Answers Explained

Part A

1. (a) $f'(4.0) \approx \dfrac{f(4.6)-f(4.0)}{4.6-4.0} = \dfrac{2.2-3.1}{0.6} = -1.5.$

$f'(4.8) \approx \dfrac{f(5)-f(4.6)}{5-4.6} = \dfrac{1.5-2.2}{0.4} = -1.75.$

(b) It appears that the rate of change of f, while negative, is increasing. This implies that the graph of f is concave upward.

(c) $L = 7.6(0.7) + 5.7(0.3) + 4.2(0.5) + 3.1(0.6) + 2.2(0.4) = 11.87.$

(d) Using disks $\Delta V = \pi r^2 \Delta x$. One possible answer uses the left endpoints of the subintervals as values of r:

$V \approx \pi(7.6)^2(0.7) + \pi(5.7)^2(0.3) + \pi(4.2)^2(0.5) + \pi(3.1)^2(0.6) + \pi(2.2)^2(0.4)$

2. (a) $12y_0 + 0.3 = 24$ yields $y_0 \approx 1.975$.

(b) Replace x by 0.3 in the equation of the curve:

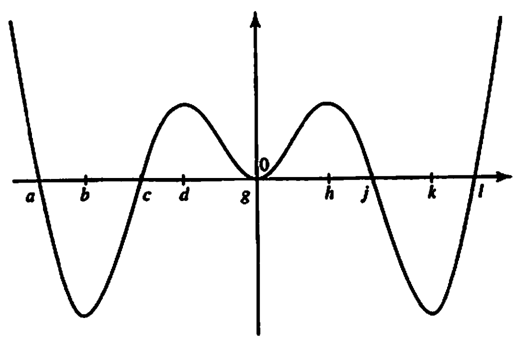
$$(0.3)^2 y_0 - (0.3) = y_0^3 - 8 \text{ or}$$

$$y_0^3 - 0.09y_0 - 7.7 = 0.$$

The calculator's solution to three decimal places is $y_0 = 1.990$.

(c) Since the true value of y_0 at $x = 0.3$ exceeds the approximation, conclude that the given curve is concave up near $x = 0$. (Therefore, it is above the line tangent at $x = 0$.)

3. Graph $f'(x) = 2x \sin x - e^{(-x^2)} + 1$ in $[-7, 7] \times [-10, 10]$.



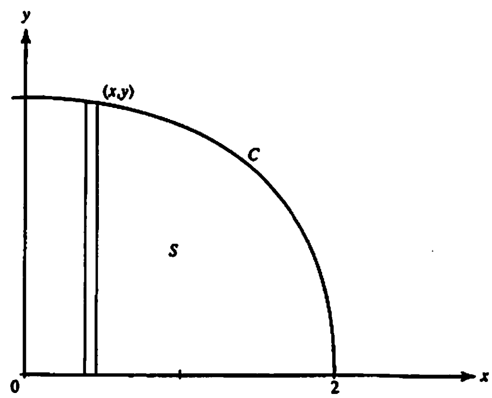
(a) Since f' is even and f contains (0, 0), f is odd and its graph is symmetric about the origin.

(b) Since f is decreasing when $f' < 0$, f decreases on the intervals (a, c) and (j, l). Use the calculator to solve $f'(x) = 0$. Conclude that f decreases on $-6.202 < x < -3.294$ and (symmetrically) on $3.294 < x < 6.202$.

(c) f has a relative maximum at $x = q$ if $f'(q) = 0$ and if f changes from increasing ($f' > 0$) to decreasing ($f' < 0$) at q. There are two relative maxima here:
at $x = a = -6.202$ and at $x = j = 3.294$.

(d) f has a point of inflection when the graph of f changes its concavity; that is, when f' changes from increasing to decreasing, as it does at points d and h, or when f' changes from decreasing to increasing, as it does at points b, g, and k. So there are five points of inflection altogether.

4. In the graph below, C is the piece of the curve lying in the first quadrant. S is the region bounded by the curve C and the coordinate axes.



(a) Graph $y = \sqrt[3]{(64-16x^2)}$ in $[0,3] \times [0,5]$. Since you want dy/dx, the slope of the tangent, where $y = 1$, use the calculator to solve

$$\sqrt[3]{64-16x^2} = 1$$

(storing the answer at B). Then evaluate the slope of the tangent to C at $y = 1$:

$$f'(B) \approx -21.182.$$

(b) Since $\Delta A = y\Delta x$, $A = \int_0^2 y\,dx \approx 6.730.$

(c) When S is rotated about the x-axis, its volume can be obtained using disks:

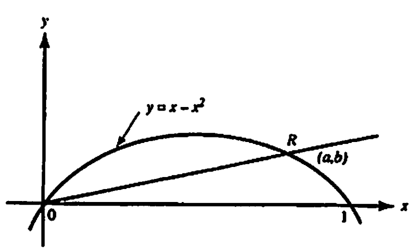
$$\Delta V = \pi R^2 \Delta x = \pi y^2\,\Delta x,$$

$$V = \pi \int_0^2 y^2\,dx$$

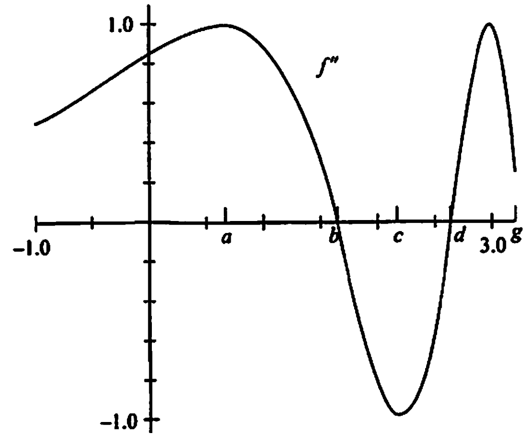
$$= \pi \int_0^2 \left(\sqrt[3]{64-16x^2}\right)^2 dx \approx 74.310.$$

5. See the figure, where R is the point (a, b), and seek a such that

$$\int_0^a \left(x - x^2 - \frac{b}{a} \cdot x\right) dx = \frac{1}{2}\int_0^1 \left(x - x^2\right) dx.$$



6. Graph $y = \sin 2^x$ in $[-1, 3.2] \times [-1, 1]$. Note that $y = f''$.

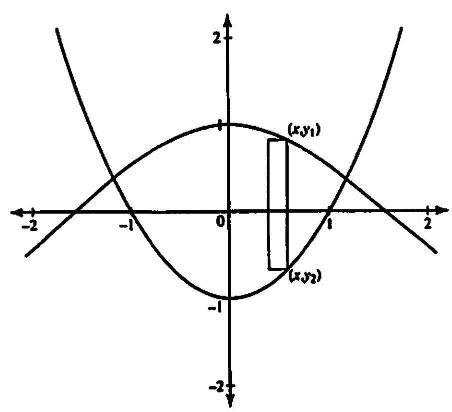


(a) The graph of f is concave downward where f'' is negative, namely, on (b, d). Use the calculator to solve $\sin 2^x = 0$, obtaining $b = 1.651$ and $d = 2.651$. The answer to (a) is therefore $1.651 < x < 2.651$.

(b) f' has a relative minimum at $x = d$, because f'' equals 0 at d, is less than 0 on (b, d), and is greater than 0 on (d, g). Thus f' has a relative minimum (from part a) at $x = 2.651$.

(c) The graph of f' has a point of inflection wherever its second derivative f''' changes from positive to negative or vice versa. This is equivalent to f'' changing from increasing to decreasing (as at a and g) or vice versa (as at c). Therefore, the graph of f' has three points of inflection on $[-1, 3.2]$.

7. Graph $f(x) = \cos x$ and $g(x) = x^2 - 1$ in $[-2, 2] \times [-2, 2]$. Here, $y_1 = f$ and $y_2 = g$.



(a) Solve $\cos x = x^2 - 1$ to find the two points of intersection: (1.177,0.384) and (−1.177,0.384).

(b) Since $\Delta A = (y_1 - y_2)\,\Delta x = [f(x) - g(x)]\,\Delta x$, the area A bounded by the two curves is

$$A = 2\int_0^{1.777} (y_1 - y_2)\,dx,$$

$$= 2\int_0^{1.777} (\cos x - (x^2 - 1))\,dx$$

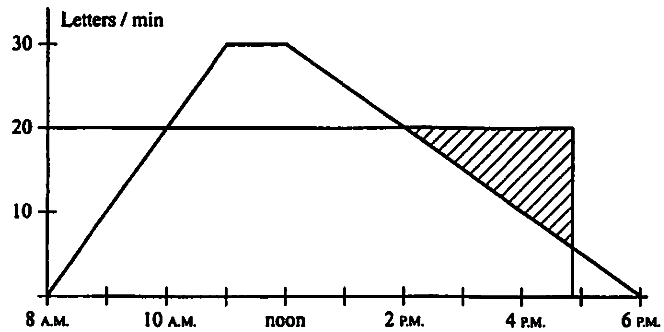
$$\approx 3.114.$$

8. (a) Use the Trapezoid Rule, with $h = 60$ min:

$$\frac{h}{2}(y_0 + 2y_0 + 2y_0 + 2y_0 + y_5) =$$

$$\frac{60}{2}(10 + 2\cdot 12 + 2\cdot 8 + 2\cdot 9 + 11) = 2370 \text{ letters.}$$

(b) Draw a horizontal line at $y = 20$ (as shown on the graph below), representing the rate at which letters are processed then.



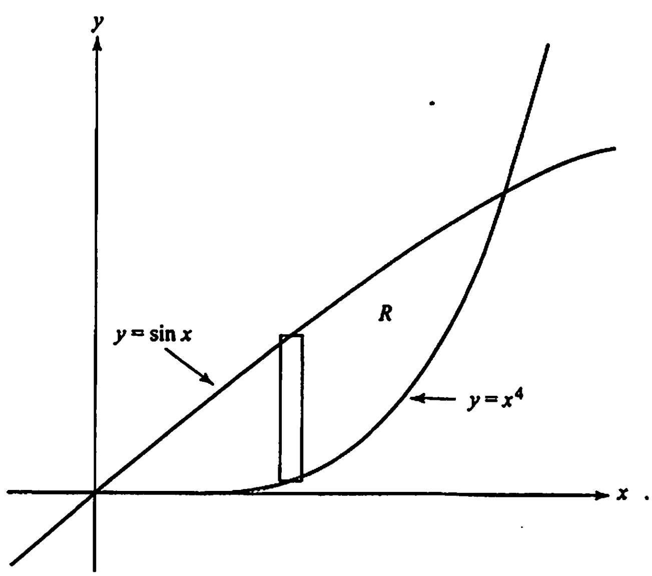
(i) Letters began to pile up when they arrived at a rate greater than that at which they were being processed, that is, at $t = 10$ A.M.

(ii) The pile was largest when the letters stopped piling up, at $t = 2$ P.M.

(iii) The number of letters in the pile is represented by the area of the small trapezoid above the horizontal line: $\frac{1}{2}(4\cdot 60 + 1\cdot 60)(10) = 1500$.

(iv) The pile began to diminish after 2 P.M., when letters were processed at a rate faster than they arrived, and vanished when the area of the shaded triangle represented 1500 letters. At 5 P.M. this area is $\frac{1}{2}(3\cdot 60)(15) = 1350$ letters, so the pile vanished shortly after 5 P.M.

9. Draw a vertical element of area as shown below.

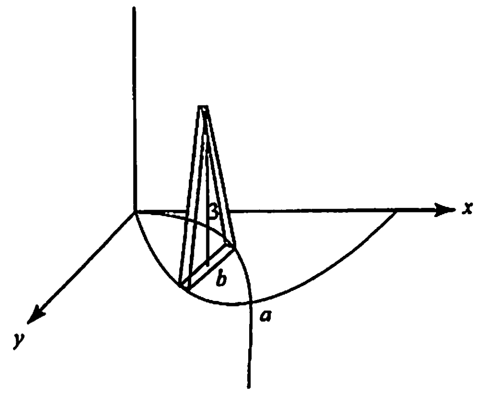


(a) Let a represent the x-value of the positive point of intersection of $y = x^4$ and $y = \sin x$. Solving $a^4 = \sin a$ with the calculator, we find $a = 0.9496$.

$$\Delta A = (y_{top} - y_{bottom})\Delta x = (\sin x - x^4)\Delta x,$$

$$A = \int_0^a (\sin x - x^4)\,dx \approx 0.264.$$

(b) Elements of volume are triangular prisms with height $h = 3$ and base $b = (\sin x - x^4)$, as shown below.



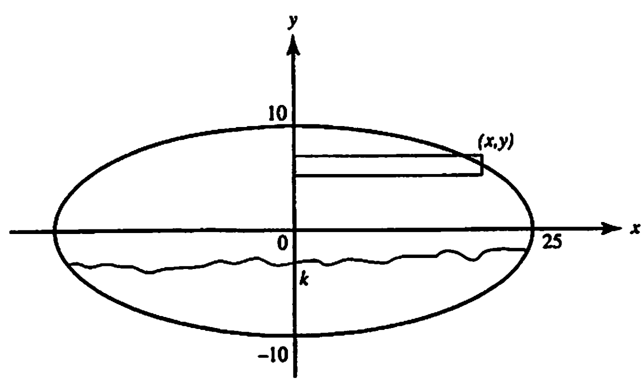
$$\Delta V = \frac{1}{2}(\sin x - x^4)(3)\Delta x,$$

$$V = \frac{3}{2}\int_0^a (\sin x - x^4)\,dx = 0.395.$$

(c) When R is rotated around the x-axis, the element generates washers. If r_1 and r_2 are the radii of the larger and smaller disks, respectively, then

$$\Delta V = \pi(r_1^2 - r_2^2)\Delta x = \pi\left((\sin x)^2 - (x^4)^2\right)\Delta x,$$

$$V = \pi\int_0^a (\sin^2 x - x^8)\,dx = 0.529.$$



10. The figure above shows an elliptical cross section of the tank. Its equation is

$$\frac{x^2}{625} + \frac{y^2}{100} = 1.$$

(a) The volume of the tank, using disks, is $V = 2\pi \int_0^{10} x^2 dy$, where the ellipse's symmetry about the x-axis has been exploited. The equation of the ellipse is equivalent to $x^2 = 6.25(100 - y^2)$, so

$$V = 12.5\pi \int_0^{10} \left(100 - y^2\right) dy.$$

Use the calculator to evaluate this integral, storing the answer as V to have it available for part (b).

The capacity of the tank is $7.48V$, or 196,000 gal of water, rounded to the nearest 1000 gal.

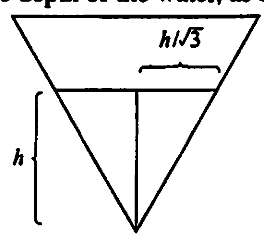
(b) Let k be the y-coordinate of the water level when the tank is one-fourth full.

Then

$$6.25\pi \int_{-10}^{k} \left(100 - y^2\right) dy = \frac{V}{4}$$

and the depth of the water is $k + 10$.

11. (a) Let h represent the depth of the water, as shown.



Then h is the altitude of an equilateral triangle, and the base $b = \frac{2h}{\sqrt{3}}$.

The volume of water is

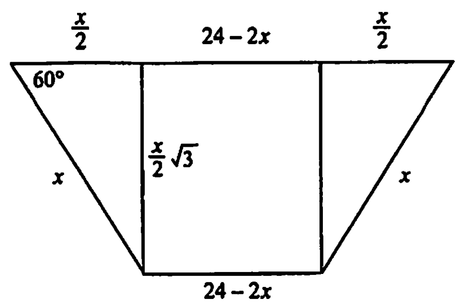
$$V = \frac{1}{2}\left(\frac{2h}{\sqrt{3}}\right)h \cdot 60 = \frac{60h^2}{\sqrt{3}} \text{ in.}^3$$

Now $\frac{dV}{dt} = \frac{120}{\sqrt{3}}h\frac{dh}{dt}$, and it is given that $\frac{dV}{dt} = 600$. Thus, when $h = 4$,

$$600 = \frac{120}{\sqrt{3}} 4 \frac{dh}{dt}, \text{ and } \frac{dh}{dt} = \frac{5\sqrt{3}}{4} \text{ in/min.}$$

(b) Let x represent the length of one of the sides, as shown.

The bases of the trapezoid are $24 - 2x$ and $24 - 2x + 2\frac{x}{2}$, and the height is $\frac{x}{2}\sqrt{3}$.



The volume of the trough (in in.^{3}) is given by

$$V=\frac{(24-2x)+(24-x)}{2}\cdot\frac{x}{2}\sqrt{3}\times 60=15\sqrt{3}(48x-3x^2) \qquad (0<x<12),$$
$$V'=15\sqrt{3}(48-6x)=0 \text{ when } x=8.$$

Since $V''=15\sqrt{3}(-6)<0$, the maximum volume is attained by folding the metal 8 inches from the edges.

12. (a) Both $\pi/4$ and the expression in brackets yield 0.7853981634, which is accurate to ten decimal places.

(b) $\tan^{-1}\frac{1}{5}=\frac{1}{5}-\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{5}\left(\frac{1}{5}\right)^5-\frac{1}{7}\left(\frac{1}{5}\right)^7=0.197396.$

$$\tan^{-1}\frac{1}{239}=\frac{1}{239}=0.004184.$$

(c) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=0.7854$; this agrees with the value of $\frac{\pi}{4}$ to four decimal places.

(d) The series

$$\tan^{-1}1 = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

converges *very* slowly. Example 56, page 438, evaluated the sum of 60 terms of the series for π (which equals $4\tan^{-1}1$). To four decimal places, we get $\pi=3.1249$, which yields 0.7812 for $\pi/4$—not accurate even to two decimal places.

13. (a) The given series is alternating. Since $\lim_{n\to\infty}\ln(n+1)=\infty$, $\lim_{n\to\infty}\frac{1}{\ln(n+1)}=0.$

Since $\ln x$ is an increasing function,

$$\ln(n+1)>\ln n \quad\text{and}\quad \frac{1}{\ln(n+1)}<\frac{1}{\ln n}.$$

The series therefore converges.

(b) Since the series converges by the Alternating Series Test, the error in using the first n terms for the sum of the whole series is less than the absolute value of the $(n + 1)$st term. Thus the error is less than $\frac{1}{\ln(n+1)}$. Solve for n using $\frac{1}{\ln(n+1)} < 0.1$:

$$\begin{aligned} \ln(n+1) &> 10, \\ (n+1) &> e^{10}, \\ n &> e^{10} - 1 > 22{,}025. \end{aligned}$$

The given series converges very slowly!

(c) The series $\sum_{2}^{\infty}(-1)^n \frac{1}{n \ln n}$ is conditionally convergent. The given alternating series converges since the nth term approaches 0 and $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$. However, the *nonnegative* series diverges by the Integral Test, since

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_2^b = \infty.$$

14. (a) Solve by separation of variables:

$$\frac{dy}{y(10-y)} = k\,dt,$$

$$\frac{1}{10}\int\left(\frac{1}{y} + \frac{1}{10-y}\right)dy = \int k\,dt,$$

$$\frac{1}{10}\ln\left(\frac{y}{10-y}\right) = kt + C,$$

$$\ln\left(\frac{10-y}{y}\right) = -10\,(kt + C).$$

Let $c = e^{-10C}$; then

$$\frac{10-y}{y} = ce^{-10kt}.$$

Now use initial condition $y = 2$ at $t = 0$:

$$\frac{8}{2} = ce^0 \text{ so } c = 4;$$

and the other condition, $y = 5$ at $t = 2$, gives

$$\frac{5}{5} = 4e^{-20k} \text{ or } k = \frac{1}{10}\ln 2.$$

(b) Since $c = 4$ and $k = \frac{1}{10}\ln 2$, then $\frac{10-y}{y} = 4e^{-10(\frac{1}{10}\ln 2)t}$.

Solving for y yields $y = \frac{10}{1 + 4 \cdot 2^{-t}}$.

(c) $8 = \dfrac{10}{1+4 \cdot 2^{-t}}$. means $1 + 4 \cdot 2^{-t} = 1.25$, so $t = 4$.

(d) $\lim\limits_{t \to \infty} \dfrac{10}{1+4 \cdot 2^{-t}} = 10$, so the value of y approaches 10.

15. (a) Since $x = \dfrac{1}{2}t$, $x(4) = \dfrac{1}{2}(4) = 2$. Since $y = 18 - 2 \cdot 2^2 = 10$, P is at (2,10).

(b) Since $y = 18 - 2x^2, \dfrac{dy}{dt} = -4x\dfrac{dx}{dt}$. Since $x = \dfrac{1}{2}t, \dfrac{dx}{dt} = \dfrac{1}{2}$. Therefore

$$\frac{dy}{dt} = -4x\frac{dx}{dt} = -4 \cdot 2 \cdot \frac{1}{2} = -4 \text{ unit/sec.}$$

(c) Let D = the object's distance from the origin. Then

$$D^2 = x^2 + y^2, \text{ and at } (2,10)\ D = \sqrt{104}.$$

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

$$2\sqrt{104}\frac{dD}{dt} = 2 \cdot 2 \cdot \frac{1}{2} + 2 \cdot 10(-4),$$

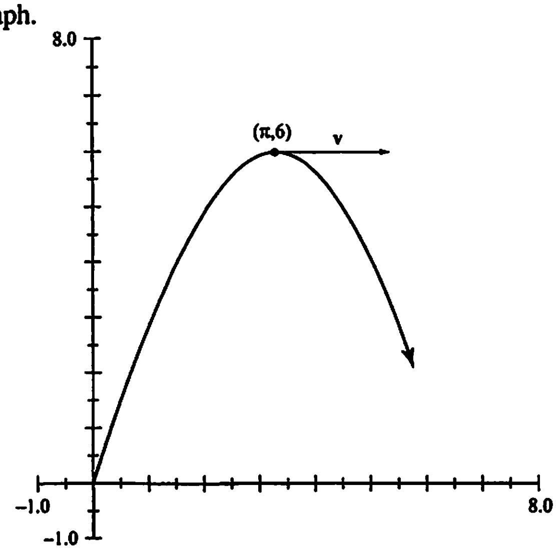
$$\frac{dD}{dt} = \frac{-78}{2\sqrt{104}} = -3.824 \text{ unit/sec.}$$

(d) The object hits the x-axis when $y = 18 - 2x^2 = 0$, or $x = 3$. Since $x = \dfrac{1}{2}t = 3, t = 6$.

(e) The length of the arc of $y = 18 - 2x^2$ for $0 \le x \le 3$ is given by

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1 + (-4x)^2}\, dx = 18.460 \text{ units.}$$

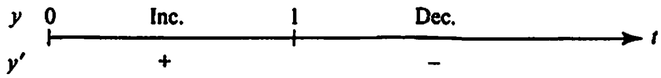
16. (a) See graph.



(b) You want to maximize $y(t)=\dfrac{12t}{t^2+1}$.

$$y'(t)=\frac{(t^2+1)(12)-12t(2t)}{(t^2+1)^2}=\frac{12(1-t)(1+t)}{(t^2+1)^2}.$$

See signs analysis.



The maximum y occurs when $t = 1$, because y changes from increasing to decreasing there.

(c) Since $x(1) = 4\arctan 1 = \pi$ and $y(1)=\dfrac{12}{1+1}=6$, the coordinates of the highest point are $(\pi,6)$.

Since $x'(t)=\dfrac{4}{1+t^2}$ and $y'(t)=\dfrac{12(1-t^2)}{(t^2+1)^2}$, so $\mathbf{v}(1)=\langle 2,0\rangle$. This vector is shown on the graph.

(d) $\lim\limits_{t\to\infty} x(t)=\lim\limits_{t\to\infty} 4\arctan t=4\left(\dfrac{\pi}{2}\right)=2\pi$, and $\lim\limits_{t\to\infty} y(t)=\lim\limits_{t\to\infty}\dfrac{12t}{t^2+1}=0$. Thus the particle approaches the point $(2\pi,0)$.

17. (a) To find the smallest rectangle with sides parallel to the x- and y-axes, you need a rectangle formed by vertical and horizontal tangents as shown in the figure. The vertical tangents are at the x-intercepts, $x = \pm 3$. The horizontal tangents are at the points where y (not r) is a maximum. You need, therefore, to maximize

$$y=r\sin\theta=(2+\cos 2\theta)\sin\theta,$$

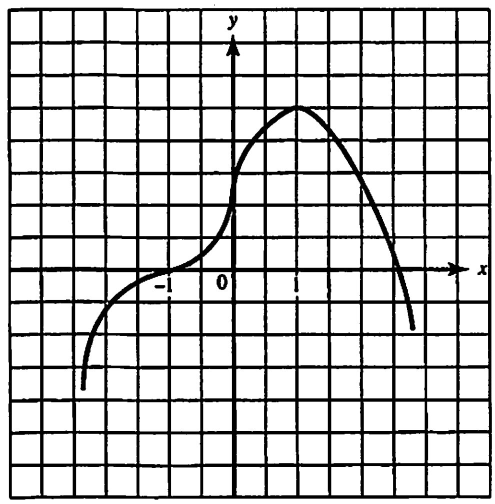
$$\frac{dy}{d\theta}=(2+\cos 2\theta)\cos\theta+\sin\theta(-2\sin 2\theta).$$

Use the calculator to find that $\dfrac{dy}{d\theta}=0$ when $\theta = 0.7854$. Therefore, $y = 1.414$, so the desired rectangle has dimensions 6×2.828.

(b) Since the polar formula for the area is $\dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} r^2\,d\theta$, the area of R (enclosed by r) is $4\cdot\dfrac{1}{2}\displaystyle\int_0^{\pi/2} r^2\,d\theta$, which is 14.137.

Part B

18. The graph shown below satisfies all five conditions. So do many others!



19.

(a) f' is defined for all x in the interval. Since f is therefore differentiable, it must also be continuous.

(b) Because $f'(2) = 0$ and f' changes from negative to positive there, f has a relative minimum at $x = 2$. To the left of $x = 9$, f' is negative, so f is decreasing as it approaches that endpoint and reaches another relative minimum there.

(c) Because f' is negative to the right of $x = -3$, f decreases from its left endpoint, indicating a relative max there. Also, $f'(2) = 0$ and f' changes from positive to negative there, so f has a relative minimum at $x = 7$.

(d) Note that $f(7) - f(-3) = \int_{-3}^{7} f'(x)\,dx$. Since there is more area above the x-axis than below the x-axis on $[-3,7]$, the integral is positive and $f(7) - f(-3) > 0$. This implies that $f(7) > f(-3)$, and that the absolute maximum occurs at $x = 7$.

(e) At $x = 2$ and also at $x = 6$, f' changes from increasing to decreasing, indicating that f changes from concave upward to concave downward at each. At $x = 4$, f' changes from decreasing to increasing, indicating that f changes from concave downward to concave upward there. Hence the graph of f has points of inflection at $x = 2$, 4, and 6.

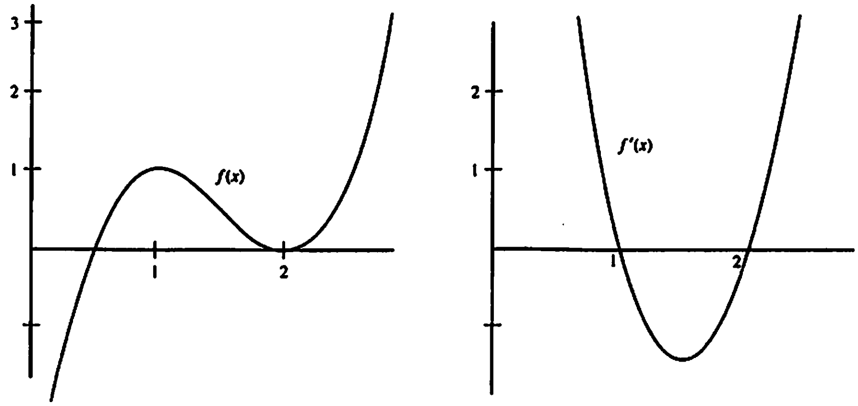
20. Draw a sketch of the region bounded above by $y_1 = 8 - 2x^2$ and below by $y_2 = x^2 - 4$, and inscribe a rectangle in this region as described in the question. If (x, y_1) and (x, y_2) are the vertices of the rectangle in quadrants I and IV, respectively, then the area

$$A = 2x(y_1 - y_2) = 2x(12 - 3x^2), \quad \text{or} \quad A(x) = 24x - 6x^3.$$

Then $A'(x) = 24 - 18x^2 = 6(4 - 3x^2)$, which equals 0 when $x = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Check to verify that $A''(x) < 0$ at this point. This assures that this value of x yields maximum area, which is given by $\frac{4\sqrt{3}}{3} \times 8$.

21. The graph of $f'(x)$ is shown here.

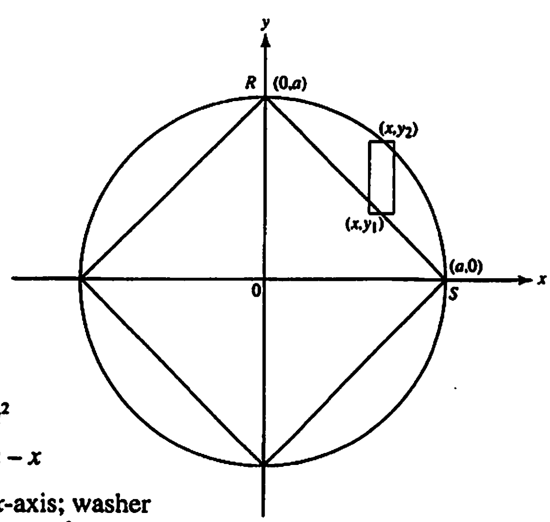


22. The rate of change in volume when the surface area is 54 ft³ is $-\frac{3}{8}$ ft³/sec.

23. See the figure. The equation of the circle is $x^2 + y^2 = a^2$; the equation of RS is $y = a - x$. If y_2 is an ordinate of the circle and y_1 of the line, then,

$$\Delta V = \pi y_2{}^2 \Delta x - \pi y_1{}^2 \Delta x,$$

$$V = 2\pi \int_0^a [(a^2 - x^2) - (a - x)^2]dx = \frac{2}{3}\pi a^3.$$



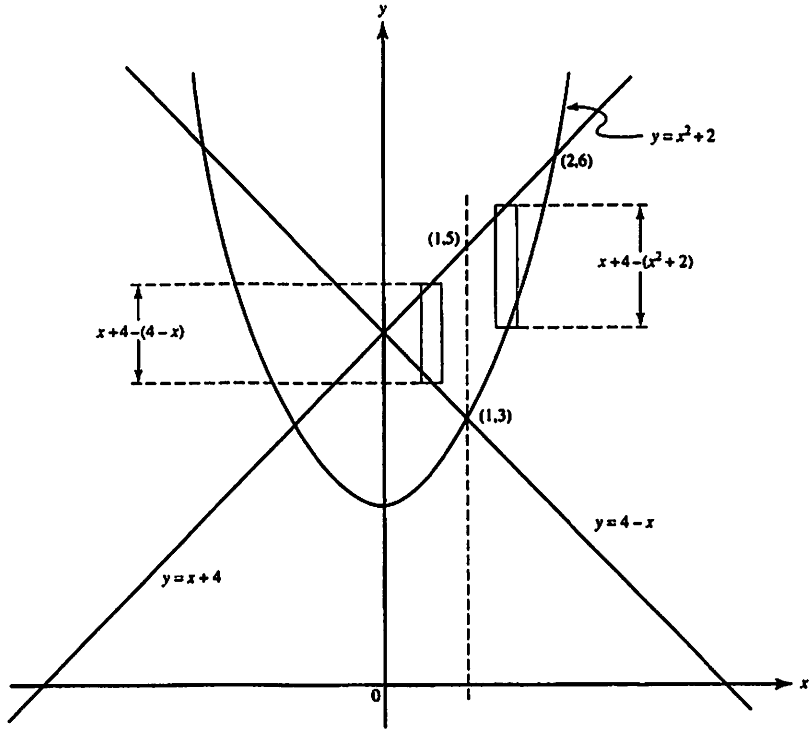
$x^2 + y_2^2 = a^2$

$y_1 = a - x$

About the x-axis; washer

$\Delta V = \pi y_2^2\, \Delta x - \pi y_1^2\, \Delta x$

24. (a) The region is sketched in the figure. The pertinent points of intersection are labeled.



(b) The required area consists of two parts. The area of the triangle is represented by $\int_1^2 [(x+4)-(4-x)]\,dx$ and is equal to 1, while the area of the region bounded at the left by $x = 1$, above by $y = x + 4$, and at the right by the parabola is represented by $\int_1^2 [(x+4)-(x^2+2)]\,dx$. This equals

$$\int_1^2 (x+2-x^2)\,dx = \frac{x^2}{2}+2x-\frac{x^3}{3}\bigg|_1^2 = \frac{7}{6}.$$

The required area, thus, equals $2\frac{1}{6}$ or $\frac{13}{6}$.

25. (a) 1975 to 1976 and 1978 to 1980.
(b) 1975 to 1977 and 1979 to 1981.
(c) 1976 to 1977 and 1980 to 1981.

26. (a) Since $a = \frac{dv}{dt} = -2v$, then, separating variables, $\frac{dv}{v} = -2dt$. Integrating gives

$$\ln v = -2t + C, \qquad (1)$$

and, since $v = 20$ when $t = 0$, $C = \ln 20$. Then (1) becomes $\ln \frac{v}{20} = -2t$ or, solving for v,

$$v = 20e^{-2t}. \qquad (2)$$

(b) Note that $v > 0$ for all t. Let s be the required distance traveled (as v decreases from 20 to 5); then

$$s = \int_{v=20}^{v=5} 20e^{-2t}\,dt = \int_{t=20}^{\ln=2} 20e^{-2t}\,dt, \qquad (3)$$

where, when $v = 20$, $t = 0$. Also, when $v = 5$, use (2) to get $\frac{1}{4} = e^{-2t}$ or $-\ln 4 = -2t$. So $t = \ln 2$. Evaluating s in (3) gives

$$s = -10e^{-2t}\Big|_0^{\ln 2} = -10\left(\frac{1}{4} - 1\right) = \frac{15}{2}.$$

27. Let (x,y) be the point in the first quadrant where the line parallel to the x-axis meets the parabola. The area of the triangle is given by

$$A = xy = x(27 - x^2) = 27x - x^3 \text{ for } 0 \le x \le 3\sqrt{3}.$$

Then $A'(x) = 27 - 3x^2 = 3(3 + x)(3 - x)$, and $A'(x) = 0$ at $x = 3$.

Since A' changes from positive to negative at $x = 3$, the area reaches its maximum there.

The maximum area is $A(3) = 3(27 - 3^2) = 54$.

28. (a) Let $f(x) = \ln(1 + x)$. Then $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$, and $f^{(4)}(x) = -\frac{3!}{(1+x)^4}$, $f^{(5)}(x) = \frac{4!}{(1+x)^5}$. At $x = 0$, $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -(3!)$, and $f^{(5)}(0) = 4!$. So

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

(b) Using the Ratio Test, you know that the series converges when

$\lim\limits_{x\to\infty}\left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| < 1$, that is, when $|x| < 1$, or $-1 < x < 1$. Thus, the radius of convergence is 1.

(c) $\ln(1.2) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5}$.

(d) Since the series converges by the Alternating Series Test, the error in the answer for (c) is less absolutely than $\frac{(0.2)^6}{6}$.

29. From the equations for x and y,

$$dx = (1 - \cos\theta)\, d\theta \quad \text{and} \quad dy = \sin\theta\, d\theta.$$

(a) The slope at any point is given by $\frac{dy}{dx}$, which here is $\frac{\sin\theta}{1-\cos\theta}$. When $\theta = \frac{2\pi}{3}$, the slope is $\frac{\sqrt{3}}{3}$.

(b) When $\theta = \frac{2\pi}{3}$, $x = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ and $y = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$. The equation of the tangent is $9y - 3\sqrt{3} \cdot x = 18 - 2\pi\sqrt{3}$.

30. Both curves are circles with centers at, respectively, (2, 0) and $\left(2, \frac{\pi}{2}\right)$; the circles intersect at $\left(2\sqrt{2}, \frac{\pi}{4}\right)$. The common area is given by

$$\int_0^{\pi/4} (4\sin\theta)^2\, d\theta \text{ or } \int_{\pi/4}^{\pi/2} (4\cos\theta)^2\, d\theta.$$

The answer is $2(\pi - 2)$.

31. (a) For $f(x) = \cos x, f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, f^{(4)}(x) = \cos x, f^{(5)}(x) = -\sin x, f^{(6)}(x) = -\cos x$. The Taylor polynomial of order 4 about 0 is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

Note that the next term of the alternating Maclaurin series for $\cos x$ is $-\frac{x^6}{6!}$.

(b) $\int_0^1 \cos x\, dx = x - \frac{x^3}{3 \cdot 2!} + \frac{x^5}{5 \cdot 4!}\Big|_0^1 = 1 - \frac{1}{6} + \frac{1}{120}$.

(c) The error in (b), convergent by the Alternating Series Test, is less absolutely than the first term dropped:

$$\text{error} < \int_0^1 \frac{x^6}{6!} dx = \frac{x^7}{7!}\Big|_0^1 = \frac{1}{7!}.$$

32. (a) Since $\frac{dy}{dt} = 2$, $y = 2t + 1$ and $x = 4t^3 + 6t^2 + 3t$.

(b) Since $\frac{d^2y}{dt^2} = 0$ and $\frac{d^2x}{dt^2} = 24t + 12$, then, when $t = 1$, $|\mathbf{a}| = 36$.

33. See the figure. The required area A is twice the sum of the following areas: that of the limaçon from 0 to $\frac{\pi}{3}$, and that of the circle from $\frac{\pi}{3}$ to $\frac{\pi}{2}$. Thus

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3}(2-\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2\,d\theta\right]$$

$$= \frac{9\pi}{4} - 3\sqrt{3}\,.$$