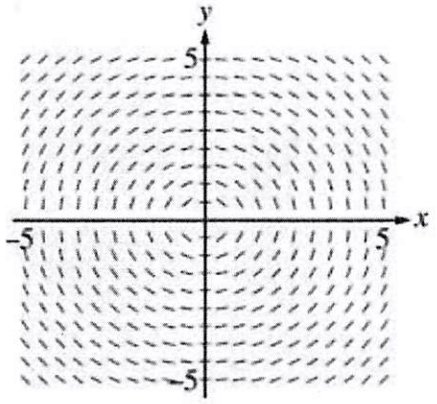
Name: AP Calc

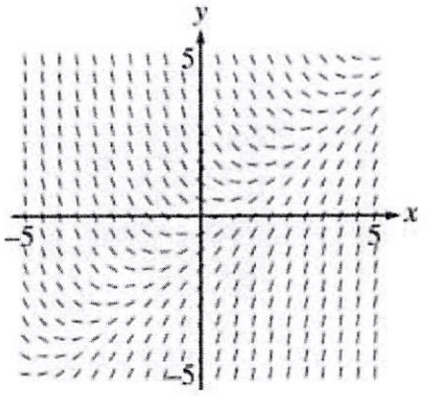
Chapter 6 TEST
Part I – No calculator

1. Which of the following is the slope field for the differential equation: $y' = \frac{x}{y}$? [1]

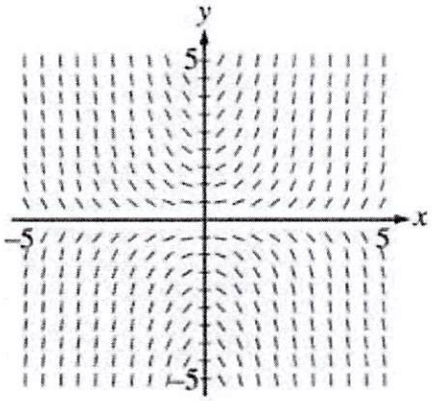
A



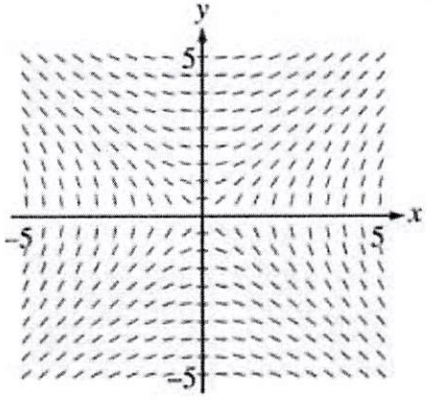
B



C



(D)



2. Show that $y = 2 + e^{-x^3}$ is a solution of the differential equation $y' + 3x^2y = 6x^2$. [1]

$y' = -3x^2e^{-x^3}$

$\therefore\ y' + 3x^2y = -3x^2e^{-x^3} + 3x^2(2 + e^{-x^3})$

$= -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3}$

$= 6x^2$ ✓

3. Solve the differential equation (that satisfies the initial condition when given): [4.5]

a) $\frac{dy}{dx} = \frac{x+\sin x}{3y^2}$

$3y^2 dy = (x+\sin x)\,dx$

$\boxed{y^3 = \frac{1}{2}x^2 - \cos x + C,\ C \in \mathbb{R}}$

1.5

b) $y' = y^2 + 1\ ,\ y(1) = 0$

$\frac{1}{y^2+1}y' = 1$

$\tan^{-1} y = x + C,\ C \in \mathbb{R}$

1.5

when $x = 1,\ y = 0$

$\tan^{-1} 0 = 1 + C$

$0 = 1 + C$

$C = -1$

$\Rightarrow \boxed{\tan^{-1} y = x - 1}$

c) $xe^{-t}\frac{dx}{dt} = 1\ ,\ x(0) = -2$

$x\,dx = e^t\,dt$

$\frac{1}{2}x^2 = e^t + C,\ C \in \mathbb{R}$

1.5

when $t = 0,\ x = -2$

$\frac{1}{2}(-2)^2 = e^0 + C$

$2 = 1 + C$

$C = 1$

$\Rightarrow \boxed{\frac{1}{2}x^2 = e^t + 1}$

4. Evaluate the following integrals: [20.5]

a) $\int \frac{x^2}{\sqrt{2+x^3}}dx = \int \frac{1}{3}\frac{1}{\sqrt{u}}du = \frac{1}{3}\cdot 2u^{1/2} + C,\ C \in \mathbb{R}$

$u = 2 + x^3$

$du = 3x^2 dx$

$\frac{1}{3}du = x^2 dx$

1.5

$= \boxed{\frac{2}{3}\sqrt{2+x^3} + C,\ C \in \mathbb{R}}$

b) $\int e^{4x}dx = \frac{1}{4}\int e^u du = \frac{1}{4}e^u + C,\ C \in \mathbb{R}$

$u = 4x$

$du = 4\,dx$

$\frac{1}{4}du = dx$

1.5

$= \boxed{\frac{1}{4}e^{4x} + C,\ C \in \mathbb{R}}$

c) $\int sec3\theta\ tan3\theta\ d\theta = \frac{1}{3}\int \sec u \tan u\, du$

1.5 $u = 3\theta$, $du = 3d\theta$, $\frac{1}{3}du = d\theta$

$= \boxed{\frac{1}{3}\sec(3\theta) + C, C \in \mathbb{R}}$

d) $\int \frac{(lnx)^2}{x} dx = \int u^2 du = \frac{1}{3}u^3 + C, C \in \mathbb{R}$

1.5 $u = \ln x$, $du = \frac{1}{x}dx$

$= \boxed{\frac{1}{3}(\ln x)^3 + C, C \in \mathbb{R}}$

e) $\int t^2 \cos(1-t^3)dt = -\frac{1}{3}\int \cos u\, du$

1.5 $u = 1 - t^3$, $du = -3t^2 dt$

$= \boxed{-\frac{1}{3}\sin(1-t^3) + C, C \in \mathbb{R}}$

f) $\int \frac{e^x}{e^x+1} dx = \int \frac{1}{u} du = \boxed{\ln|e^x+1| + C, C \in \mathbb{R}}$

1.5 $u = e^x + 1$, $du = e^x dx$

g) $\int \frac{e^x+1}{e^x} dx = \int (1 + e^{-x}) dx$

1.5 $= \boxed{x - e^{-x} + C, C \in \mathbb{R}}$

h) $\int \cos^3 x\, dx = \int \cos x \cdot \cos^2 x\, dx = \int \cos x (1 - \sin^2 x) dx$

$= \int \cos x\, dx - \int \cos x \cdot \sin^2 x\, dx$

2 $= \boxed{\sin x - \frac{1}{3}(\sin x)^3 + C, C \in \mathbb{R}}$

i) $\int x\sqrt{x+2}dx$

$= \int (u-2)\sqrt{u}\, du = \int (u^{3/2} - 2u^{1/2}) du$

2 $u = x+2$, $du = dx$, $x = u-2$

$= \boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C, C \in \mathbb{R}}$

j) $\int \frac{2}{x^2-6x+10}dx = \int \frac{2}{(x-3)^2+1}dx = 2\int \frac{1}{1+u^2}du$

$u = x-3$
$du = dx$

$= \boxed{2\tan^{-1}(x-3)+C,\ C\in\mathbb{R}}$

2

k) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}x - \frac{1}{2}\int \cos 2x\,dx$

$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x)+C,\ C\in\mathbb{R}}$

2

l) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{u}du$

$u = \cos x$
$du = -\sin x\,dx$

$= \boxed{-\ln|\cos x|+C,\ C\in\mathbb{R}}$

2

5. Evaluate the following definite integrals: [6]

a) $\int_0^1 (2x-1)^{100}dx = \frac{1}{2}\int_{-1}^{1} u^{100}du = \frac{1}{2}\cdot\frac{1}{101}u^{101}\Big]_{-1}^{1}$

$u = 2x-1$
$du = 2dx$

$= \frac{1}{202}\left(1-(-1)^{101}\right) = \boxed{\frac{1}{101}}$

2

b) $\int_0^{1/2} \frac{1}{1+4x^2}dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}du$

$u = 2x$
$du = 2dx$

$= \frac{1}{2}\tan^{-1}u\Big]_0^1 = \frac{1}{2}\left(\frac{\pi}{4}-0\right) = \boxed{\frac{\pi}{8}}$

2

c) $\int_0^3 \frac{dx}{2x+3} = \frac{1}{2}\int_3^9 \frac{1}{u}du = \frac{1}{2}\ln|u|\Big]_3^9$

$u = 2x+3$
$du = 2dx$

$= \frac{1}{2}(\ln 9 - \ln 3)$

$= \boxed{\frac{1}{2}\ln 3}$

2

Chapter 6 TEST
Part II – Calculator Allowed

6. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. Suppose that a roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

a) If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45min? [2]

$\frac{dT}{dt} = k(T-75)$ $T|_{t=0} = 185$ $T|_{t=\frac{1}{2}} = 150$ $T|_{t=3/4} = ?$

Let $u = T-75$ $\frac{du}{dt} = \frac{dT}{dt}$ $\frac{du}{dt} = ku$

$\Rightarrow u = u_0 e^{kt}$

when $t=0$, $T=185$ i.e. $u=110$

$\Rightarrow T-75 = 110e^{kt}$

$T = 110e^{kt} + 75$

when $t=\frac{1}{2}$, $T=150$

$\Rightarrow 150 = 110e^{k/2} + 75$

$\frac{75}{110} = e^{k/2}$

$\ln\left(\frac{15}{22}\right) = \frac{k}{2}$

$k = 2\ln\left(\frac{15}{22}\right)$

$T = 110 e^{\ln\left(\frac{15}{22}\right)\cdot 2t} + 75$

$T = 110\cdot\left(\frac{15}{22}\right)^{2t} + 75$

when $t = 3/4$:

$T = 110\left(\frac{15}{22}\right)^{3/2} + 75$

$T \simeq 137°F$

b) When will the turkey have cooled to 100°F? [1]

$100 = 110\left(\frac{15}{22}\right)^{2t} + 75$

$\frac{25}{110} = \left(\frac{15}{22}\right)^{2t}$

$\frac{5}{22} = \left(\frac{15}{22}\right)^{2t}$

$2t\ln\left(\frac{15}{22}\right) = \ln\left(\frac{5}{22}\right)$

$t = \frac{\ln(5/22)}{2\ln(15/22)}$

$t \simeq 1.93 \Rightarrow$ 1h56min

7. How long will it take an investment to double in value if the interest rate is 6% compounded continuously? [1]

$V = V_0 e^{0.06t}$

$2 = e^{0.06t}$

$\ln 2 = 0.06t$

$t = \frac{1}{0.06}\ln 2$

$t \simeq 11.55$ years

8. A tank contains 20kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour? [3]

$\frac{dS}{dt} = 0.03 \times 25 - \frac{S}{5000} \times 25$ $\quad S|_{t=0} = 20$ $\quad S|_{t=30} = ?$

$S' = 0.75 - \frac{1}{200} S$

$200 S' = 150 - S$

$\frac{1}{150 - S} S' = \frac{1}{200}$

$\int \frac{1}{150 - S} dS = \int \frac{1}{200} dt$

$u = 150 - S$
$du = -dS$

$-\ln|150 - S| = \frac{1}{200} t + C$

* when $t = 0$, $S = 20$
$-\ln 130 = C$

$\Rightarrow -\ln|150 - S| = \frac{1}{200} t - \ln 130$

$\ln|150 - S| = -\frac{1}{200} t + \ln 130$

$|150 - S| = e^{-\frac{1}{200} t + \ln 130}$

$150 - S = \pm 130 \cdot e^{-\frac{1}{200} t}$ $\quad$ when $t = 0$, $S = 20$

$150 - S = 130 e^{-\frac{1}{200} t}$

$S = 150 - 130 e^{-\frac{1}{200} t}$

$S|_{t=30} = 150 - 130 e^{-\frac{3}{20}} \simeq \boxed{38.1 \text{ kg}}$