Proof that $\lim_{x\to 0}\frac{\sin x}{x}=1$

First, we need to find $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$. In order to do this we need to restrict θ so that $0<\theta<\frac{\pi}{2}$. Why are we able to do this? when θ is approaching 0 from the right, the values of θ will end up in $(0;\frac{\pi}{2})$ ...

a) Find the area of $\triangle OAP$.

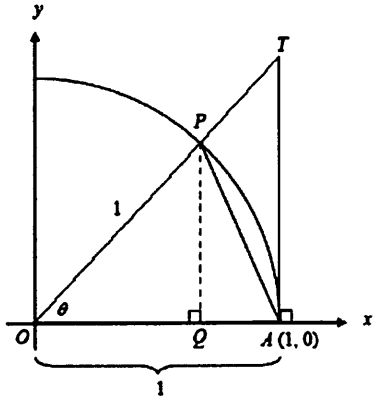
- $\sin\theta = PQ$
- $\triangle OAP = \frac{PQ\times OA}{2} = \frac{\sin\theta}{2}$

b) Find the area of sector OAP.

angle	area
2π	π
θ	$\frac{\theta}{2}$

c) Find the area of $\triangle OAT$.

- $\tan\theta = AT$
- $\triangle OAT = \frac{AT\times OA}{2} = \frac{\tan\theta}{2}$



d) Set up an inequality with the three areas from parts *a*, *b*, and *c*.

$$\triangle OAP < \text{sec } OAP < OAT$$

e) Divide all three parts by $\frac{1}{2}\sin\theta$. Why do the inequality signs stay the same?

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2} \Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \qquad \left(\frac{\sin\theta}{2}>0\right)$$

f) Make the middle term $\frac{\sin\theta}{\theta}$. *Hint*: If your middle term doesn't look anything like this, start over! ☺

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

g) Use the Sandwich Theorem to show that $\lim_{x\to 0^+}\frac{\sin\theta}{\theta}=1$.

$\lim_{x\to 0^+}\cos\theta = 1$

Sandwich $\Rightarrow \lim_{x\to 0^+}\frac{\sin\theta}{\theta}=1$

h) Show that $f(\theta)=\frac{\sin\theta}{\theta}$ is an even function.

$$f(-\theta)=\frac{\sin(-\theta)}{-\theta}=\frac{-\sin\theta}{-\theta}=\frac{\sin\theta}{\theta}$$

i) Since $f(\theta)=\frac{\sin\theta}{\theta}$ is an even function, what can you conclude about $\lim_{x\to 0^+}\frac{\sin\theta}{\theta}$?

$$\lim_{x\to 0^+}\frac{\sin\theta}{\theta} = \lim_{x\to 0^-}\frac{\sin\theta}{\theta} \quad \text{(symetry)}$$

j) Explain why we can conclude that $\lim_{x\to 0}\frac{\sin\theta}{\theta}=1$.

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