6.1 – Antiderivatives and Slope Fields

Differential equations:

A **differential equation** is an equation containing a derivative. To solve equations, we usually use an inverse operation to undo what is happening to x. To undo a derivative, we will take an antiderivative. But, we need to be cautious, because a function has many possible antiderivatives, so we will have a constant that can only be determined if we have some more info (like a specific value that the function needs to have…).

The **order** of a differential equation is the order of the highest derivative involved in the equation.

This year, we will only learn one method to solve differential equation: by **separation of variables**. It is not always possible…

Examples: solve:

1) $\frac{dy}{dx} = sinx$ if $y(0) = 2$

$y = -\cos x + C$

$y(0) = -\cos 0 + C$

$2 = -1 + C$

$C = 3$ $\boxed{y = -\cos x + 3}$

2) $y' + 6x^2 = e^x$

$y' = e^x - 6x^2$

$\boxed{y = e^x - 2x^3 + C, \ C \in \mathbb{R}}$

3) $y' = 2xy$ if $y(3) = 1$

$\frac{y'}{y} = 2x$

$\ln|y| = x^2 + C$

$|y| = e^{x^2+C}$

$y = \pm e^{x^2+C}$

since $y(3) > 0$, then $y = e^{x^2+C}$

$y(3) = 1 \Rightarrow 1 = e^{9+C}$

$C = -9$

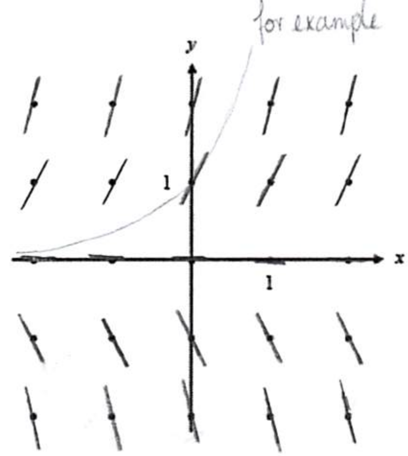
$\boxed{y = e^{x^2-9}}$

Slope Fields:

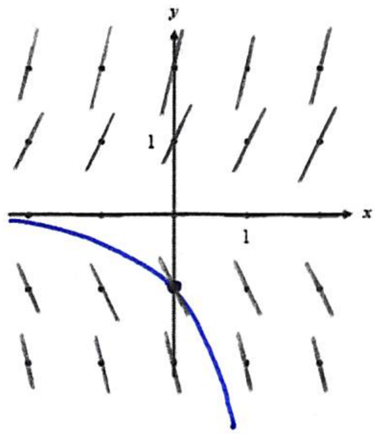
A **slope field** for the first order differential equation $y' = f(x, y)$ is a plot of short line segments with slope $f(x,y)$ for a lattice of points (x, y) in the plane.

Example: for $\frac{dy}{dx} = 2y$

for example



Suppose now that (0;-1) is on the solution of the previous differential equation. By following the slopes, draw on the previous diagram what you think the particular solution looks like (the graph should follow the pattern of the slope field, but may go between the points rather than through them…)



Solve the previous differential equation by separating the variables, and compare with your graph.

$\frac{y'}{y} = 2$

$\ln|y| = 2x + C$, $C \in \mathbb{R}$

$|y| = e^{2x+C}$

$y = \pm e^{2x+C}$

$y(0) = -1 < 0 \Rightarrow y = -e^{2x+C}$

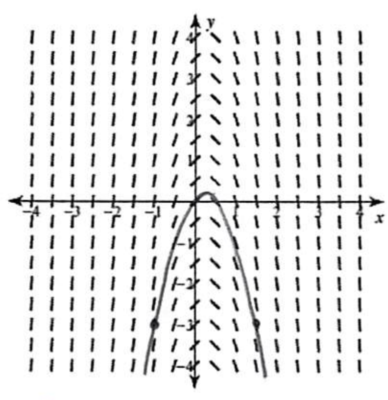
$-1 = -e^{C}$

$C = 0$

$\Rightarrow \boxed{y = -e^{2x}}$ (coherent with our graph)

Example: Determine the particular solution of the following differential equation, and graph it on the slope field provided.

$\frac{dy}{dx} = -4x + 1$, $y(-1) = -3$



$y' = -4x + 1$

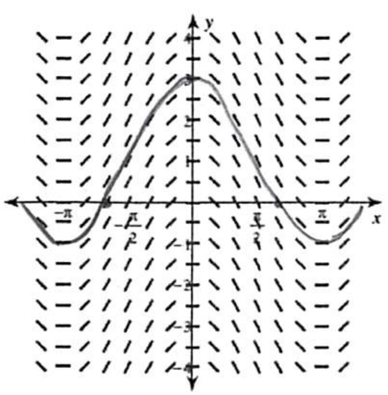
$y = -2x^2 + x + C$, $C \in \mathbb{R}$

$-3 = -2 - 1 + C$

$C = 0$

$\boxed{y = -2x^2 + x}$

$\frac{dy}{dx} = -2\sin x$, $y\left(\frac{2\pi}{3}\right) = 0$



$y' = -2\sin x$

$y = 2\cos x + C$

$0 = 2\cos\left(\frac{2\pi}{3}\right) + C$

$0 = -1 + C$

$C = 1$

$\boxed{y = 2\cos x + 1}$

6.2 - Integration by Substitution

We're trying to undo the chain rule…

Example: Let $f(x) = (3x^2 + 1)^5$ then, $f'(x) = 5(3x^2 + 1)^4 \times 6x$ or $f'(x) = 30x(3x^2 + 1)^4$

Then, by definition, we should have: $\int 30x(3x^2 + 1)^4 dx = (3x^2 + 1)^5 + C$

How can we find the result on our own? Every time we see a composite function to integrate, it must be multiplied by the derivative of the "inside" function in order to be integrated.

Examples:

a) $\int (x^2 - 1)^3 2x\, dx$

let $u = x^2 - 1$ $du = 2x dx$

$\int (x^2-1)^3 . 2x dx = \int u^3 du$

$= \frac{u^4}{4} + C$

$= \boxed{\frac{1}{4}(x^2-1)^4 + C}$

b) $\int 3x^2\sqrt{x^3+2}\, dx$ $= \int \sqrt{u} . du$

let $u = x^3 + 2$ $du = 3x^2 dx$

$= \frac{2}{3} u^{3/2} + C$

$= \boxed{\frac{2}{3}(x^3+2)^{3/2} + C}$

Be careful, when substituting by u, all the x must disappear for it to work…

Example 3: $\int x^3\sqrt{x^4+2}\, dx$ $= \int \frac{1}{4}(4x^3)\sqrt{x^4+2}\, dx$

let $u = x^4 + 2$

$du = 4x^3 . dx$

$= \frac{1}{4}\int \sqrt{u}\, du$ $= \frac{1}{4} \times \frac{2}{3} u^{3/2} + C$ $= \boxed{\frac{1}{6}(x^4+2)^{3/2} + C}$, $C \in \mathbb{R}$

Example 4: $\int \sin^2(3x)\cos(3x)\, dx$ $= \int u^2 . \frac{1}{3} du$ $= \frac{1}{3} \frac{u^3}{3} + C$

let $u = \sin 3x$

$du = \cos 3x \times 3 dx$

$= \boxed{\frac{1}{9}\sin^3 3x + C}$

Example 5: $\int \frac{x}{x^2+2}\, dx$ $= \int \frac{1}{2} \frac{1}{u} du$ $= \frac{1}{2} \ln|u| + C$

let $u = x^2 + 2$

$du = 2x dx$

$= \boxed{\frac{1}{2}\ln(x^2+2) + C}$

Definite Integrals with substitution: You have 2 options:

#1: Leave the limits in terms of the original variable and integrate like you did for the indefinite integrals. Once you have returned all variables back to the original letter, you can plug in the upper limits and lower limits.

#2: Using the rule for the change of variables, change the limits with the same rule … then you never need to return to the original variable.

Examples:

1) $\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_1^0 = \frac{1}{3}$

$u = 1-x^2$
$du = -2x\,dx$

when $x=1$, $u=0$

when $x=0$, $u=1$

$\boxed{\text{or}}$

$$= -\frac{1}{2}\int_{x=0}^{x=1} \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_{x=0}^{x=1} = -\frac{1}{3}(1-x^2)^{3/2}\Big]_0^1 = \frac{1}{3}$$

2) $\int_{-\pi}^{\pi}\frac{\cos x}{\sqrt{4+3\sin x}}dx = \frac{1}{3}\int_4^4 \frac{du}{\sqrt{u}} = 0$

$u = 4+3\sin x$
$du = 3\cos x\,dx$

$\boxed{\text{or}}$

$$= \frac{1}{3}\int_{x=-\pi}^{x=\pi} u^{-1/2}du = \frac{1}{3}\cdot 2\sqrt{u}\Big]_{x=-\pi}^{x=\pi} = \frac{2}{3}\sqrt{4+3\sin x}\Big]_{-\pi}^{\pi}$$

$$= \frac{2}{3}\sqrt{4+3\underbrace{\sin\pi}_{0}} - \frac{2}{3}\sqrt{4+3\underbrace{\sin(-\pi)}_{0}}$$

$$= 0$$

3) $\int_0^{\pi/4}\frac{\tan^3\theta}{\cos^2\theta}d\theta = \int_0^1 u^3 du$

(let $u = \tan\theta$
$du = \sec^2\theta\, d\theta$)

$$= \frac{u^4}{4}\Big]_0^1$$

$$= \frac{1}{4}$$

Here are some examples with trigonometric functions:

First, a few identities from trigonometry that you may or may not remember.

1. $1+\tan^2 x = \sec^2 x$ ... everyone remembers this one, right?!

2. $\cos(2x) = 2\cos^2 x - 1$ ... which can be rewritten to $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

3. $\cos(2x) = 1 - 2\sin^2 x$ ... which can be rewritten to $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

Can you integrate all of these functions? The first 4 should already be known.

$\int \sin x\, dx$ $= -\cos x + C$

$\int \cos x\, dx$ $= \sin x + C$

$\int \sec^2 x\, dx$ $= \tan x + C$

$\int \csc^2 x\, dx$ $= -\cot x + C$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

let $u = \cos x$
$du = -\sin x\, dx$

$$= -\int \frac{1}{u}\, du$$

$$= -\ln|\cos x| + C$$

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$$

let $u = \sin x$
$du = \cos x\, dx$

$$= \int \frac{1}{u}\, du$$

$$= \ln|\sin x| + C$$

$$\int \sin^2 x\, dx = \int \frac{1-\cos(2x)}{2}\, dx$$

$$= \int \frac{1}{2}\, dx - \frac{1}{2}\int \cos 2x \,.\, dx$$

let $u = 2x$
$du = 2dx$

$$= \int \frac{1}{2}\, dx - \frac{1}{4}\int \cos u \,.\, du$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$$

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx$$

$$= \int \frac{1}{2}\, dx + \frac{1}{4}\int \cos 2x \,.\, 2dx$$

$$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}$$

Here are some examples where substitution works, but not directly…

$$\int x^2\sqrt{x-1}\,dx = \int (u+1)^2\sqrt{u}\,du$$

let $u = x-1$ why?
$du = dx$
$x = u+1$

$$= \int (u^2+2u+1)\sqrt{u}\,du$$

$$= \int \left(u^{5/2}+2u^{3/2}+u^{1/2}\right)du$$

$$= \frac{2}{7}u^{7/2} + 2\times\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}$$

$$\int x\sqrt{4-x}\,dx = -\int (4+u)\sqrt{u}\,.du$$

let $u = 4-x$
$du = -dx$
$x = 4+u$

$$= -\int \left(4u^{1/2}+u^{3/2}\right)du$$

$$= -4\times\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} + C$$

$$= \boxed{-\frac{8}{3}(4-x)^{3/2} - \frac{2}{5}(4-x)^{5/2} + C}$$

$$\int \frac{y}{\sqrt{2y+1}}dy = \frac{1}{2}\int \frac{u-1}{2}\cdot\frac{1}{\sqrt{u}}\,du$$

let $u = 2y+1$
$du = 2dy$
$\frac{u-1}{2} = y$

$$= \frac{1}{4}\int \left(u^{1/2} - u^{-1/2}\right)du$$

$$= \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$

$$= \boxed{\frac{1}{6}(2y+1)^{3/2} - \frac{1}{2}(2y+1)^{1/2} + C}$$

$$\int \cos^3 x\,dx = \int \cos x\,.\cos^2 x\,.dx$$

$$= \int \cos x\,(1-\sin^2 x)\,dx$$

let $u = \sin x$
$du = \cos x\,.dx$

$$= \int \cos x\,.dx - \int \sin^2 x\,.\cos x\,dx$$

$$= \int \cos x\,.dx - \int u^2 du$$

$$= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$$

When substitution doesn't work, you need to transform your expression into something where it works…

Long Division … when the numerator has a degree greater than or equal to the denominator

Example 9: $\int \frac{x^2-1}{x^2+1}\,dx$ $= \int\left(1-\frac{2}{x^2+1}\right)dx = \boxed{x - 2\tan^{-1}x + C}$

$\frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$

Expand … when the "inside" doesn't have a derivative on the outside, try expanding the function

Example 10: $\int(\sin x+\cos x)^2\,dx$ $= \int(\sin^2 x + 2\sin x.\cos x + \cos^2 x)\,dx$

$= \int(1 + 2\sin x.\cos x)\,dx$ $= \boxed{x + \sin^2 x + C}$

let $u = \sin x$
$du = \cos x.dx$

$= x + 2\int u.du$

or

$x - \cos^2 x + C$ …

$= x + 2\frac{u^2}{2} + C$

Complete the Square … useful when you have a x^2 and x term in the denominator but no x term in the numerator.

Example 11: $\int \frac{2\,dx}{x^2-6x+10}$ $= \int \frac{2}{(x-3)^2+1}\,dx = \int \frac{2}{u^2+1}\,du$

let $u = x-3$
$du = dx$

$= 2\tan^{-1}u + C$

$= \boxed{2\tan^{-1}(x-3) + C}$

Separate the numerator … when you have more than one term in the numerator

Example 12: $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$ $= \int \frac{3x}{\sqrt{1-x^2}}\,dx + \int \frac{2}{\sqrt{1-x^2}}\,dx$

$= -\frac{3}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx + 2\sin^{-1}x$

$u = 1-x^2$
$du = -2x\,dx$

$= -\frac{3}{2} \times 2u^{1/2} + 2\sin^{-1}x + C$ $= \boxed{-3\sqrt{1-x^2} + 2\sin^{-1}x + C}$

6.4 – Exponential Growth and Decay

Exponential Growth and Decay Model

If y changes at a rate proportional to the amount present ($\frac{dy}{dt} = ky$) and $y = y_0$ when $t = 0$, then

$$\boxed{y = y_0 e^{kt}}$$

where k is the **proportional constant**.

Exponential **growth** occurs when $k > 0$, and exponential **decay** occurs when $k < 0$.

$y' = ky$

$\frac{y'}{y} = k$

$\ln|y| = kt + c$

$|y| = e^{kt+c}$

$y = \pm e^c . e^{kt}$

$y = y_0 e^{kt}$ with $y_0 = \pm e^c$

y_0 is also the value of y when $t = 0$.

Example 1: The rate of change of y is proportional to y. When $t = 0$, $y = 2$. When $t = 2$, $y = 4$. What is the value of y when $t = 3$?

$y' = ky \implies y = y_0 e^{kt}$

- when $t = 0$, $y = 2$

  so $y_0 = 2$

- when $t = 2$, $y = 4$

  so $4 = 2e^{2k}$

  $2 = e^{2k}$

  $k = \frac{\ln 2}{2} \implies y = 2e^{\frac{\ln 2}{2}t}$

  $y = 2 . 2^{t/2}$

when $t = 3$, $y = 2 . 2^{3/2}$ $\boxed{y = 2^{5/2}}$

Example 2: **Newton's Law of Cooling**: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature in the surrounding medium. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3 °F. One hour later, the temperature of the body is 89.0 °F. The temperature of the room has been maintained at a constant 68 °F.

(a) Assuming the temperature, T, of the body obeys Newton's Law of Cooling, write a differential equation for T.

Let's consider that $t=0$ at 9AM. (t: time in hours)

$T' = k(T-68)$ $\quad T_0 = 90.3°F \quad T(1) = 89.0°$

or, with $y = T-68$, $y' = T'$ $\Rightarrow y' = ky \quad y_0 = 22.3 \quad y(1) = 21$

(b) Solve the differential equation to estimate the time the murder occurred.

$T - 68 = 22.3e^{kt}$

$T = 22.3e^{kt} + 68$

• $T(1) = 22.3e^{k} + 68$

$89 = 22.3e^{k} + 68$

$21 = 22.3e^{k}$

$e^k = \frac{21}{22.3}$

$k = \ln\left(\frac{21}{22.3}\right)$

$$\boxed{T = 22.3\left(\frac{21}{22.3}\right)^t + 68}$$

Example 3: Compounding Interest continuously:
Suppose you deposit \$800 in an account that pays 6.3% annual interest. How much will you have 8 years later if the interest is compounded continuously?

$A = A_0 e^{0.063t}$

$A_0 = 800$

$A(8) = 800e^{0.063(8)}$

$$\boxed{A(8) \simeq \$1324.26}$$