Answer Key

AB PRACTICE EXAMINATION 3

Part A

1. E	**8.** E	**15.** C	**22.** E
2. C	**9.** E	**16.** E	**23.** B
3. C	**10.** E	**17.** E	**24.** D
4. A	**11.** B	**18.** B	**25.** B
5. C	**12.** B	**19.** D	**26.** C
6. C	**13.** D	**20.** E	**27.** A
7. D	**14.** A	**21.** D	**28.** A

Part B

29. A	**34.** C	**38.** B	**42.** B
30. E	**35.** E	**39.** A	**43.** D
31. B	**36.** E	**40.** A	**44.** E
32. C	**37.** C	**41.** C	**45.** D
33. B			

ANSWERS EXPLAINED

Multiple-Choice

Part A

1. (E) Here, $\lim_{x\to 2^-}[x]=1$, while $\lim_{x\to 2^+}[x]=2$.

2. (C) The given limit equals $f'\left(\frac{\pi}{2}\right)$, where $f(x)=\sin x$.

3. (C) Since $f(x) = x \ln x$,

   $f'(x) = 1 + \ln x,\quad f''(x) = \frac{1}{x},\quad$ and $\quad f'''(x) = -\frac{1}{x^2}.$

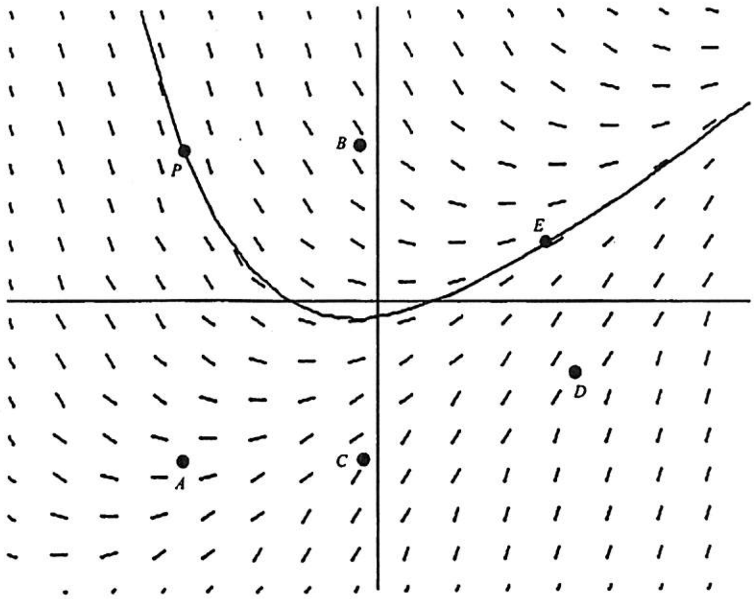
4. (A) Differentiate implicitly to get $4x-4y^3\frac{dy}{dx}=0$. Substitute $(-1, 1)$ to find $\frac{dy}{dx}=-1$, the slope at this point, and write the equation of the tangent: $y - 1 = -1(x + 1)$.

5. (C) $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x - 1)(x - 2)$. To determine the signs of $f'(x)$, inspect the sign at any point in each of the intervals $x < 0$, $0 < x < 1$, $1 < x < 2$, and $x > 2$. The function increases whenever $f'(x)>0$.

6. (C) The integral is equivalent to $\frac{1}{2}\int\frac{2\cos x\,dx}{4+2\sin x}=\frac{1}{2}\int\frac{du}{u}$, where $u = 4 + 2\sin x$.

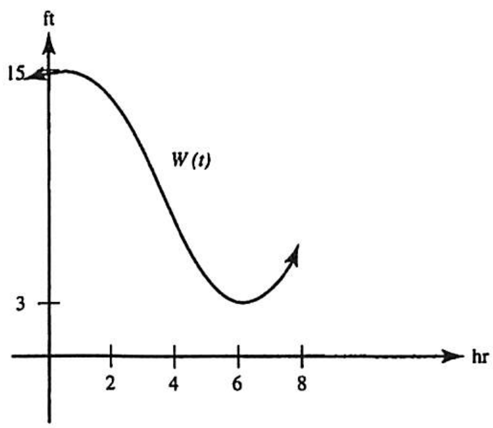
7. (D) Here $y'=\frac{1-\ln x}{x^2}$, which is zero for $x = e$. Since the sign of y' changes from positive to negative as x increases through e, this critical value yields a relative maximum. Note that $f(e)=\frac{1}{e}$.

8. (E) Since $v=\frac{ds}{dt}=5t^4+6t^2=t^2(5t^2 + 6)$ is always positive, there are no reversals in motion along the line.



9. (E) The slope field suggests the curve shown above as a particular solution.

10. (E) Since $f(x) = F'(x) = \dfrac{1}{1-x^2}$, f is discontinuous at $x = 1$; the domain of F is therefore $x > 1$. On [2, 5] $f(x) < 0$, so $\int_5^2 f > 0$. $F''(x) = f'(x) = \dfrac{2x}{(1-x^2)^2}$, which is positive for $x > 1$.



11. (B) In the graph above, $W(t)$, the water level at time t, is a cosine function with amplitude 6 ft and period 12 hr:

$$W(t) = 6\cos\left(\frac{\pi}{6}t\right) + 9\,\text{ft},$$

$$W'(t) = -\pi\sin\left(\frac{\pi}{6}t\right)\text{ft/hr}.$$

Hence, $W'(4) = -\pi \sin\left(\dfrac{2\pi}{3}\right) = -\dfrac{\pi\sqrt{3}}{2}$ ft/hr.

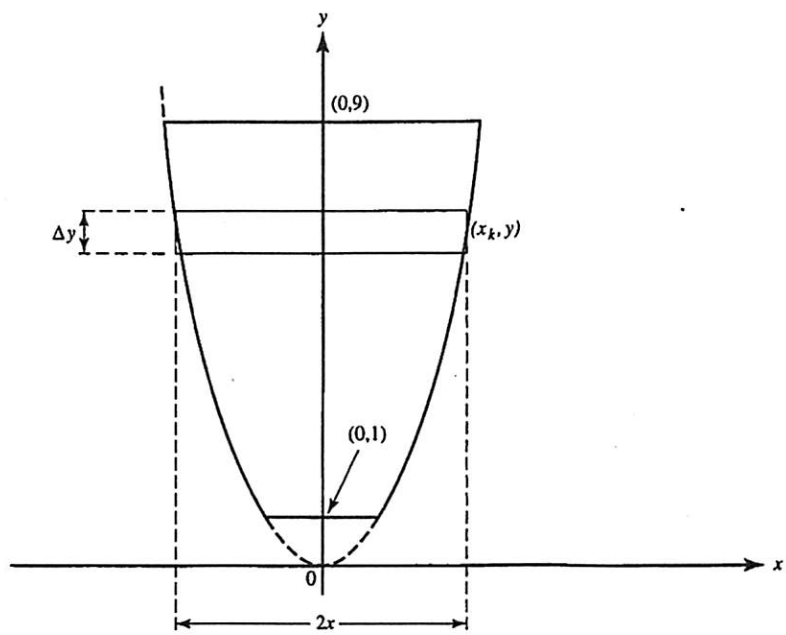
12. (B) Solve the differential equation $\frac{dy}{dx}=x^2$, getting $y=\frac{x^3}{3}+C$. Use $x=-1$, $y=2$ to determine $C=\frac{7}{3}$.

13. (D) $\int \frac{e^u du}{1+(e^u)^2} = \tan^{-1}(e^u)+C$

14. (A) $f'(100) \simeq \frac{f(102)-f(100)}{102-100} = \frac{2.0086-2}{2}$.

15. (C) $G'(2)=4$, so $G(x) \simeq 4(x-2)+5$.

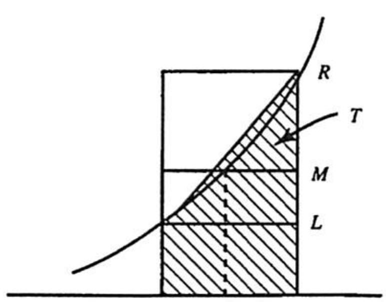
16. (E) $A = 2\int_1^9 x\,dy = 2\int_1^9 \sqrt{y}\,dy = \frac{104}{3}$. See the figure below.



17. (E) Note that $\lim_{x\to 0} f(x) = f(0) = 1$.

18. (B) Note that (0, 0) is on the graph, as are (1, 2) and (−1, −2). So only (B) and (E) are possible. Since $\lim_{x\to+\infty} y = \lim_{x\to-\infty} y = 0$, only (B) is correct.

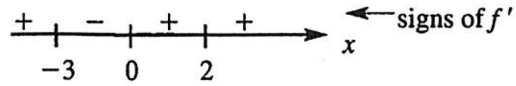
19. (D) See the figure.



20. **(E)** $\frac{x+3}{x^2-9}=\frac{x+3}{(x+3)(x-3)}=\frac{1}{x-3}$; $\lim\limits_{x\to3^+}\frac{1}{x-3}=+\infty$; $\lim\limits_{x\to3^-}\frac{1}{x-3}=-\infty$.

21. **(D)** In (D), $f(x)$ is not defined at $x=0$. Verify that each of the other functions satisfies both conditions of the Mean Value Theorem.

22. **(E)** The signs within the intervals bounded by the critical points are given below.



Since f changes from increasing to decreasing at $x=-3$, f has a local maximum at -3. Also, f has a local minimum at $x=0$, because it is decreasing to the left of zero and increasing to the right.

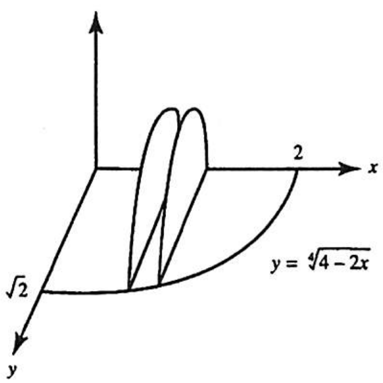
23. **(B)** Since $\ln\frac{x}{\sqrt{x^2+1}}=\ln x-\frac{1}{2}\ln(x^2+1)$, then

$$\frac{dy}{dx}=\frac{1}{x}-\frac{1}{2}\cdot\frac{2x}{x^2+1}=\frac{1}{x(x^2+1)}.$$

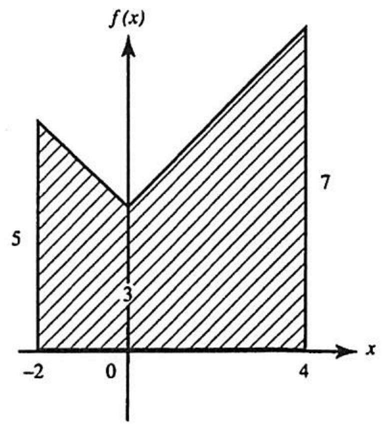
24. **(D)** $\int_{-2}^{0}f\doteq\int_{0}^{2}f=-\int_{2}^{4}f$, but $\int_{0}^{4}f=0$.

25. **(B)** As seen from the figure, $\Delta V=\frac{1}{2}\pi r^2\Delta x$, where $y=2r$,

$$V=\frac{\pi}{2}\int_0^2\left(\frac{y}{2}\right)^2dx=\frac{\pi}{8}\int_0^2\sqrt{4-2x}\,dx.$$



26. **(C)** From the figure below, $\frac{\int_{-2}^{4} f}{6} = \frac{\frac{5+3}{2}\cdot 2+\frac{3+7}{2}\cdot 4}{6} = \frac{28}{6}$.

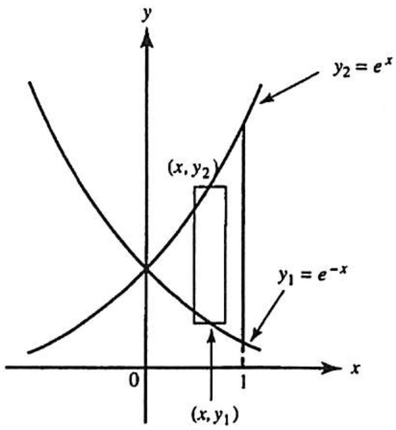


27. **(A)** Since the degrees of numerator and denominator are the same, the limit as $x \to \infty$ is the ratio of the coefficients of the terms of highest degree: $\frac{-2}{4}$.

28. **(A)** We see from the figure that $\Delta A = (y_2 - y_1)\Delta x$;

$$A = \int_0^1 (e^x - e^{-x})\,dx$$

$$= (e^x + e^{-x})\Big|_0^1 = e + \frac{1}{e} - 2.$$



Part B

29. **(A)** Let $u = x^2 + 2$. Then

$$\frac{d}{du}\int_0^u \sqrt{1+\cos t}\, dt = \sqrt{1+\cos u}$$

and

$$\frac{d}{dx}\int_0^u \sqrt{1+\cos t}\, dt = \sqrt{1+\cos u}\,\frac{du}{dx} = \sqrt{1+\cos(x^2+2)}\cdot(2x).$$

30. **(E)** $\frac{dh}{dt}$ will increase above the half-full level (that is, the height of the water will rise more rapidly) as the area of the cross section diminishes.

31. **(B)** Since $v = ks = \frac{ds}{dt}$, then $a = \frac{d^2s}{dt^2} = k\frac{ds}{dt} = kv = k^2s$.

32. **(C)** $\frac{dH}{H-70} = -0.05\,dt.\quad \ln|H-70| = -0.05t + C$

$$H - 70 = ce^{-0.05t}$$

$$H(x) = 70 + ce^{-0.05t}$$

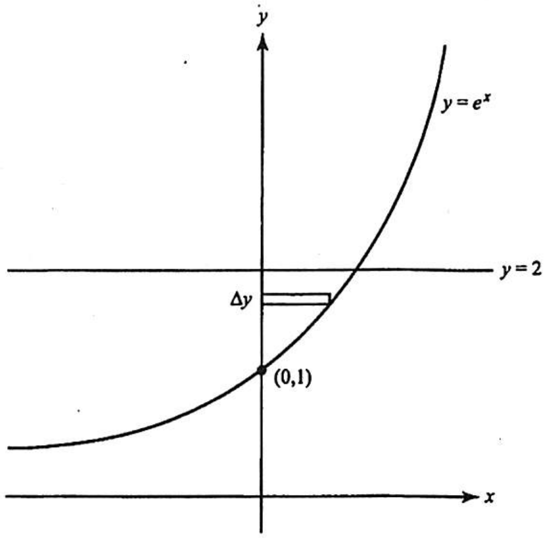
The initial condition $H(0) = 120$ shows $c = 50$. Evaluate $H(10)$.

33. **(B)** Let P be the amount after t years. It is given that $\frac{dP}{dt} = 320\,e^{0.08t}$. The solution of this differential equation is $P = 4000e^{0.08t} + C$, where $P(0) = 4000$ yields $C = 0$. The answer is $4000e^{(0.08)\cdot 10}$.

34. **(C)** The inverse of $y = 1 + e^x$ is $x = 1 + e^y$ or $y = \ln(x-1)$; $(x-1)$ must be positive.

35. **(E)** Speed is the magnitude of velocity: $|v(8)| = 4$.

36. **(E)** For $3 < t < 6$ the object travels to the right $\frac{1}{2}(3)(2) = 3$ units. At $t = 7$ it has returned 1 unit to the left; by $t = 8$, 4 units to the left.

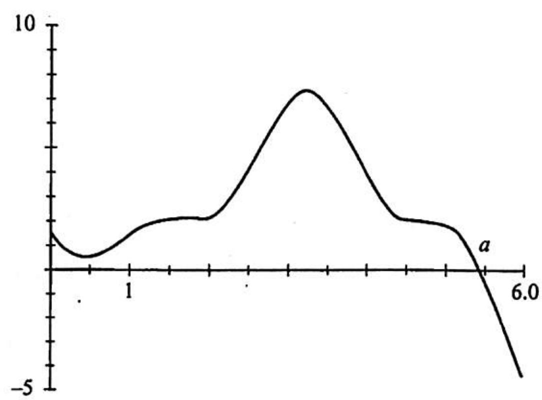


37. **(C)** Use disks: $\Delta V = \pi r^2 \Delta y = \pi x^2 \Delta y$, where $x = \ln y$. Use your calculator to evaluate $V = \pi\int_1^2 (\ln y)^2\, dy$.

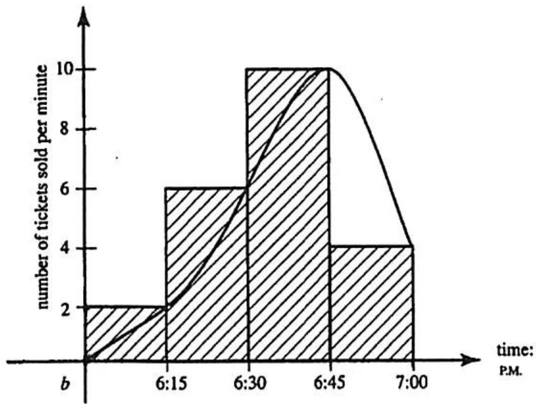
38. **(B)** If $u = \sqrt{x-2}$, then $u^2 = x - 2$, $x = u^2 + 2$, $dx = 2u\, du$. When $x = 3$, $u = 1$; when $x = 6$, $u = 2$.

39. **(A)** The tangent line passes through points (8,1) and (0,3). Its slope, $\frac{1-3}{8-0}$, is $f'(8)$.

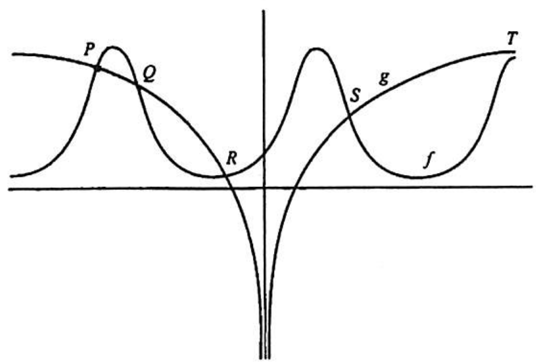
40. **(A)** Graph f'' in $[0,6] \times [-5,10]$. The sign of f'' changes only at $x = a$, as seen in the figure.



41. **(C)** In the graph below, the first rectangle shows 2 tickets sold per minute for 15 min, or 30 tickets. Similarly, the total is $2(15) + 6(15) + 10(15) + 4(15)$.



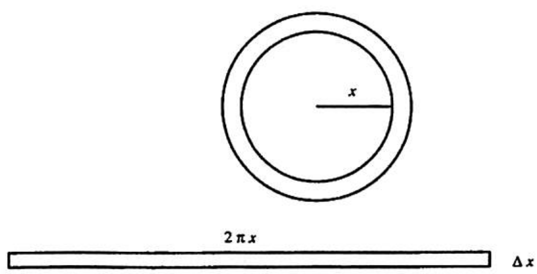
42. **(B)** Graph both functions in $[-8,8] \times [-5,5]$. At point of intersection Q, both are decreasing. Tracing reveals $x \approx -4$ at Q. If you zoom in on the curves at $x = T$, you will note that they do not actually intersect there.



43. **(D)** Counterexamples are, respectively, for (A), $f(x) = |x|$, $c = 0$; for (B), $f(x) = x^3$, $c = 0$; for (C), $f(x) = x^4$, $c = 0$; for (E), $f(x) = x^2$ on $(-1, 1)$.

44. **(E)** $f'(x) > 0$; the curve shows that f' is defined for all $a < x < b$, so f is differentiable and therefore continuous.

45. **(D)** Consider the blast area as a set of concentric rings; one is shown in the figure. The area of this ring, which represents the region x meters from the center of the blast, may be approximated by the area of the rectangle shown. Since the number of particles in the ring is the area times the density, $\Delta P = 2\pi x \cdot \Delta x \cdot N(x)$. To find the total number of fragments within 20 m of the point of the explosion, integrate: $2\pi \int_0^{20} x \frac{2x}{1+x^{3/2}}\, dx \approx 711.575$.



Free-Response

Part A

AB1. (a) Since $x^2y - 3y^2 = 48$,

$$x^2\frac{dy}{dx} + 2xy - 6y\frac{dy}{dx} = 0,$$
$$(x^2 - 6y)\frac{dy}{dx} = -2xy,$$
$$\frac{dy}{dx} = \frac{2xy}{6y - x^2}.$$

(b) At (5,3), $\frac{dy}{dx} = \frac{2 \cdot 5 \cdot 3}{6 \cdot 3 - 5^2} = -\frac{30}{7}$, so the equation of the tangent line is

$$y - 3 = -\frac{30}{7}(x - 5).$$

(c) $y - 3 = -\frac{30}{7}(4.93 - 5) = 0.3$, so $y = 3.3$.

(d) Horizontal tangent lines have $\frac{dy}{dx} = \frac{2xy}{6y - x^2} = 0$. This could happen only if $2xy = 0$, which means that $x = 0$ or $y = 0$.
If $x = 0$, $0y - 3y^2 = 48$, which has no real solutions.
If $y = 0$, $x^2 \cdot 0 - 3 \cdot 0^2 = 48$, which is impossible. Therefore, there are no horizontal tangents.

AB/BC2. (a) $W'(16) \approx \frac{W(20) - W(16)}{20 - 16} = \frac{33 - 35}{4} = -\frac{1}{2}$ ft/hr

(OR $\frac{W(16) - W(12)}{16 - 12} = \frac{35 - 37}{4}$ OR $\frac{W(20) - W(12)}{20 - 12} = \frac{33 - 37}{8}$).

(b) The average value of a function is the integral across the given interval divided by the interval width. Here $\text{Avg}(W) = \frac{\int_0^{24} W(t)\,dt}{24 - 0}$. Estimate the value of the integral using trapezoid rule T with values from the table and $\Delta t = 4$:

$$T = \frac{\Delta t}{2}(W(0) + 2W(4) + 2W(8) + 2W(12) + 2W(16) + 2W(20) + W(24))$$
$$= \frac{4}{2}(32 + 2\times 36 + 2\times 38 + 2\times 37 + 2\times 35 + 2\times 33 + 32)$$
$$= 844.$$

Hence

$$\text{Avg}(W) \approx \frac{844}{24} = 35.167\text{ ft}.$$

AB Practice Examination 3

(c) For $F(t) = 35 - 3\cos\left(\frac{t+3}{4}\right)$, use your calculator to evaluate $F'(16) \approx -0.749$. After 16 hr, the river depth is dropping at the rate of 0.749 ft/hr.

(d) $\text{Avg}(F) = \dfrac{\int_0^{24} F(t)\,dt}{24-0} \approx 35.116\,\text{ft}.$

Part B

AB/BC 3. (a) $\Delta A = (g(x) - f(x))\,\Delta x$, so:

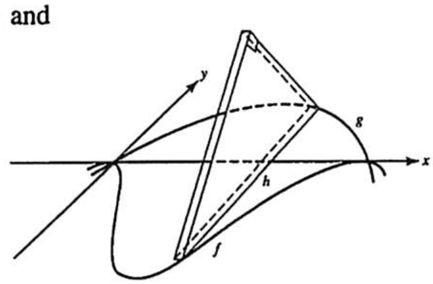
$$A = \int_0^2 (g(x) - f(x))\,dx = \int_0^2 (x(2-x) - (\cos \pi x - 1)\,dx$$

$$= \int_0^2 (2x - x^2 - \cos \pi x + 1)\,dx = \left(x^2 - \frac{x^3}{3} - \frac{1}{\pi}\sin \pi x + x\right)\Big|_0^2$$

$$= \left(2^2 - \frac{2^3}{3} - \frac{1}{\pi}\sin 2\pi + 2\right) - \left(0^2 - \frac{0^3}{3} - \frac{1}{\pi}\sin 0 + 0\right)$$

$$= \frac{10}{3}.$$

(b) Let h = the hypotenuse of an isosceles right triangle, as shown in the figure. Then each leg of the triangle is $\frac{h}{\sqrt{2}}$ and its area is $\frac{1}{2} \cdot \frac{h}{\sqrt{2}} \cdot \frac{h}{\sqrt{2}} = \frac{h^2}{4}$.



An element of volume is

$$\Delta V = \frac{h^2}{4}\Delta x = \frac{(g(x) - f(x))^2}{4}\Delta x,$$

and thus $V = \frac{1}{4}\int_0^2 (x(2-x) - (\cos \pi x - 1))^2\,dx$.

(c) Washers; $\Delta V = \pi(r_1^2 - r_2^2)\Delta x$ where:

$r_1 = 3 - f(x) = 3 - (\cos \pi x - 1)$,

$r_2 = 3 - g(x) = 3 - x(2 - 1)$.

So $V = \pi\int_0^2 [(3 - (\cos \pi x - 1))^2 - (3 - x(2-x))^2]\,dx.$

AB/BC 4. (a) $v_P(t) = 6\left(\sqrt{1+8t} - 1\right)$, so $v(0) = 0$ and $v(10) = 48$.

The average acceleration is $\frac{\Delta v}{\Delta t} = \frac{48-0}{10-0} = \frac{24}{5}\,\text{ft/sec}^2$.

Acceleration $a(t) = v'(t) = 6 \cdot \frac{1}{2}(1+8t)^{-\frac{1}{2}}(8)\,\text{ft/sec}^2$.

$$\frac{24}{\sqrt{1+8t}} = \frac{24}{5} \text{ when } t = 3\,\text{sec}.$$

(b) Since Q's acceleration, for all t in $0 \le t \le 5$, is the slope of its velocity graph, $a = \frac{20-0}{5-0} = 4\,\text{ft/sec}^2$.

(c) Find the distance each auto has traveled. For P, the distance is

$$\int_0^{10} 6\left(\sqrt{1+8t}-1\right)$$

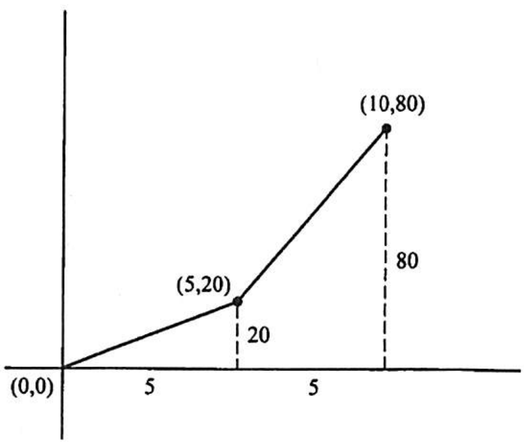
$$6\left[\frac{1}{8}\int_0^{10}\sqrt{1+8t}\cdot 8\,dt-\int_0^{10}dt\right]$$

$$6\left(\frac{1}{8}\cdot\frac{2}{3}(1+8t)^{\frac{3}{2}}-t\right)\Bigg|_0^{10},$$

$$6\left(\frac{1}{12}\left(81^{\frac{3}{2}}-1^{\frac{3}{2}}\right)-10\right)=304\text{ ft.}$$

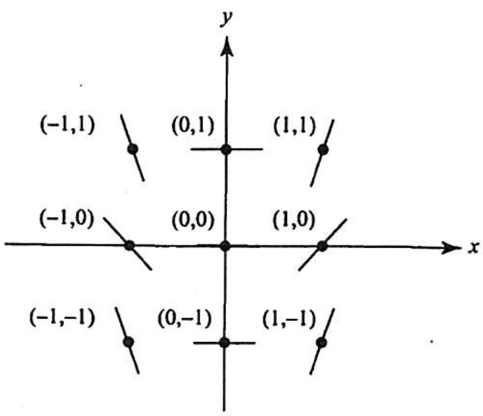
For auto Q, the distance is the total area of the triangle and trapezoid under the velocity graph shown below, namely,

$$\frac{1}{2}(5\cdot 20)+\frac{1}{2}(20+80)(5)=300\,\text{ft.}$$



Auto P won the race.

AB5. (a) Using the differential equation, evaluate the derivative at each point, then sketch a short segment having that slope. For example, at $(-1,-1)$, $\frac{dy}{dx}=2(-1)((-1)^2+1)-4$; draw a steeply decreasing segment at $(-1,-1)$.
Repeat this process at each of the other points. The result follows.



(b) The differential equation $\frac{dy}{dx} = 2x(y^2 + 1)$ is separable.

$$\int \frac{dy}{y^2+1} = \int 2x\, dx$$
$$\arctan(y) = x^2 + c$$
$$y = \tan(x^2 + c)$$

It is given that f passes through (0,1), so $1 = \tan(0^2 + c)$ and $c = \frac{\pi}{4}$.

The solution is $f(x) = \tan\left(x^2 + \frac{\pi}{4}\right)$.

The particular solution must be differentiable on an interval containing the initial point (0,1). The tangent function has vertical asymptotes at $x = \pm\frac{\pi}{2}$, hence:

$-\frac{\pi}{2} < x^2 + \frac{\pi}{4} < \frac{\pi}{2}$. (Since $x^2 \geq 0$, we ignore the left inequality.)

$$x^2 < \frac{\pi}{4}$$

$$|x| < \frac{\sqrt{\pi}}{2}$$

AB 6. (a) $\int_0^2 f(t)\,dt = F(2) = 4.$

(b) One estimate might be $\int_2^7 f(t)\,dt = F(7) - F(2) = 2 - 4 = -2.$

(c) $f(x) = F'(x)$; $F'(x) = 0$ at $x = 4$.

(d) $f'(x) = F''(x)$. F'' is negative when F is concave downward, which is true for the entire interval $0 < x < 8$.

(e) $G(x) = \int_2^x f(t)\,dt$

$$= \int_0^x f(t)\,dt - \int_0^2 f(t)\,dt$$
$$= F(x) - 4.$$

Then the graph of G is the graph of F translated downward 4 units.