More practice on the Sandwich Theorem (or Squeezing Theorem)

Determine the following limits :

a) $\lim_{x \to 0} x \sin \frac{1}{x}$

$-1 \leq \sin\frac{1}{x} \leq 1$ for all $x \in \mathbb{R} \setminus \{0\}$

$-|x| \leq x \sin\frac{1}{x} \leq |x|$

$\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$ Squeezing Theorem $\Rightarrow \lim_{x \to 0} x\sin\frac{1}{x} = 0$

b) $\lim_{x \to 0} x^2 \sin \frac{1}{x}$

$-1 \leq \sin\frac{1}{x} \leq 1$

$-x^2 \leq x^2\sin\frac{1}{x} \leq x^2$

$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$ Squeezing Theorem $\Rightarrow \lim_{x \to 0} x^2\sin\frac{1}{x} = 0$

c) $\lim_{x \to 0} x \left(1 - \cos \frac{1}{x}\right)$

$-1 \leq \cos\frac{1}{x} \leq 1$

$0 \leq 1 - \cos\frac{1}{x} \leq 2$

$0 \leq x\left(1-\cos\frac{1}{x}\right) \leq 2|x|$

$\lim_{x \to 0} 2|x| = 0$

Squeezing Theorem $\Rightarrow \lim_{x \to 0} x\left(1-\cos\frac{1}{x}\right) = 0$

d) $\lim_{x \to 0} x^2 \left(1 - \cos \frac{1}{x}\right)$

$-1 \leq \cos\frac{1}{x} \leq 1$

$0 \leq 1 - \cos\frac{1}{x} \leq 2$

$0 \leq x^2\left(1-\cos\frac{1}{x}\right) \leq 2x^2$

$\lim_{x \to 0} 2x^2 = 0$ $\Rightarrow$ squeezing Theorem $\Rightarrow \lim_{x \to 0} x^2\left(1-\cos\frac{1}{x}\right) = 0$

e) $\lim_{x\to 0} \sqrt{x^3+3x^2}\sin\frac{1}{x}$

$-1 \le \sin\frac{1}{x} \le 1$

$-\sqrt{x^3+3x^2} \le \sin\frac{1}{x} \le \sqrt{x^3+3x^2}$ for all $x \in [-3, +\infty)$

$\lim_{x\to 0} -\sqrt{x^3+3x^2} = \lim_{x\to 0} \sqrt{x^3+3x^2} = 0$ Squeezing Theorem

$\Rightarrow \lim_{x\to 0} \sqrt{x^3+3x^2}\sin\left(\frac{1}{x}\right) = 0$

f) Suppose $-x^2+1 \le g(x) \le x^2+1$ for all x in an open interval containing 0.
Find $\lim_{x\to 0} g(x)$.

$\lim_{x\to 0} (-x^2+1) = \lim_{x\to 0} x^2+1 = 1$

Squeezing theorem

$\lim_{x\to 0} g(x) = 1$

g) Suppose $-(x-2)^2-3 \le g(x) \le (x-2)^2-3$ for all x in an open interval containing 2.
Find $\lim_{x\to 2} g(x)$.

$\lim_{x\to 2} -(x-2)^2-3 = \lim_{x\to 2} (x-2)^2-3 = -3$

Squeezing Theorem

$\lim_{x\to 2} g(x) = -3$

h) Suppose $\cos x \le g(x) \le 1$ for all x in an open interval containing 0.
Find $\lim_{x\to 0} g(x)$.

$\lim_{x\to 0} \cos x = 1$

Squeezing Theorem

$\lim_{x\to 0} g(x) = 1$

i) Suppose $-x^2 + 1 \le g(x) \le \sec x$ for all x in an open interval containing 0. Find $\lim_{x \to 0} g(x)$.

$\lim_{x \to 0} -x^2 + 1 = \lim_{x \to 0} \sec x = 1$

Squeezing Theorem : $\lim_{x \to 0} g(x) = 1$

More practice on the IVT

1)

$$F(t)=\begin{cases} 2^{t+1}-2 & \text{for } 0\le t<7 \\ \frac{800t-4800}{t-4} & \text{for } t\ge 7\end{cases}$$

The amount of money raised during a fund-raising campaign is modeled by the function F defined above, where $F(t)$ is measured in United States dollars and t is the time in days since the campaign began.

(a) Find $\lim_{t\to\infty} F(t)$. Explain the meaning of $\lim_{t\to\infty} F(t)$ in the context of the problem.

(b) Is the function F continuous at $t=7$? Justify your answer.

(c) The amount of money raised during a competing fund-raising campaign is modeled by the function M defined by $M(t)=\frac{240(2^t-1)}{(2^t+36)}$, where $M(t)$ is measured in United States dollars and t is the time in days since that campaign began. According to this model, is there a time t, for $0\le t\le 2$, at which the amount of money raised is 10 dollars? Justify your answer.

(a) $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \frac{800t-4800}{t-4} = 800$

As time increases, the amount raised approaches \$800.

(b) $\lim_{t\to 7^-} F(t) = \lim_{t\to 7^-} (2^{t+1}-2) = 2^8-2=254$

$F(7)=\lim_{t\to 7^+} F(t) = \lim_{t\to 7^+} \frac{800t-4800}{t-4} = \frac{800}{3} \neq 254$

} not continuous at 7

(c) M is continuous on $[0,2]$ $M(0)=0$ $M(2)=18$ $10\in[0,18]$

By the IVT, there must be at least one time $t\in[0,2]$ such that $M(t)=10$

2)

Let f be the function given by $f(x)=\frac{9+2xe^{-\frac{x}{4}}}{\cos\left(\frac{x}{2}\right)}$. The Intermediate Value Theorem applied to f on the closed interval $[24, 28]$ guarantees a solution in $[24, 28]$ to which of the following equations?

(A) $f(x)=0$

(B) $f(x)=9.090$

(C) ✓ $f(x)=12.235$

(D) $f(x)=76.999$

$f(24)\simeq 10.806$ $f(28)\simeq 66.193$

$12.235\in[10.806, 66.193]$

3)

$$f(x) = \frac{x^4+x^3+x^2+x+1}{380-\ln\left(\frac{x^2+1}{2}\right)}$$

Let f be the function defined above. The Intermediate Value Theorem applied to f on the closed interval $[10, 12]$ guarantees a solution in $[10, 12]$ to which of the following equations?

(A) $f(x) = 0$

(B) $f(x) = 27.372$

(C) $f(x) = 42.421$ ✓

(D) $f(x) = 67.205$

$f(10) \simeq 29.544$

$f(12) \simeq 66.208$

$42.421 \in [29.544, 66.208]$

4)

Let f be a twice-differentiable function such that $f(2)=5$ and $f(5)=2$. Let g be the function given by $g(x)=f(f(x))$.

Let $h(x)=f(x)-x$. Explain why there must be a value r for $2<r<5$ such that $h(r)=0$.

$h(2) = f(2) - 2 = 5 - 2 = 3$

$h(5) = f(5) - 5 = 2 - 5 = -3$

Since $h(2) > 0 > h(5)$ IVT $\Rightarrow$ there must be ...

5)

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	−20	−30	−20	−14	−10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

For $0<t<60$, must there be a time t when $v(t)=-5$? Justify your answer.

$v(35) = -10$

$v(50) = 0$

$v(35) < -5 < v(50)$

IVT $\Rightarrow$ yes

6)

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

Train *A* runs back and forth on an east-west section of railroad track. Train *A*'s velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

Do the data in the table support the conclusion that train *A*'s velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

V_A is continuous function of time

$V_A(8) = -120 < -100 < V_A(5) = 40$

IVT $\Rightarrow$ there is a time t, $5 < t < 8$ such that $V_A(t) = -100$

7)

$$R(t) = \begin{cases} 41\sqrt{\frac{t}{20}} & \text{for } 0 \le t < 20 \\ \frac{100t^2+1000}{t^2+600} & \text{for } t \ge 20 \end{cases}$$

The total accumulated revenue a company has received up to time t is modeled by the function R defined above, where $R(t)$ is measured in millions of United States dollars and t is the time in years since 1980.

(a) Find $\lim_{t\to\infty} R(t)$. Explain the meaning of $\lim_{t\to\infty} R(t)$ in the context of the problem.

(b) Is the function R continuous at $t = 20$? Justify your answer.

(c) The company's total accumulated expenses up to time t is modeled by the function E defined by $E(t) = 3\log_2(12t+4)$, where $E(t)$ is measured in millions of United States dollars and t is the time in years since 1980. According to these models, is there a time t, for $0 \le t \le 5$, at which the total accumulated profit, $R(t) - E(t)$, is equal to 0 ? Justify your answer.

(a) $\lim_{t\to+\infty} R(t) = \lim_{t\to+\infty} \frac{100t^2+1000}{t^2+600} = 100$

As time increases, the total accumulated revenue of the company approaches \$100 million.

(b) $\lim_{t\to 20^-} R(t) = \lim_{t\to 20^-} 41\sqrt{\frac{t}{20}} = 41$ $\quad \lim_{t\to 20^+} R(t) = \lim_{t\to 20^+} \frac{100t^2+1000}{t^2+600} = 41$

$R(20) = 41$

Since $\lim_{t\to 20} R(t) = R(20)$, then yes R is continuous at 20

(c) R and E are continuous for $0 \le t \le 5$ $\therefore$ $R - E$ is continuous on $[0,5]$

$R(0) - E(0) = -6$ $\quad R(5) - E(5) = 2.5$ $\quad$ since $0 \in (-6, 2.5)$

by IVT, there must be at least one time $t \in [0,5]$ such that $R(t) - E(t) = 0$