EXAMPLE 9 Applying Euler's Method

Let f be the function that satisfies the initial value problem in Example 6 (that is, $dy/dx = x + y$ and $f(2) = 0$). Use Euler's method and increments of $\Delta x = 0.2$ to approximate $f(3)$.

SOLUTION

We use Euler's Method to construct an approximation of the curve from $x = 2$ to x = 3 by pasting together five small linearization segments (Figure 6.7). Each segment will ext[end] from a point (x, y) to a point $(x + \Delta x, y + \Delta y)$, where $\Delta x = 0.2$ and $\Delta y = (dy/dx)\Delta x$. The following table shows how we construct each new point from the previous one.

(x, y)	$dy/dx = x + y$	Δx	$\Delta y = (dy/dx)\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 0)	2	0.2	0.4	(2.2, 0.4)
(2.2, 0.4)	2.6	0.2	0.52	(2.4, 0.92)
(2.4, 0.92)	3.32	0.2	0.664	(2.6, 1.584)
(2.6, 1.584)	4.184	0.2	0.8368	(2.8, 2.4208)
(2.8, 2.4208)	5.2208	0.2	1.04416	(3, 3.46496)

Euler's Method leads us to an approximation $f(3) \approx 3.46496$, which we would more rea[sonably] report as $f(3) \approx 3.465$. *Now try Exercise*

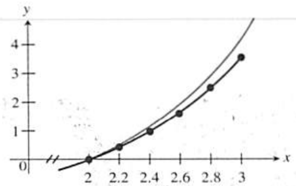


Figure 6.7 Euler's Method is used to construct an approximate solution to an initial value problem between $x = 2$ and $x = 3$. (Example 9)

You can see from Figure 6.7 that Euler's Method leads to an underestimate when the curve is concave up, just as it will lead to an overestimate when the curve is concave do[wn]. You can also see that the error increases as the distance from the original point incre[ases]. In fact, the true value of $f(3)$ is about 4.155, so the approximation error is about 16.6%. We could increase the accuracy by taking smaller increments; a reasonable option if [we] have a calculator program to do the work. For example, 100 increments of 0.01 give an [es]timate of 4.1144, cutting the error to about 1%.

EXAMPLE 10 Moving Backward with Euler's Method

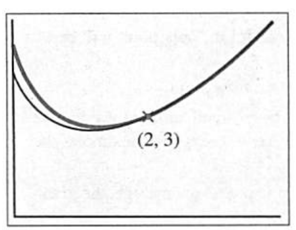
If $dy/dx = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's Method with five equal steps to approximate y when $x = 1.5$.

SOLUTION

Starting at $x = 2$, we need five equal steps of $\Delta x = -0.1$.

(x, y)	$dy/dx = 2x - y$	Δx	$\Delta y = (dy/dx)\Delta x$	$(x + \Delta x, y +$
(2, 3)	1	−0.1	−0.1	(1.9, 2.9)
(1.9, 2.9)	0.9	−0.1	−0.09	(1.8, 2.81)
(1.8, 2.81)	0.79	−0.1	−0.079	(1.7, 2.731)
(1.7, 2.731)	0.669	−0.1	−0.0669	(1.6, 2.6641)
(1.6, 2.6641)	0.5359	−0.1	−0.05359	(1.5, 2.6105

The value at $x = 1.5$ is approximately 2.61. (The actual value is about 2.649, so the percentage error in this case is about 1.4%.) *Now try Exercise*



[0, 4] by [0, 6]

Figure 6.8 A grapher program using Euler's Method and increments of 0.1 produced this approximation to the solution curve for the initial value problem in Example 10. The actual solution curve is shown in red.

If we program a grapher to do the work of finding the points, Euler's Method can be used to graph (approximately) the solution to an initial value problem without actu[ally] solving it. For example, a graphing calculator program starting with the initial value pr[ob]lem in Example 9 produced the graph in Figure 6.8 using increments of 0.1. The graph of the actual solution is shown in red. Notice that Euler's Method does a better job of appr[ox]imating the curve when the curve is nearly straight, as should be expected.

Euler's Method is one example of a *numerical method* for solving differential equations. The table of values is the *numerical solution*. The analysis of error in a numerical solution and the investigation of methods to reduce it are important, but appropriate for a more advanced course (which would also describe more accurate numerical methods than the one shown here).

Quick Review 6.1

In Exercises 1–8, determine whether or not the function y satisfies the differential equation.

$\frac{dy}{dx} = y$ $\quad y = e^x$

$\frac{dy}{dx} = 4y$ $\quad y = e^{4x}$

$\frac{dy}{dx} = 2xy$ $\quad y = x^2e^x$

$\frac{dy}{dx} = 2xy$ $\quad y = e^{x^2}$

$\frac{dy}{dx} = 2xy$ $\quad y = e^{x^2} + 5$

6. $\frac{dy}{dx} = \frac{1}{y}$ $\quad y = \sqrt{2x}$

7. $\frac{dy}{dx} = y \tan x$ $\quad y = \sec x$

8. $\frac{dy}{dx} = y^2$ $\quad y = x^{-1}$

In Exercises 9–12, find the constant C.

9. $y = 3x^2 + 4x + C$ and $y = 2$ when $x = 1$
10. $y = 2\sin x - 3\cos x + C$ and $y = 4$ when $x = 0$
11. $y = e^{2x} + \sec x + C$ and $y = 5$ when $x = 0$
12. $y = \tan^{-1} x + \ln(2x - 1) + C$ and $y = \pi$ when $x = 1$

Section 6.1 Exercises

In Exercises 1–10, find the general solution to the exact differential equation.

$\frac{dy}{dx} = 5x^4 - \sec^2 x$

$\frac{dy}{dx} = \sec x \tan x - e^x$

$\frac{dy}{dx} = \sin x - e^{-x} + 8x^3$

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2}\ (x > 0)$

$\frac{dy}{dx} = 5^x \ln 5 + \frac{1}{x^2 + 1}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{x}}$

$\frac{dy}{dt} = 3t^2 \cos(t^3)$

$\frac{dy}{dt} = (\cos t)\, e^{\sin t}$

$\frac{du}{dx} = (\sec^2 x^5)(5x^4)$

$\frac{dy}{du} = 4(\sin u)^3(\cos u)$

In Exercises 11–20, solve the initial value problem explicitly.

$\frac{dy}{dx} = 3\sin x$ and $y = 2$ when $x = 0$

$\frac{dy}{dx} = 2e^x - \cos x$ and $y = 3$ when $x = 0$

13. $\frac{du}{dx} = 7x^6 - 3x^2 + 5$ and $u = 1$ when $x = 1$

14. $\frac{dA}{dx} = 10x^9 + 5x^4 - 2x + 4$ and $A = 6$ when $x = 1$

15. $\frac{dy}{dx} = -\frac{1}{x^2} - \frac{3}{x^4} + 12$ and $y = 3$ when $x = 1$

16. $\frac{dy}{dx} = 5\sec^2 x - \frac{3}{2}\sqrt{x}$ and $y = 7$ when $x = 0$

17. $\frac{dy}{dt} = \frac{1}{1 + t^2} + 2^t \ln 2$ and $y = 3$ when $t = 0$

18. $\frac{dx}{dt} = \frac{1}{t} - \frac{1}{t^2} + 6$ and $x = 0$ when $t = 1$

19. $\frac{dv}{dt} = 4\sec t \tan t + e^t + 6t$ and $v = 5$ when $t = 0$

20. $\frac{ds}{dt} = t(3t - 2)$ and $s = 0$ when $t = 1$

In Exercises 21–24, solve the initial value problem using the Fundamental Theorem. (Your answer will contain a definite integral.)

21. $\frac{dy}{dx} = \sin(x^2)$ and $y = 5$ when $x = 1$

22. $\frac{du}{dx} = \sqrt{2 + \cos x}$ and $u = -3$ when $x = 0$

23. $F'(x) = e^{\cos x}$ and $F(2) = 9$

24. $G'(s) = \sqrt[3]{\tan s}$ and $G(0) = 4$

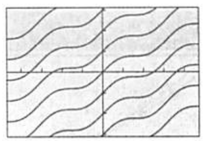
In Exercises 25–28, match the differential equation with the graph of a family of functions (a)–(d) that solve it. Use slope analysis, not your graphing calculator.

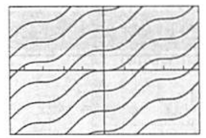
25. $\frac{dy}{dx} = (\sin x)^2$

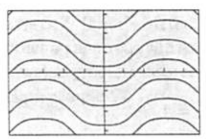
26. $\frac{dy}{dx} = (\sin x)^3$

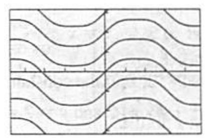
27. $\frac{dy}{dx} = (\cos x)^2$

28. $\frac{dy}{dx} = (\cos x)^3$

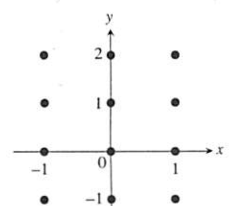

(a)


(b)


(c)


(d)

In Exercises 29–34, construct a slope field for the differential equation. In each case, copy the graph at the right and draw tiny segments through the twelve lattice points shown in the graph. Use slope analysis, not your graphing calculator.



29. $\frac{dy}{dx} = x$

30. $\frac{dy}{dx} = y$

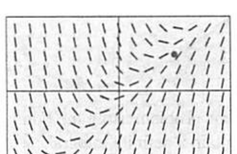
31. $\frac{dy}{dx} = 2x + y$

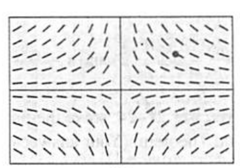
32. $\frac{dy}{dx} = 2x - y$

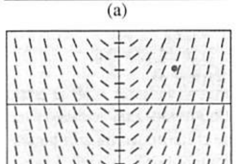
33. $\frac{dy}{dx} = x + 2y$

34. $\frac{dy}{dx} = x - 2y$

In Exercises 35–40, match the differential equation with the appropriate slope field. Then use the slope field to sketch the graph of the particular solution through the highlighted point (3, 2). (All slope fields are shown in the window $[-6, 6]$ by $[-4, 4]$.)


(a)


(b)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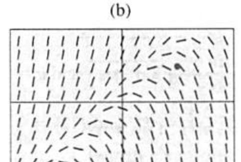
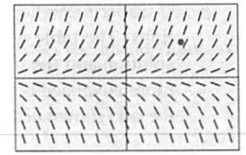
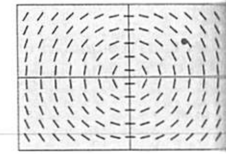

(d)


(e)


(f)

35. $\frac{dy}{dx} = x$

36. $\frac{dy}{dx} = y$

37. $\frac{dy}{dx} = x - y$

38. $\frac{dy}{dx} = y - x$

39. $\frac{dy}{dx} = -\frac{y}{x}$

40. $\frac{dy}{dx} = -\frac{x}{y}$

In Exercises 41–44, use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$.

41. $\frac{dy}{dx} = x - 1$ and $y = 2$ when $x = 1$

42. $\frac{dy}{dx} = y - 1$ and $y = 3$ when $x = 1$

43. $\frac{dy}{dx} = y - x$ and $y = 2$ when $x = 1$

44. $\frac{dy}{dx} = 2x - y$ and $y = 0$ when $x = 1$

In Exercises 45–48, use Euler's Method with increments of $\Delta x = -0.1$ to approximate the value of y when $x = 1.7$.

45. $\frac{dy}{dx} = 2 - x$ and $y = 1$ when $x = 2$

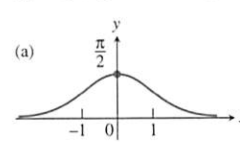
46. $\frac{dy}{dx} = 1 + y$ and $y = 0$ when $x = 2$

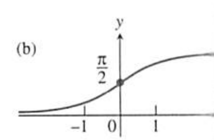
47. $\frac{dy}{dx} = x - y$ and $y = 2$ when $x = 2$

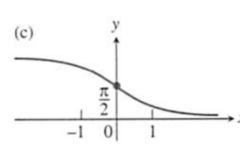
48. $\frac{dy}{dx} = x - 2y$ and $y = 1$ when $x = 2$

In Exercises 49 and 50, **(a)** determine which graph shows the solution of the initial value problem without actually solving the problem. **(b) Writing to Learn** Explain how you eliminated two of the possibilities.

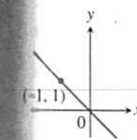
49. $\frac{dy}{dx} = \frac{1}{1 + x^2}, \quad y(0) = \frac{\pi}{2}$

(a)


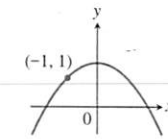
(b)


(c)


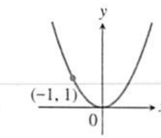
$\frac{dy}{dx} = -x, \quad y(-1) = 1$



(a)

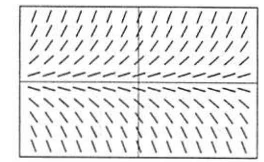


(b)

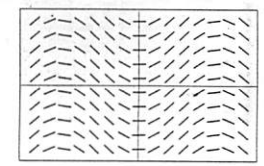


(c)

Writing to Learn Explain why $y = x^2$ could not be a solution to the differential equation with slope field shown below.


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

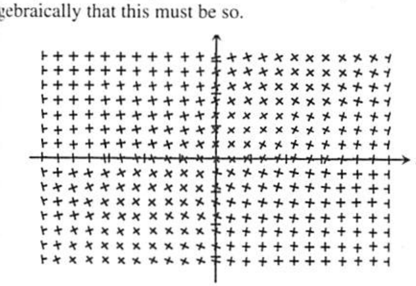
Writing to Learn Explain why $y = \sin x$ could not be a solution to the differential equation with slope field shown below.


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

Percentage Error Let $y = f(x)$ be the solution to the initial value problem $dy/dx = 2x + 1$ such that $f(1) = 3$. Find the percentage error if Euler's Method with $\Delta x = 0.1$ is used to approximate $f(1.4)$.

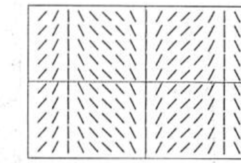
Percentage Error Let $y = f(x)$ be the solution to the initial value problem $dy/dx = 2x - 1$ such that $f(2) = 3$. Find the percentage error if Euler's Method with $\Delta x = -0.1$ is used to approximate $f(1.6)$.

Perpendicular Slope Fields The figure below shows the slope fields for the differential equations $dy/dx = e^{(x-y)/2}$ and $dy/dx = -e^{(y-x)/2}$ superimposed on the same grid. It appears that the slope lines are perpendicular wherever they intersect. Prove algebraically that this must be so.

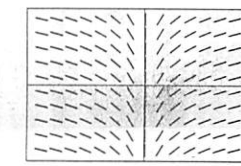


56. ***Perpendicular Slope Fields*** If the slope fields for the differential equations $dy/dx = \sec x$ and $dy/dx = g(x)$ are perpendicular (as in Exercise 55), find $g(x)$.

57. ***Plowing Through a Slope Field*** The slope field for the differential equation $dy/dx = \csc x$ is shown below. Find a function that will be *perpendicular* to every line it crosses in the slope field. (*Hint:* First find a differential equation that will produce a perpendicular slope field.)


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

58. ***Plowing Through a Slope Field*** The slope field for the differential equation $dy/dx = 1/x$ is shown below. Find a function that will be *perpendicular* to every line it crosses in the slope field. (*Hint:* First find a differential equation that will produce a perpendicular slope field.)


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

59. **True or False** Any two solutions to the differential equation $dy/dx = 5$ are parallel lines. Justify your answer.

60. **True or False** If $f(x)$ is a solution to $dy/dx = 2x$, then $f^{-1}(x)$ is a solution to $dy/dx = 2y$. Justify your answer.

61. **Multiple Choice** A slope field for the differential equation $dy/dx = 42 - y$ will show

(A) a line with slope −1 and y-intercept 42.

(B) a vertical asymptote at $x = 42$.

(C) a horizontal asymptote at $y = 42$.

(D) a family of parabolas opening downward.

(E) a family of parabolas opening to the left.

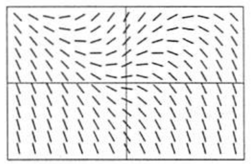
62. **Multiple Choice** For which of the following differential equations will a slope field show nothing but negative slopes in the fourth quadrant?

(A) $\frac{dy}{dx} = -\frac{x}{y}$ (B) $\frac{dy}{dx} = xy + 5$ (C) $\frac{dy}{dx} = xy^2 - 2$

(D) $\frac{dy}{dx} = \frac{x^3}{y^2}$ (E) $\frac{dy}{dx} = \frac{y}{x^2} - 3$

63. **Multiple Choice** If $dy/dx = 2xy$ and $y = 1$ when $x = 0$, then $y =$
(A) y^{2x} (B) e^{x^2} (C) x^2y (D) $x^2y + 1$ (E) $\frac{x^2y^2}{2} + 1$

64. **Multiple Choice** Which of the following differential equations would produce the slope field shown below?
(A) $\frac{dy}{dx} = y - |x|$ (B) $\frac{dy}{dx} = |y| - x$
(C) $\frac{dy}{dx} = |y - x|$ (D) $\frac{dy}{dx} = |y + x|$
(E) $\frac{dy}{dx} = |y| - |x|$



[−3, 3] by [−1.98, 1.98]

Explorations

65. ***Solving Differential Equations*** Let $\frac{dy}{dx} = x - \frac{1}{x^2}$.

(a) Find a solution to the differential equation in the interval $(0, \ )$ that satisfies $y(1) = 2$.

(b) Find a solution to the differential equation in the interval $(-\infty, 0)$ that satisfies $y(-1) = 1$.

(c) Show that the following piecewise function is a solution to the differential equation for any values of C_1 and C_2.

$$y = \begin{cases} \frac{1}{x} + \frac{x^2}{2} + C_1, & x < 0 \\ \frac{1}{x} + \frac{x^2}{2} + C_2, & x > 0 \end{cases}$$

(d) Choose values for C_1 and C_2 so that the solution in part (c) agrees with the solutions in parts (a) and (b).

(e) Choose values for C_1 and C_2 so that the solution in part (c) satisfies $y(2) = -1$ and $y(-2) = 2$.

66. ***Solving Differential Equations*** Let $\frac{dy}{dx} = \frac{1}{x}$.

(a) Show that $y = \ln x + C$ is a solution to the differential equation in the interval $(0, \infty)$.

(b) Show that $y = \ln(-x) + C$ is a solution to the differential equation in the interval $(-\infty, 0)$.

(c) **Writing to Learn** Explain why $y = \ln|x| + C$ is a solution to the differential equation in the domain $(-\infty, 0) \cup (0, \infty)$.

(d) Show that the function

$$y = \begin{cases} \ln(-x) + C_1, & x < 0 \\ \ln x + C_2, & x > 0 \end{cases}$$

is a solution to the differential equation for any values of C_1 and C_2.

Extending the Ideas

67. ***Second-Order Differential Equations*** Find the general solution to each of the following second-order differential equations by first finding dy/dx and then finding y. The general solution will have two unknown constants.

(a) $\frac{d^2y}{dx^2} = 12x + 4$ (b) $\frac{d^2y}{dx^2} = e^x + \sin x$ (c) $\frac{d^2y}{dx^2} = x^3 +$

68. ***Second-Order Differential Equations*** Find the specific solution to each of the following second-order initial value problems by first finding dy/dx and then finding y.

(a) $\frac{d^2y}{dx^2} = 24x^2 - 10$. When $x = 1$, $\frac{dy}{dx} = 3$ and $y = 5$.

(b) $\frac{d^2y}{dx^2} = \cos x - \sin x$. When $x = 0$, $\frac{dy}{dx} = 2$ and $y = 0$

(c) $\frac{d^2y}{dx^2} = e^x - x$. When $x = 0$, $\frac{dy}{dx} = 0$ and $y = 1$.

69. ***Differential Equation Potpourri*** For each of the following differential equations, find at least one particular solution. You will need to call on past experience with functions you have differentiated. For a greater challenge, find the general solution.

(a) $y' = x$ (b) $y' = -x$ (c) $y' = y$
(d) $y' = -y$ (e) $y' = xy$

70. ***Second-Order Potpourri*** For each of the following second-order differential equations, find at least one particular solution. You will need to call on past experience with functions you have differentiated. For a significantly greater challenge, find the general solution (which will involve two unknown constants).

(a) $y'' = x$ (b) $y'' = -x$ (c) $y'' = -\sin x$
(d) $y'' = y$ (e) $y'' = -y$

6.2 Antidifferentiation by Substitution

What you'll learn about

- Indefinite Integrals
- Leibniz Notation and Antiderivatives
- Substitution in Indefinite Integrals
- Substitution in Definite Integrals

. . . and why

Antidifferentiation techniques were historically crucial for applying the results of calculus.

Indefinite Integrals

If $y = f(x)$ we can denote the derivative of f by either dy/dx or $f'(x)$. What can we use to denote the *antiderivative* of f? We have seen that the general solution to the differential equation $dy/dx = f(x)$ actually consists of an infinite family of functions of the form $F(x) + C$, where $F'(x) = f(x)$. Both the name for this family of functions and the symbol we use to denote it are closely related to the definite integral because of the Fundamental Theorem of Calculus.

DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

As in Chapter 5, the symbol $\int$ is an **integral sign,** the function f is the **integrand** of the integral, and x is the **variable of integration.**

Notice that an indefinite integral is not at all like a definite integral, despite the similarities in notation and name. A definite integral is a *number*, the limit of a sequence of Riemann sums. An indefinite integral is a *family of functions* having a common derivative. If the Fundamental Theorem of Calculus had not provided such a dramatic link between antiderivatives and integration, we would surely be using a different name and symbol for the general antiderivative today.

EXAMPLE 1 Evaluating an Indefinite Integral

Evaluate $\int (x^2 - \sin x)\, dx$.

SOLUTION

Evaluating this definite integral is just like solving the differential equation $dy/dx = x^2 - \sin x$. Our past experience with derivatives leads us to conclude that

$$\int (x^2 - \sin x)dx = \frac{x^3}{3} + \cos x + C$$

(as you can check by differentiating).

Now try Exercise 3.

You have actually been finding antiderivatives since Section 5.3, so Example 1 should hardly have seemed new. Indeed, each derivative formula in Chapter 3 could be turned around to yield a corresponding indefinite integral formula. We list some of the most useful such indefinite integral formulas below. Be sure to familiarize yourself with these before moving on to the next section, in which function composition becomes an issue. (Incidentally, it is in anticipation of the next section that we give some of these formulas in terms of the variable u rather than x.)

(c) $\int \cos^3 x\,dx = \int (\cos^2 x)\cos x\,dx$

$= \int (1 - \sin^2 x)\cos x\,dx$

$= \int (1 - u^2)\,du$ Let $u = \sin x$ and $du = \cos x\,dx$.

$= u - \dfrac{u^3}{3} + C$

$= \sin x - \dfrac{\sin^3 x}{3} + C$ Re-substitute after antidifferentiating.

Now try Exercise 4

Substitution in Definite Integrals

Antiderivatives play an important role when we evaluate a definite integral by the Fundamental Theorem of Calculus, and so, consequently, does substitution. In fact, if we make full use of our substitution of variables and change the interval of integration to match the u-substitution in the integrand, we can avoid the "resubstitution" step in the previous four examples.

EXAMPLE 8 Evaluating a Definite Integral by Substitution

Evaluate $\displaystyle\int_0^{\pi/3} \tan x \sec^2 x\,dx.$

SOLUTION

Let $u = \tan x$ and $du = \sec^2 x\,dx$.

Note also that $u(0) = \tan 0 = 0$ and $u(\pi/3) = \tan(\pi/3) = \sqrt{3}$.

So

$$\int_0^{\pi/3} \tan x \sec^2 x\,dx = \int_0^{\sqrt{3}} u\,du \quad \text{Substitute } u\text{-interval for } x\text{-interval.}$$

$$= \left.\frac{u^2}{2}\right|_0^{\sqrt{3}}$$

$$= \frac{3}{2} - 0 = \frac{3}{2}$$

Now try Exercise 61

EXAMPLE 9 That Absolute Value Again

Evaluate $\displaystyle\int_0^1 \frac{x}{x^2 - 4}\,dx.$

SOLUTION

Let $u = x^2 - 4$ and $du = 2x\,dx$. Then $u(0) = 0^2 - 4 = -4$ and $u(1) = 1^2 - 4 = -3$,

continued

So

$$\int_0^1 \frac{x}{x^2 - 4}\,dx = \frac{1}{2}\int_0^1 \frac{2x\,dx}{x^2 - 4}$$

$$= \frac{1}{2}\int_{-4}^{-3} \frac{du}{u} \quad \text{Substitute } u\text{-interval for } x\text{-interval.}$$

$$= \left.\frac{1}{2}\ln|u|\right|_{-4}^{-3}$$

$$= \frac{1}{2}(\ln 3 - \ln 4) = \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

Notice that $\ln u$ would not have existed over the interval of integration $[-4, -3]$. The absolute value in the antiderivative is important. ***Now try Exercise 63.***

Finally, consider this historical note. The technique of u-substitution derived its importance from the fact that it was a powerful tool for antidifferentiation. Antidifferentiation derived its importance from the Fundamental Theorem, which established it as the way to evaluate definite integrals. Definite integrals derived their importance from real-world applications. While the applications are no less important today, the fact that the definite integrals can be easily evaluated by technology has made the world less reliant on antidifferentiation, and hence less reliant on u-substitution. Consequently, you have seen in this book only a sampling of the substitution tricks calculus students would have routinely studied in the past. You may see more of them in a differential equations course.

Quick Review 6.2 *(For help, go to Sections 3.6 and 3.9.)*

In Exercises 1 and 2, evaluate the definite integral.

1. $\displaystyle\int_0^2 x^4\,dx$
2. $\displaystyle\int_1^5 \sqrt{x-1}\,dx$

In Exercises 3–10, find dy/dx.

3. $y = \displaystyle\int_2^x 3^t\,dt$
4. $y = \displaystyle\int_0^x 3^t\,dt$
5. $y = (x^3 - 2x^2 + 3)^4$
6. $y = \sin^2(4x - 5)$
7. $y = \ln \cos x$
8. $y = \ln \sin x$
9. $y = \ln(\sec x + \tan x)$
10. $y = \ln(\csc x + \cot x)$

Section 6.2 Exercises

In Exercises 1–6, find the indefinite integral.

1. $\displaystyle\int (\cos x - 3x^2)\,dx$
2. $\displaystyle\int x^{-2}\,dx$
3. $\displaystyle\int \left(t^2 - \frac{1}{t^2}\right)dt$
4. $\displaystyle\int \frac{dt}{t^2+1}$
5. $\displaystyle\int (3x^4 - 2x^{-3} + \sec^2 x)\,dx$
6. $\displaystyle\int (2e^x + \sec x \tan x - \sqrt{x})\,dx$

In Exercises 7–12, use differentiation to verify the antiderivative formula.

7. $\displaystyle\int \csc^2 u\,du = -\cot u + C$
8. $\displaystyle\int \csc u \cot u = -\csc u + C$
9. $\displaystyle\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$
10. $\displaystyle\int 5^x\,dx = \frac{1}{\ln 5}5^x + C$
11. $\displaystyle\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$
12. $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$

In Exercises 13–16, verify that $\int f(u)\,du \neq \int f(u)\,dx$

13. $f(u) = \sqrt{u}$ and $u = x^2\ (x > 0)$

14. $f(u) = u^2$ and $u = x^5$

15. $f(u) = e^u$ and $u = 7x$

16. $f(u) = \sin u$ and $u = 4x$

In Exercises 17–24, use the indicated substitution to evaluate the integral. Confirm your answer by differentiation.

17. $\int \sin 3x\,dx, \quad u = 3x$

18. $\int x\cos(2x^2)\,dx, \quad u = 2x^2$

19. $\int \sec 2x \tan 2x\,dx, \quad u = 2x$

20. $\int 28(7x-2)^3\,dx, \quad u = 7x - 2$

21. $\int \frac{dx}{x^2+9}, \quad u = \frac{x}{3}$

22. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

23. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$

24. $\int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy, \quad u = y^4 + 4y^2 + 1$

In Exercises 25–46, use substitution to evaluate the integral.

25. $\int \frac{dx}{(1-x)^2}$

26. $\int \sec^2(x+2)\,dx$

27. $\int \sqrt{\tan x}\,\sec^2 x\,dx$

28. $\int \sec\left(\theta + \frac{\pi}{2}\right)\tan\left(\theta + \frac{\pi}{2}\right)d\theta$

29. $\int \tan(4x+2)\,dx$

30. $\int 3(\sin x)^{-2}\,dx$

31. $\int \cos(3z+4)\,dz$

32. $\int \sqrt{\cot x}\,\csc^2 x\,dx$

33. $\int \frac{\ln^6 x}{x}\,dx$

34. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

35. $\int s^{1/3}\cos(s^{4/3} - 8)\,ds$

36. $\int \frac{dx}{\sin^2 3x}$

37. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

38. $\int \frac{6\cos t}{(2+\sin t)^2}\,dt$

39. $\int \frac{dx}{x\ln x}$

40. $\int \tan^2 x \sec^2 x\,dx$

41. $\int \frac{x\,dx}{x^2+1}$

42. $\int \frac{40\,dx}{x^2+25}$

43. $\int \frac{dx}{\cot 3x}$

44. $\int \frac{dx}{\sqrt{5x+8}}$

45. $\int \sec x\,dx$ (*Hint:* Multiply the integrand by
$$\frac{\sec x + \tan x}{\sec x + \tan x}$$
and then use a substitution to integrate the result.)

46. $\int \csc x\,dx$ (*Hint:* Multiply the integrand by
$$\frac{\csc x + \cot x}{\csc x + \cot x}$$
and then use a substitution to integrate the result.)

In Exercises 47–52, use the given trigonometric identity to set up a u-substitution and then evaluate the indefinite integral.

47. $\int \sin^3 2x\,dx, \quad \sin^2 2x = 1 - \cos^2 2x$

48. $\int \sec^4 x\,dx, \quad \sec^2 x = 1 + \tan^2 x$

49. $\int 2\sin^2 x\,dx, \quad \cos 2x = 1 - 2(\sin(x))^2$

50. $\int 4\cos^2 x\,dx, \quad \cos 2x = 2(\cos x)^2 - 1$

51. $\int \tan^4 x\,dx, \quad \tan^2 x = \sec^2 x - 1$

52. $\int (\cos^4 x - \sin^4 x)dx, \quad \cos 2x = \cos^2 x - \sin^2 x$

In Exercises 53–66, make a u-substitution and integrate from $u(a)$ to $u(b)$.

53. $\int_0^3 \sqrt{y+1}\,dy$

54. $\int_0^1 r\sqrt{1-r^2}\,dr$

55. $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

56. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$

57. $\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\,d\theta$

58. $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$

59. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$

60. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

61. $\int_0^7 \frac{dx}{x+2}$

62. $\int_2^5 \frac{dx}{2x-3}$

63. $\int_1^2 \frac{dt}{t-3}$

64. $\int_{\pi/4}^{3\pi/4} \cot x\,dx$

65. $\int_{-1}^3 \frac{x\,dx}{x^2+1}$

66. $\int_0^2 \frac{e^x\,dx}{3+e^x}$

…o Routes to the Integral In Exercises 67 and 68, make a …stitution $u = \cdots$ (an expression in x), $du = \cdots$. Then

(a) integrate with respect to u from $u(a)$ to $u(b)$.

(b) find an antiderivative with respect to u, replace u by the expression in x, then evaluate from a to b.

67. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

68. $\int_{\pi/6}^{\pi/3} (1 - \cos 3x)\sin 3x\,dx$

69. Show that
$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5$$
is the solution to the initial value problem
$$\frac{dy}{dx} = \tan x, \quad f(3) = 5.$$
(See the discussion following Example 4, Section 5.4.)

70. Show that
$$y = \ln\left|\frac{\sin x}{\sin 2}\right| + 6$$
is the solution to the initial value problem
$$\frac{dy}{dx} = \cot x, \quad f(2) = 6.$$

…andardized Test Questions

You should solve the following problems without using a graphing calculator.

71. **True or False** By u-substitution, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx = \int_0^{\pi/4} u^3\,du$. Justify your answer.

72. **True or False** If f is positive and differentiable on $[a, b]$, then $\int_a^b \frac{f'(x)dx}{f(x)} = \ln\left(\frac{f(b)}{f(a)}\right)$. Justify your answer.

73. **Multiple Choice** $\int \tan x\,dx =$

(A) $\frac{\tan^2 x}{2} + C$

(B) $\ln|\cot x| + C$

(C) $\ln|\cos x| + C$

(D) $-\ln|\cos x| + C$

(E) $-\ln|\cot x| + C$

74. **Multiple Choice** $\int_0^2 e^{2x}\,dx =$

(A) $\frac{e^4}{2}$ (B) $e^4 - 1$ (C) $e^4 - 2$ (D) $2e^4 - 2$ (E) $\frac{e^4 - 1}{2}$

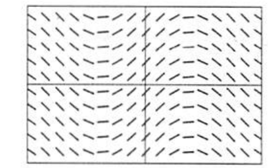
75. **Multiple Choice** If $\int_3^5 f(x-a)\,dx = 7$ where a is a constant, then $\int_{3-a}^{5-a} f(x)\,dx =$

(A) $7 + a$ (B) 7 (C) $7 - a$ (D) $a - 7$ (E) -7

76. **Multiple Choice** If the differential equation $dy/dx = f(x)$ leads to the slope field shown below, which of the following could be $\int f(x)\,dx$?

(A) $\sin x + C$ (B) $\cos x + C$ (C) $-\sin x + C$

(D) $-\cos x + C$ (E) $\frac{\sin^2 x}{2} + C$



Explorations

77. ***Constant of Integration*** Consider the integral
$$\int \sqrt{x+1}\,dx.$$

(a) Show that $\int \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{3/2} + C$.

(b) **Writing to Learn** Explain why
$$y_1 = \int_0^x \sqrt{t+1}\,dt \quad \text{and} \quad y_2 = \int_3^x \sqrt{t+1}\,dt$$
are antiderivatives of $\sqrt{x+1}$.

(c) Use a table of values for $y_1 - y_2$ to find the value of C fo… which $y_1 = y_2 + C$.

(d) **Writing to Learn** Give a convincing argument that
$$C = \int_0^3 \sqrt{x+1}\,dx.$$

78. **Group Activity *Making Connections*** Suppose that
$$\int f(x)\,dx = F(x) + C.$$

(a) Explain how you can use the derivative of $F(x) + C$ to confirm the integration is correct.

(b) Explain how you can use a slope field of f and the graph o… $y = F(x)$ to support your evaluation of the integral.

(c) Explain how you can use the graphs of $y_1 = F(x)$ and $y_2 = \int_0^x f(t)\,dt$ to support your evaluation of the integral.

(d) Explain how you can use a table of values for $y_1 - y_2$, y_1 and y_2 defined as in part (c), to support your evaluation of th… integral.

(e) Explain how you can use graphs of f and NDER of $F(x)$ to support your evaluation of the integral.

(f) Illustrate parts (a)–(e) for $f(x) = \frac{x}{\sqrt{x^2+1}}$.

79. ***Different Solutions?*** Consider the integral $\int 2 \sin x \cos x\, dx$.

(a) Evaluate the integral using the substitution $u = \sin x$.

(b) Evaluate the integral using the substitution $u = \cos x$.

(c) **Writing to Learn** Explain why the different-looking answers in parts (a) and (b) are actually equivalent.

80. ***Different Solutions?*** Consider the integral $\int 2 \sec^2 x \tan x\, dx$.

(a) Evaluate the integral using the substitution $u = \tan x$.

(b) Evaluate the integral using the substitution $u = \sec x$.

(c) **Writing to Learn** Explain why the different-looking answers in parts (a) and (b) are actually equivalent.

Extending the Ideas

81. ***Trigonometric Substitution*** Suppose $u = \sin^{-1} x$. Then $\cos u > 0$.

(a) Use the substitution $x = \sin u$, $dx = \cos u\, du$ to show that

$$\int \frac{dx}{\sqrt{1-x^2}} = \int 1\, du.$$

(b) Evaluate $\int 1\, du$ to show that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$

82. ***Trigonometric Substitution*** Suppose $u = \tan^{-1} x$.

(a) Use the substitution $x = \tan u$, $dx = \sec^2 u\, du$ to show that

$$\int \frac{dx}{1+x^2} = \int 1\, du.$$

(b) Evaluate $\int 1\, du$ to show that $\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$

83. ***Trigonometric Substitution*** Suppose $\sqrt{x} = \sin y$.

(a) Use the substitution $x = \sin^2 y$, $dx = 2 \sin y \cos y\, dy$ to show that

$$\int_0^{1/2} \frac{\sqrt{x}\, dx}{\sqrt{1-x}} = \int_0^{\pi/4} 2 \sin^2 y\, dy.$$

(b) Use the identity given in Exercise 49 to evaluate the definite integral without a calculator.

84. ***Trigonometric Substitution*** Suppose $u = \tan^{-1} x$.

(a) Use the substitution $x = \tan u$, $dx = \sec^2 u\, du$ to show that

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} = \int_0^{\pi/3} \sec u\, du.$$

(b) Use the hint in Exercise 45 to evaluate the definite integral without a calculator.

6.3 Antidifferentiation by Parts

What you'll learn about

- Product Rule in Integral Form
- Solving for the Unknown Integral
- Tabular Integration
- Inverse Trigonometric and Logarithmic Functions

. . . and why

The Product Rule relates to derivatives as the technique of parts relates to antiderivatives.

Product Rule in Integral Form

When u and v are differentiable functions of x, the Product Rule for differentiation tells us that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides with respect to x and rearranging leads to the integral equation

$$\int \left(u\frac{dv}{dx}\right) dx = \int \left(\frac{d}{dx}(uv)\right) dx - \int \left(v\frac{du}{dx}\right) dx$$

$$= uv - \int \left(v\frac{du}{dx}\right) dx.$$

When this equation is written in the simpler differential notation we obtain the following formula.

Integration by Parts Formula

$$\int u\, dv = uv - \int v\, du$$

This formula expresses one integral, $\int u\, dv$, in terms of a second integral, $\int v\, du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. This is the reason for the importance of the formula. When faced with an integral that we cannot handle analytically, we can replace it by one with which we might have more success.

LIPET

If you are wondering what to choose for u, here is what we usually do. Our first choice is a natural logarithm (L), if there is one. If there isn't, we look for an inverse trigonometric function (I). If there isn't one of these either, look for a polynomial (P). Barring that, look for an exponential (E) or a trigonometric function (T). That's the preference order: **L I P E T.**

In general, we want u to be something that simplifies when differentiated, and dv to be something that remains manageable when integrated.

EXAMPLE 1 Using Integration by Parts

Evaluate $\int x \cos x\, dx$.

SOLUTION

We use the formula $\int u\, dv = uv - \int v\, du$ with

$$u = x, \qquad dv = \cos x\, dx.$$

To complete the formula, we take the differential of u and find the simplest antiderivative of $\cos x$.

$$du = dx \qquad v = \sin x$$

Then,

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

Now try Exercise 1.

Let's examine the choices available for u and v in Example 1.

$$\int \sin^{-1} x\,dx = (\sin^{-1} x)(x) - \int (x)\left(\frac{1}{\sqrt{1-x^2}}\right) dx \qquad \int u\,dv = uv - \int v\,du$$

$$= x \sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= x \sin^{-1} x + \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} \qquad \text{Set up } u\text{-substitution.}$$

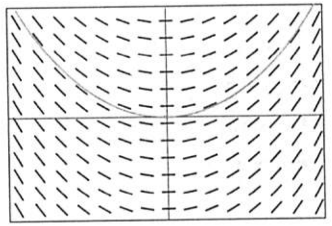
$$= x \sin^{-1} x + \frac{1}{2}\int u^{-1/2} du \qquad \text{Let } u = 1 - x^2,\ du = -2x\,dx.$$

$$= x \sin^{-1} x + u^{1/2} + C$$

$$= x \sin^{-1} x + \sqrt{1 - x^2} + C \qquad \text{Re-substitute.}$$

Applying the initial condition $y = 0$ when $x = 0$, we conclude that the particular solution is $y = x \sin^{-1} x + \sqrt{1 - x^2} - 1$.

A graph of $y = x \sin^{-1} x + \sqrt{1 - x^2} - 1$ conforms nicely to the slope field for $dy/dx = \sin^{-1} x$, as shown in Figure 6.10.



[−1, 1] by [−0.5, 0.5]

Figure 6.10 The solution to the initial value problem in Example 8 conforms nicely to the slope field of the differential equation. (Example 8)

Quick Review 6.3 *(For help, go to Sections 3.8 and 3.9.)*

In Exercises 1–4, find dy/dx.

1. $y = x^3 \sin 2x$
2. $y = e^{2x} \ln (3x + 1)$
3. $y = \tan^{-1} 2x$
4. $y = \sin^{-1} (x + 3)$

In Exercises 5 and 6, solve for x in terms of y.

5. $y = \tan^{-1} 3x$
6. $y = \cos^{-1} (x + 1)$
7. Find the area under the arch of the curve $y = \sin \pi x$ from $x = 0$ to $x = 1$.
8. Solve the differential equation $dy/dx = e^{2x}$.
9. Solve the initial value problem $dy/dx = x + \sin x$, $y(0) = 2$.
10. Use differentiation to confirm the integration formula

$$\int e^x \sin x\,dx = \frac{1}{2} e^x (\sin x - \cos x).$$

Section 6.3 Exercises

In Exercises 1–10, find the indefinite integral.

1. $\int x \sin x\,dx$
2. $\int x\,e^x\,dx$
3. $\int 3t\,e^{2t}\,dt$
4. $\int 2t \cos(3t)\,dt$
5. $\int x^2 \cos x\,dx$
6. $\int x^2 e^{-x}\,dx$
7. $\int 3x^2 e^{2x}\,dx$
8. $\int x^2 \cos\left(\frac{x}{2}\right) dx$
9. $\int y \ln y\,dy$
10. $\int t^2 \ln t\,dt$

In Exercises 11–16, solve the initial value problem. Confirm your answer by checking that it conforms to the slope field of the differential equation.

11. $\frac{dy}{dx} = (x + 2) \sin x$ and $y = 2$ when $x = 0$
12. $\frac{dy}{dx} = 2xe^{-x}$ and $y = 3$ when $x = 0$
13. $\frac{du}{dx} = x \sec^2 x$ and $u = 1$ when $x = 0$
14. $\frac{dz}{dx} = x^3 \ln x$ and $z = 5$ when $x = 1$
15. $\frac{dy}{dx} = x\sqrt{x - 1}$ and $y = 2$ when $x = 1$
16. $\frac{dy}{dx} = 2x\sqrt{x + 2}$ and $y = 0$ when $x = -1$

In Exercises 17–20, use parts and solve for the unknown integral.

17. $\int e^x \sin x\,dx$
18. $\int e^{-x} \cos x\,dx$
19. $\int e^x \cos 2x\,dx$
20. $\int e^{-x} \sin 2x\,dx$

In Exercises 21–24, use tabular integration to find the antiderivative.

21. $\int x^4 e^{-x}\,dx$
22. $\int (x^2 - 5x)e^x\,dx$
23. $\int x^3 e^{-2x}\,dx$
24. $\int x^3 \cos 2x\,dx$

In Exercises 25–28, evaluate the integral analytically. Support your answer using NINT.

25. $\int_0^{\pi/2} x^2 \sin 2x\,dx$
26. $\int_0^{\pi/2} x^3 \cos 2x\,dx$
27. $\int_{-2}^{3} e^{2x} \cos 3x\,dx$
28. $\int_{-3}^{2} e^{-2x} \sin 2x\,dx$

In Exercises 29–32, solve the differential equation.

29. $\frac{dy}{dx} = x^2 e^{4x}$
30. $\frac{dy}{dx} = x^2 \ln x$
31. $\frac{dy}{d\theta} = \theta \sec^{-1} \theta, \quad \theta > 1$
32. $\frac{dy}{d\theta} = \theta \sec \theta \tan \theta$

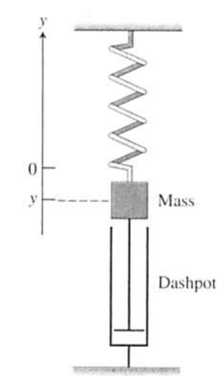
33. ***Finding Area*** Find the area of the region enclosed by the x-axis and the curve $y = x \sin x$ for
 (a) $0 \le x \le \pi$, (b) $\pi \le x \le 2\pi$, (c) $0 \le x \le 2\pi$.

34. ***Finding Area*** Find the area of the region enclosed by the y-axis and the curves $y = x^2$ and $y = (x^2 + x + 1)e^{-x}$.

35. ***Average Value*** A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

36. **True or False** If $f'(x) = g(x)$, then $\int x\,g(x)\,dx = x\,f(x) - \int f(x)\,dx$. Justify your answer.

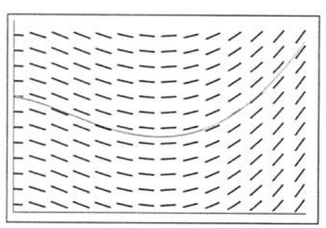
37. **True or False** If $f'(x) = g(x)$, then $\int x^2\,g(x)\,dx = x^2 f(x) - 2\int x\,f(x)\,dx$. Justify your answer.

38. **Multiple Choice** If $\int x^2 \cos x\,dx = h(x) - \int 2x \sin x\,dx$, then $h(x) =$
 (A) $2 \sin x + 2x \cos x + C$
 (B) $x^2 \sin x + C$
 (C) $2x \cos x - x^2 \sin x + C$
 (D) $4 \cos x - 2x \sin x + C$
 (E) $(2 - x^2) \cos x - 4 \sin x + C$

39. **Multiple Choice** $\int x \sin(5x)\,dx =$
 (A) $-x \cos (5x) + \sin (5x) + C$
 (B) $-\frac{x}{5}\cos (5x) + \frac{1}{25}\sin (5x) + C$
 (C) $-\frac{x}{5}\cos (5x) + \frac{1}{5}\sin (5x) + C$
 (D) $\frac{x}{5}\cos (5x) + \frac{1}{25}\sin (5x) + C$
 (E) $5x \cos (5x) - \sin (5x) + C$

40. **Multiple Choice** $\int x \csc^2 x\,dx =$
 (A) $\frac{x^2 \csc^3 x}{6} + C$
 (B) $x \cot x - \ln |\sin x| + C$
 (C) $-x \cot x + \ln |\sin x| + C$
 (D) $-x \cot x - \ln |\sin x| + C$
 (E) $-x \sec^2 x - \tan x + C$

41. **Multiple Choice** The graph of $y = f(x)$ conforms to the slope field for the differential equation $dy/dx = 4x \ln x$, as shown in the graph below. Which of the following could be $f(x)$?
 (A) $2x^2 (\ln x)^2 + 3$
 (B) $x^3 \ln x + 3$
 (C) $2x^2 \ln x - x^2 + 3$
 (D) $(2x^2 + 3) \ln x - 1$
 (E) $2x (\ln x)^2 - \frac{4}{3}(\ln x)^3 + 3$



[0, 2] by [0, 5]

Explorations

42. Consider the integral $\int x^n e^x\,dx$. Use integration by parts to evaluate the integral if

(a) $n = 1$.

(b) $n = 2$.

(c) $n = 3$.

(d) Conjecture the value of the integral for any positive integer n.

(e) **Writing to Learn** Give a convincing argument that your conjecture in part (d) is true.

In Exercises 43–46, evaluate the integral by using a substitution prior to integration by parts.

43. $\int \sin\sqrt{x}\,dx$ **44.** $\int e^{\sqrt{3x+9}}\,dx$

45. $\int x^7 e^{x^2}\,dx$ **46.** $\int \sin(\ln r)\,dr$

In Exercises 47–50, use integration by parts to establish the *reduction formula.*

47. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

48. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$

49. $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,dx,\quad a \neq 0$

50. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

Extending the Ideas

51. ***Integrating Inverse Functions*** Assume that the function f has an inverse.

(a) Show that $\int f^{-1}(x)\,dx = \int y f'(y)\,dy$. (*Hint:* Use the substitution $y = f^{-1}(x)$.)

(b) Use integration by parts on the second integral in part (a) to show that

$$\int f^{-1}(x)\,dx = \int y f'(y)\,dy = x f^{-1}(x) - \int f(y)\,dy.$$

52. ***Integrating Inverse Functions*** Assume that the function f has an inverse. Use integration by parts directly to show that

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx.$$

In Exercises 53–56, evaluate the integral using

(a) the technique of Exercise 51.

(b) the technique of Exercise 52.

(c) Show that the expressions (with $C = 0$) obtained in parts (a) and (b) are the same.

53. $\int \sin^{-1} x\,dx$

54. $\int \tan^{-1} x\,dx$

55. $\int \cos^{-1} x\,dx$

56. $\int \log_2 x\,dx$

Quick Quiz for AP* Preparation: Sections 6.1–6.3

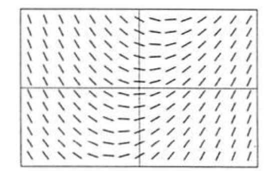
You should solve the following problems without using a graphing calculator.

1. Multiple Choice Which of the following differential equations would produce the slope field shown below?

(A) $\dfrac{dy}{dx} = y - 3x$ (B) $\dfrac{dy}{dx} = y - \dfrac{x}{3}$

(C) $\dfrac{dy}{dx} = y + \dfrac{x}{3}$ (D) $\dfrac{dy}{dx} = y + \dfrac{x}{3}$

(E) $\dfrac{dy}{dx} = x - \dfrac{y}{3}$



2. Multiple Choice If the substitution $\sqrt{x} = \sin y$ is made in the integrand of $\int_0^{1/2} \dfrac{\sqrt{x}}{\sqrt{1-x}}\,dx$, the resulting integral is

(A) $\int_0^{1/2} \sin^2 y\,dy$ (B) $2\int_0^{1/2} \dfrac{\sin^2 y}{\cos y}\,dy$

(C) $2\int_0^{\pi/4} \sin^2 y\,dy$ (D) $\int_0^{\pi/4} \sin^2 y\,dy$

(E) $2\int_0^{\pi/6} \sin^2 y\,dy$

3. Multiple Choice $\int x e^{2x}\,dx =$

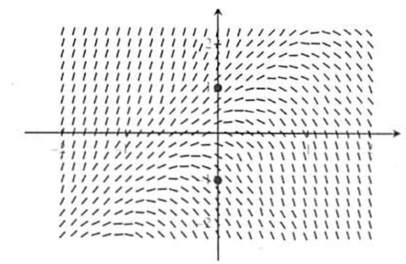
(A) $\dfrac{xe^{2x}}{2} - \dfrac{e^{2x}}{4} + C$ (B) $\dfrac{xe^{2x}}{2} - \dfrac{e^{2x}}{2} + C$

(C) $\dfrac{xe^{2x}}{2} + \dfrac{e^{2x}}{4} + C$ (D) $\dfrac{xe^{2x}}{2} + \dfrac{e^{2x}}{2} + C$

(E) $\dfrac{x^2 e^{2x}}{4} + C$

4. Free Response Consider the differential equation $dy/dx = 2y - 4x$.

(a) The slope field for the differential equation is shown below. Sketch the solution curve that passes through the point (0, 1) and sketch the solution curve that goes through the point (0, −1).



(b) There is a value of b for which $y = 2x + b$ is a solution to the differential equation. Find this value of b. Justify your answer.

(c) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. It appears from the slope field that g has a local maximum at the point (0, 0). Using the differential equation, prove analytically that this is so.

Table 6.1 Experimental Data

Time (sec)	T (°C)	$T - T_s$ (°C)
2	64.8	60.3
5	49.0	44.5
10	31.4	26.9
15	22.0	17.5
20	16.5	12.0
25	14.2	9.7
30	12.0	7.5

EXAMPLE 7 Using Newton's Law of Cooling

A temperature probe (thermometer) is removed from a cup of coffee and placed in water that has a temperature of $T_s = 4.5°C$. Temperature readings T, as recorded in Table 6.1, are taken after 2 sec, 5 sec, and every 5 sec thereafter. Estimate

(a) the coffee's temperature at the time the temperature probe was removed.

(b) the time when the temperature probe reading will be 8°C.

SOLUTION

Model According to Newton's Law of Cooling, $T - T_s = (T_0 - T_s)e^{-kt}$, where $T_s = 4.5$ and T_0 is the temperature of the coffee (probe reading) at $t = 0$.

We use exponential regression to find that

$$T - 4.5 = 61.66(0.9277^t)$$

is a model for the $(t, T - T_s) = (t, T - 4.5)$ data.

Thus,

$$T = 4.5 + 61.66(0.9277^t)$$

is a model for the (t, T) data.

Figure 6.12a shows the graph of the model superimposed on a scatter plot of the (t, T) data.

(a) At time $t = 0$, when the probe was removed, the temperature was

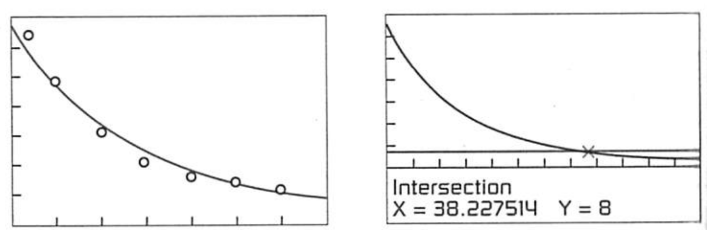
$$T = 4.5 + 61.66(0.9277^0) \approx 66.16°C.$$

(b) Solve Graphically Figure 6.12b shows that the graphs of

$$y = 8 \quad \text{and} \quad y = T = 4.5 + 61.66(0.9277^t)$$

intersect at about $t = 38$.

Interpret The temperature of the coffee was about 66.2°C when the temperature probe was removed. The temperature probe will reach 8°C about 38 sec after it is removed from the coffee and placed in the water.



[0, 35] by [0, 70]
(a)

[0, 60] by [−20, 70]
(b)

Figure 6.12 (Example 7)

Now try Exercise 31.

Quick Review 6.4

You should solve the following problems without using a graphing calculator.

In Exercises 1 and 2, rewrite the equation in exponential form or logarithmic form.

1. $\ln a = b$
2. $e^c = d$

In Exercises 3–8, solve the equation.

3. $\ln(x + 3) = 2$
4. $100e^{2x} = 600$
5. $0.85^x = 2.5$
6. $2^{k+1} = 3^k$
7. $1.1^t = 10$
8. $e^{-2t} = \frac{1}{4}$

In Exercises 9 and 10, solve for y.

9. $\ln(y + 1) = 2x - 3$
10. $\ln|y + 2| = 3t - 1$

Section 6.4 Exercises

In Exercises 1–10, use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

1. $\frac{dy}{dx} = \frac{x}{y}$ and $y = 2$ when $x = 1$
2. $\frac{dy}{dx} = -\frac{x}{y}$ and $y = 3$ when $x = 4$
3. $\frac{dy}{dx} = \frac{y}{x}$ and $y = 2$ when $x = 2$
4. $\frac{dy}{dx} = 2xy$ and $y = 3$ when $x = 0$
5. $\frac{dy}{dx} = (y + 5)(x + 2)$ and $y = 1$ when $x = 0$
6. $\frac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$
7. $\frac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$
8. $\frac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$
9. $\frac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$
10. $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$

In Exercises 11–14, find the solution of the differential equation $dy/dt = ky$, k a constant, that satisfies the given conditions.

11. $k = 1.5$, $y(0) = 100$
12. $k = -0.5$, $y(0) = 200$
13. $y(0) = 50$, $y(5) = 100$
14. $y(0) = 60$, $y(10) = 30$

In Exercises 15–18, complete the table for an investment if interest is compounded continuously.

	Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
15.	1000	8.6		
16.	2000		15	
17.		5.25		2898.44
18.	1200			10,405.37

In Exercises 19 and 20, find the amount of time required for a $2000 investment to double if the annual interest rate r is compounded **(a)** annually, **(b)** monthly, **(c)** quarterly, and **(d)** continuously.

19. $r = 4.75\%$
20. $r = 8.25\%$

21. ***Half-Life*** The radioactive decay of Sm-151 (an isotope of samarium) can be modeled by the differential equation $dy/dt = -0.0077y$, where t is measured in years. Find the half-life of Sm-151.

22. ***Half-Life*** An isotope of neptunium (Np-240) has a half-life of 65 minutes. If the decay of Np-240 is modeled by the differential equation $dy/dt = -ky$, where t is measured in minutes, what is the decay constant k?

23. ***Growth of Cholera Bacteria*** Suppose that the cholera bacteria in a colony grows unchecked according to the Law of Exponential Change. The colony starts with 1 bacterium and doubles in number every half hour.

    (a) How many bacteria will the colony contain at the end of 24 h?

    (b) Writing to Learn Use part (a) to explain why a person who feels well in the morning may be dangerously ill by evening even though, in an infected person, many bacteria are destroyed.

24. ***Bacteria Growth*** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

25. ***Radon-222*** The decay equation for radon-222 gas is known to be $y = y_0e^{-0.18t}$, with t in days. About how long will it take the amount of radon in a sealed sample of air to decay to 90% of its original value?

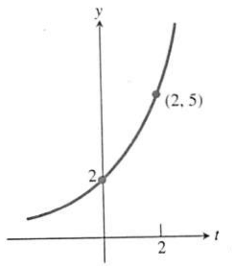
26. ***Polonium-210*** The number of radioactive atoms remaining after t days in a sample of polonium-210 that starts with y_0 radioactive atoms is $y = y_0e^{-0.005t}$.

    (a) Find the element's half-life.

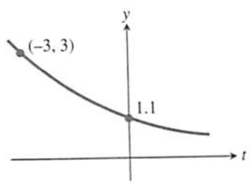
(b) Your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives have disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

In Exercises 27 and 28, find the exponential function $y = y_0 e^{kt}$ whose graph passes through the two points.

27.



28.



29. ***Mean Life of Radioactive Nuclei*** Physicists using the radioactive decay equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon-222 nucleus is about $1/0.18 \approx 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in any sample will disintegrate within three mean lifetimes, that is, by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

30. ***Finding the Original Temperature of a Beam*** An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min its temperature was 50°F. Use Newton's Law of Cooling to estimate the beam's initial temperature.

31. ***Cooling Soup*** Suppose that a cup of soup cooled from 90°C to 60°C in 10 min in a room whose temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.

(a) How much longer would it take the soup to cool to 35°C?

(b) Instead of being left to stand in the room, the cup of 90°C soup is put into a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

32. ***Cooling Silver*** The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

(a) 15 minutes from now?

(b) 2 hours from now?

(c) When will the silver be 10°C above room temperature?

33. ***Temperature Experiment*** A temperature probe is removed from a cup of coffee and placed in water whose temperature (T_s) is 10°C. The data in Table 6.2 were collected over the next 30 sec with a CBL™ temperature probe.

Table 6.2 Experimental Data

Time (sec)	T (°C)	$T - T_s$ (°C)
2	80.47	70.47
5	69.39	59.39
10	49.66	39.66
15	35.26	25.26
20	28.15	18.15
25	23.56	13.56
30	20.62	10.62

(a) Find an exponential regression equation for the $(t, T - T_s)$ data.

(b) Use the regression equation in part (a) to find a model for the (t, T) data. Superimpose the graph of the model on a scatter plot of the (t, T) data.

(c) Estimate when the temperature probe will read 12°C.

(d) Estimate the coffee's temperature when the temperature probe was removed.

34. ***A Very Cool Experiment*** A temperature probe is removed from a cup of hot chocolate and placed in ice water (temperature $T_s = 0°C$). The data in Table 6.3 were collected over the next 30 seconds.

Table 6.3 Experimental Data

Time (sec)	Temperature (°C)
2	74.68
5	61.99
10	34.89
15	21.95
20	15.36
25	11.89
30	10.02

(a) **Writing to Learn** Explain why temperature in this experiment can be modeled as an exponential function of time.

(b) Use exponential regression to find the best exponential model. Superimpose a graph of the model on a scatter plot of the *(time, temperature)* data.

(c) Estimate when the probe will reach 5°C.

(d) Estimate the temperature of the hot chocolate when the probe was removed.

35. ***Dating Crater Lake*** The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

Carbon-14 Dating Measurement Sensitivity To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, answer the following questions about this hypothetical situation.

(a) A fossilized bone found in central Illinois in the year A.D. 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

(b) Repeat part (a) assuming 18% instead of 17%.

(c) Repeat part (a) assuming 16% instead of 17%.

What is the half-life of a substance that decays to 1/3 of its original radioactive amount in 5 years?

A savings account earning compound interest triples in value in 10 years. How long will it take for the original investment to quadruple?

The Inversion of Sugar The processing of raw sugar has an "inversion" step that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 h, how much raw sugar will remain after another 14 h?

Oil Depletion Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present level?

Atmospheric Pressure Earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 lb/in^2) and that the pressure at an altitude of 20 km is 90 millibars.

(a) Solve the initial value problem

Differential equation: $\dfrac{dp}{dh} = kp$,

Initial condition: $p = p_0$ when $h = 0$,

to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

(b) What is the atmospheric pressure at $h = 50$ km?

(c) At what altitude does the pressure equal 900 millibars?

First Order Chemical Reactions In some chemical reactions the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when y is measured in grams and t is measured in hours. If there are 100 grams of a δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

Discharging Capacitor Voltage Suppose that electricity is draining from a capacitor at a rate proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

(a) Solve this differential equation for V, using V_0 to denote the value of V when $t = 0$.

(b) How long will it take the voltage to drop to 10% of its original value?

44. ***John Napier's Answer*** John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question, "What happens if you invest an amount of money at 100% yearly interest, compounded continuously?"

(a) **Writing to Learn** What does happen? Explain.

(b) How long does it take to triple your money?

(c) **Writing to Learn** How much can you earn in a year?

45. ***Benjamin Franklin's Will*** The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> … with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. … If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. … The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. … At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the receipt of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

(a) What annual rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

(b) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131?

46. ***Rules of 70 and 72*** The rules state that it takes about $70/i$ or $72/i$ years for money to double at i percent, compounded continuously, using whichever of 70 or 72 is easier to divide by i.

(a) Show that it takes $t = (\ln 2)/r$ years for money to double if it is invested at annual interest rate r (in decimal form) compounded continuously.

(b) Graph the functions

$$y_1 = \frac{\ln 2}{r}, \quad y_2 = \frac{70}{100r}, \quad \text{and} \quad y_3 = \frac{72}{100r}$$

in the $[0, 0.1]$ by $[0, 100]$ viewing window.

(c) Writing to Learn Explain why these two rules of thumb for mental computation are reasonable.

(d) Use the rules to estimate how long it takes to double money at 5% compounded continuously.

(e) Invent a rule for estimating the number of years needed to triple your money.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

47. True or False If $dy/dx = ky$, then $y = e^{kx} + C$. Justify your answer.

48. True or False The general solution to $dy/dt = 2y$ can be written in the form $y = C(3^{kt})$ for some constants C and k. Justify your answer.

49. Multiple Choice A bank account earning continuously compounded interest doubles in value in 7.0 years. At the same interest rate, how long would it take the value of the account to triple?

(A) 4.4 years (B) 9.8 years (C) 10.5 years
(D) 11.1 years (E) 21.0 years

50. Multiple Choice A sample of Ce-143 (an isotope of cerium) loses 99% of its radioactive matter in 199 hours. What is the half-life of Ce-143?

(A) 4 hours (B) 6 hours (C) 30 hours
(D) 100.5 hours (E) 143 hours

51. Multiple Choice In which of the following models is dy/dt directly proportional to y?

I. $y = e^{kt} + C$
II. $y = Ce^{kt}$
III. $y = 2^{8kt}$

(A) I only (B) II only (C) I and II only
(D) II and III only (E) I, II, and III

52. Multiple Choice An apple pie comes out of the oven at 425°F and is placed on a counter in a 68°F room to cool. In 30 minutes it has cooled to 195°F. According to Newton's Law of Cooling, how many additional minutes must pass before it cools to 100°F?

(A) 12.4 (B) 15.4 (C) 25.0 (D) 35.0 (E) 40.0

Explorations

53. ***Resistance Proportional to Velocity*** It is reasonable to assume that the air resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The resisting force on an object of mass m moving with velocity v is thus $-kv$ for some positive constant k.

(a) Use the law *Force* = *Mass* × *Acceleration* to show that the velocity of an object slowed by air resistance (and no other forces) satisfies the differential equation

$$m\frac{dy}{dt} = -kv.$$

(b) Solve the differential equation to show that $v = v_0 e^{-(k/m)t}$, where v_0 is the velocity of the object at time $t = 0$.

(c) If k is the same for two objects of different masses, which one will slow to half its starting velocity in the shortest time? Justify your answer.

54. ***Coasting to a Stop*** Assume that the resistance encountered by a moving object is proportional to the object's velocity so that its velocity is $v = v_0 e^{-(k/m)t}$.

(a) Integrate the velocity function with respect to t to obtain the distance function s. Assume that $s(0) = 0$ and show that

$$s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right).$$

(b) Show that the total coasting distance traveled by the object as it coasts to a complete stop is $v_0 m/k$.

55. ***Coasting to a Stop*** Table 6.4 shows the distance s (meters) coasted on in-line skates in terms of time t (seconds) by Kelly Schmitzer. Find a model for her position in the form given in Exercise 54(a) and superimpose its graph on a scatter plot of the data. Her initial velocity was $v_0 = 0.80$ m/sec, her mass $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m.

Table 6.4 Kelly Schmitzer Skating Data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	1.5	0.89	3.1	1.30
0.1	0.07	1.7	0.97	3.3	1.31
0.3	0.22	1.9	1.05	3.5	1.32
0.5	0.36	2.1	1.11	3.7	1.32
0.7	0.49	2.3	1.17	3.9	1.32
0.9	0.60	2.5	1.22	4.1	1.32
1.1	0.71	2.7	1.25	4.3	1.32
1.3	0.81	2.9	1.28	4.5	1.32

Source: Valerie Sharrits, St. Francis de Sales H.S., Columbus, OH.

56. ***Coasting to a Stop*** Table 6.5 shows the distance s (meters) coasted on in-line skates in t seconds by Johnathon Krueger. Find a model for his position in the form given in Exercise 54(a) and superimpose its graph on a scatter plot of the data. His initial velocity was $v_0 = 0.86$ m/sec, his mass $m = 30.84$ kg (he weighed 68 lb), and his total coasting distance 0.97 m.

Table 6.5 Johnathon Krueger Skating Data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	0.93	0.61	1.86	0.93
0.13	0.08	1.06	0.68	2.00	0.94
0.27	0.19	1.20	0.74	2.13	0.95
0.40	0.28	1.33	0.79	2.26	0.96
0.53	0.36	1.46	0.83	2.39	0.96
0.67	0.45	1.60	0.87	2.53	0.97
0.80	0.53	1.73	0.90	2.66	0.97

Source: Valerie Sharrits, St. Francis de Sales H.S., Columbus, OH.

Extending the Ideas

57. ***Continuously Compounded Interest***

(a) Use tables to give a numerical argument that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

Support your argument graphically.

(b) For several different values of r, give numerical and graphical evidence that

$$\lim_{x\to\infty}\left(1 + \frac{r}{x}\right)^x = e^r.$$

(c) Writing to Learn Explain why compounding interest over smaller and smaller periods of time leads to the concept of interest compounded continuously.

58. ***Skydiving*** If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity $v(t)$ is modeled by the initial value problem

Differential equation: $m\frac{dv}{dt} = mg - kv^2$,

Initial condition: $v(0) = 0$,

where t represents time in seconds, g is the acceleration due to gravity, and k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that variation in the air's density will not affect the outcome.)

(a) Show that the function

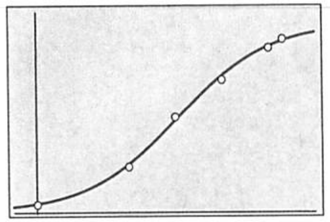
$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{e^{at} - e^{-at}}{e^{at} + e^{-at}},$$

where $a = \sqrt{gk/m}$, is a solution of the initial value problem.

(b) Find the body's limiting velocity, $\lim_{t\to\infty} v(t)$.

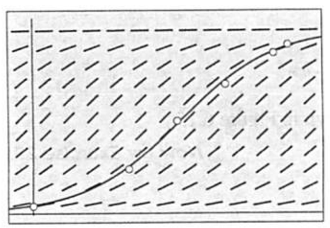
(c) For a 160-lb skydiver ($mg = 160$), and with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity in feet per second? in miles per hour?

Skydivers can vary their limiting velocities by changing the amount of body area opposing the fall. Their velocities can vary from 94 to 321 miles per hour.



[−5, 60] by [−3600, 337000]

Figure 6.16 The logistic regression curve fitted to the data for population growth in Aurora, CO from 1950 to 2003. (Example 6)



[−5, 60] by [−3600, 337000]

Figure 6.17 The slope field for the differential equation derived from the regression curve fits the data and the regression curve nicely. (Example 6)

Graphing Calculator Logistics

Unfortunately, some graphing calculators allow for a *vertical shift* when fitting a logistic curve to a set of data points. The regression equation for such a curve would have the form

$$y = \frac{c}{1 + ae^{-bx}} + d.$$

While the curve might fit the data better, this function cannot be a solution to the logistic differential equation if d is not zero. Since our definition of a logistic function begins with the differential equation, we will consistently use only logistic regression equations of the form

$$y = \frac{c}{1 + ae^{-bx}}.$$

EXAMPLE 6 Using Logistic Regression

Table 6.6 shows the population of Aurora, CO for selected years between 1950 and 2003.

Table 6.6 Population of Aurora, CO

Years after 1950	Population
0	11,421
20	74,974
30	158,588
40	222,103
50	275,923
53	290,418

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a logistic curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what will the Aurora population approach in the long run?

(c) Based on the regression equation, when will the population of Aurora first exceed 300,000 people?

(d) Write a logistic differential equation in the form $dP/dt = kP(M - P)$ that models the growth of the Aurora data in Table 6.6.

SOLUTION

(a) The regression equation is

$$P = \frac{316440.7}{1 + 23.577e^{-0.1026t}}.$$

The graph is shown superimposed on the scatter plot in Figure 6.16. The fit is almost perfect.

(b) Approximately 316,441 people. (The carrying capacity is the numerator of the regression equation.)

(c) Set

$$\frac{316440.7}{1 + 23.577e^{-0.1026t}} = 300{,}000.$$

The regression line crosses the 300,000 mark sometime in the 59th year, that is, in 2009.

(d) We see from the regression equation that $M = 316440.7$ and $Mk = 0.1026$. Therefore $k \approx 3.24 \times 10^{-7}$. The logistic growth model is

$$\frac{dP}{dt} = (3.24 \times 10^{-7})\,P(316440.7 - P).$$

Figure 6.17 shows the slope field for this differential equation superimposed on the scatter plot and the regression equation. ***Now try Exercise***

We caution readers once again not to assume that logistic models work perfectly in real-world, population-growth problems; there are too many unpredictable variables that can and will change the growth conditions over time.

Quick Review 6.5 *(For help, go to Sections 2.2 and 2.3.)*

In Exercises 1–4, use the polynomial division algorithm (as in Example 2 of this section) to write the rational function in the form $Q(x) + \frac{R(x)}{D(x)}$, where the degree of R is less than the degree of D.

1. $\frac{x^2}{x - 1}$
2. $\frac{x^2}{x^2 - 4}$
3. $\frac{x^2 + x + 1}{x^2 + x - 2}$
4. $\frac{x^3 - 5}{x^2 - 1}$

In Exercises 5–10, let $f(x) = \frac{60}{1 + 5e^{-0.1x}}$.

5. Find where f is continuous.
6. Find $\lim_{x\to\infty} f(x)$.
7. Find $\lim_{x\to-\infty} f(x)$.
8. Find the y-intercept of the graph of f.
9. Find all horizontal asymptotes of the graph of f.
10. Draw the graph of $y = f(x)$.

Section 6.5 Exercises

In Exercises 1–4, find the values of A and B that complete the partial fraction decomposition.

1. $\frac{x - 12}{x^2 - 4x} = \frac{A}{x} + \frac{B}{x - 4}$
2. $\frac{2x + 16}{x^2 + x - 6} = \frac{A}{x + 3} + \frac{B}{x - 2}$
3. $\frac{16 - x}{x^2 + 3x - 10} = \frac{A}{x - 2} + \frac{B}{x + 5}$
4. $\frac{3}{x^2 - 9} = \frac{A}{x - 3} + \frac{B}{x + 3}$

In Exercises 5–14, evaluate the integral.

5. $\int \frac{x - 12}{x^2 - 4x}\,dx$
6. $\int \frac{2x + 16}{x^2 + x - 6}\,dx$
7. $\int \frac{2x^3}{x^2 - 4}\,dx$
8. $\int \frac{x^2 - 6}{x^2 - 9}\,dx$
9. $\int \frac{2\,dx}{x^2 + 1}$
10. $\int \frac{3\,dx}{x^2 + 9}$
11. $\int \frac{7\,dx}{2x^2 - 5x - 3}$
12. $\int \frac{1 - 3x}{3x^2 - 5x + 2}\,dx$
13. $\int \frac{8x - 7}{2x^2 - x - 3}\,dx$
14. $\int \frac{5x + 14}{x^2 + 7x}\,dx$

In Exercises 15–18, solve the differential equation.

15. $\frac{dy}{dx} = \frac{2x - 6}{x^2 - 2x}$
16. $\frac{du}{dx} = \frac{2}{x^2 - 1}$
17. $F'(x) = \frac{2}{x^3 - x}$
18. $G'(t) = \frac{2t^3}{t^3 - t}$

In Exercises 19–22, find the integral *without* using the technique of partial fractions.

19. $\int \frac{2x}{x^2 - 4}\,dx$
20. $\int \frac{4x - 3}{2x^2 - 3x + 1}\,dx$
21. $\int \frac{x^2 + x - 1}{x^2 - x}\,dx$
22. $\int \frac{2x^3}{x^2 - 1}\,dx$

In Exercises 23–26, the logistic equation describes the growth of a population P, where t is measured in years. In each case, find **(a)** the carrying capacity of the population, **(b)** the size of the population when it is growing the fastest, and **(c)** the rate at which the population is growing when it is growing the fastest.

23. $\frac{dP}{dt} = 0.006P(200 - P)$
24. $\frac{dP}{dt} = 0.0008P(700 - P)$
25. $\frac{dP}{dt} = 0.0002P(1200 - P)$
26. $\frac{dP}{dt} = 10^{-5}P(5000 - P)$

In Exercises 27–30, solve the initial value problem using partial fractions. Use a graphing utility to generate a slope field for the differential equation and verify that the solution conforms to the slope field.

27. $\frac{dP}{dt} = 0.006P(200 - P)$ and $P = 8$ when $t = 0$.
28. $\frac{dP}{dt} = 0.0008P(700 - P)$ and $P = 10$ when $t = 0$.
29. $\frac{dP}{dt} = 0.0002P(1200 - P)$ and $P = 20$ when $t = 0$.
30. $\frac{dP}{dt} = 10^{-5}P(5000 - P)$ and $P = 50$ when $t = 0$.

In Exercises 31 and 32, a population function is given.

(a) Show that the function is a solution of a logistic differential equation. Identify k and the carrying capacity.

(b) Writing to Learn Estimate $P(0)$. Explain its meaning in the context of the problem.

31. ***Rabbit Population*** A population of rabbits is given by the formula

$$P(t) = \frac{1000}{1 + e^{4.8 - 0.7t}},$$

where t is the number of months after a few rabbits are released.

32. ***Spread of Measles*** The number of students infected by measles in a certain school is given by the formula

$$P(t) = \frac{200}{1 + e^{5.3 - t}},$$

where t is the number of days after students are first exposed to an infected student.

33. *Guppy Population* A 2000-gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0015P(150 - P),$$

where time t is in weeks.

(a) Find a formula for the guppy population in terms of t.

(b) How long will it take for the guppy population to be 100? 125?

34. *Gorilla Population* A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0004P(250 - P),$$

where time t is in years.

(a) Find a formula for the gorilla population in terms of t.

(b) How long will it take for the gorilla population to reach the carrying capacity of the preserve?

35. *Logistic Differential Equation* Show that the solution of the differential equation

$$\frac{dP}{dt} = kP(M - P) \quad \text{is} \quad P = \frac{M}{1 + Ae^{-Mkt}},$$

where A is a constant determined by an appropriate initial condition.

36. *Limited Growth Equation* Another differential equation that models limited growth of a population P in an environment with carrying capacity M is $dP/dt = k(M - P)$ (where $k > 0$ and $M > 0$).

(a) Show that $P = M - Ae^{-kt}$, where A is a constant determined by an appropriate initial condition.

(b) What is $\lim_{t\to\infty} P(t)$?

(c) For what time $t \geq 0$ is the population growing the fastest?

(d) Writing to Learn How does the growth curve in this model differ from the growth curve in the logistic model?

37. *Population Growth* Table 6.7 shows the population of Laredo, Texas for selected years between 1950 and 2003.

Table 6.7 Population of Laredo, TX

Years after 1950	Population
0	10,571
20	81,437
30	138,857
40	180,650
50	215,794
53	218,027

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data a superimpose it on a scatter plot of population against years aft 1950.

(b) Based on the regression equation, what number will the Laredo population approach in the long run?

(c) Based on the regression equation, when will the Laredo po ulation first exceed 225,000 people?

(d) Write a logistic differential equation in the form $dP/dt =$ $kP(M - P)$ that models the growth of the Laredo data in Table 6.7.

38. *Population Growth* Table 6.8 shows the population of Virginia Beach, VA for selected years between 1950 and 2003.

Table 6.8 Population of Virginia Beach

Years after 1950	Population
0	5,390
20	172,106
30	262,199
40	393,069
50	425,257
53	439,467

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data an superimpose it on a scatter plot of population against years aft 1950.

(b) Based on the regression equation, what number will the Vir ginia Beach population approach in the long run?

(c) Based on the regression equation, when will the Virginia Beach population first exceed 450,000 people?

(d) Write a logistic differential equation in the form $dP/dt =$ $kP(M - P)$ that models the growth of the Virginia Beach data in Table 6.8.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

39. True or False For small values of t, the solution to logistic differential equation $dP/dt = kP(100 - P)$ that passes through the point (0, 10) resembles the solution to the differential equa tion $dP/dt = kP$ that passes through the point (0, 10). Justify your answer.

40. True or False The graph of any solution to the differential equation $dP/dt = kP(100 - P)$ has asymptotes $y = 0$ and $y = 100$. Justify your answer.

41. Multiple Choice The spread of a disease through a commu nity can be modeled with the logistic equation

$$\frac{dy}{dt} = \frac{600}{1 + 59e^{-0.1t}},$$

where y is the number of people infected after t days. How ma people are infected when the disease is spreading the fastest?

(A) 10 (B) 59 (C) 60 (D) 300 (E) 600

Multiple Choice The spread of a disease through a commu- nity can be modeled with the logistic equation

$$\frac{dy}{dt} = \frac{0.9}{1 + 45e^{-0.15t}},$$

where y is the proportion of people infected after t days. According to the model, what percentage of the people in the commu- nity will not become infected?

(A) 2% (B) 10% (C) 15% (D) 45% (E) 90%

Multiple Choice $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx =$

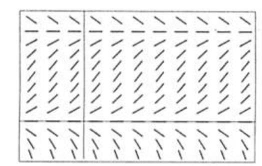
(A) $-\frac{33}{20}$ (B) $-\frac{9}{20}$ (C) $\ln\left(\frac{5}{2}\right)$ (D) $\ln\left(\frac{8}{5}\right)$ (E) $\ln\left(\frac{2}{5}\right)$

Multiple Choice Which of the following differential equa- tions would produce the slope field shown below?

(A) $\frac{dy}{dx} = 0.01x(120 - x)$ (B) $\frac{dy}{dx} = 0.01y(120 - y)$

(C) $\frac{dy}{dx} = 0.01y(100 - x)$ (D) $\frac{dy}{dx} = \frac{120}{1 + 60e^{-1.2x}}$

(E) $\frac{dy}{dx} = \frac{120}{1 + 60e^{-1.2y}}$



[−3, 8] by [−50, 150]

Explorations

Extinct Populations One theory states that if the size of a population falls below a minimum m, the population will become extinct. This condition leads to the *extended* logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right) = \frac{k}{M}(M - P)(P - m),$$

with $k > 0$ the proportionality constant and M the population maximum.

(a) Show that dP/dt is positive for $m < P < M$ and negative if $P < m$ or $P > M$.

(b) Let $m = 100$, $M = 1200$, and assume that $m < P < M$. Show that the differential equation can be rewritten in the form

$$\left[\frac{1}{1200 - P} + \frac{1}{P - 100}\right]\frac{dP}{dt} = \frac{11}{12}k.$$

Use a procedure similar to that used in Example 5 in Section 6.5 to solve this differential equation.

(c) Find the solution to part (b) that satisfies $P(0) = 300$.

(d) Superimpose the graph of the solution in part (c) with $k = 0.1$ on a slope field of the differential equation.

(e) Solve the general extended differential equation with the restriction $m < P < M$.

46. *Integral Tables* Antiderivatives of various generic functions can be found as formulas in *integral tables*. See if you can derive the formulas that would appear in an integral table for the following functions. (Here, a is an arbitrary constant.)

(a) $\int \frac{dx}{a^2 + x^2}$ **(b)** $\int \frac{dx}{a^2 - x^2}$ **(c)** $\int \frac{dx}{(a + x)^2}$

Extending the Ideas

47. *Partial Fractions with Repeated Linear Factors*

If

$$f(x) = \frac{P(x)}{(x - r)^m}$$

is a rational function with the degree of P less than m, then the partial fraction decomposition of f is

$$f(x) = \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

For example,

$$\frac{4x}{(x - 2)^2} = \frac{4}{x - 2} + \frac{8}{(x - 2)^2}.$$

Use partial fractions to find the following integrals:

(a) $\int \frac{5x}{(x + 3)^2}\,dx$

(b) $\int \frac{5x}{(x + 3)^3}\,dx$ (*Hint:* Use part (a).)

48. *More on Repeated Linear Factors* The Heaviside Method is not very effective at finding the unknown numerators for partial fraction decompositions with repeated linear factors, but here is another way to find them.

(a) If $\frac{x^2 + 3x + 5}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}$, show that $A(x - 1)^2 + B(x - 1) + C = x^2 + 3x + 5.$

(b) Expand and equate coefficients of like terms to show that $A = 1$, $-2A + B = 3$, and $A - B + C = 5$. Then find A, B, and C.

(c) Use partial fractions to evaluate $\int \frac{x^2 + 3x + 5}{(x - 1)^3}\,dx.$

Quick Quiz: Sections 6.4 and 6.5

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** The rate at which acreage is being consumed by a plot of kudzu is proportional to the number of acres already consumed at time t. If there are 2 acres consumed when $t = 1$ and 3 acres consumed when $t = 5$, how many acres will be consumed when $t = 8$?

   (A) 3.750 (B) 4.000 (C) 4.066 (D) 4.132 (E) 4.600

2. **Multiple Choice** Let $F(x)$ be an antiderivative of $\cos(x^2)$. If $F(1) = 0$, then $F(5) =$

   (A) −0.099 (B) −0.153 (C) −0.293 (D) −0.992 (E) −1.833

3. **Multiple Choice** $\int \frac{dx}{(x-1)(x+3)} =$

   (A) $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$ (B) $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right| + C$

   (C) $\frac{1}{2}\ln|(x-1)(x+3)| + C$ (D) $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right| + C$

   (E) $\ln|(x-1)(x+3)| + C$

4. **Free Response** A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{10}\right).$$

   (a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

   (b) If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

   (c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{10}\right).$$

   Find $Y(t)$ if $Y(0) = 3$.

   (d) For the function Y found in part (c), what is $\lim_{t\to\infty} Y(t)$?

Chapter 6 Key Terms

antidifferentiation by parts (p. 341)
antidifferentiation by substitution (p. 334)
arbitrary constant of integration (p. 331)
carbon-14 dating (p. 354)
carrying capacity (p. 367)
compounded continuously (p. 352)
constant of integration (p. 331)
continuous interest rate (p. 352)
decay constant (p. 351)
differential equation (p.321)
direction field (p. 323)
Euler's Method (p. 325)
evaluate an integral (p. 331)
exact differential equation (p. 321)
general solution to a differential equation (p. 321)
growth constant (p. 351)
first-order differential equation (p. 321)
first-order linear differential equation (p. 324)
graphical solution of a differential equation (p. 322)
half-life (p. 352)
Heaviside method (p. 363)
indefinite integral (p. 331)
initial condition (p. 321)
initial value problem (p. 321)
integral sign (p. 331)
integrand (p. 331)
integration by parts (p. 341)
Law of Exponential Change (p. 351)
Leibniz notation for integrals (p. 333)
logistic differential equation (p. 365)
logistic growth model (p. 367)
logistic regression (p. 365)
Newton's Law of Cooling (p. 354)
numerical method (p. 327)
numerical solution of a differential equation (p. 327)
order of a differential equation (p. 321)
partial fraction decomposition (p. 363)
particular solution (p. 321)
proper rational function (p. 363)
properties of indefinite integrals (p. 332)
radioactive (p. 352)
radioactive decay (p. 352)
resistance proportional to velocity (p. 360)
second-order differential equation (p. 330)
separable differential equations (p. 350)
separation of variables (p. 350)
slope field (p. 323)
solution of a differential equation (p. 321)
substitution in definite integrals (p. 336)
tabular integration (p. 344)
variable of integration (p. 331)

Chapter 6 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–10, evaluate the integral analytically. Then use NINT to support your result.

1. $\int_0^{\pi/3} \sec^2\theta\, d\theta$
2. $\int_1^2 \left(x + \frac{1}{x^2}\right) dx$
3. $\int_0^1 \frac{36\, dx}{(2x+1)^3}$
4. $\int_{-1}^1 2x\sin(1 - x^2)\, dx$
5. $\int_0^{\pi/2} 5\sin^{3/2} x\cos x\, dx$
6. $\int_{1/2}^4 \frac{x^2 + 3x}{x}\, dx$
7. $\int_0^{\pi/4} e^{\tan x}\sec^2 x\, dx$
8. $\int_1^e \frac{\sqrt{\ln r}}{r}\, dr$
9. $\int_0^1 \frac{x}{x^2 + 5x + 6}\, dx$
10. $\int_1^2 \frac{2x+6}{x^2 - 3x}\, dx$

In Exercises 11–24, evaluate the integral.

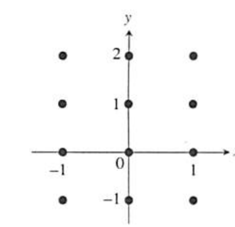
11. $\int \frac{\cos x}{2 - \sin x}\, dx$
12. $\int \frac{dx}{\sqrt[3]{3x+4}}$
13. $\int \frac{t\, dt}{t^2 + 5}$
14. $\int \frac{1}{\theta^2}\sec\frac{1}{\theta}\tan\frac{1}{\theta}\, d\theta$
15. $\int \frac{\tan(\ln y)}{y}\, dy$
16. $\int e^x \sec(e^x)\, dx$
17. $\int \frac{dx}{x\ln x}$
18. $\int \frac{dt}{t\sqrt{t}}$
19. $\int x^3\cos x\, dx$
20. $\int x^4\ln x\, dx$
21. $\int e^{3x}\sin x\, dx$
22. $\int x^2 e^{-3x}\, dx$
23. $\int \frac{25}{x^2 - 25}\, dx$
24. $\int \frac{5x+2}{2x^2 + x - 1}\, dx$

In Exercises 25–34, solve the initial value problem analytically. Support your solution by overlaying its graph on a slope field of the differential equation.

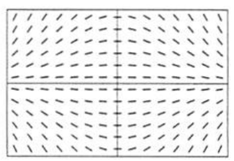
25. $\frac{dy}{dx} = 1 + x + \frac{x^2}{2}, \quad y(0) = 1$
26. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$
27. $\frac{dy}{dt} = \frac{1}{t+4}, \quad y(-3) = 2$
28. $\frac{dy}{d\theta} = \csc 2\theta\cot 2\theta, \quad y(\pi/4) = 1$
29. $\frac{d^2y}{dx^2} = 2x - \frac{1}{x^2}, \quad x > 0, \quad y'(1) = 1, \quad y(1) = 0$
30. $\frac{d^3r}{dt^3} = -\cos t, \quad r''(0) = r'(0) = r(0) = -1$
31. $\frac{dy}{dx} = y + 2, \quad y(0) = 2$
32. $\frac{dy}{dx} = (2x+1)(y+1), \quad y(-1) = 1$
33. $\frac{dy}{dt} = y(1 - y), \quad y(0) = 0.1$
34. $\frac{dy}{dx} = 0.001y(100 - y), \quad y(0) = 5$
35. Find an integral equation $y = \int_a^x f(t)dt$ such that $dy/dx = \sin^3 x$ and $y = 5$ when $x = 4$.
36. Find an integral equation $y = \int_a^x f(t)dt$ such that $dy/dx = \sqrt{1 + x^4}$ and $y = 2$ when $x = 1$.

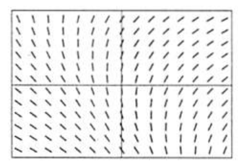
In Exercises 37 and 38, construct a slope field for the differential equation. In each case, copy the graph shown and draw tiny segments through the twelve lattice points shown in the graph. Use slope analysis, not your graphing calculator.

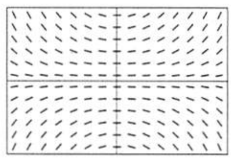
37. $\frac{dy}{dx} = -x$
38. $\frac{dy}{dx} = 1 - y$

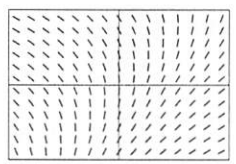


In Exercises 39–42, match the differential equation with the appropriate slope field. (All slope fields are shown in the window [−6, 6] by [−4, 4].)


(a)


(b)


(c)


(d)

39. $\frac{dy}{dx} = \frac{5}{x+y}$
40. $\frac{dy}{dx} = \frac{5}{x-y}$
41. $\frac{dy}{dx} = \frac{xy}{10}$
42. $\frac{dy}{dx} = -\frac{xy}{10}$

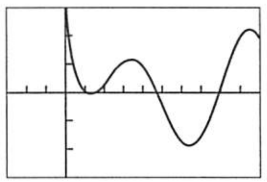
43. Suppose $dy/dx = x + y - 1$ and $y = 1$ when $x = 1$. Use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$.

44. Suppose $dy/dx = x - y$ and $y = 2$ when $x = 1$. Use Euler's Method with increments of $\Delta x = -0.1$ to approximate the value of y when $x = 0.7$.

In Exercises 45 and 46, match the indefinite integral with the graph of one of the antiderivatives of the integrand.

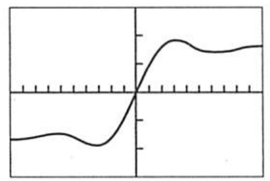
45. $\int \frac{\sin x}{x}\, dx$

46. $\int e^{-x^2}\, dx$



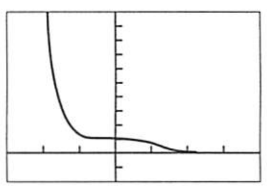
[−3, 10] by [−3, 3]

(a)



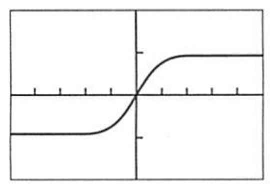
[−10, 10] by [−3, 3]

(b)



[−3, 4] by [−2, 10]

(c)



[−5, 5] by [−2, 2]

(d)

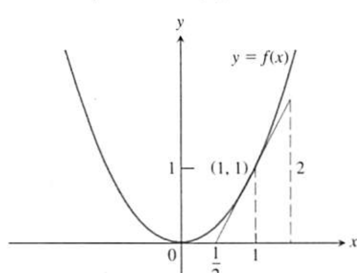
47. *Writing to Learn* The figure shows the graph of the function $y = f(x)$ that is the solution of one of the following initial value problems. Which one? How do you know?

i. $dy/dx = 2x, \quad y(1) = 0$

ii. $dy/dx = x^2, \quad y(1) = 1$

iii. $dy/dx = 2x + 2, \quad y(1) = 1$

iv. $dy/dx = 2x, \quad y(1) = 1$



48. *Writing to Learn* Does the following initial value problem have a solution? Explain.

$$\frac{d^2y}{dx^2} = 0, \quad y'(0) = 1, \quad y(0) = 0$$

49. *Moving Particle* The acceleration of a particle moving along a coordinate line is

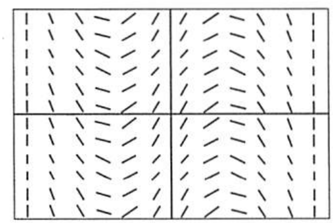
$$\frac{d^2s}{dt^2} = 2 + 6t \text{ m/sec}^2.$$

At $t = 0$ the velocity is 4 m/sec.

(a) Find the velocity as a function of time t.

(b) How far does the particle move during the first second of its trip, from $t = 0$ to $t = 1$?

50. *Sketching Solutions* Draw a possible graph for the function $y = f(x)$ with slope field given in the figure that satisfies the initial condition $y(0) = 0$.



[−10, 10] by [−10, 10]

51. *Californium-252* What costs $27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only about 8 g of it have been made in the western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per second.

(a) What is the value of k in the decay equation for this isotope?

(b) What is the isotope's mean life? (See Exercise 19, Section 6.4.)

52. *Cooling a Pie* A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a 40°F breezy porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

53. *Finding Temperature* A pan of warm water (46°C) was put into a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 minutes after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

54. *Art Forgery* A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

55. *Carbon-14* What is the age of a sample of charcoal in which 90% of the carbon-14 that was originally present has decayed?

56. *Appreciation* A violin made in 1785 by John Betts, one of England's finest violin makers, cost $250 in 1924 and sold for $7500 in 1988. Assuming a constant relative rate of appreciation, what was that rate?

57. *Working Underwater* The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL,$$

where k is a constant. As a diver you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below a tenth of the surface value. About how deep can you expect to work without artificial light?

58. *Transport through a Cell Membrane* Under certain conditions, the result of the movement of a dissolved substance across a cell's membrane is described by the equation

$$\frac{dy}{dt} = k\frac{A}{V}(c - y),$$

where y is the concentration of the substance inside the cell, and dy/dt is the rate with which y changes over time. The letters k, A, V, and c stand for constants, k being the *permeability coefficient* (a property of the membrane), A the surface area of the membrane, V the cell's volume, and c the concentration of the substance outside the cell. The equation says that the rate at which the concentration changes within the cell is proportional to the difference between it and the outside concentration.

(a) Solve the equation for $y(t)$, using $y_0 = y(0)$.

(b) Find the steady-state concentration, $\lim_{t\to\infty} y(t)$.

59. *Logistic Equation* The spread of flu in a certain school is given by the formula

$$P(t) = \frac{150}{1 + e^{4.3-t}},$$

where t is the number of days after students are first exposed to infected students.

(a) Show that the function is a solution of a logistic differential equation. Identify k and the carrying capacity.

(b) *Writing to Learn* Estimate $P(0)$. Explain its meaning in the context of the problem.

(c) Estimate the number of days it will take for a total of 125 students to become infected.

60. *Confirming a Solution* Show that

$$y = \int_0^x \sin(t^2)\, dt + x^3 + x + 2$$

is the solution of the initial value problem.

Differential equation: $y'' = 2x\cos(x^2) + 6x$

Initial conditions: $y'(0) = 1, \quad y(0) = 2$

61. *Finding an Exact Solution* Use analytic methods to find the exact solution to

$$\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right), \quad P(0) = 50.$$

62. *Supporting a Solution* Give two ways to provide graphical support for the integral formula

$$\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

63. *Doubling Time* Find the amount of time required for $10,000 to double if the 6.3% annual interest is compounded **(a)** annually, **(b)** continuously.

64. *Constant of Integration* Let

$$f(x) = \int_0^x u(t)\, dt \quad \text{and} \quad g(x) = \int_3^x u(t)\, dt.$$

(a) Show that f and g are antiderivatives of $u(x)$.

(b) Find a constant C so that $f(x) = g(x) + C$.

65. *Population Growth* Table 6.9 shows the population of Anchorage, AK for selected years between 1950 and 2003.

Table 6.9 Population of Anchorage, AK

Years after 1950	Population
0	11,254
20	48,081
30	174,431
53	270,951

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what number will the Anchorage population approach in the long run?

(c) Write a logistic differential equation in the form $dp/dt = kP(M - P)$ that models the growth of the Anchorage data in Table 6.9.

(d) *Writing to Learn* The population of Anchorage in 2000 was 260,283. If this point is included in the data, how does it affect carrying capacity predicted by the regression equation? Is there reason to be concerned about our model?

66. *Temperature Experiment* A temperature probe is removed from a cup of hot chocolate and placed in water whose temperature (T_s) is 0°C. The data in Table 6.10 were collected over the next 30 sec with a CBL™ temperature probe.

Table 6.10 Experimental Data

Time t (sec)	T (°C)
2	74.68
5	61.99
10	34.89
15	21.95
20	15.36
25	11.89
30	10.02

(a) Find an exponential regression equation for the (t, T) data. Superimpose its graph on a scatter plot of the data.

(b) Estimate when the temperature probe will read 40°C.

(c) Estimate the hot chocolate's temperature when the temperature probe was removed.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

67. The spread of a rumor through a small town is modeled by $dy/dt = 1.2y(1 - y)$, where y is the proportion of the townspeople who have heard the rumor at time t in days. At time $t = 0$, ten percent of the townspeople have heard the rumor.

(a) What proportion of the townspeople have heard the rumor when it is spreading the fastest?

(b) Find y explicitly as a function of t.

(c) At what time t is the rumor spreading the fastest?

68. A population P of wolves at time t years ($t \geq 0$) is increasing at a rate directly proportional to $600 - P$, where the constant of proportionality is k.

(a) If $P(0) = 200$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 500$, find k.

(c) Find $\lim_{t\to\infty} P(t)$.

69. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $dv/dt = -2(v + 17)$, with initial condition $v(0) = -47$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

Calculus at Work

I have a Bachelor's and Master's degree in Aerospace Engineering from the University of California at Davis. I started my professional career as a Facility Engineer managing productivity and maintenance projects in the Unitary Project Wind Tunnel facility at NASA Ames Research Center. I used calculus and differential equations in fluid mechanic analyses of the tunnels. I then moved to the position of Test Manager, still using some fluid mechanics and other mechanical engineering analysis tools to solve problems. For example, the lift and drag forces acting on an airplane wing can be determined by integrating the known pressure distribution on the wing.

I am currently a NASA On-Site Systems Engineer for the Lunar Prospector spacecraft project, at Lockheed Martin Missiles and Space in Sunnyvale, California. Differential equations and integration are used to design some of the flight hardware for the spacecraft. I work on ensuring that the different systems of the spacecraft are adequately integrated together to meet the specified design requirements. This often means doing some analysis to determine if the systems will function properly and within the constraints of the space environment. Some of these analyses require use of differential equations and integration to determine the most exact results, within some margin of error.

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Sunnyvale, CA