7.1 – Integrals as a Net Change

Distance versus Displacement

We have already seen how the position of an object can be found by finding the integral of the velocity function. The change in position is a *displacement.* To see the difference between distance and displacement, consider the following saying:

"Two steps forward and one step back"

Example 1: What is the total distance traveled? 3 steps What is the total displacement? 1 step (forward)

To find displacement, we only need to find $\int_a^b v(t)\,dt$.

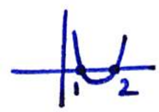
In order to find your new location, we say that your *new position = initial position + displacement.*

To find total distance we use $\int_a^b |v(t)|\,dt$ or find when the object is moving in the negative direction, break the integral into pieces and subtract the value of the integral for the area under the curve.

Example 2: Suppose the velocity of a particle moving along the x – axis is given by $v(t) = 6t^2 - 18t + 12$ when $0 \le t \le 2$.

a) When is the particle moving to the right? When is it moving left? When is it stopped?

- moving to the right when $v(t) > 0$. $v(t) = 6(t-2)(t-1)$ i.e for $t \in [0,1)$



- moving to the left when $v(t) < 0$, i.e for $t \in (1,2)$
- stopped when $t=1$ and $t=2$.

b) Find the particle's displacement for the time interval.

$$\int_0^2 (6t^2 - 18t + 12)\,dt = 2t^3 - 9t^2 + 12t\Big]_0^2 = 4 \text{ units.}$$

c) Find the particle's total distance traveled by setting up ONE integral and using your calculator.

$$\int_0^2 |6t^2 - 18t + 12|\,dt = 6 \text{ units}$$

d) Find the particle's total distance traveled without using absolute value.

$$\int_0^1 (6t^2 - 18t + 12)\,dt + \int_1^2 (-6t^2 + 8t - 12)\,dt = 2t^3 - 9t^2 + 12t\Big]_0^1 - 2t^3 + 9t^2 - 12t\Big]_1^2$$

$$= 5 - 4 + 5$$

$$= 6$$

Consumption over Time

Velocity is not the only rate in which you can integrate to get a total. In fact if you were given a function that gave the number of tickets per hour that the police wrote each day, and you wanted to find the total number of tickets in a 24-hour period, you could integrate.

Example 3: The tide removes sand from Sandy Point Beach at a rate modeled by the function R given by

$$R(t)=2+5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t)=\frac{15t}{1+3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

$$\int_0^6 \left(2+5\sin\left(\frac{4\pi t}{25}\right)\right)dt = \left[2t - \frac{125}{4\pi}\cos\left(\frac{4\pi t}{25}\right)\right]_0^6$$

$$= 12 - \frac{125}{4\pi}\cos\left(\frac{24\pi}{25}\right) + \frac{125}{4\pi} \quad \text{cubic yards}$$

b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

$$Y(t) = -\int_0^t 2+5\sin\left(\frac{4\pi x}{25}\right)dx + \int_0^t \frac{15x}{1+3x}\,dx + 2500 \quad \text{cubic yards}$$

c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

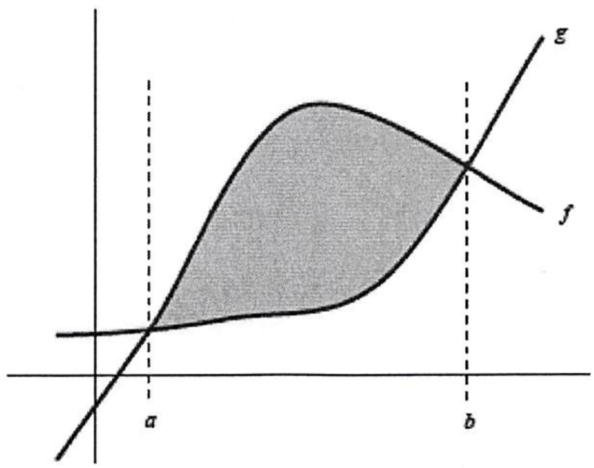
$$Y'(4) = -R(4) + S(4) = -2-5\sin\left(\frac{16\pi}{25}\right) + \frac{60}{13} = \frac{34}{13} - 5\sin\left(\frac{16\pi}{25}\right) \quad \text{cubic yards/hour}$$

7.2 – Areas in the Plane

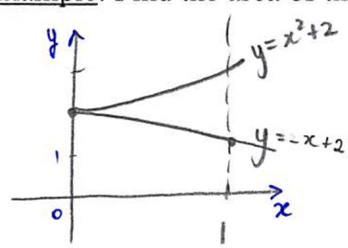
Area of a Region Between Two Curves

If f and g are continuous on [a, b] and $g(x) \leq f(x)$ for all x in [a, b], then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$



Example: Find the area of the region bounded by the graphs of $y = x^2 + 2$, ~~$y = x$~~, $x = 0$ and $x = 1$.
$y = -x + 2$



$$A = \int_0^1 \left(x^2 + 2 - (-x + 2)\right) dx$$

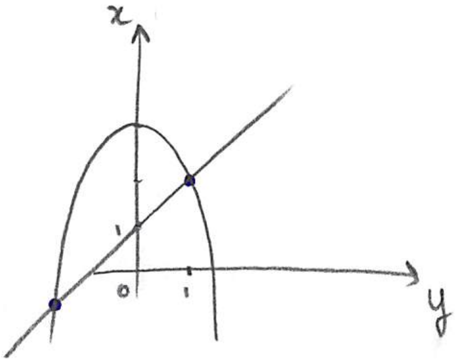
$$= \int_0^1 (x^2 + x)\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} \Big]_0^1$$

$$= \frac{5}{6}$$

Note: It is very important to start with a rough graph in order to know which one is "above"…

Example: Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.



Intersections: $3 - y^2 = y + 1$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y = -2 \quad y = 1$

$$A = \int_{-2}^{1} \left(3 - y^2 - (y+1)\right) dy$$

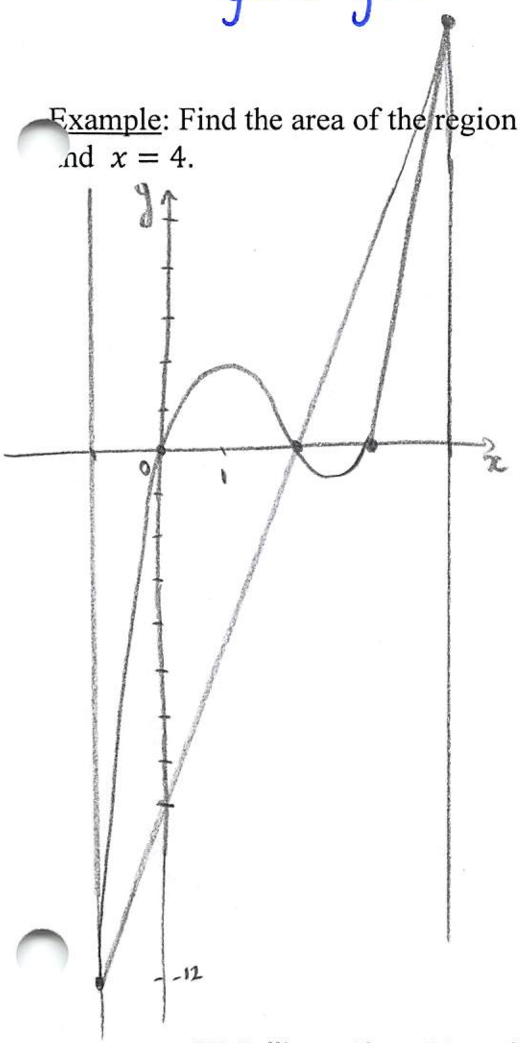
$$= \int_{-2}^{1} (-y^2 - y + 2)\, dy$$

$$= -\frac{y^3}{3} - \frac{y^2}{2} + 2y \Big]_{-2}^{1}$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + \frac{4}{2} + 4$$

$$= \boxed{\frac{9}{2}}$$

Example: Find the area of the region bounded by the graphs of $y = x(x-2)(x-3)$, $y = 4x - 8$, $x = -1$ and $x = 4$.



$$A = \int_{-1}^{2} \left(x(x-2)(x-3) - (4x-8)\right) dx + \int_{2}^{4} \left(4x - 8 - x(x-2)(x-3)\right) dx$$

$$= \int_{-1}^{2} (x^3 - 5x^2 + 2x + 8)\, dx + \int_{2}^{4} (-x^3 + 5x^2 - 2x - 8)\, dx$$

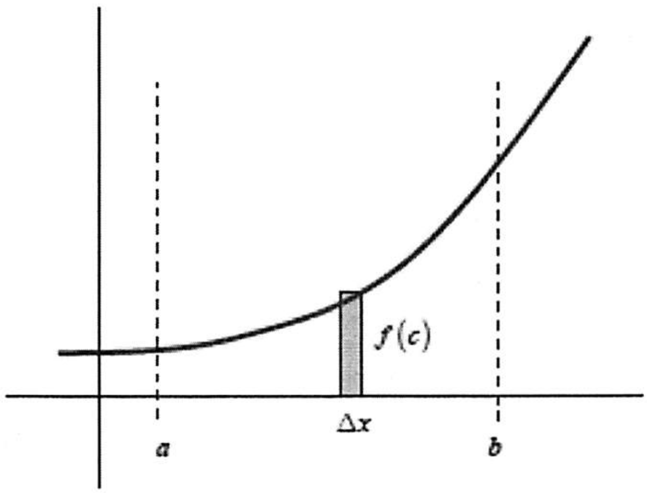
$$= \frac{x^4}{4} - \frac{5}{3}x^3 + x^2 + 8x \Big]_{-1}^{2} + -\frac{1}{4}x^4 + \frac{5}{3}x^3 - x^2 - 8x \Big]_{2}^{4}$$

$$= 4 - \frac{40}{3} + 4 + 16 - \frac{1}{4} - \frac{5}{3} - 1 + 8 - 64 + \frac{320}{3} - 16 - 32 + 4 - \frac{40}{3} + 4 + 16$$

$$= \boxed{+\frac{253}{12}}$$

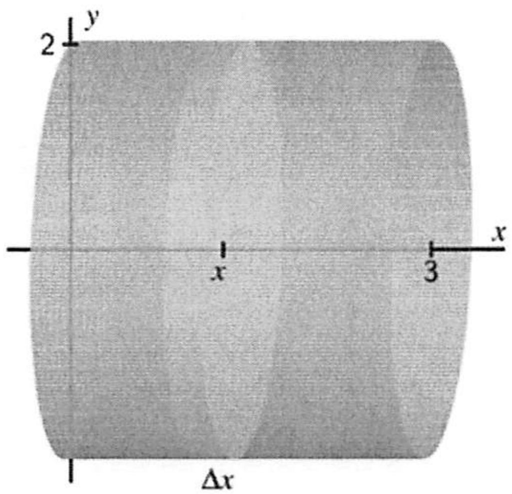
7.3 - Volumes

We already know that the integral can be seen as a sum of thin strips with values $f(x)$ and width dx.



$$Area = \int_a^b f(x)dx$$

The same way, it can calculate volumes if we add thin slices with a base area $A(x)$ and width dx.

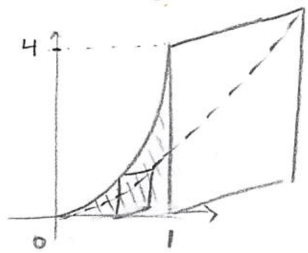


$$Volume = \int_a^b A(x)dx$$

We will need to determine the area of a cross section (as a function of x most of the time, but sometimes y) and just integrate it…

In the previous example, $A(x) = 4\pi$, and you get: $V = \int_0^3 4\pi dx$

Example:
The base of a solid is the region in the first quadrant enclosed by the parabola $y = 4x^2$, the line $x = 1$, and the x-axis. Each plane section of the solid perpendicular to the x-axis is a square. Determine the volume of the solid.

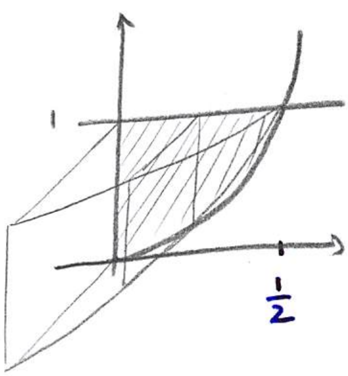


$$V = \int_0^1 (4x^2)^2 \, dx$$

$$= 16 \left. \frac{x^5}{5} \right]_0^1$$

$$\boxed{V = \frac{16}{5}}$$

Example:
The base of a solid is the region enclosed by the parabola $y = 4x^2$, the line $y = 1$, and the y-axis.
Each plane section of the solid perpendicular to the x-axis is a square. Determine the volume of the solid.



Intersection : $4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = \pm \frac{1}{2} \Rightarrow x = \frac{1}{2}$

$$V = \int_0^{\frac{1}{2}} (1 - 4x^2)^2 \, dx$$

$$= \int_0^{\frac{1}{2}} (1 - 8x^2 + 16x^4) \, dx$$

$$= x - \frac{8}{3}x^3 + \frac{16}{5}x^5 \Big]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{10}$$

$$\boxed{V = \frac{4}{15}}$$

$= [illegible]$

$= [illegible]$

$= \int [illegible]$

$V = \int [illegible]$

[illegible]

[illegible] : [illegible]

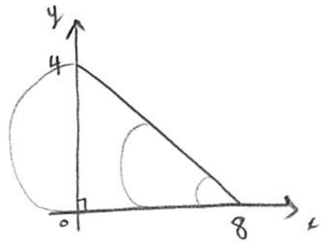
$A = [illegible]$

$= [illegible]$

$A = \int [illegible]$

Example:
The base of a solid is a region in the first quadrant bounded by the x-axis and the line $x + 2y = 8$.
If the cross-section of the solid perpendicular to the x-axis are semicircles. What is the volume of the solid to the nearest hundredth?

$y = -\frac{1}{2}x + 4$



$$V = \int_0^8 \frac{1}{2}\pi \left(-\frac{1}{4}x + 2\right)^2 dx$$

$$= \frac{1}{2}\pi \int_0^8 \left(\frac{x^2}{16} - x + 4\right) dx$$

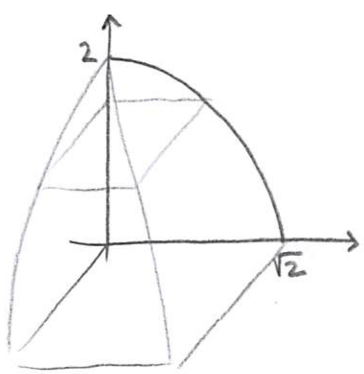
$$= \frac{1}{2}\pi \left[\frac{x^3}{48} - \frac{x^2}{2} + 4x\right]_0^8$$

$$\boxed{V = \frac{16\pi}{3}}$$

Example: The base of a solid is the region in the first quadrant enclosed by the graph of $y = 2 - x^2$ and the coordinate axes. If every cross section of the solid perpendicular to the y-axis is a square, what integral gives you the volume of the solid?

$x^2 = 2 - y$

$x = \sqrt{2 - y}$ (because we're in the 1st quadrant)



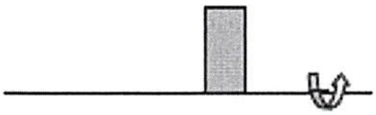
$$V = \int_0^2 (2 - y)\, dy$$

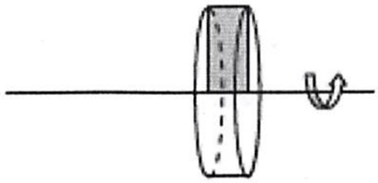
$$= 2y - \frac{y^2}{2}\Big]_0^2$$

$$= 4 - 2$$

$$\boxed{V = 2}$$

Solids of Revolution:





Example: What is the volume of the cylinder formed by revolving the graph of $y = 3$ around the x-axis on the interval [0;4]?

$$V = \int_0^4 \pi(3)^2 dx \qquad V = \pi\int_0^4 3^2 dx$$

$$= 9\pi x\Big]_0^4 = \boxed{36\pi}$$

Note: when we rotate a curve around an axis, the cross-section is always a circle. Therefore, $A(x) = \pi(f(x))^2$. $f(x)$ is then the radius of the cross-section, and we usually note it $R(x)$.

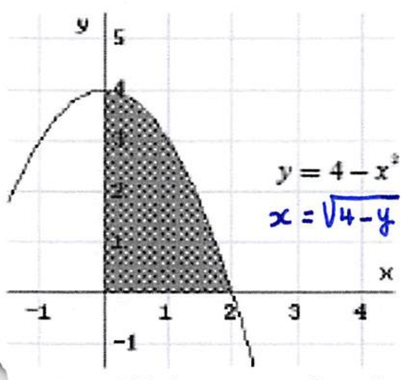
The Disc Method

To find the volume of a solid of revolution with the *disc method*, use one of the following;

HORIZONTAL AXIS OF REVOLUTION	VERTICAL AXIS OF REVOLUTION
$V = \pi\int_a^b [R(x)]^2\, dx$	$V = \pi\int_a^b [R(y)]^2\, dy$

where $R(x)$ and $R(y)$ are the "heights" of your representative rectangular strips.

Example: a) Determine the volume of the solid formed by revolving the following region about the x-axis.



b) Same question about the y-axis.

a) $V = \pi\int_0^2 (4-x^2)^2 dx$

$= \pi\int_0^2 (16 - 8x^2 + x^4) dx$

$= \pi\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$

$= \pi\left(32 - \frac{64}{3} + \frac{32}{5}\right)$

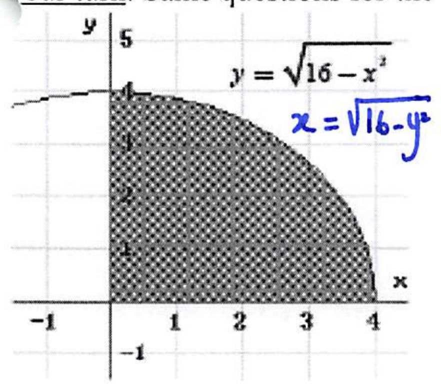
$\boxed{V = \frac{256\pi}{15}}$

b) $V = \pi\int_0^4 (4-y)\,dy$

$= \pi\left(4y - \frac{y^2}{2}\right]_0^4$

$\boxed{V = 8\pi}$

Your turn: Same questions for the following region:



a) $V = \pi \int_0^4 (16-x^2)\,dx$

$= \pi \left(16x - \frac{x^3}{3}\right]_0^4$

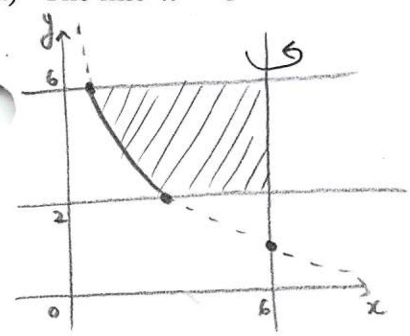
$\boxed{V = \frac{128\pi}{3}}$

b) $V = \pi \int_0^4 (16-y^2)\,dy$

$\boxed{V = \frac{128\pi}{3}}$

Example: Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $xy = 6,\ y = 2,\ y = 6$ and $x = 6$ about the indicated lines.

a) The line $x = 6$



$x = \frac{6}{y}$

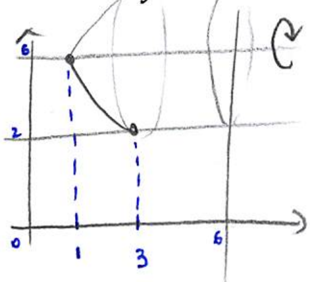
$V = \pi \int_2^6 \left(6 - \frac{6}{y}\right)^2 dy$

$= \pi \int_2^6 \left(36 - \frac{72}{y} + \frac{36}{y^2}\right) dy$

$= \pi \left(36y + 72 \ln y - \frac{36}{y}\right]_2^6$

$\boxed{V = (-72 \ln 3 + 156)\pi}$

b) The line $y = 6$



$y = \frac{6}{x}$

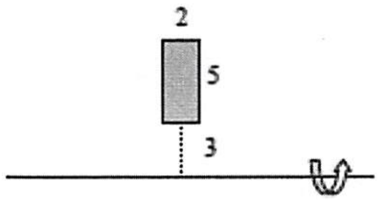
$V = \pi \int_1^3 \left(6 - \frac{6}{x}\right)^2 dx + \pi \int_3^6 4^2\,dx$

$= \pi \left(36x + 72 \ln x - \frac{36}{x}\right]_1^3 + 16\pi x\Big]_3^6$

$= (-72 \ln 3 + 96)\pi + 48\pi$

$\boxed{V = (-72 \ln 3 + 144)\pi}$

The Washer method:



The volume of the solid formed by revolving a region around the axis using the washer method is given by

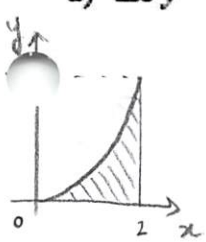
$$\pi\int_a^b \left[R^2 - r^2\right] dx$$

Examples: Determine the integral that represents the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated lines:

$$y = 2x^2; \quad y = 0; \quad x = 2$$

a) the y – axis

$x = \sqrt{\frac{y}{2}}$



$$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right) dy$$

$$= \pi\left[4y - \frac{y^2}{4}\right]_0^8$$

$$\boxed{V = 16\pi}$$

b) the x – axis

$$V = \pi\int_0^2 4x^4\,dx$$

$$= \pi\left[\frac{4}{5}x^5\right]_0^2$$

$$\boxed{V = \frac{128\pi}{5}}$$

c) the line $y = 8$

$$V = \pi\int_0^2 \left(8^2 - (8 - 2x^2)^2\right) dx$$

$$= \pi\int_0^2 \left(32x^2 - 4x^4\right) dx$$

$$= \pi\left[\frac{32x^3}{3} - \frac{4}{5}x^5\right]_0^2$$

$$\boxed{V = \frac{896\pi}{15}}$$

d) the line $x = 2$

$$V = \pi\int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi\int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$$

$$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8$$

$$\boxed{V = \frac{16\pi}{3}}$$