Answer Key

AB PRACTICE EXAMINATION 2

Part A

1. E	**8.** B	**15.** B	**22.** B
2. A	**9.** B	**16.** E	**23.** E
3. E	**10.** B	**17.** C	**24.** E
4. D	**11.** C	**18.** B	**25.** B
5. B	**12.** A	**19.** B	**26.** A
6. D	**13.** B	**20.** C	**27.** E
7. B	**14.** D	**21.** B	**28.** A

Part B

29. B	**34.** C	**38.** D	**42.** D
30. C	**35.** C	**39.** E	**43.** C
31. E	**36.** D	**40.** E	**44.** E
32. D	**37.** B	**41.** A	**45.** B
33. D			

ANSWERS EXPLAINED

Multiple-Choice

Part A

1. **(E)** $\frac{x^2-2}{4-x^2} \rightarrow +\infty$ as $x \rightarrow 2$.

2. **(A)** Divide both numerator and denominator by $\sqrt{x}$; $\lim_{x\to\infty} \frac{1-\frac{4}{\sqrt{x}}}{\frac{4}{\sqrt{x}}-3} = -\frac{1}{3}$.

3. **(E)** Since $e^{\ln u} = u$, $y = 1$.

4. **(D)** $f(0) = 3$, and $f'(x) = \frac{1}{2}(9 + \sin 2x)^{-1/2} \cdot (2\cos 2x)$, so $f'(0) = \frac{1}{3}$; $y \simeq \frac{1}{3}x + 3$.

5. **(B)** $\int_0^1 \frac{60}{1+t^2}\,dt = 60 \arctan t \Big|_0^1 = 60 \arctan 1 = 60 \cdot \frac{\pi}{4}$.

6. **(D)** Here $y' = 3\sin^2(1-2x)\cos(1-2x) \cdot (-2)$.

7. **(B)** $\frac{d}{dx}(x^2 e^{x^{-1}}) = x^2 e^{x^{-1}}\left(-\frac{1}{x^2}\right) + 2xe^{x^{-1}}$.

8. **(B)** Let s be the distance from the origin: then

$$s = \sqrt{x^2+y^2} \quad \text{and} \quad \frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}.$$

Since $\frac{dy}{dt} = 2x\frac{dx}{dt}$ and $\frac{dx}{dt} = \frac{3}{2}$, $\frac{dy}{dt} = 3x$. Substituting yields $\frac{ds}{dt} = \frac{3\sqrt{5}}{2}$.

9. **(B)** For $f(x) = \sqrt{x}$, this limit represents $f'(25)$.

10. **(B)** $V = \int_0^1 y^2\,dx = \int_0^1 \sqrt{1-x^2}\,dx$. This definite integral represents the area of a quadrant of the circle $x^2 + y^2 = 1$, hence $V = \frac{\pi}{4}$.

11. **(C)** $-\frac{1}{2}\int (9-x^2)^{-1/2}(-2x\,dx) = -\frac{1}{2}\frac{(9-x^2)^{\frac{1}{2}}}{1/2} + C$.

AB Practice Examination 2

12. **(A)** The integral is rewritten as

$$\int \frac{(y-1)^2}{dy}\,dy = \frac{1}{2}\int \frac{y^2-2y+1}{y}\,dy,$$

$$= -\frac{1}{2}\int\left(y-2+\frac{1}{y}\right)dy,$$

$$= -\frac{1}{2}\left(\frac{y^2}{2}-2y+\ln|y|\right)+C.$$

13. **(B)** $\int_{\pi/6}^{\pi/2} \cot x\,dx = \ln \sin x \Big|_{\pi/6}^{\pi/2} = 0 - \ln\frac{1}{2}$.

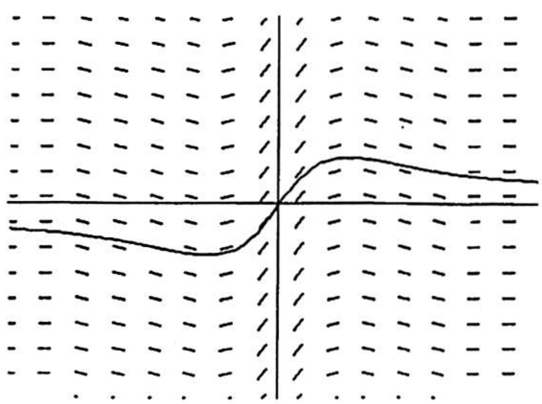
14. **(D)** Note:

```
     0            1            2            3
     |------------|------------|------------|
f' { |<----------positive---------->|<--negative-->|
   { |<increasing>|<---------decreasing---------->|
f  { |<---------increasing--------->|<decreasing>|
   { |<concave up>|<-------concave down------->|
```

15. **(B)** The winning times are positive, decreasing, and concave upward.

16. **(E)** $G(x) = H(x) + \int_0^2 f(t)\,dt$, where $\int_0^2 f(t)dt$ represents the area of a trapezoid.

17. **(C)** $f'(x) = 0$ for $x = 1$ and $f''(1) > 0$.

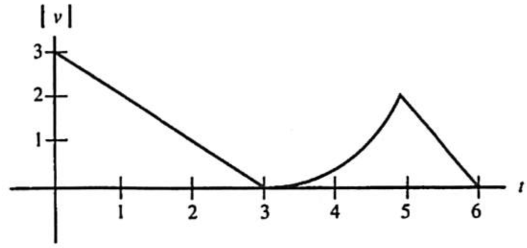


18. **(B)** Solution curves appear to represent odd functions with a horizontal asymptote. In the figure above, the curve in (B) of the question has been superimposed on the slope field.

19. **(B)** Note that

$$\lim_{x\to\infty} xe^x = \infty,\ \lim_{x\to\infty} \frac{e^x}{x} = \infty,\ \lim_{x\to-\infty} \frac{x}{x^2+1} = 0, \text{ and } \frac{x^2}{x^3+1} \geq 0 \text{ for } x > -1.$$

20. **(C)** v is not differentiable at $t = 3$ or $t = 5$.



21. **(B)** Speed is the magnitude of velocity; its graph is shown above.

22. **(B)** The average rate of change of velocity is $\frac{v(5)-v(0)}{5-0} = \frac{-2-3}{5}$.

23. **(E)** The curve has vertical asymptotes at $x = 2$ and $x = -2$ and a horizontal asymptote at $y = -2$.

24. **(E)** The function is not defined at $x = -2$; $\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$. Defining $f(-2) = 4$ will make f continuous at $x = -2$, but f will still be discontinuous at $x = 1$.

25. **(B)** Since $\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}$,

$$\left(f^{-1}\right)'(y) = \frac{1}{5x^4+3} \quad \text{and} \quad \left(f^{-1}\right)'(2) = \frac{1}{5\cdot1+3} = \frac{1}{8}.$$

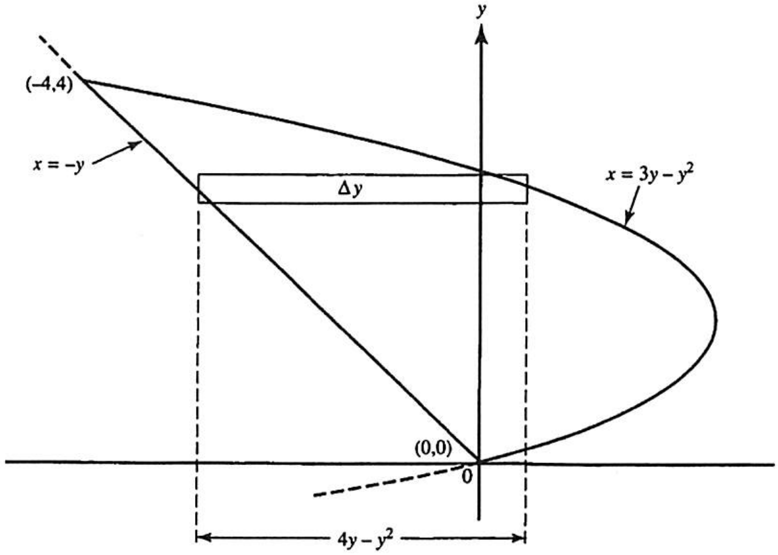
26. **(A)** $\int_1^e \frac{\ln^3 x}{x}\,dx = \int_1^e (\ln x)^3\left(\frac{1}{x}dx\right) = \frac{1}{4}\ln^4 x\Big|_1^e = \frac{1}{4}(\ln^4 e - 0) = \frac{1}{4}.$

27. **(E)** $\ln(4 + x^2) = \ln(4 + (-x)^2)$; $y' = \frac{2x}{4+x^2}$; $y'' = \frac{-2(x^2-4)}{(4+x^2)^2}$.

28. **(A)** $f(x) = \frac{d}{dx}(x \sin \pi x) = \pi x \cos \pi x + \sin \pi x.$

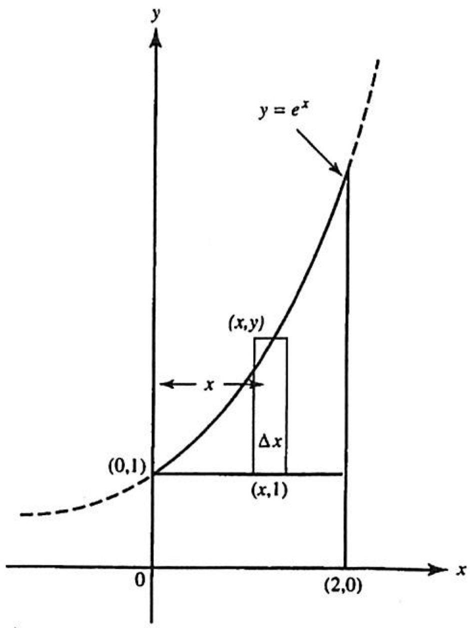
Part B

29. **(B)** See the figure below. $A = \int_0^4 [3y - y^2 - (-y)]\, dy = \int_0^4 (4y - y^2)\, dy$.



30. **(C)** See the figure below. About the x-axis: Washer. $\Delta V = \pi(y^2 - 1^2)\,\Delta x$,

$$V = \pi \int_0^2 (e^{2x} - 1)\, dx.$$



31. (E) We solve the differential equation $\frac{ds}{dt} = 12s^{\frac{1}{2}}$ by separation:

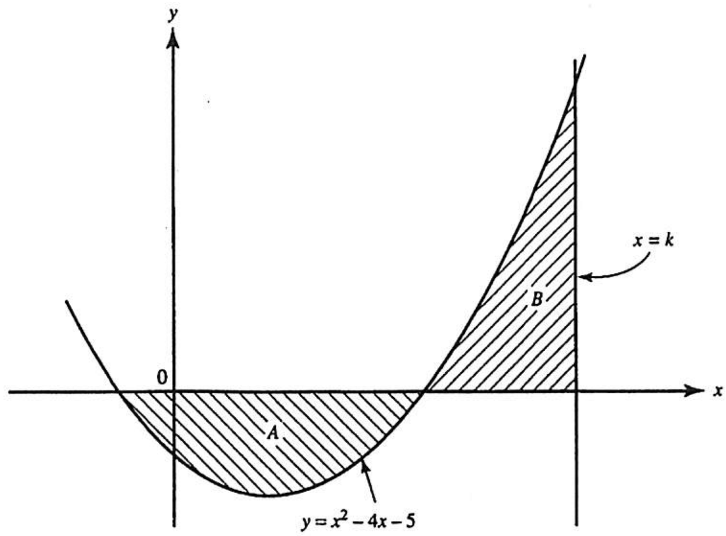
$$\int s^{-\frac{1}{2}}\,ds = 12\int dt$$

$$2s^{\frac{1}{2}} = 12t + C$$

$$\sqrt{s} = 6t + C$$

If $s = 1$ when $t = 0$, we have $C = 1$; hence, $\sqrt{s} = 6t + 1$ so $\sqrt{s} = 7$ when $t = 1$.

32. (D)



(This figure is not drawn to scale.)

The roots of $f(x) = x^2 - 4x - 5 = (x-5)(x+1)$ are $x = -1$ and 5. Since areas A and B are equal, therefore $\int_{-1}^{k} f(x)\,dx = 0$. Thus,

$$\left(\frac{x^3}{3} - 2x^2 - 5x\right)\Bigg|_{-1}^{k} = \left(\frac{k^3}{3} - 2k^2 - 5k\right) - \left(-\frac{1}{3} - 2 + 5\right)$$

$$= \frac{k^3}{3} - 2k^2 - 5k - \frac{8}{3} = 0.$$

Solving on a calculator gives k (or x) equal to 8.

33. (D) If N is the number of bacteria at time t, then $N = 200e^{kt}$. It is given that $3 = e^{10k}$. When $t = 24$, $N = 200e^{24k}$. Therefore $N = 200(e^{10k})^{2.4} = 200(3)^{2.4} \simeq 2793$ bacteria.

34. (C) Since $t = \frac{x+1}{2}$, $dt = \frac{1}{2}dx$. For $x = 2t - 1$, $t = 3$ yields $x = 5$ and $t = 5$ yields $x = 9$.

35. **(C)** Using implicit differentiation on the equation

$$x^3 + xy - y^2 = 10$$

yields

$$3x^2 + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0,$$

$$3x^2 + y = (2y - x)\frac{dy}{dx},$$

and

$$\frac{dy}{dx} = \frac{3x^2 + y}{2y - x}.$$

The tangent is vertical when $\frac{dy}{dx}$ is undefined; that is, when $2y - x = 0$.

Replacing y by $\frac{x}{2}$ in (1) gives

$$x^3 + \frac{x^2}{2} - \frac{x^2}{4} = 10$$

or

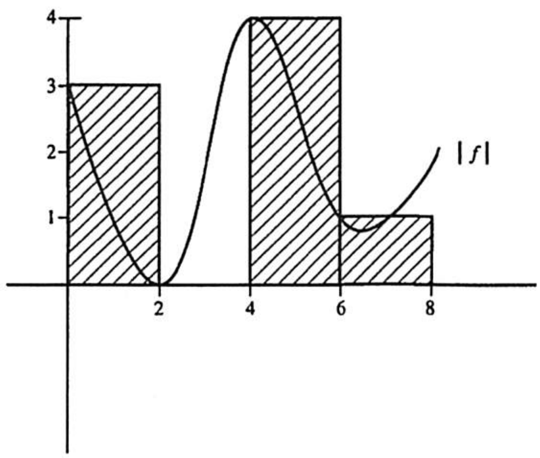
$$4x^3 + x^2 = 40.$$

Let $y_1 = 4x^3 + x^2 - 40$. Inspection of the equation $y_1 = f(x) = 0$ reveals that there is a root near $x = 2$. Solving on a calculator yields $x = 2.074$.

36. **(D)** $G'(x) = f(3x - 1) \cdot 3$.

37. **(B)** Since f changes from positive to negative at $t = 3$, G' does also where $3x - 1 = 3$.

38. **(D)** Using your calculator, evaluate $y'(2)$.



39. **(E)** $2(3) + 2(0) + 2(4) + 2(1)$.

See the figure above.

40. **(E)** $\frac{d}{dx}(f^2(x)) = 2f(x)f'(x),$

$$\frac{d^2}{dx^2}(f^2(x)) = 2[f(x)f''(x) + f'(x)f''(x)]$$
$$= 2[ff'' + (f')^2].$$

At $x = 3$, the answer is $2[2(-2) + 5^2] = 42$.

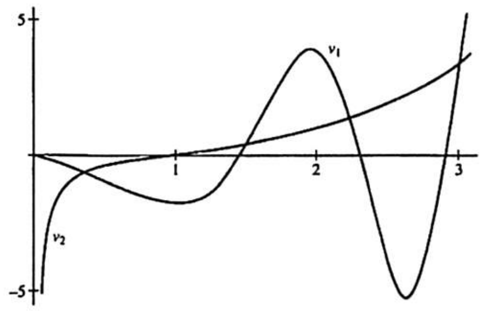
41. **(A)** The object is at rest when $v(t) = \ln(2 - t^2) = 0$; that occurs when $2 - t^2 = 1$, so $t = 1$. The acceleration is $a(t) = v'(t) = \frac{-2t}{2-t^2}$; $a(1) = \frac{-2(1)}{2-1^2}$.

42. **(D)** $\frac{df}{dt} = (x^2 + 1)\frac{dx}{dt}$. Find x when $\frac{df}{dt} = 10\frac{dx}{dt}$.

$$10\frac{dx}{dt} = (x^2 + 1)\frac{dx}{dt}$$

implies that $x = 3$.

43. **(C)** $\frac{dS}{dt}$ represents the rate of change of the surface area; if y is inversely proportional to x, then, $y = \frac{k}{x}$.



44. **(E)** The velocity functions are

$$v_1 = -2t\sin(t^2 + 1)$$

and

$$v_2 = \frac{2t(e^t) - 2e^t}{(2t)^2} = \frac{e^t(t-1)}{2t^2}.$$

Graph both functions in $[0, 3] \times [-5, 5]$. The graphs intersect four times during the first 3 sec, as shown in the figure above.

45. **(B)** $\dfrac{\int_{100}^{200} 50e^{-0.015t}\,dt}{100} \simeq 5.778$ lb.

Free-Response

Part A

AB/BC1. (a) $T=\left(\frac{7.6+5.7}{2}\right)(0.7)+\left(\frac{5.7+4.2}{2}\right)(0.3)+\left(\frac{4.2+3.8}{2}\right)(0.5)+$

$\left(\frac{3.8+2.2}{2}\right)(0.6)+\left(\frac{2.2+1.6}{2}\right)(0.4)=10.7.$

(b) $\frac{\Delta y}{\Delta x}=\frac{7.6-1.6}{2.5-5.0}=-2.4.$

(c) $f'(2.5)\approx\frac{5.7-7.6}{3.2-2.5}=-2.714.$

(d) To work with $g(x)=f^{-1}(x)$, interchange x and y:

x	7.6	5.7	4.2	3.8	2.2	1.6
$g(x)$	2.5	3.2	3.5	4.0	4.5	5.0

Now $g'(4)\approx\frac{4.0-3.5}{3.8-4.2}=-1.25$ OR $\frac{4.5-4.0}{2.2-3.8}=-0.313$ OR $\frac{4.5-3.5}{2.2-4.2}=-0.5.$

AB 2. Let M = the temperature of the milk at time t. Then

$$\frac{dM}{dt}=k(68-M).$$

The differential equation is separable:

$$\int\frac{dM}{68-M}=\int k\,dt,$$

$$-\ln|68-M|=kt+C \quad \text{(note that } 68-M>0\text{)},$$

$$\ln(68-M)=-(kt+C),$$

$$68-M=e^{-(kt+C)},$$

$$M=68-ce^{-kt},$$

where $c=e^{-C}$.
Find c, using the fact that $M=40°$ when $t=0$:

$$40=68-ce^{0} \quad \text{means} \quad c=28.$$

Find k, using the fact that $M=43°$ when $t=3$:

$$43=68-28e^{-3k},$$

$$e^{-3k}=\frac{25}{28},$$

$$k=-\frac{1}{3}\ln\frac{25}{28}.$$

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Hence $M = 68 - 28e^{\frac{1}{3}\ln\frac{25}{28}t}$.

Now find t when $M = 60°$:

$$60 = 68 - 28e^{\frac{1}{3}\ln\frac{25}{28}t},$$
$$e^{\frac{1}{3}\ln\frac{25}{28}t} = \frac{8}{28},$$
$$t = \frac{\ln\frac{8}{28}}{\frac{1}{3}\ln\frac{25}{28}} = 33.163.$$

Since the phone rang at $t = 3$, you have 30 min to solve the problem.

Part B

AB 3. (a)

$$\int_0^k \frac{18}{9+x^2}\,dx = \pi,$$
$$3\cdot\frac{18}{9}\int_0^k \frac{\frac{1}{3}dx}{1+(\frac{x}{3})^2} = \pi,$$
$$6 \text{ arctan } \frac{x}{3}\Big|_0^k = \pi,$$
$$6 \text{ arctan } \frac{k}{3} - 6 \text{ arctan } \frac{0}{3} = \pi,$$
$$\frac{k}{3} = \tan\frac{\pi}{6},$$
$$k = \sqrt{3},$$

See the figure on page 556.

(b) The average value of a function on an interval is the area under the graph of the function divided by the interval width, here $\frac{\pi}{\sqrt{3}}$.

(c) From part (a) you know that the area of the region is given by $\int_0^k \frac{18}{9+x^2}\,dx = 6 \text{ arctan } \frac{k}{3}$. Since $\lim_{k\to\infty} 6 \text{ arctan } \frac{k}{3} = 6\left(\frac{\pi}{2}\right) = 3\pi$, as k increases the area of the region approaches 3π.

AB/BC 4. (a) The rectangular slices have base y, height $5y$, and thickness along the x-axis:

$$\Delta V = (y)(5y)\Delta x = 5y^2\Delta x = 5\left(\sqrt{8\sin\left(\frac{\pi x}{6}\right)}\right)\Delta x$$

$$V = 40\int_0^1 \sin\left(\frac{\pi x}{6}\right)dx = 40\,\frac{6}{\pi}\int_0^1 \sin\left(\frac{\pi x}{6}\right)\left(\frac{\pi}{6}dx\right)$$

$$= -\frac{240}{\pi}\cos\left(\frac{\pi x}{6}\right)\Big|_0^1 = -\frac{240}{\pi}\left(\cos\left(\frac{\pi}{6}\right) - \cos(0)\right)$$

$$= -\frac{240}{\pi}\left(\frac{\sqrt{3}}{2} - 1\right)$$

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(b) The disks have radius x and thickness along the y-axis:

$$\Delta V = \pi x^2 \Delta y, \text{ so } V = \pi\int_0^2 x^2\,dy$$

Now we solve for x in terms of y:

$$y = \sqrt{8\sin\left(\frac{\pi x}{6}\right)}, \text{ so } y^2 = 8\sin\left(\frac{\pi x}{6}\right) \text{ and } \frac{y^2}{8} = \sin\left(\frac{\pi x}{6}\right).$$

Then $\arcsin\frac{y^2}{8} = \left(\frac{\pi x}{6}\right)$, which gives us $x = \frac{6}{\pi}\arcsin\frac{y^2}{8}$.

Therefore $V = \pi\int_0^2 \left(\frac{6}{\pi}\arcsin\frac{y^2}{8}\right)^2 dy$.

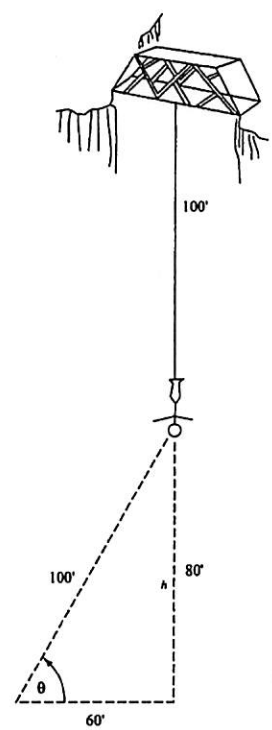
(NOTE: Although the shells method is not a required AP topic, another correct integral for this volume is $V = 2\pi\int_0^1 x\left(2 - \sqrt{8\sin\frac{\pi x}{6}}\right)dx$.)

AB/BC 5. (a) The volume of the cord is $V = \pi r^2 h$. Differentiate with respect to time, then substitute known values. (Be sure to use consistent units; here, all measurements have been converted to inches.)

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right),$$

$$0 = \pi\left(\left(\tfrac{1}{2}\right)^2 \cdot 480 + 2\cdot\tfrac{1}{2}\cdot 1200\frac{dr}{dt}\right),$$

$$\frac{dr}{dt} = -\frac{1}{10}\text{ in/sec}.$$



(b) Let θ represent the angle of elevation and h the height, as shown.

$$\tan\theta = \frac{h}{60}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{60}\frac{dh}{dt}$$

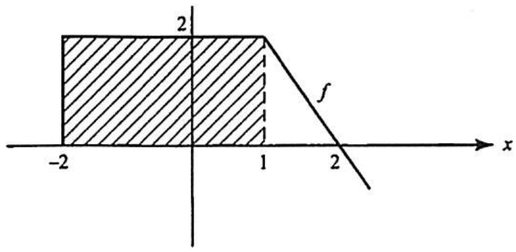
When $h = 80$, your distance to the jumper is 100 ft, as shown.

Then

$$\left(\frac{100}{60}\right)^2\frac{d\theta}{dt} = \frac{1}{60}(-40),$$

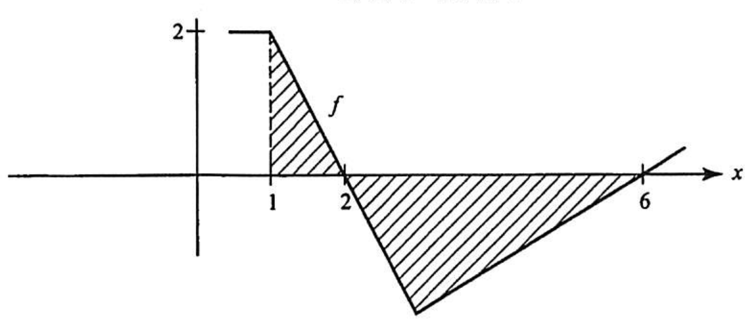
$$\frac{d\theta}{dt} = -\frac{6}{25}\text{ rad/sec}.$$

AB/BC 6. (a) $F(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt =$ the negative of the area of the shaded rectangle in the figure. Hence $F(-2) = -(3)(2) = -6$.

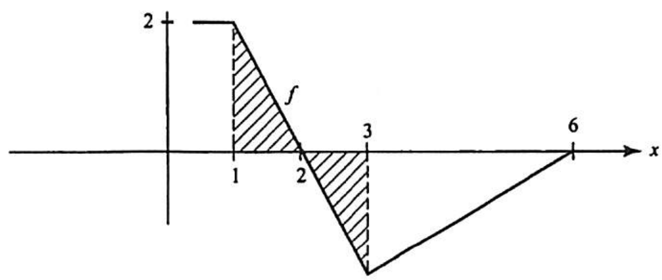


$F(6) = \int_1^6 f(t)\,dt$ is represented by the shaded triangles in the figure.

$$\int_1^6 f(t)\,dt = \int_1^2 f(t)\,dt + \int_2^6 f(t)\,dt$$
$$= \tfrac{1}{2}(1)(2) - \tfrac{1}{2}(4)(2) = -3.$$



(b) $\int_1^1 f(t)\,dt = 0$, so $F(x) = 0$ at $x = 1$. $\int_1^3 f(t)\,dt = 0$ because the regions above and below the x-axis have the same area. Hence $F(x) = 0$ at $x = 3$.



(c) F is increasing where $F' = f$ is positive: $-2 \le x < 2$.

(d) The maximum value of F occurs at $x = 2$, where $F' = f$ changes from positive to negative. $F(2) = \int_1^2 f(t)\,dt = \tfrac{1}{2}(1)(2) = 1$.

The minimum value of F must occur at one of the endpoints. Since $F(-2) = -6$ and $F(6) = -3$, the minimum is at $x = -2$.

(e) F has points of inflection where F'' changes sign, as occurs where $F' = f$ goes from decreasing to increasing, at $x = 3$.