Unit I Chapter 2 Rates of Change and Limits
Big Ideas

1. **Taking Limits (finite limits, limits involving infinity)**

1. $\lim_{x\to 2}(x^2-3x) = -2$

2. $\lim_{x\to 2}\left(\frac{x}{x+1}\right) = \frac{2}{3}$

3. $\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = 4$

4. $\lim_{x\to 2}\sqrt{1-x} =$ DNE

5. $\lim_{x\to 2}(\text{int}(x)) =$ DNE $\quad \lim_{x\to 2^+}\text{Int}(x) = 2 \quad \lim_{x\to 2^-}\text{Int}(x) = 1$

6. $\lim_{x\to \frac{3}{2}}(\text{int}(x)) = 1$

7. $\lim_{x\to 0}\frac{|x|}{x} =$ DNE $\quad \lim_{x\to 0^+}\frac{|x|}{x} = 1 \quad \lim_{x\to 0^-}\frac{|x|}{x} = -1$

8. $\lim_{x\to\infty}\frac{5x^3-2x+3}{4x^3+7x^2} = \frac{5}{4}$

9. $\lim_{x\to-\infty}\frac{5x^3-2x+3}{4x^2+7x} = \lim_{x\to-\infty}\frac{5}{4}x = -\infty$

10. $\lim_{x\to\infty}\frac{5x^4-2x+3}{4x^3+7x^2} = \lim_{x\to+\infty}\frac{5}{4}x = +\infty$

11. $\lim_{x\to\infty} \frac{1}{x} = 0$

12. $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1$

2. Important Limits to Know

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\qquad$ $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ $\qquad$ $\lim_{x\to\infty} \frac{1}{x} = 0$

1. $\lim_{x\to 0} \frac{\sin x}{x^2 - 3x} = \lim_{x\to 0} \left(\cancel{\frac{\sin x}{x}}^{\to 1} \cdot \frac{1}{x-3} \right) = -\frac{1}{3}$

2. $\lim_{x\to\infty} \frac{\sin x + 3x + 1}{x} = \lim_{x\to\infty} \left(\cancel{\frac{\sin x}{x}}^{\to 0} + 3 + \cancel{\frac{1}{x}}^{\to 0} \right) = 3$

3. One-sided and Two-sided Limits (Piecewise-defined Functions)

1. Invent a graph and determine the right-hand and left-hand limits at two or more important points

2. $f(x) = \begin{cases} x^2 - 2, & x < 1 \\ \frac{1}{x} + 3, & x \ge 1 \end{cases}$ $\qquad$ $\lim_{x\to 1+} f(x) = 4$ $\qquad$ $\lim_{x\to 1-} f(x) = -1$

4. **Horizontal Asymptotes**

1. $y = \dfrac{2x^3 - 3x + 1}{4 - 6x - 5x^3}$ $\lim\limits_{x \to \pm\infty} \dfrac{2x^3 - 3x + 1}{4 - 6x - 5x^3} = \lim\limits_{x \to \pm\infty} \dfrac{2x^3}{-5x^3} = -\dfrac{2}{5}$

H.A $y = -\frac{2}{5}$ towards $+\infty$ and $-\infty$

2. $y = \dfrac{3x^2}{4 + 5x^3}$ $\lim\limits_{x \to \pm\infty} \dfrac{3x^2}{4 + 5x^3} = \lim\limits_{x \to \pm\infty} \dfrac{3x^2}{5x^3} = 0$

H.A $y = 0$ towards $+\infty$ and $-\infty$

5. **Vertical Asymptotes**

1. $y = \dfrac{1}{x+2}$ $\lim\limits_{x \to -2^-} \dfrac{1}{x+2} = -\infty$ $\lim\limits_{x \to -2^+} \dfrac{1}{x+2} = +\infty$

V.A $x = -2$

2. $y = \dfrac{1}{x^2 - 9}$ $\lim\limits_{x \to -3^-} \dfrac{1}{x^2-9} = +\infty$ $\lim\limits_{x \to -3^+} \dfrac{1}{x^2-9} = -\infty$

V.A : $x = 3$
V.A : $x = -3$

$\lim\limits_{x \to 3^-} \dfrac{1}{x^2-9} = -\infty$ $\lim\limits_{x \to 3^+} \dfrac{1}{x^2-9} = +\infty$

3. $y = \dfrac{x+1}{x^2 - 1}$

$y = \dfrac{1}{x-1}$

$\left.\begin{array}{l} \lim\limits_{x \to 1^-} \dfrac{x+1}{x^2-1} = -\infty \\ \lim\limits_{x \to 1^+} \dfrac{x+1}{x^2-1} = +\infty \end{array}\right\}$ V.A $x = 1$

(Note: $\lim\limits_{x \to -1} \dfrac{x+1}{x^2-1} = -\frac{1}{2}$ (hole))

4. $y = \dfrac{5}{2x^2 - 3x}$

$y = \dfrac{5}{x(2x-3)}$ V.A $x = 0$ and V.A: $x = 3/2$

6. **End Behaviour Models**

1. $y = \dfrac{2x^3 - 3x + 1}{4 - 6x - 5x^3}$ $y = -\frac{2}{5}$ on both ends

2. $y = \dfrac{3x^2}{4 + 5x^3}$ $y = 0$ on both ends

3. $y = \dfrac{3x^5}{4 + 5x^3}$ $\lim\limits_{x \to \pm\infty} \dfrac{3x^5}{4 + 5x^3} = +\infty$

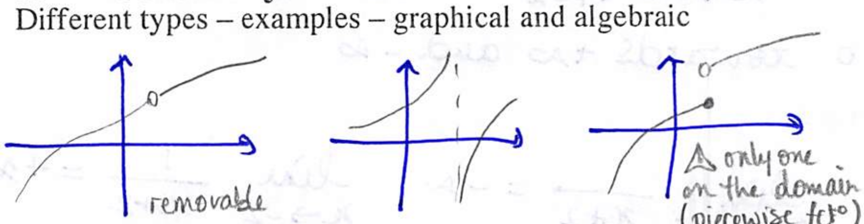
(asymptote ... $y = \frac{3}{5}x^2$)

4. $y = x + e^x$ $\quad \lim_{x \to +\infty} (x + e^x) = +\infty \quad \lim_{x \to -\infty} (x + \cancel{e^x}) = -\infty$

5. graphical

7. Continuity

Different types – examples – graphical and algebraic



algebraic: $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

Determine all points of discontinuity (which ones are removable?)

$\uparrow$ holes only

1. $y = \dfrac{x}{x^2 - x} = \dfrac{x}{x(x-1)}$

disc. at 0 (removable)

disc. at 1 (non removable : V.A) $\}$ none on the domain.

2. $f(x) = \begin{cases} x^2 - 2, & x \le 1 \\ 3x + 1, & x > 1 \end{cases}$

f is continuous on $(-\infty, 1)$ and on $(1, +\infty)$

$\lim_{x \to 1^-} f(x) = f(1) = -1 \qquad \lim_{x \to 1^+} f(x) = 4 \qquad \Rightarrow$ <u>NOT</u> continuous at 1 (jump : non removable)

3. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \ne 1 \\ 4, & x = 1 \end{cases}$

f is continuous on $(-\infty, 1)$ and $(1, +\infty)$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = 2 \qquad f(1) \ne 2 \qquad \Rightarrow$ <u>NOT</u> continuous at 1 (jump : non removable)

8. Extended Function – making a function continuous

Determine the value of k which will make this function continuous

$$f(x) = \begin{cases} \dfrac{x-4}{\sqrt{x} - 2}, & x \ne 4 \\ k, & x = 4 \end{cases}$$

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x \to 4} (\sqrt{x} + 2) = 4$$

$$\Rightarrow \boxed{k = 4}$$

9. Intermediate Value Theorem for continuous functions (using it)

Determine a number which is 1 less than its cube

let $f(x) = x^3 - x$ [if f is positive, then $x < x^3$]

wrong question

10. Average Rate of change – secant lines

Find the average rate of change of the function $f(x) = 1 + x^2$ over the interval $[1,3]$

$$\frac{f(3)-f(1)}{3-1} = \frac{10-2}{2} = \boxed{4}$$

11. Instantaneous Rate of Change – tangent lines – normal lines

1. Find the instantaneous rate of change of the function $f(x) = 1 + x^2$ at $x = 1$.

$$\frac{f(1+h)-f(1)}{h} = \frac{1+(1+h)^2-(1+1)}{h}$$

$$= \frac{1+h^2+2h+1-2}{h}$$

$$= h+2$$

$$\lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \boxed{2}$$

2. Determine the slope of the tangent line of $f(x) = \frac{1}{x}$ at $x = 2$ Then, determine the equation of the tangent line and the normal line at this point.

- slope of the tangent line at 2 :

$$\frac{f(2+h) - f(2)}{h} = \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \frac{2-(2+h)}{2(2+h)} \cdot \frac{1}{h}$$

$$= -\frac{1}{2(2+h)} \xrightarrow[h \to 0]{} \boxed{-\frac{1}{4}}$$

- equation of the tangent line at $x=2$

Point : $(2, \frac{1}{2})$ slope: $-\frac{1}{4}$

$$\Rightarrow \boxed{y - \frac{1}{2} = -\frac{1}{4}(x-2)}$$

- equation of the normal line at $x=2$

Point : $(2, \frac{1}{2})$ slope : 4

$$\Rightarrow y - \frac{1}{2} = 4(x-2)$$

Review Questions to Do:

Text	**Pages 91 – 93 #1 – 13 odd, 15 – 26 all, #27 – 47 odd**
Barrons	**Set 2 Pages 37 – 41 #1 – 35 all (omit 5, 13, 33)**