Answer Key

AB PRACTICE EXAMINATION 1

Part A

1. C	**8.** A	**15.** D	**22.** C
2. C	**9.** A	**16.** A	**23.** B
3. B	**10.** E	**17.** B	**24.** A
4. B	**11.** C	**18.** B	**25.** D
5. B	**12.** D	**19.** B	**26.** B
6. E	**13.** A	**20.** B	**27.** B
7. E	**14.** E	**21.** A	**28.** C

Part B

29. B	**34.** D	**38.** E	**42.** D
30. E	**35.** D	**39.** E	**43.** E
31. E	**36.** A	**40.** A	**44.** B
32. D	**37.** D	**41.** B	**45.** C
33. D			

ANSWERS EXPLAINED

Multiple-Choice

Part A

1. (C) Use the Rational Function Theorem on page 96.

2. (C) Note that $\lim_{h\to 0} \frac{\ln(2+h)-\ln 2}{h} = f'(2)$, where $f(x) = \ln x$.

3. (B) Since $y' = -2xe^{-x^2}$, therefore $y'' = -2(x \cdot e^{-x^2} \cdot (-2x) + e^{-x^2})$. Replace x by 0.

4. (B) $\frac{f(4)-f(1)}{4-1} = \frac{6-2}{4-1} = \frac{4}{3}$.

5. (B) $h'(3) = g'(f(3)) \cdot f'(3) = g'(4) \cdot f'(3) = \frac{1}{2} \cdot 2$.

6. (E) Since $f'(x)$ exists for all x, it must equal 0 for any x_0 for which f is a relative maximum, and it must also change sign from positive to negative as x increases through x_0. For the given derivative, no x satisfies both of these conditions.

7. (E) By the Quotient Rule (formula (6) on page 113),

$$\frac{dy}{dx} = \frac{(2-5x)(1)-(x-3)(-5)}{(2-5x)^2}.$$

8. (A) Here, $f'(x)$ is $e^{-x}(1-x)$; f has maximum value when $x = 1$.

9. (A) Note that (1) on a horizontal line the slope segments are all parallel, so the slopes there are all the same and $\frac{dy}{dx}$ must depend only on y; (2) along the x-axis (where $y = 0$) the slopes are infinite; and (3) as y increases, the slope decreases.

10. (E) Acceleration is the derivative (the slope) of velocity v; v is largest on $8 < t < 9$.

11. (C) Velocity v is the derivative of position; because $v > 0$ until $t = 6$ and $v < 0$ thereafter, the position increases until $t = 6$ and then decreases; since the area bounded by the curve above the axis is larger than the area below the axis, the object is farthest from its starting point at $t = 6$.

12. (D) From $t = 5$ to $t = 8$, the displacement (the integral of velocity) can be found by evaluating definite integrals based on the areas of two triangles: $\frac{1}{2}(1)(2) - \frac{1}{2}(2)(4) = -3$. Thus, if K is the object's position at $t = 5$, then $K - 3 = 10$ at $t = 8$.

13. (A) The integral is of the form $\int u^3 du$; evaluate $\frac{1}{4}\sin^4 \alpha \Big|_{\pi/4}^{\pi/2}$.

14. **(E)** $-\int_0^1 (3-e^x)^{-2}(-e^x dx) = \left.\frac{1}{3-e^x}\right|_0^1 = \frac{e-1}{2(3-e)}.$

15. **(D)** $f'(2.1) \simeq \frac{f(2.2)-f(2.0)}{2.2-2.0}.$

16. **(A)** $f(x) = e^{-x}$ is decreasing and concave upward.

17. **(B)** Implicit differentiation yields $2yy' = 1$; so $\frac{dy}{dx} = \frac{1-3x^2}{2y}$. At a vertical tangent, $\frac{dy}{dx}$ is undefined; y must therefore equal 0 and the numerator be non-zero. The original equation with $y = 0$ is $0 = x - x^3$, which has three solutions.

18. **(B)** Let $t = x - 1$; then $t = -1$ when $x = 0$, $t = 5$ when $x = 6$, and $dt = dx$.

19. **(B)** The required area, A, is given by the integral

$$2\int_0^1 \left(4 - \frac{4}{1+x^2}\right)dx = 2(4x - 4\tan^{-1} x)\Big|_0^1 = 2\left(4 - 4\cdot\frac{\pi}{4}\right).$$

20. **(B)** The average value is $\frac{1}{10-0}\int_0^{10} f(x)dx$. The definite integral represents the sum of the areas of a trapezoid and a rectangle: $\frac{1}{2}(8 + 3)(6) = 4(7) = 61$.

21. **(A)** Solve the differential equation $\frac{dy}{dx} = 2y$ by separation of variables: $\frac{dy}{y} = 2dx$ yields $y = ce^{2x}$. The initial condition yields $1 = ce^{2\cdot 2}$; so $c = e^{-4}$ and $y = e^{2x-4}$.

22. **(C)** Changes in values of f'' show that f''' is constant; hence f'' is linear, f' is quadratic, and f must be cubic.

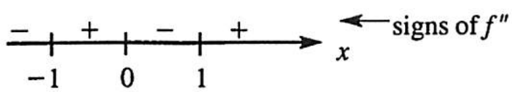
23. **(B)** By implicit differentiation, $3x^2 + x\sec^2 y\frac{dy}{dx} + \tan y = 0$. At (3,0), $\frac{dy}{dx} = -9$; so the equation of the tangent line at (3,0) is $y = -9(x-3)$.

24. **(A)** $(h^{1/2})' = 2h'$ implies $\frac{1}{2}h^{-1/2} = 2$.

25. **(D)** The graph shown has the x-axis as a horizontal asymptote.

26. **(B)** Since $\lim_{x\to 1} f(x) = 1$, to render $f(x)$ continuous at $x = 1$ $f(1)$ must be defined to be 1.

27. **(B)** $f'(x) = 15x^4 - 30x^2$; $f''(x) = 60x^3 - 60x = 60x(x + 1)(x - 1)$; this equals 0 when $x = -1$, 0, or 1. Here are the signs within the intervals:



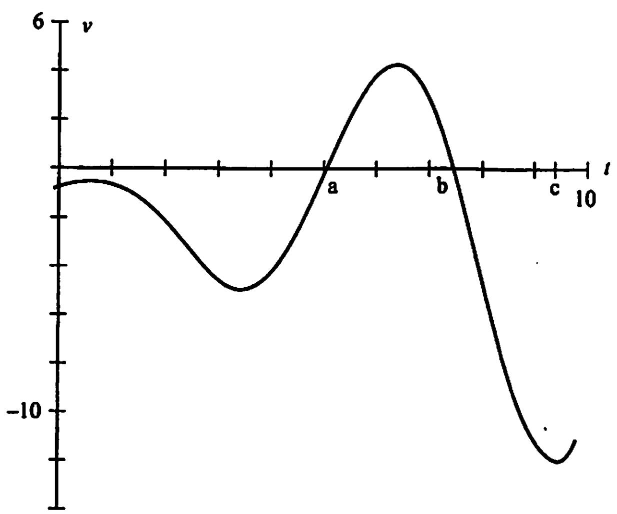
The graph of f changes concavity at $x = -1$, 0, and 1.

28. (C) Note that $f'(x) = \frac{4+x}{x^2+4}$, so f has a critical value at $x = -4$. As x passes through -4, the sign of f' changes from $-$ to $+$, so f has a local minimum at $x = -4$.

Part B

29. (B) We are given that (1) $f'(a) > 0$; (2) $f''(a) < 0$; and (3) $G'(a) < 0$. Since $G'(x) = 2f(x) \cdot f'(x)$, therefore $G'(a) = 2f(a) \cdot f'(a)$. Conditions (1) and (3) imply that (4) $f(a) < 0$. Since $G''(x) = 2[f(x) \cdot f''(x) + (f'(x))^2]$, therefore $G''(a) = 2[f(a)f''(a) + (f'(a))^2]$. Then the sign of $G''(a)$ is $2[(-) \cdot (-) + (+)]$ or positive, where the minus signs in the parentheses follow from conditions (4) and (2).

30. (E) Since $f'(x) = 1 - \frac{c}{x^2}$, it equals 0 for $x = \pm\sqrt{c}$. When $x = 3$, $c = 9$; this yields a minimum since $f''(3) > 0$.



31. (E) Use your calculator to graph velocity against time. Speed is the absolute value of velocity. The greatest deviation from $v = 0$ is at $t = c$. With a calculator, $c = 9.538$.

32. (D) Because f' changes from increasing to decreasing, f'' changes from positive to negative and thus the graph of f changes concavity.

33. (D) $H(3) = \int_0^3 f(t)dt$. We evaluate this definite integral by finding the area of a trapezoid (negative) and a triangle: $H(3) = -\frac{1}{2}(2 + 1)(2) + \frac{1}{2}(1)(2) = -2$, so the tangent line passes through the point $(3,-2)$. The slope of the line is $H'(3) = f(3) = 2$, so an equation of the line is $y - (-2) = 2(x - 3)$.

34. **(D)** The distance is approximately 14(6) + 8(2) + 3(4).

35. **(D)** $\int_0^8 R(x)\,dx = 166.396.$

36. **(A)** Selecting an answer for this question from your calculator graph is unwise. In some windows the graph may appear continuous; in others there may seem to be cusps, or a vertical asymptote. Put the calculator aside. Find

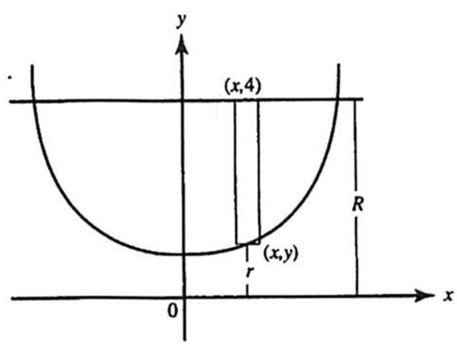
$$\lim_{x\to 1^+}\left(\arctan\left(\frac{1}{\ln x}\right)\right)=\frac{\pi}{2} \text{ and } \lim_{x\to 1^-}\left(\arctan\left(\frac{1}{\ln x}\right)\right)=-\frac{\pi}{2}.$$

These limits indicate the presence of a jump discontinuity in the function at $x = 1$.

37. **(D)** $\frac{d}{du}\int_0^u \frac{1}{1+x^2}\,dx = \frac{1}{1+u^2}$. When $u = t^2$,

$$\frac{d}{dt}\int_0^u \frac{1}{1+x^2}\,dx = \frac{1}{1+u^2}\frac{du}{dt} = \frac{1}{1+t^4}(2t).$$

38. **(E)** $\int(\sqrt{x}-2)x^2\,dx = \int(x^{5/2}-2x^2)\,dx = \frac{2}{7}x^{7/2}-\frac{2}{3}x^3+C.$



39. **(E)** In the figure above, S is the region bounded by $y = \sec x$, the y axis, and $y = 4$. Send region S about the x-axis. Use washers; then $\Delta V = \pi(R^2 - r^2)\,\Delta x$. Symmetry allows you to double the volume generated by the first quadrant of S, so V is

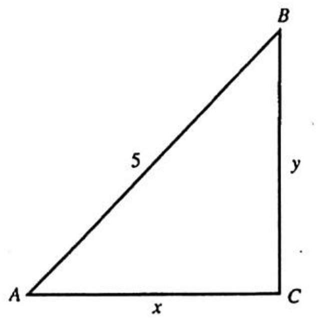
$$2\pi\int_0^{\arccos\frac{1}{4}}(16-\sec^2 x)\,dx.$$

A calculator yields 108.177.

40. **(A)** The curve falls when $f'(x) < 0$ and is concave up when $f''(x) > 0$.

41. **(B)** $g'(y) = \frac{1}{f'(x)} = \frac{1}{5x^4}$. To find $g'(0)$, find x such that $f(x) = 0$. By inspection,

$x = -1$, so $g'(0) = \frac{1}{5(-1)^4} = \frac{1}{5}.$



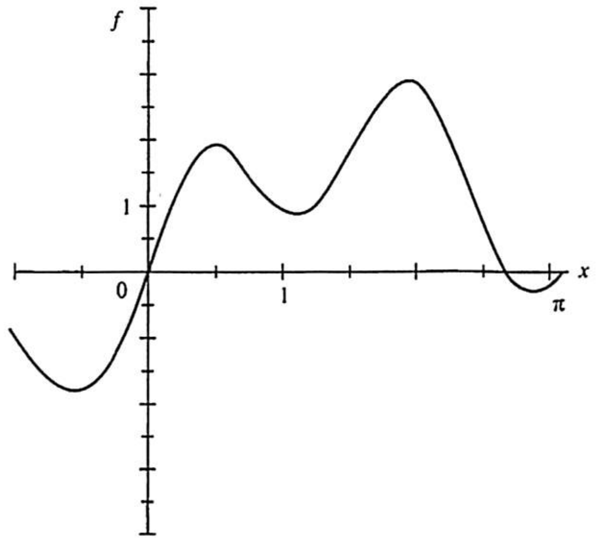
42. **(D)** It is given that $\frac{dx}{dt} = -2$; you want $\frac{dA}{dt}$, where $A = \frac{1}{2}xy$.

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) = \frac{1}{2}\left[3 \cdot \frac{dy}{dt} + y \cdot (-2)\right].$$

Since $y^2 = 25 - x^2$, it follows that $2y\frac{dy}{dt} = -2x\frac{dx}{dt}$ and, when $x = 3$, $y = 4$

and $\frac{dy}{dt} = \frac{3}{2}$.

Then $\frac{dA}{dt} = -\frac{7}{4}$.



The function $f(x) = 2 \sin x + \sin 4x$ is graphed above.

43. **(E)** Since $f(0) = f(\pi)$ and f is both continuous and differentiable, Rolle's Theorem predicts at least one c in the interval such that $f'(c) = 0$.

There are four points in $[0,\pi]$ of the calculator graph above where the tangent is horizontal.

44. **(B)** Since $\frac{dr}{dt} = k$, a positive constant, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi r^2 k = cr^2$, where c is a positive constant. Then $\frac{d^2V}{dt^2} = 2cr\frac{dr}{dt} = 2crk$, which is also positive.

45. **(C)** If $Q(t)$ is the amount of contaminant in the tank at time t and Q_0 is the initial amount, then

$$\frac{dQ}{dt} = kQ \text{ and } Q(t) = Q_0 e^{kt}.$$

Since $Q(1) = 0.8Q_0$, $0.8Q_0 = Q_0 e^{k \cdot 1}$, $0.8 = e^k$, and

$$Q(t) = Q_0(0.8)^t.$$

We seek t when $Q(t) = 0.02Q_0$. Thus,

$$0.02Q_0 = Q_0(0.8)^t$$

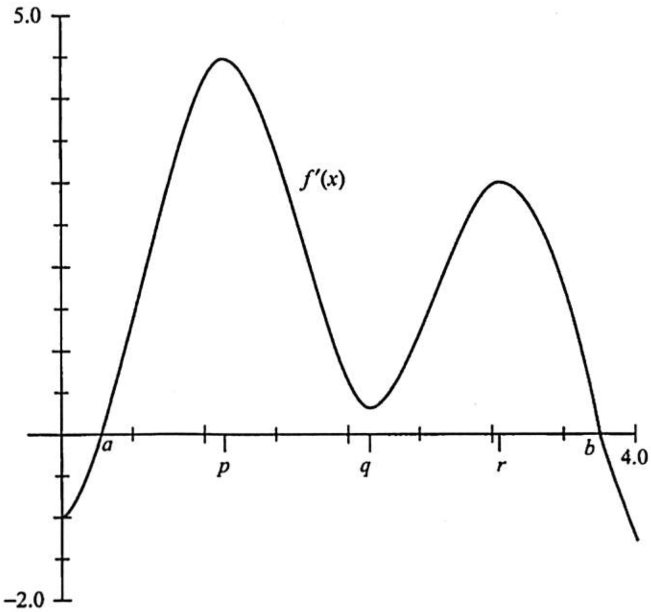
and

$$t \simeq 17.53 \text{ min.}$$

Free-Response

Part A

AB/BC 1. (a) This is the graph of $f'(x)$.



(b) f is increasing when $f'(x) > 0$. The graph shows this to be true in the interval $a < x < b$. Use the calculator to find a and b (where $e^x - 2\cos 3x = 0$); then $a = 0.283 < x < 3.760 = b$.

(c) See signs analysis.

	0		a				b		4
f		dec			inc			dec	
f'		–			+			–	

Since f decreases to the right of endpoint $x = 0$, f has a local maximum at $x = 0$. There is another local maximum at $x = 3.760$, because f changes from increasing to decreasing there.

(d) See signs analysis.

	0		p		q		r		4
f'		inc		dec		inc		dec	
f''		+		–		+		–	
f		conc up		conc down		conc up		conc down	

Since the graph of f changes concavity at p, q, and r, there are three points of inflection.

AB2. (a) Since April 1 is 3 months from January 1 and June 30 is 3 months later, we form the sum for the interval [3,6]:

$$\left(\frac{2.32+3.12}{2}\right)\cdot 1+\left(\frac{3.12+3.78}{2}\right)\cdot 1+\left(\frac{3.78+4.90}{2}\right)\cdot 1 = 10.51$$

We estimate the company sold 1051 software units during the second quarter.

(b) $S(t) = 1.2(2)^{t/3}$

(c) $\int_3^6 1.2(2^{t/3})dt = 10.387$. The model's estimate of 1039 sales is slightly lower, but the two are in close agreement.

(d) $\frac{1}{12}\int_0^{12} 1.2(2^{t/3})dt = 6.492$; the model predicts an average sales rate of 649.2 units per month from January 1, 2012, through December 31, 2012.

Part B

AB 3. (a) At (2,5), $\dfrac{dy}{dx} = \dfrac{6(2^2)-4}{5} = 4$, so the tangent line is $y - 5 = 4(x - 2)$.

Solving for y yields $f(x) \approx 5 + 4(x - 2)$.

(b) $f(2.1) \approx 5 + 4(2.1 - 2) = 5.4$.

(c) The differential equation $\dfrac{dy}{dx} = \dfrac{6x^2 - 4}{5}$ is separable:

$$\int y\,dy = \int (6x^2 - 4)dx,$$

$$\frac{y^2}{2} = 2x^3 - 4x + C,$$

$$y = \pm\sqrt{4x^2 - 8x + c}\text{, where } c = 2C.$$

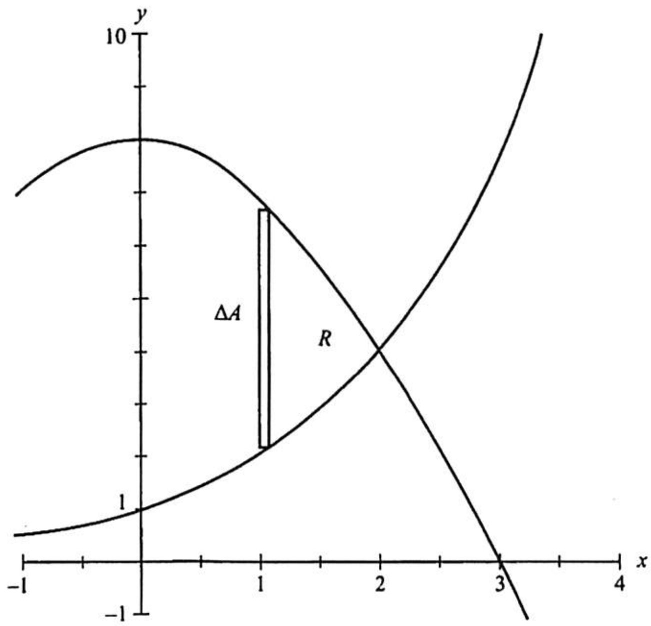
Since f passes through (2,5), it must be true that $5 = \pm\sqrt{4(2^3) - 8(2) + c}$.

Thus $c = 9$, and the positive root is used.

The solution is $f(x) = \sqrt{4x^3 - 8x + 9}$.

(d) $f(2.1) = \sqrt{4(2.1^3) - 8(2.1) + 9} = 5.408$.

AB 4. (a) Draw a vertical element of area, as shown.



$$\Delta A = (y_{top} - y_{bottom})\,\Delta x = \left(8\cos\frac{\pi x}{6} - 2^x\right)\Delta x,$$

$$A = \int_0^2 \left(8\cos\frac{\pi x}{6} - 2^x\right) dx$$

$$= \frac{6}{\pi}\cdot 8\int_0^2 \cos\frac{\pi x}{6}\,dx - \int_0^2 2^x\,dx$$

$$= \frac{48}{\pi}\cdot \sin\frac{\pi x}{6}\Big|_0^2 - \frac{2^x}{\ln 2}\Big|_0^2$$

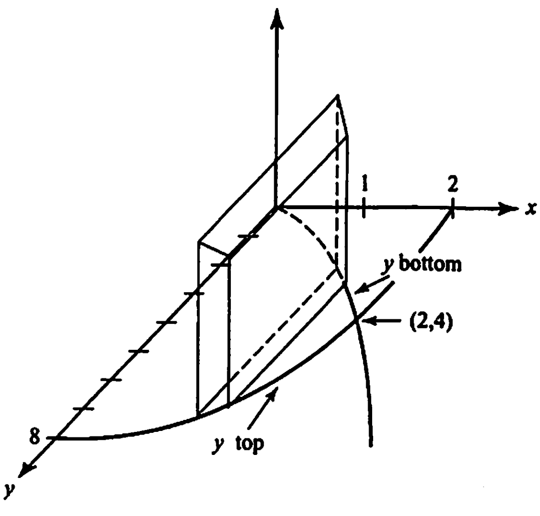
$$= \frac{48}{\pi}\left(\sin\frac{\pi}{3} - \sin 0\right) - \left(\frac{2^2}{\ln 2} - \frac{2^0}{\ln 2}\right)$$

$$= \frac{24\sqrt{3}}{\pi} - \frac{3}{\ln 2}.$$

(b) (i) Use washers; then

$$\Delta V = (r_2^2 - r_1^2)\,\Delta x = \pi(y^2_{top} - y^2_{bottom})\,\Delta x,$$

$$V = \pi\int_0^2 \left[\left(8\cos\frac{\pi x}{6}\right)^2 - (2^x)^2\right] dx.$$



(ii) See the figure above.

$$\Delta V = s^2\Delta x = (y_{top} - y_{bottom})^2\,\Delta x,$$

$$V = \int_0^2 \left(8\cos\frac{\pi x}{6} - 2^x\right)^2 dx.$$

AB/BC 5. (a)
$$f(x) = e^{2x}(x^2-2),$$
$$f'(x) = e^{2x}(2x) + 2e^{2x}(x^2-2)$$
$$= 2e^{2x}(x+2)(x-1)$$
$$= 0 \text{ at } x = -2, 1.$$

f is decreasing where $f'(x) < 0$, which occurs for $-2 < x < 1$.

(b) f is decreasing on the interval $-2 < x < 1$, so there is a minimum at $(1, -e^2)$. Note that, as x approaches $\pm\infty$, $f(x) = e^{2x}(x^2-2)$ is always positive. Hence $(1, -e^2)$ is the global minimum.

(c) As x approaches $+\infty$, $f(x) = e^{2x}(x^2-2)$ also approaches $+\infty$. There is no global maximum.

AB/BC 6. (a) $S = 4\pi r^2$, so $\dfrac{dS}{dt} = 8\pi r \dfrac{dr}{dt}$. Substitute given values; then

$$-10 = 8\pi(6)\frac{dr}{dt}, \text{ so } \frac{dr}{dt} = -\frac{5}{24\pi}\text{cm/min.}$$

Since $V = \dfrac{4}{3}\pi r^3$, therefore $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. Substituting known values gives $\dfrac{dV}{dt} = 4\pi(6^2) \cdot \dfrac{-5}{24\pi} = -30 \text{ cm}^3\text{/min.}$

(b) Regions of consistent density are concentric spherical shells. The volume of each shell is approximated by its surface area ($4\pi x^2$) times its thickness (Δx). The weight of each shell is its density times its volume (g/cm^3·cm^3). If, when the snowball is 12 cm in diameter, ΔG is the weight of a spherical shell x cm from the center, then $\Delta G = \dfrac{1}{1+\sqrt{x}} \cdot 4\pi x^2 \Delta x$, and the integral to find the weight of the snowball is

$$G = \int_0^6 \frac{1}{1+\sqrt{x}} \cdot 4\pi x^2 dx.$$