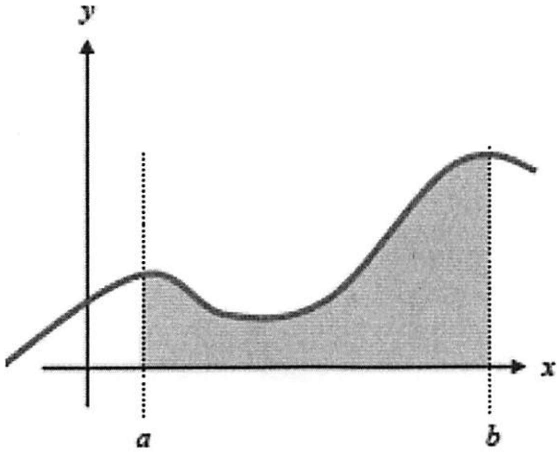
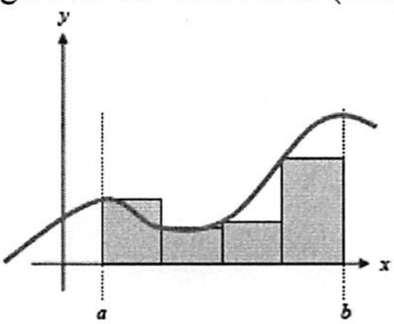
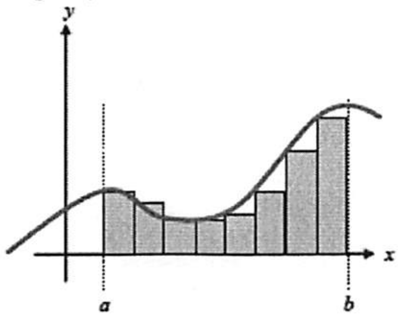
5 - Chapter overview

In this chapter we will see that for some situations, we need to calculate the area between the *x*-axis and a curve.



We will learn how to approach it geometrically (Riemann Sums) and realize that a limit will give us the exact area (The definite integral).





$$\int_a^b f(x)\,dx$$

We will then realize that the definite integral is closely linked to antiderivatives.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

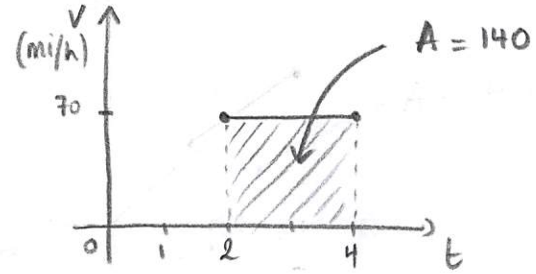
We will work more deeply on antiderivatives in chapter 6…

5.1 – Estimating with Finite Sums

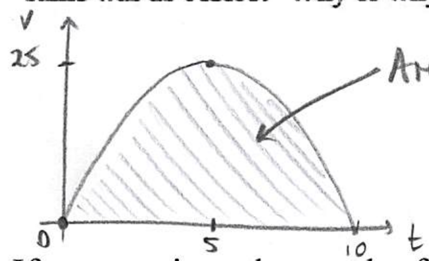
Example 1: Suppose from the 2nd to 4th hour of your road trip, you travel with the cruise control set to exactly 70 miles per hour for that two hour stretch. How far have you traveled during this time?

$70 \times 2 = 140$ miles.

Example 2: Sketch a graph modeling the situation in the above example. Geometrically, how can we indicate the total distance traveled?



Example 3: What if the velocity was NOT constant. Say, for instance the velocity in miles per hour is given by the function $v(t) = 10t - t^2$, where t is in hours, and we wanted to know the total distance traveled during the first 10 hours. Sketch this graph below. Geometrically speaking, do you think we can find the total distance traveled in the same was as before? Why or why not?

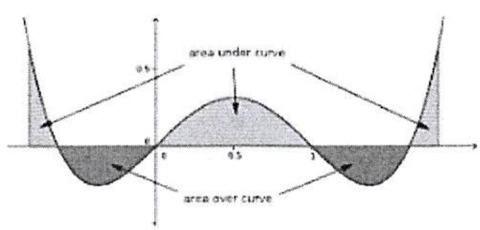


Area "under" the curve... no formula.

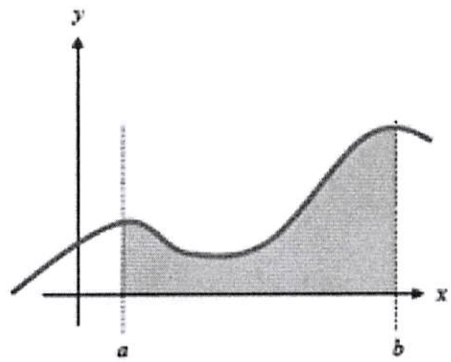
If we are given the graph of a rate of change (like velocity in miles per hour) we will be able to find the **accumulated change over an interval** (like total distance traveled in miles) by finding the area under the curve.

We will now see different ways to approximate the area under a curve (or more specifically the area between the x-axis and the curve between 2 values of x).

Note that it will be an algebraic area (which means that it can be positive or negative) as opposed to a geometric area that is always positive. It will be negative if the curve is below the x-axis, and positive if it is above.



The Area Problem and the Rectangular Approximation Method (RAM)
(a.k.a. Riemann Sums)



Suppose we wanted to know the area of the region bounded by a curve, the x – axis, and the lines $x = a$ and $x = b$, as shown at the right.

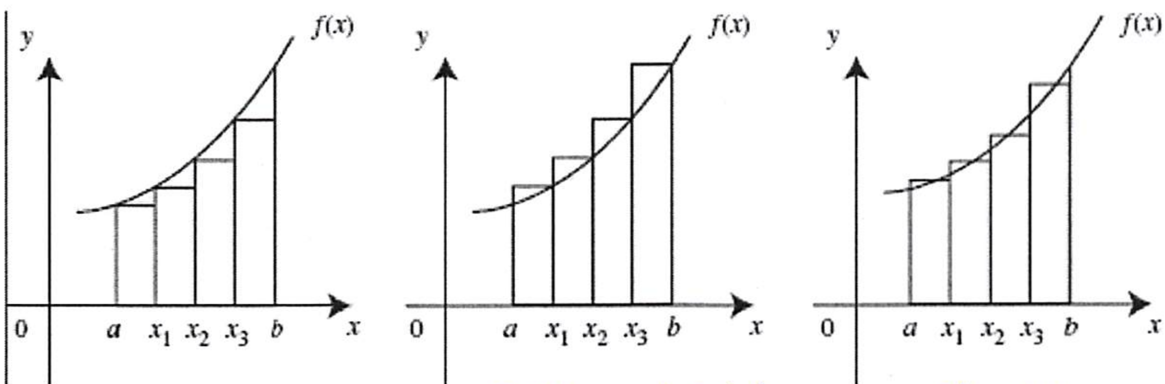
The first step is to divide the interval from a to b into subintervals.
(The examples below show 4 and 8 subintervals, respectively.)

After dividing the given interval into subintervals, we can then draw rectangles using the width of each subinterval as the base.

The *height of each rectangle is determined by the function value at a point in the specific subinterval*, and can be determined using 3 different methods.

We could use the left endpoint of each subinterval (called LRAM), the right endpoint of each subinterval (RRAM), or the midpoint of each subinterval (MRAM).

Example 4: Which method is shown in the two graphs below?



Left end point   Right end point   midpoint

Example 5: The total area under the curve then is approximately equal to the total area of all the rectangles. Which of the graphs above gives a better approximation of the area under the curve? Why? How could it be further improved?

midpoint this time.

It can be improved by increasing the number of subintervals.

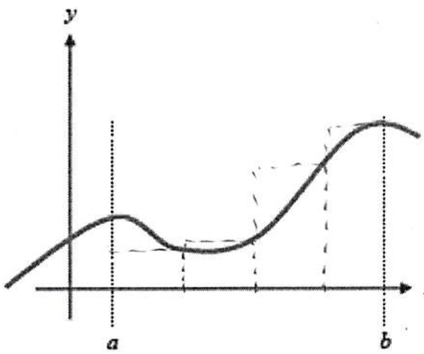
Summary of the Process: A sketch is almost mandatory!

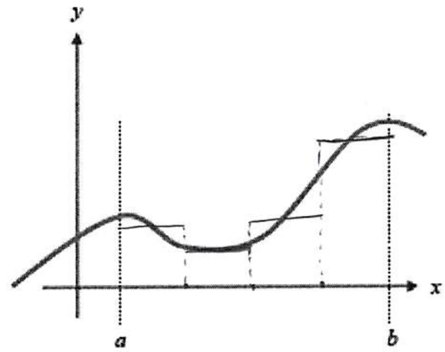
Step 1: Divide (or Partition) the interval into n subintervals.

Step 2: Create n rectangles whose base equals the width of each subinterval and whose height is determined by the function value at the left endpoint, the right endpoint, or the midpoint of the subinterval.

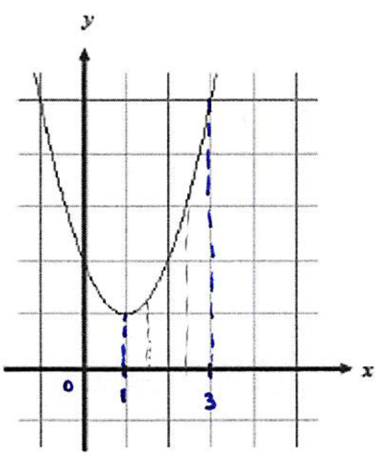
Step 3: Find the area of all n rectangles and add them together.

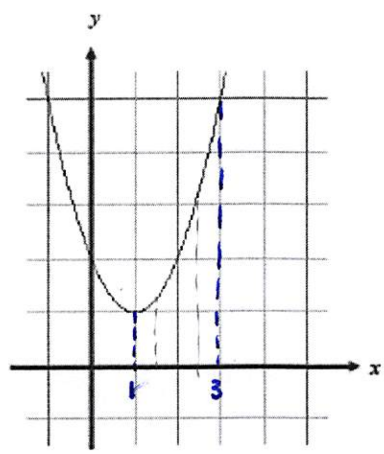
Example 6: Illustrate the use of RRAM and MRAM on the graphs below. (use 4 rectangles)

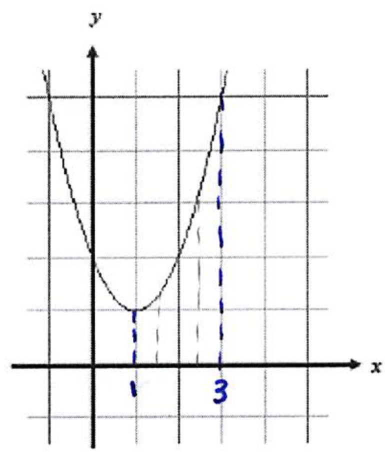




Example 7: Use 4 rectangles to approximate the area under the graph of $y = x^2 - 2x + 2$ from $x = 1$ to $x = 3$. Use LRAM, RRAM, and then MRAM.







Example 8: Using your rectangles as a guide, find each approximation.

a) LRAM = $1 \times 0.5 + 1.25 \times 0.5 + 2 \times 0.5 + 3.25 \times 0.5$
$= 3.75$

$f(1) = 1$ $\quad f(2) = 2$
$f(1.5) = 1.25$ $\quad f(2.5) = 3.25$

b) RRAM = $1.25 \times 0.5 + 2 \times 0.5 + 3.25 \times 0.5 + 5 \times 0.5$
$= 5.75$

$f(3) = 5$

c) MRAM = $1.0625 \times 0.5 + 1.5625 \times 0.5$
$+ 2.5625 \times 0.5 + 4.0625 \times 0.5$
$= 4.625$

$f(1.25) = 1.0625$
$f(1.75) = 1.5625$
$f(2.25) = 2.5625$
$f(2.75) = 4.0625$

Example 9: It is not necessary to have a graph to estimate the area. Suppose the table below shows the velocity of a model train engine moving along a track for 10 seconds.

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

a) Using a left Riemann Sum with 10 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

$D \simeq 0+12+22+10+5+13+11+6+2+6$

$\boxed{D \simeq 87 \text{ in}}$

b) Using a Midpoint Riemann Sum with 5 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

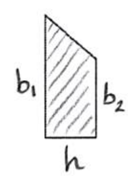
$D \simeq 12\times 2 + 10\times 2 + 13\times 2 + 6\times 2 + 6\times 2$

$\boxed{D \simeq 94 \text{ in}}$

The Trapezoidal Rule (Really §5.5)

While rectangles make a fairly good approximation, it's easy to see that we're going to need a lot of them to provide a good estimate. We can find a better estimate in less time if we use trapezoids. If we were to partition the interval into subintervals like we did before, we can use each subinterval to create a trapezoid if we just connect the function values of the left and right endpoints. Before we begin, let's make sure you understand the area formula for a trapezoid.

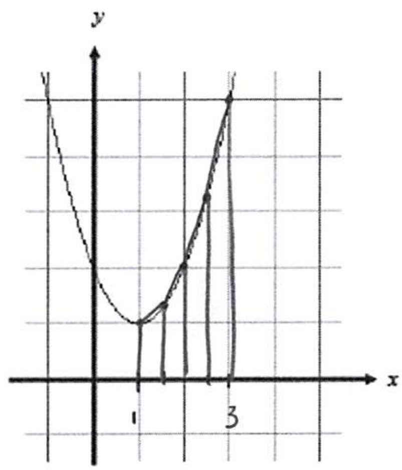
Area of a Trapezoid: $A = \frac{1}{2} \cdot h \cdot (b_1 + b_2)$



Example 10: Use 4 trapezoids to approximate the area under the curve $y = x^2 - 2x + 2$ from $x = 1$ to $x = 3$. Sketch the trapezoids.

$A \simeq \frac{1}{2}\times 0.5(1+1.25) + \frac{1}{2}\times 0.5(1.25+2)$
$+ \frac{1}{2}\times 0.5(2+3.25) + \frac{1}{2}\times 0.5(3.25+5)$

$A \simeq 4.75$



5.2 - Definite Integrals

The trapezoid approximation is a great one, but it's still not giving us an exact value of the area… No matter which approximation we use, the more subintervals we use, the better the approximation.
The exact value would be if we could have an infinity of subintervals with an infinitesimally small width… The way to do it, would be to divide the interval into n subintervals and take the limit of the approximation as n approaches infinity.

Definition of the Definite Integral:

The definite integral of $f(x)$ over $[a,b]$ is the limit of Riemann Sums:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

where n is the number of subintervals, $f(x_i)$ is any value of the function on the i^{th} subinterval, and Δx is the length of the subintervals (assuming they are all the same).

If the subintervals are not all the same, we write:

Upper limit of integration

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

Lower limit of integration

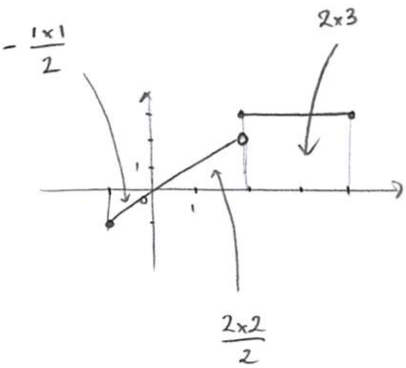
where $\|P\|$ is the maximum width of the subintervals, $f(c_i)$ is any value of the function on the i^{th} subinterval, and Δx_i is the length of the i^{th} subinterval.

Where the limit exists, we say that $f(x)$ is **integrable** over $[a,b]$.

Note: The sigma notation Σ that means "sum" is replaced by $\int$ which can also be interpreted as "sum" but for an infinity of infinitely thin "sticks" on the graph…

Examples from the AP exam:

1)



$$f(x) = \begin{cases} x & \text{for } x < 2 \\ 3 & \text{for } x \geq 2 \end{cases}$$

If f is the function defined above, then $\int_{-1}^{4} f(x)\,dx$ is

(A) $\frac{9}{2}$

(B) $\frac{15}{2}$

(C) $\frac{17}{2}$

(D) undefined

$A = -\frac{1}{2} + 2 + 6$

value of the function on the kth subinterval.

width of the interval

2)

Which of the following integral expressions is equal to $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1+\frac{3k}{n}}\cdot\frac{1}{n}\right)$?

width of the subinterval

(A) $\int_0^1 \sqrt{1+3x}\,dx$

(B) $\int_0^3 \sqrt{1+x}\,dx$

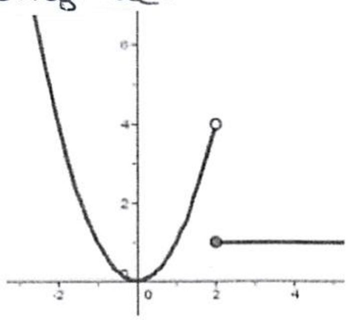
(C) $\int_1^4 \sqrt{x}\,dx$

(D) $\frac{1}{3}\int_0^3 \sqrt{x}\,dx$

$f(x) = \sqrt{1+3x}$ RRAM

On the exam, you will be asked to calculate a Riemann sum with 4 or 5 subintervals to make sure you understand how it works, or you will be asked to recognize an integral as a limit of Riemann sums (like in the previous example), but that's it. You won't be asked to actually find that limit yourself. If you need to evaluate a definite integral, you will do it using usual/easy geometric shapes (like in the following examples).

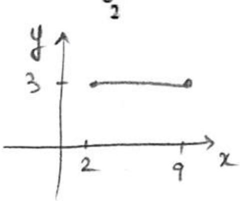
Important: If the function is continuous over $[a,b]$, then the function is integrable.
But even if the function is not continuous over $[a,b]$, but if the discontinuity is a hole or jump it is still integrable.



Important (again): The area is an algebraic area that will be positive if the curve is above the x-axis, and negative if the curve is below the x-axis.

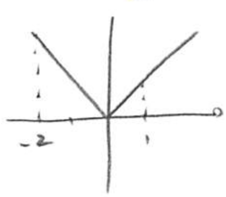
Examples: Evaluate

a) $\int_2^9 3\,dx = 7\times 3 = 21$



b) $\int_{-2}^{1} |x|\,dx = \frac{2\times 2}{2} + \frac{1\times 1}{2}$

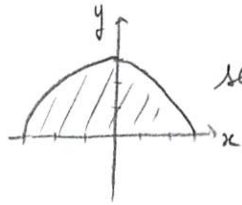
$= \frac{5}{2}$



c) $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{1}{2}\pi(3)^2$

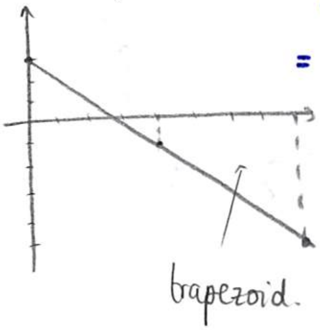
$= \frac{9\pi}{2}$

semicircle.

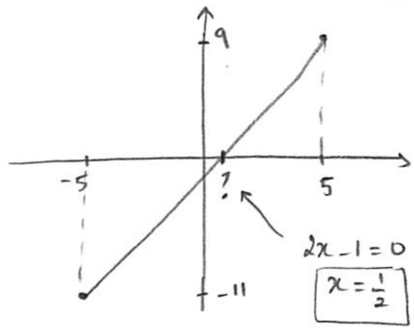


d) $\int_4^8 (3-x)\,dx = -\frac{1}{2} \times 4 (1+5)$

$= -12$



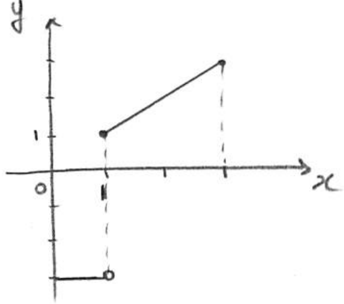
e) $\int_{-5}^{5} (2x-1)dx = \int_{-5}^{1/2} (2x-1)dx + \int_{1/2}^{5} (2x-1)dx$



$= -\frac{1}{2} \times 5.5 \times 11 + \frac{1}{2} \times 4.5 \times 9$

$= -10$

f) $\int_0^3 f(x)\,dx$ with $f(x) = \begin{cases} x \text{ if } x \geq 1 \\ -3 \text{ if } x < 1 \end{cases}$



$\int_0^3 f(x)dx = \int_0^1 -3\,dx + \int_1^3 x\,dx$

$= -1 \times 3 + \frac{1}{2} \times 2 \times (1+3)$

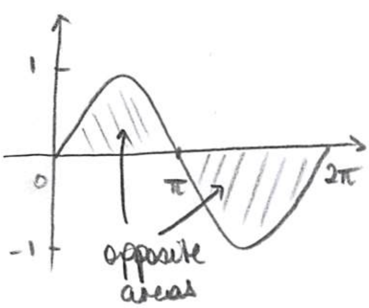
$= 1$

Examples: Given that $\int_0^{\pi} sinx\; dx = 2$, use what you know about the sine function to evaluate the following integrals:

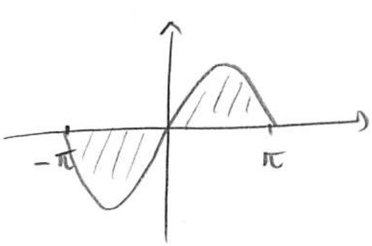
a) $\int_{\pi}^{2\pi} \sin x\, dx = -2$

b) $\int_{0}^{2\pi} \sin x\, dx = 0$

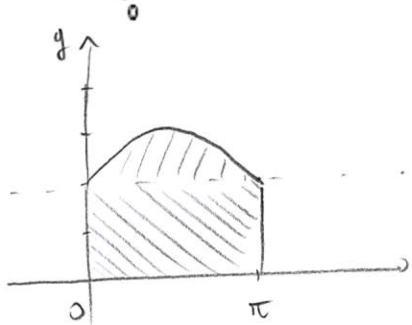
c) $\int_{0}^{\pi/2} \sin x\, dx = 1$



d) $\int_{-\pi}^{\pi} \sin x\, dx = 0$



e) $\int_{0}^{\pi} (2+\sin x)\, dx = 2\pi + 2$

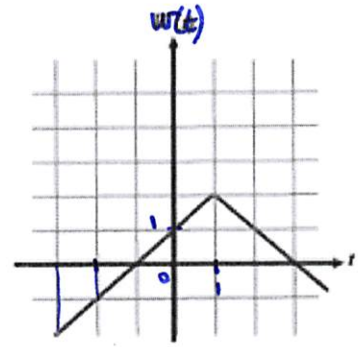


Example: Let $g(x) = \int_{-2}^{x} w(t)dt$, where the graph of $w(t)$ is given by the graph below.

Determine: a) $g(0) = \int_{-2}^{0} w(t)\,dt = 0$

b) $g(2) = \int_{-2}^{2} w(t)\,dt = 3$

c) $g(-3) = \int_{-2}^{-3} w(t)\,dt = ??$ TBD...



Note: In this last example, the definite integral was defining a new function… It happens when the lower limit and/or the upper limit depends on a variable (which has to be different than the variable used in the integrand)

How to Use Your Calculator to Find a Definite Integral:

The syntax for using your calculator is as follows: fnInt(*function, x, lower bound, upper bound*)

1. Press MATH
2. Press 9: fnInt(
3. Follow the syntax above

   … enter the function, x, lower bound, upper bound (be sure to enter a comma between each)

Example 4: Evaluate $\int_1^3 (x^2 - 2x + 2)\,dx = \frac{14}{3}$

Example 5: Evaluate $3 + 2\int_0^{\pi/4} \tan x\,dx \simeq 4.39$

You can also do the same thing from the graphing screen.

Example 6: Graph $y = \sqrt{x}$ on a standard viewing window. Evaluate $\int_1^8 \sqrt{x}\,dx$.

Press 2nd TRACE (which is CALC), 7: ∫ f(x)dx, enter Lower Bound as 1, enter Upper Bound as 8.

$\int_1^8 \sqrt{x}\,dx \simeq 14.42$

… the down side to using this method is that you MUST be able to set your window to SEE everything.

5.3 – Definite Integrals and Antiderivatives

Rules for Definite Integrals

1. Order of Integration: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   If you reverse the *order* of integration you get the opposite answer.

2. Zero: $\int_a^a f(x)\,dx = 0$

   This should make sense if you think about the "area" of a rectangle with no width.

3. Constant Multiple: If k is any constant, then $\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

   Taking the constant out of the integral many times makes it simpler to integrate.

4. Sum and Difference: $\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   This allows you to integrate functions that are added or subtracted separately. Notice, there are NO rules here for two functions that are multiplied or divided … that comes later!

5. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

   Pay close attention to the limits of integration … this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.

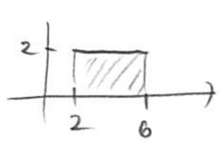
Example 1: Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, find the following:

a) $\int_2^6 [f(x)+g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx = 10 - 2 = 8$

b) $\int_2^6 [g(x)-f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx = -2 - 10 = -12$

c) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3\times 10 = 30$

d) $\int_2^6 (f(x)+2)\,dx = \int_2^6 f(x)\,dx + \int_2^6 2\,.\,dx = 10 + 8 = 18$



Example 2: Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find the following:

a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 13$

b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

c) $\int_5^5 f(x)\,dx = 0$

d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 30$

The Average Value of a Function

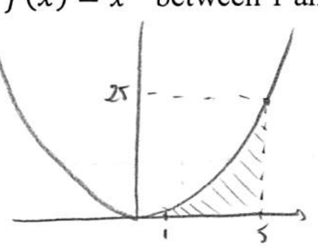
If f is integrable on $[a, b]$, its average value on $[a, b]$ is given by

$$\text{AVERAGE VALUE} = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \ldots \text{ or } \ldots \quad \text{AVERAGE VALUE} = \frac{\int_a^b f(x)\,dx}{b-a}$$

The *average value of a function* is just … "the integral over the interval".

Example: Determine the average value of the function $f(x) = x^2$ between 1 and 5.

$\text{Avg} = \frac{1}{5-1}\int_1^5 f(x)\,dx$



$= \frac{1}{4} \times \frac{124}{3}$ ← calc for now.

$= \frac{31}{3}$

Here is another look at the Mean Value Theorem with integral notations…

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

…Once again, we have a theorem that tells us a value of c exists, but the theorem doesn't actually <u>find</u> it for us!

5.4 – Fundamental Theorem of Calculus

The relationship between the definite integral and the antiderivatives is called:

The Fundamental Theorem of Calculus [The Evaluation Part]

If f is continuous at every point of $[a, b]$,

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F(x) is an antiderivative of f (x).

Reminder: F is an **antiderivative** of f if $F'(x) = f(x)$.

The set of all antiderivatives of a function $f(x)$ is denoted $\int f(x)dx$ and is called the **Integral** of f.

A function doesn't usually have a single antiderivative. All antiderivatives differ by a constant.

$\int f(x)dx = F(x) + C$, where C is called the constant of integration.

C determines which antiderivative you're talking about…. You leave it random if you're talking about all of the antiderivatives in general.

To determine an antiderivative, you need to imagine what function would have the appropriate derivative…

You need to memorize most usual functions' antiderivatives:

Integral Formulas

1. $\int a\,dx = ax + C$, where a is a constant
2. Power Rule for x^n when $n \neq -1$: $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$
3. Rule for x^n when $n = -1$: $\int \dfrac{1}{x}\,dx = \ln|x| + C$
4. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$, where a is a constant
5. $\int e^x\,dx = e^x + C$
6. $\int \sin(x)\,dx = -\cos(x) + C$
7. $\int \cos(x)\,dx = \sin(x) + C$
8. $\int \sec^2(x)\,dx = \tan(x) + C$
9. $\int \csc^2(x)\,dx = -\cot(x) + C$
10. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$
11. $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
12. $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$
13. $\int \dfrac{dx}{1+x^2} = \tan^{-1}(x) + C$
14. $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(x) + C$

Using these formulas, you can now find the integral of sums and differences of functions, but **NOT** products, quotients or compositions of functions! We will learn some techniques for products and compositions in the next section…

Examples: Evaluate:

1) $\int(-x^{-3}+\sqrt[3]{x}-1+e^x)dx$

$= \frac{1}{2}x^{-2}+\frac{3}{4}x^{4/3}-x+e^x+C$

2) $\int\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+1+x+x^2\right)dx$

$= \int(x^{-3}+x^{-2}+x^{-1}+1+x+x^2)dx$

$= -\frac{1}{2}x^{-2}-x^{-1}+\ln|x|+x+\frac{1}{2}x^2+\frac{1}{3}x^3+C$

Examples: Use the evaluation part of the Fundamental Theorem of Calculus to evaluate each expression:

a) $\int_0^3 x^2\,dx = \left.\frac{x^3}{3}+C\right]_0^3$

$= \frac{3^3}{3}+\cancel{C}-\frac{0^3}{3}-\cancel{C}$

$= 9$

b) $\int_{\pi/2}^{\pi}(1+\cos x)dx = \left.x+\sin x+C\right]_{\pi/2}^{\pi}$

$= \pi+0+\cancel{C}-\left(\frac{\pi}{2}+1+\cancel{C}\right)$

$= \frac{\pi}{2}-1$

We notice that C always gets cancelled… might as well choose C = 0 …

c) $\int_{-1}^{2} 3^x\,dx = \left.\frac{3^x}{\ln 3}\right]_{-1}^{2}$

$= \frac{3^2}{\ln 3}-\frac{3^{-1}}{\ln 3}$

$= \frac{1}{\ln 3}\left(9-\frac{1}{3}\right)$

$= \frac{1}{\ln 3}\cdot\frac{26}{3}$

d) $\int_4^9 f'(x)dx = f(9)-f(4)$

e) $\int_a^x f'(t)dt$ where a is a constant.

$= f(x)-f(a)$

f) $\frac{d}{dx}\left[\int_a^x f'(t)dt\right]$, where a is a constant.

$= \frac{d}{dx}\left(f(x)-f(a)\right)$

$= f'(x)$

The Fundamental Theorem of Calculus [Part #1 ... Simple]

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$$

Example 6: If $g(x)=\int_{-2}^{x} w(t)dt$, then $g'(x)=$?

$g'(x)=w(x)$

Example 7: $\frac{d}{dx}\left[\int_3^x \left(5t^2-6t+1\right)dt\right]$

$=5x^2-6x+1$

The Fundamental Theorem of Calculus [Part 1 ... Extended]

$$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right]=f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x)$$

Example 9: Find $\frac{d}{dx}\left[\int_{x^2}^{3x} f(t)\,dt\right]$

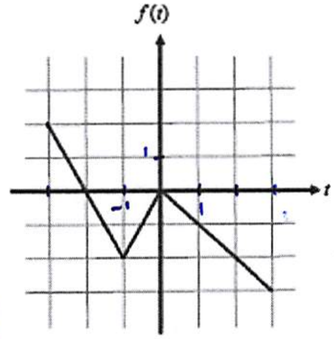
$=f(3x)\times 3-f(x^2)\times 2x$

Example 10: Let $g(x)=\int_{5x}^{3x^2}\sqrt{1+t^3}\,dt$. Find $g'(x)$.

$g'(x)=\sqrt{1+(3x^2)^3}\times 6x-\sqrt{1+(5x)^3}\times 5$

$=\sqrt{1+27x^6}\times 6x-\sqrt{1+125x^3}\times 5$

Example 16: Suppose the function below is the graph of $f(t)$ and $g(x)=\int_{-1}^{x} f(t)\,dt$.



a) Complete the table:

x	-3	-2	-1	0	1	2	3
$g(x)$	0	1	0	-1	$-3/2$	-3	$-11/2$

b) What are the intervals on which g is increasing or decreasing? Justify each response.

$g'(x) = f(x)$ (Fundamental Theorem of Calculus)

x	-3		-2		0		3
$f(x)$		+	0	−	0	−	

g is increasing on $(-3;-2)$ g is decreasing on $(-2;3)$.

c) What are the intervals on which g is concave up or concave down? Justify each response.

$g''(x) = f'(x)$

x	-3		-1		0		3
$g''(x)$		−	‖	+	‖	−	

g is concave up on $(-1;0)$

g is concave down on $(-3;-1)$ and $(0,3)$

d) For what value of x does g have a relative maximum? Justify your response.

g has a relative maximum when g' goes from positive to negative.

when $x=-2$

e) For what value of x does g have an inflection point? Justify your response.

g has an inflection point when g'' changes signs.

when $x=-1$ and when $x=0$

f) Graph $g(x)$