance traveled by the sled.

Writing to Learn Your friend knows how to compute integrals but never could understand what difference the "dx" makes, claiming that it is irrelevant. How would you explain to your friend why it is necessary?

The function

$$f(x) = \begin{cases} x^2, & x \ge 0 \\ x - 2, & x < 0 \end{cases}$$

is discontinuous at 0, but integrable on $[-4, 4]$. Find $\int_{-4}^{4} f(x)\,dx$.

Show that $0 \le \int_{0}^{1} \sqrt{1 + \sin^2 x}\,dx \le \sqrt{2}$.

Find the average value of

(a) $y = \sqrt{x}$ over the interval [0, 4].

(b) $y = a\sqrt{x}$ over the interval $[0, a]$.

Exercises 39–42, find dy/dx.

$y = \int_{2}^{x} \sqrt{2 + \cos^3 t}\,dt$

40. $y = \int_{2}^{7x^2} \sqrt{2 + \cos^3 t}\,dt$

$y = \int_{x}^{1} \frac{6}{3 + t^4}\,dt$

42. $y = \int_{x}^{2x} \frac{1}{t^2 + 1}\,dt$

Printing Costs Including start-up costs, it costs a printer \$50 to print 25 copies of a newsletter, after which the marginal cost at x copies is

$$\frac{dc}{dx} = \frac{2}{\sqrt{x}} \text{ dollars per copy.}$$

Find the total cost of printing 2500 newsletters.

Average Daily Inventory Rich Wholesale Foods, a manufacturer of cookies, stores its cases of cookies in an air-conditioned warehouse for shipment every 14 days. Rich tries to keep 600 cases on reserve to meet occasional peaks in demand, so a typical 14-day inventory function is $I(t) =$ $600 + 600t$, $0 \le t \le 14$. The holding cost for each case is 4¢ per day. Find Rich's average daily inventory and average daily holding cost (that is, the average of $I(x)$ for the 14-day period, and this average multiplied by the holding cost).

45. Solve for x: $\int_{0}^{x} (t^3 - 2t + 3)\,dt = 4$.

46. Suppose $f(x)$ has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of

$$g(x) = \int_{0}^{x} f(t)\,dt?$$

(a) g is a differentiable function of x.

(b) g is a continuous function of x.

(c) The graph of g has a horizontal tangent line at $x = 1$.

(d) g has a local maximum at $x = 1$.

(e) g has a local minimum at $x = 1$.

(f) The graph of g has an inflection point at $x = 1$.

(g) The graph of dg/dx crosses the x-axis at $x = 1$.

47. Suppose $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_{0}^{1} \sqrt{1 + x^4}\,dx$ in terms of F.

48. Express the function $y(x)$ with

$$\frac{dy}{dx} = \frac{\sin x}{x} \quad \text{and} \quad y(5) = 3$$

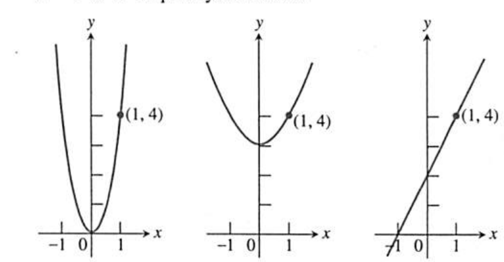
as a definite integral.

49. Show that $y = x^2 + \int_{1}^{x} 1/t\,dt + 1$ satisfies both of the following conditions:

i. $y'' = 2 - \frac{1}{x^2}$

ii. $y = 2$ and $y' = 3$ when $x = 1$.

50. **Writing to Learn** Which of the following is the graph of the function whose derivative is $dy/dx = 2x$ and whose value at $x = 1$ is 4? Explain your answer.



(a) (b) (c)

51. ***Fuel Efficiency*** An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 minutes for a full hour of travel.

time	gal/h	time	gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

(a) Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

(b) If the automobile covered 60 miles in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

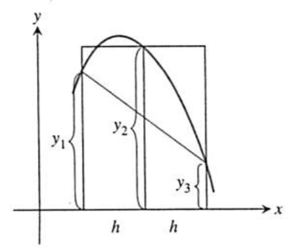
52. ***Skydiving*** Skydivers A and B are in a helicopter hovering at 6400 feet. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 feet and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening her parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (with acceleration -32 ft/sec^2) before their parachutes open.

(a) At what altitude does A's parachute open?

(b) At what altitude does B's parachute open?

(c) Which skydiver lands first?

53. ***Relating Simpson's Rule, MRAM, and T*** The figure below shows an interval of length $2h$ with a trapezoid, a midpoint rectangle, and a parabolic region on it.



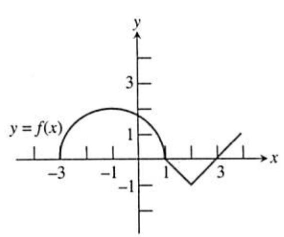
(a) Show that the area of the trapezoid plus twice the area of the rectangle equals

$$h(y_1 + 4y_2 + y_3).$$

(b) Use the result in part (a) to prove that

$$S_{2n} = \frac{2 \cdot \text{MRAM}_n + T_n}{3}.$$

54. The graph of a function f consists of a semicircle and two line segments as shown below.



Let $g(x) = \int_1^x f(t)\,dt$.

(a) Find $g(1)$.

(b) Find $g(3)$.

(c) Find $g(-1)$.

(d) Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.

(e) Write an equation for the line tangent to the graph of g at $x = -1$.

(f) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.

(g) Find the range of g.

55. What is the total area under the curve $y = e^{-x^2/2}$?

The graph approaches the x-axis as an asymptote both to the left ... the right, but quickly enough so that the total area is a finite num... In fact,

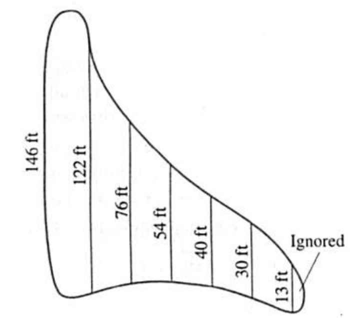
$$\text{NINT}\,(e^{-x^2/2}, x, -10, 10)$$

computes all but a negligible amount of the area.

(a) Find this number on your calculator. Verify that NINT $(e^{-x^2/2}, x, -20, 20)$ does not increase the number enough for the calculator to distinguish the difference.

(b) This area has an interesting relationship to π. Perform vario... (simple) algebraic operations on the number to discover what it...

56. ***Filling a Swamp*** A town wants to drain and fill the small polluted swamp shown below. The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?



Horizontal spacing = 20 ft

57. ***Household Electricity*** We model the voltage V in our home with the sine function

$$V = V_{\max} \sin(120\,\pi t),$$

which expresses V in volts as a function of time t in seconds. The function runs through 60 cycles each second. The number $V_{\max}$ is the *peak voltage*.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage over a 1-second interval:

$$V_{\text{rms}} = \sqrt{(V^2)_{\text{av}}}\,.$$

The subscript "rms" stands for "root mean square." It turns out ...

$$V_{\text{rms}} = \frac{V_{\max}}{\sqrt{2}}.$$

The familiar phrase "115 volts ac" means that the rms voltage is 115. The peak voltage, obtained from Equation 1 as $V_{\max} = 115\sqrt{2}$, is about 163 volts.

(a) Find the average value of V^2 over a 1-sec interval. Then find V_{rms}, and verify Equation 1.

(b) The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?

*Examination Preparation

You may use a graphing calculator to solve the following problems.

...The rate at which water flows out of a pipe is given by a differentiable function R of time t. The table below records the rate at 4-hour intervals for a 24-hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
4	10.3
8	10.9
12	11.1
16	10.9
20	10.5
24	9.6

(a) Use the Trapezoid Rule with 6 subdivisions of equal length to approximate $\int_0^{24} R(t)dt$. Explain the meaning of your answer in terms of water flow, using correct units.

(b) Is there some time t between 0 and 24 such that $R'(t) = 0$? Justify your answer.

(c) Suppose the rate of water flow is approximated by $Q(t) = 0.01(950 + 25x - x^2)$. Use $Q(t)$ to approximate the average rate of water flow during the 24-hour period. Indicate units of measure.

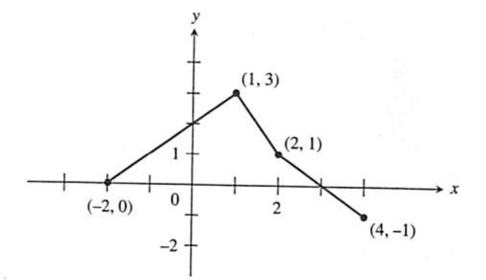
59. Let f be a differentiable function with the following properties.

i. $f'(x) = ax^2 + bx$ **ii.** $f'(1) = -6$ and $f''(x) = 6$

iii. $\int_1^2 f(x)\,dx = 14$

Find $f(x)$. Show your work.

60. The graph of the function f, consisting of three line segments, is shown below.



Let $g(x) = \int_1^x f(t)dt$.

(a) Compute $g(4)$ and $g(-2)$.

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 2$.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. Which of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

Calculus at Work

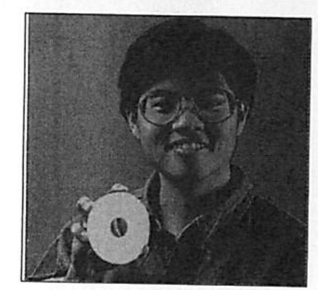
...have a degree in Mechanical Engineering with a minor in Psychology. I am a Research and Development Engineer at ...omag, which designs and manufactures ...rd disks in Santa Clara, California. My ...b is to test the durability and reliability of ...e disks, measuring the rest friction between the read/write heads and the disk ...rface, which is called "Contact-Start-...op" testing.

...se calculus to evaluate the moment of ...ertia of different disk stacks, which con...t of disks on a spindle, separated by ...acer rings. Because the rings vary in ...e as well as material, the mass of each ring must be determined. For such problems, I refer to my college calculus textbook and its tables of summations and integrals. For instance, I use:

Moment of Inertia =

$$\left[\sum_{i=1}^{n} M_i L_i^2\right] + \frac{1}{3} M_{rod} L_{rod}^2$$

where i = components 1 to n;
M_i = mass of component i such as the disk and/or ring stack;
L_i = distance of component i from a reference point;
M_{rod} = mass of the spindle that rotates;
L_{rod} = length of the spindle.



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