AP Calculus
6.1 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. A general solution to a differential equation will have a constant in the solution.

2. Find the general solution to the differential equations below: (need more practice? … page 327 #2 and #4)

a) $\frac{dy}{dx} = 5x^4 + \sec^2 x$

$y = x^5 + \tan x + C, \ C \in \mathbb{R}$

b) $\frac{dy}{dx} = \sin x - e^{-x} + 8x^3$

$y = -\cos x + e^{-x} + 2x^4 + C, \ C \in \mathbb{R}$

c) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{x}}$

$y = \sin^{-1} x - 2\sqrt{x} + C, \ C \in \mathbb{R}$

d) $\frac{dy}{dx} = 5^x \ln(5) - \frac{1}{x^2+1}$

$y = 5^x - \tan^{-1} x + C, \ C \in \mathbb{R}$

3. Find the particular solution $y = f(x)$ using the given initial condition. How are these different than solutions in the last question? (need more practice? … p327 #11, 12, 14, 16, 17, 20)

a) $\frac{dy}{dx} = -\frac{1}{x^2} - \frac{3}{x^4} + 12$ and $y = 3$ when $x = 1$.

$y = \frac{1}{x} + \frac{1}{x^3} + 12x + C, \ C \in \mathbb{R}$

- $3 = 1 + 1 + 12 + C$

$C = -11$

$\Rightarrow \boxed{y = \frac{1}{x} + \frac{1}{x^3} + 12x - 11}$

b) $\frac{dx}{dt} = \frac{1}{t} - \frac{1}{t^2} + 6$ and $x = 0$ when $t = 1$.

$x = \ln|t| + \frac{1}{t} + 6t + C, \ C \in \mathbb{R}$

- $0 = \ln 1 + 1 + 6 + C$

$C = -7$

$\Rightarrow \boxed{x = \ln|t| + \frac{1}{t} + 6t - 7}$

c) $\frac{du}{dx} = 7x^6 - 3x^2 + 5$ and $u = 1$ when $x = 1$.

$u = x^7 - x^3 + 5x + C, \ C \in \mathbb{R}$

- $1 = 1 - 1 + 5 + C$

$C = -4$

$\Rightarrow \boxed{u = x^7 - x^3 + 5x - 4}$

d) $\frac{dv}{dt} = 4\sec t \tan t + e^t + 6t$ and $v = 5$ when $t = 0$.

$v = 4\sec t + e^t + 3t^2 + C, \ C \in \mathbb{R}$

- $5 = 4\sec 0 + e^0 + 3(0)^2 + C$

$5 = 4 + 1 + C$

$C = 0$

$\Rightarrow \boxed{v = 4\sec t + e^t + 3t^2}$

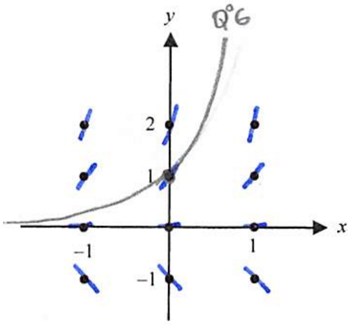
From your textbook … page 322 …
"An initial condition determines a particular solution by requiring that a solution curve pass through a given point. If the curve is continuous, this pins down the solution on the entire domain. If the curve is discontinuous, the initial condition only pins down the continuous *piece of the curve* that passes through the given point. In this case, the domain of the solution must be specified."

4. Which (if any) of the examples in question 3, require you to specify a domain? What are those domains?

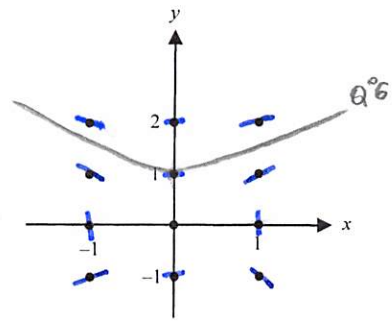
a) $D = (0, +\infty)$ b) $D = (0, +\infty)$ d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

5. Construct a slope field for each differential equation. Draw tiny segments through the twelve lattice points shown in the graph.

a) $\dfrac{dy}{dx} = 2y$

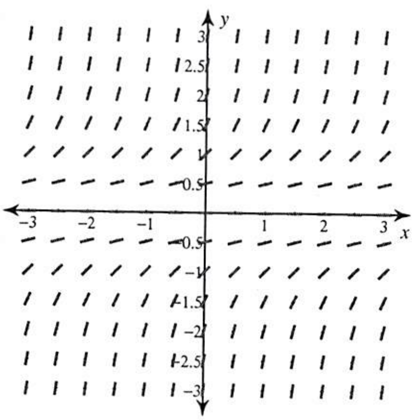


b) $\dfrac{dy}{dx} = \dfrac{x}{2y}$



6. For each slope field above, sketch the solution curve that passes through the point (0, 1).

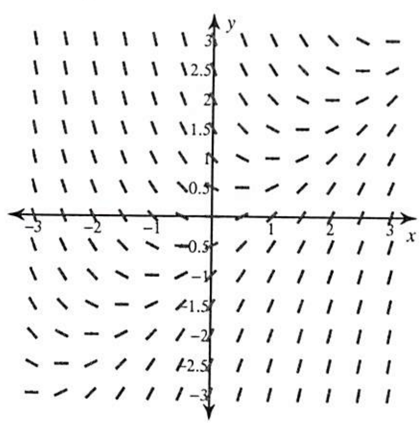
7. **For each problem, find a differential equation that could be represented with the given slope field.** a)



A) $\dfrac{dy}{dx} = -\dfrac{1}{x}$ B) $\dfrac{dy}{dx} = -\dfrac{1}{y}$

C) $\dfrac{dy}{dx} = 1$ D) $\dfrac{dy}{dx} = y^2$

b)



A) $\dfrac{dy}{dx} = x + y$ B) $\dfrac{dy}{dx} = x - y$

C) $\dfrac{dy}{dx} = xy$ D) $\dfrac{dy}{dx} = -xy$

8. Use separation of variables to solve each differential equation. Indicate the domain over which the solution is valid.

a) $\frac{dy}{dx} = \frac{x}{y}$ and $y = 2$ when $x = 1$.

$y'y = x$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$

$y^2 = x^2 + 2C$

$y = \pm\sqrt{x^2 + 2C}$

• if $x = 1$, $y = 2 > 0$

$y = \sqrt{x^2 + 2C}$

$2 = \sqrt{1 + 2C}$

$4 = 1 + 2C$

$2C = 3$

$C = 3/2$

Test ok.

$\boxed{y = \sqrt{x^2 + 3}}$

b) $\frac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$.

$\frac{y'}{y^2} = -2x$

$-\frac{1}{y} = -x^2 + C$

$y = \frac{1}{x^2 - C}$

• if $x = 1$, $y = \frac{1}{4}$

$\frac{1}{4} = \frac{1}{1 - C}$

$C = -3$

$\boxed{y = \frac{1}{x^2 + 3}}$

9. Solve the initial value problem $\frac{d^2y}{dx^2} = 2 - 6x$ given that $y(0) = 1$ and $y'(0) = 4$.

$y'' = 2 - 6x$

$y' = 2x - 3x^2 + C \quad , C \in \mathbb{R}$

$y = x^2 - x^3 + Cx + D \quad , D \in \mathbb{R}$

• $y(0) = D$ • $y'(0) = C$

$D = 1$ $C = 4$

$\boxed{y = x^2 - x^3 + 4x + 1}$

10. Solve the differential equation $\frac{dy}{dt} = ky$ by using separation of variables and assuming k is a constant.
(Solving this differential equation means getting an equation of the form "$y = \ldots$")

$y = y_0 e^{kx}$ (from the lesson) or... $\frac{y'}{y} = k$

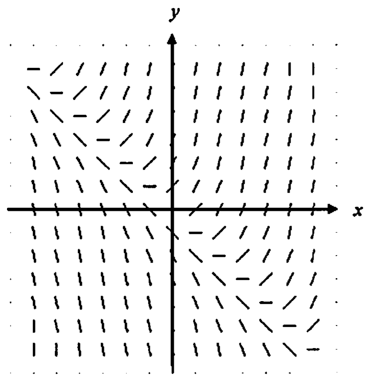
$\ln|y| = kx + C$

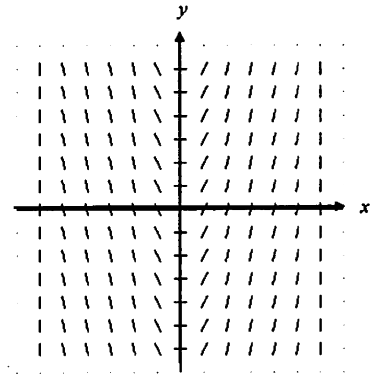
$|y| = e^{kx + C}$

$y = \pm e^{kx + C}$

$y = \underbrace{\pm e^{C}}_{y_0} e^{kx}$

11. [MATCHING] Connect each of the six slope fields shown below to their differential equations. Explain each choice.





$\frac{dy}{dx} = x - y$

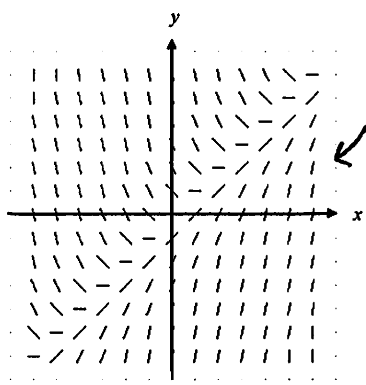
$\frac{dy}{dx} = 2x$

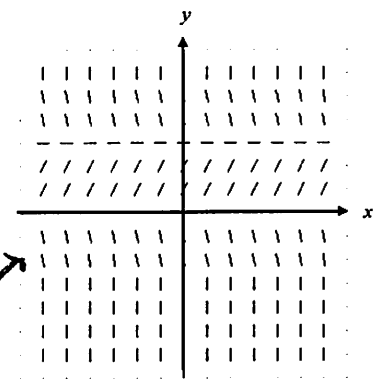
$\frac{dy}{dx} = 1 + y$

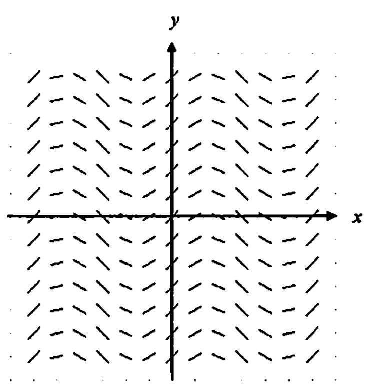
$\frac{dy}{dx} = \cos x$

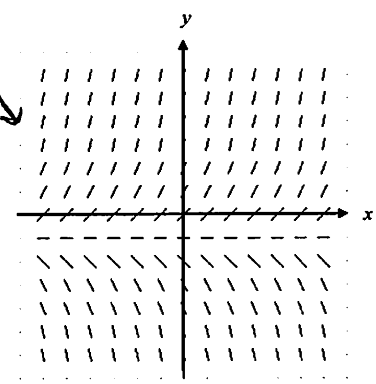
$\frac{dy}{dx} = x + y$

$\frac{dy}{dx} = y(3 - y)$









AP Calculus
6.2 Worksheet Day 1

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Indefinite Integrals

1. $\int \frac{2x}{\sqrt{x^2+6}}\,dx = \int u^{-1/2}\,du$

let $u = x^2+6$
$du = 2x\,dx$

$= 2u^{1/2} + C$

$= 2\sqrt{x^2+6} + C$

2. $\int \frac{e^x}{e^x+4}\,dx = \int \frac{1}{u}\,du$

let $u = e^x+4$
$du = e^x dx$

$= \ln|u| + C$

$= \ln(e^x+4) + C$

3. $\int \frac{e^x}{1+2e^x}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$

let $u = 1+2e^x$
$du = 2e^x dx$

$= \frac{1}{2}\ln|1+2e^x| + C$

4. $\int \sec^2(2x)\,dx = \frac{1}{2}\int \sec^2 u\,du$

let $u = 2x$
$du = 2dx$

$= \frac{1}{2}\tan(2x) + C$

5. $\int \sec^2(3x)e^{\tan(3x)}\,dx = \frac{1}{3}\int e^u\,du$

let $u = \tan 3x$
$du = \sec^2 3x \cdot 3dx$

$= \frac{1}{3}e^{\tan 3x} + C$

6. $\int \frac{x}{2x^2+1}\,dx = \frac{1}{4}\int \frac{1}{u}\,du$

let $u = 2x^2+1$
$du = 4x\,dx$

$= \frac{1}{4}\ln(2x^2+1) + C$

7. $\int e^x(2+e^x)^{1/2}\,dx$ let $u = 2+e^x$
$du = e^x dx$

$= \int u^{1/2}\,du$

$= \frac{2}{3}(2+e^x)^{3/2} + C$

8. $\int x^2\cos(x^3)\,dx$ let $u = x^3$
$du = 3x^2dx$

$= \frac{1}{3}\int \cos u\,du$

$= \frac{1}{3}\sin x^3 + C$

9. $\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$ let $u = \tan x$
$du = \sec^2 x\,dx$

$= \int u^{-1/2}\,du$

$= 2\sqrt{\tan x} + C$

10. $\int \frac{\tan^{-1}x}{1+x^2}\,dx$ let $u = \tan^{-1}x$
$du = \frac{1}{1+x^2}dx$

$= \int u\,du$

$= \frac{1}{2}(\tan^{-1}x)^2 + C$

11. $\int \csc^2(3x+5)\,dx$ let $u = 3x+5$
$du = 3dx$

$= \frac{1}{3}\int \csc^2 u\,du$

$= -\frac{1}{3}\cot(3x+5) + C$

12. $\int \frac{x+1}{(x^2+2x+7)^3}\,dx$ let $u = x^2+2x+7$
$du = (2x+2)dx$
$du = 2(x+1)dx$

$= \frac{1}{2}\int \frac{1}{u^3}\,du$

$= \frac{1}{2} \times \frac{u^{-2}}{-2} + C$

$= \frac{1}{-4(x^2+2x+7)^2} + C$

AP Calculus
6.2 Worksheet Day 2

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Algebraic Techniques To Integration … Straight substitution works well when there is one part of the problem that is a derivative of the rest of the problem. When this doesn't occur, you may have to "massage" the problem to fit into a form that can be integrated from a rule or by using substitution. The more of these you do, the better you will get at recognizing which method will work. For now, use the following hints to help you get started:

1. Long Division … You should use this when you see … a rational expression with a higher degree on the numerator than on the denominator.
2. Complete the Square … You should use this when you see … there is a quadratic on the denominator and its derivative is not on the numerator.
3. Separate the Numerator … You should use this when you see … each separate fraction can be integrated.
4. Expand … You should use this when you see … when you have a power of an expression without its derivative outside.

1. $\int \frac{x^5 - 35x}{x^2+6}\,dx$

$= \int \left(x^3 - 6x + \frac{x}{x^2+6}\right) dx$

$= \boxed{\frac{x^4}{4} - 3x^2 + \frac{1}{2}\ln|x^2+6| + C}$

2. $\int \frac{x^2}{x+1}\,dx = \int \left(x - 1 + \frac{1}{x+1}\right) dx$

$= \boxed{\frac{x^2}{2} - x + \ln|x+1| + C}$

3. $\int \frac{dx}{x^2-4x+4} = \int \frac{1}{(x-2)^2}\,dx$

$= \boxed{-\frac{1}{x-2} + C}$

4. $\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{1}{\sqrt{-(x-2)^2+1}}\,dx$

let $u = x-2$
$du = dx$

$= \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \boxed{\sin^{-1}(x-2) + C}$

7. $\int \frac{e^x+4}{e^x}\,dx$

$= \int (1+4e^{-x})\,dx$

$= \boxed{x - 4e^{-x} + C,\ C\in\mathbb{R}}$

8. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

$= \int \frac{x}{2x\sqrt{x-1}}\,dx + \int \frac{2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

$= \int \frac{1}{2\sqrt{x-1}}\,dx + \int \frac{1}{x}\,dx$

$= \int \frac{1}{2\sqrt{u}}\,du + \int \frac{1}{x}\,dx$ with $u = x-1$, $du = dx$

$= \boxed{\sqrt{x-1} + \ln|x| + C,\ C\in\mathbb{R}}$

9. $\int (x^3-7)^2\,dx$

$= \int (x^6 - 14x^3 + 49)\,dx$

$= \boxed{\frac{x^7}{7} - \frac{7}{2}x^4 + 49x + C,\ C\in\mathbb{R}}$

10. $\int \frac{5-e^x}{e^{2x}}\,dx$

$= \int (5e^{-2x} - e^{-x})\,dx$

$= \boxed{-\frac{5}{2}e^{-2x} + e^{-x} + C,\ C\in\mathbb{R}}$

11. [No Calculator] … Consider a differentiable function f having domain all positive real numbers, and for which it is known that $f'(x)=(4-x)x^{-3}$ for $x>0$.

a) Find the x-coordinate of the critical point of f. Determine whether or not the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer.

for $x>0$, $f'(x)=0 \Leftrightarrow x=4$. So the x-coord. of the critical point is 4.

for $0<x<4$: $f'(x)>0$ and for $x>4$: $f'(x)<0$: Therfore relative max.

b) Find all intervals on which the graph of f is concave down. Justify your answer.

$$f''(x) = -x^{-3} - 3x^{-4}(4-x) = \frac{-x-3(4-x)}{x^4} = \frac{2x-12}{x^4} = \frac{2(x-6)}{x^4}$$

• f is concave down on $(0,6)$

c) Given that $f(1) = 2$, determine the function f.

$$f'(x) = \frac{4}{x^3} - \frac{1}{x^2}$$

$$f(x) = \frac{4}{-2x^2} + \frac{1}{x} + C$$

• $f(1) = -2 + 1 + C$

$2 = -1 + C$

$C = 3$

$$\Rightarrow \boxed{f(x) = -\frac{2}{x^2} + \frac{1}{x} + 3}$$

or $f(x) = \dfrac{x-2}{x^2} + 3$ or …

The following two integrals involve a "twist" to the normal substitution method.

After you make your normal substitution for u, you have not accounted for all of the integrand … replace the remaining x's by solving your substitution rule for x in terms of u.

12. $\int (x+1)\sqrt{2-x}\,dx = -\int (3-u)\sqrt{u}\,du$

let $u = 2-x$

$du = -dx$

$x = 2-u$

$= -\int 3\sqrt{u}\,du + \int u^{3/2}\,du$

$= -3 \cdot \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$

$= \frac{2}{5}u^{5/2} - 2u^{3/2} + C$

$= \frac{2}{5}(2-x)^{5/2} - 2(2-x)^{3/2} + C$

13. $\int \frac{2x+1}{\sqrt{x+4}}\,dx = \int \frac{2u-7}{\sqrt{u}}\,du$

let $u = x+4$

$du = dx$

$x = u-4$

$= 2\int u^{1/2}\,du - 7\int u^{-1/2}\,du$

$= 2 \cdot \frac{2}{3}u^{3/2} - 7 \cdot 2u^{1/2} + C$

$= \frac{4}{3}u^{3/2} - 14u^{1/2} + C$

$= \frac{4}{3}(x+4)^{3/2} - 14(x+4)^{1/2} + C$

14. $\int x^3\sqrt{x^2+3}dx$ $= \frac{1}{2}\int 2x \cdot x^2\sqrt{x^2+3}\,dx = \frac{1}{2}\int (u-3)\sqrt{u}\,du$

let $u = x^2+3$ $\quad x^2 = u-3$

$du = 2x\,dx$

$= \frac{1}{2}\int \left(u^{3/2} - 3u^{1/2}\right)du$

$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2} - 3\frac{u^{3/2}}{3/2}\right) + C, \quad C \in \mathbb{R}$

$= \boxed{\frac{1}{5}(x^2+3)^{5/2} - (x^2+3)^{3/2} + C}$

4. Integrate each indefinite integral using any method possible. All Mixed UP!

a) $\int \cos(3x)e^{\sin(3x)}\,dx$

$u = \sin 3x$
$du = 3\cos 3x\,dx$

$= \frac{1}{3}\int e^u\,du$

$= \frac{1}{3}e^{\sin 3x} + C$

b) $\int \frac{1}{x\ln(3x)}\,dx$

$u = \ln 3x$
$du = \frac{1}{x}\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|\ln 3x| + C$

c) $\int \frac{\sin(3x)}{1+\cos(3x)}\,dx = -\frac{1}{3}\int \frac{1}{u}\,du$

$u = 1+\cos 3x$
$du = -3\sin 3x\,dx$

$= -\frac{1}{3}\ln|1+\cos 3x| + C$

d) $\int \frac{1}{\sqrt{1-9x^2}}\,dx$

$u = 3x$
$du = 3\,dx$

$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{3}\sin^{-1}(3x) + C$

e) $\int x\csc(3x^2)\cot(3x^2)\,dx$

$u = 3x^2$
$du = 6x\,dx$

$= \frac{1}{6}\int \csc u \cdot \cot u \cdot du$

$= \frac{1}{6}(-\csc 3x^2) + C$

$= -\frac{1}{6}\csc 3x^2 + C$

f) $\int x\sqrt{4+x}\,dx$

let $u = 4+x$
$x = u-4$
$du = dx$

$= \int (u-4)\sqrt{u}\,du$

$= \int \left(u^{3/2} - 4u^{1/2}\right)du$

$= \frac{2}{5}(4+x)^{5/2} - \frac{8}{3}(4+x)^{3/2} + C,\ C\in\mathbb{R}$

g) $\int \frac{x}{(2-3x)^3}\,dx$

let $u = 2-3x$
$du = -3\,dx$
$x = \frac{2-u}{3}$

$= -\frac{1}{9}\int \frac{2-u}{u^3}\,du$

$= -\frac{1}{9}\int \left(2u^{-3} - u^{-2}\right)du$

$= -\frac{1}{9}\left(-u^{-2} + u^{-1}\right) + C,\ C\in\mathbb{R}$

$= \frac{1}{9(2-3x)^2} - \frac{1}{9(2-3x)} + C$

AP Calculus
6.2 Worksheet Day 3

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1. Integrate each of the following:

a) $\int \sin^2(4x)\,dx = \int \frac{1-\cos 8x}{2}\,dx$

let $u = 8x$
$du = 8dx$

$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos 8x\,dx$

$= \boxed{\frac{1}{2}x - \frac{1}{16}\sin 8x + C}$

b) $\int \left(8\tan t + \cos^2(8t)\right)dt = 8\int \frac{\sin t}{\cos t}\,dt + \int \frac{1+\cos 16t}{2}\,dt$

let $u = \cos t$
$du = -\sin t\,dt$

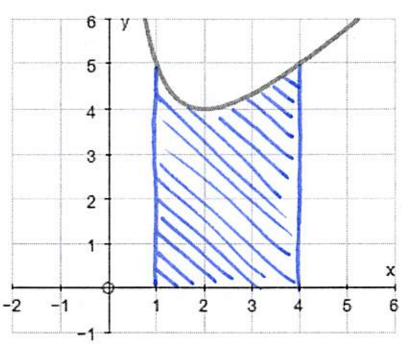
$= -8\int \frac{1}{u}\,du + \int \frac{1}{2}\,dt + \frac{1}{2} \times \frac{1}{16}\int \cos v \cdot dv$

let $v = 16t$
$dv = 16dt$

$= \boxed{-8\ln|\cos t| + \frac{1}{2}t + \frac{1}{32}\sin 16t + C}$

2. [No Calculator] A graph of each function is given. ***Shade*** each region bounded by the graphs of the equations, then find the area of that region.

a) $y = \dfrac{x^2+4}{x}$, $x = 1, x = 4$, and $y = 0$



$A = \int_1^4 \frac{x^2+4}{x}\,dx$

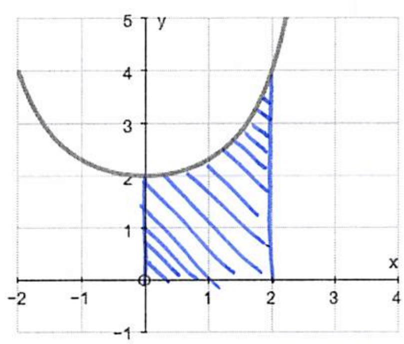
$= \int_1^4 x\,dx + \int_1^4 \frac{4}{x}\,dx$

$= \frac{x^2}{2} + 4\ln x \Big]_1^4$

$= 8 + 4\ln 4 - \frac{1}{2}$

$= \frac{15}{2} + 4\ln 4$

b) $y = 2\sec\left(\dfrac{\pi x}{6}\right)$, $x = 0, x = 2, y = 0$



$A = \int_0^2 2\sec\left(\frac{\pi x}{6}\right)dx$

6.2 – worksheet day 3

4. Integrate each indefinite integral using any method possible. All Mixed UP!

a) $\int \cos(3x)e^{\sin(3x)}\,dx$

$u = \sin 3x$
$du = 3\cos 3x\,dx$

$= \frac{1}{3}\int e^u\,du$

$= \frac{1}{3}e^{\sin 3x} + C,\ C\in\mathbb{R}$

b) $\int \frac{1}{x\ln(3x)}\,dx$

$u = \ln(3x)$
$du = \frac{1}{x}dx$

$= \int \frac{1}{u}\,du$

$= \ln|\ln(3x)| + C,\ C\in\mathbb{R}$

c) $\int \frac{\sin(3x)}{1+\cos(3x)}\,dx$

$u = 1+\cos 3x$
$du = -3\sin 3x\,dx$

$= -\frac{1}{3}\int \frac{1}{u}\,du$

$= -\frac{1}{3}\ln|1+\cos 3x| + C$
$C\in\mathbb{R}$

d) $\int \frac{1}{\sqrt{1-9x^2}}\,dx$

$u = 3x$
$u^2 = 9x^2$
$du = 3dx$

$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{3}\sin^{-1}(3x) + C,\ C\in\mathbb{R}$

e) $\int x\csc(3x^2)\cot(3x^2)\,dx$

$u = 3x^2$
$du = 6x\,dx$

$= \frac{1}{6}\int \csc u.\cot u.\,du$

$= \frac{1}{6}\left(-\csc(3x^2)\right) + C,\ C\in\mathbb{R}$

$= -\frac{1}{6}\csc(3x^2) + C$

f) $\int x^2\sqrt{4+x}\,dx$

$u = 4+x$
$x = u-4$
$dx = du$

$= \int (u-4)^2\sqrt{u}\,du$

$= \int\left(u^{5/2} - 8u^{3/2} + 16u^{1/2}\right)du$

$= \frac{2}{7}(4+x)^{7/2} - \frac{16}{5}(4+x)^{5/2} + \frac{32}{3}(4+x)^{3/2}$
$+C$

g) $\int \frac{x^2}{(2-x)^3}\,dx$

$u = 2-x$
$x = 2-u$
$dx = -du$

$= -\int \frac{(2-u)^2}{u^3}\,du$

$= -\int\left(4u^{-3} - 4u^{-2} + u^{-1}\right)du$

$= 2u^{-2} + 4u^{-1} + \ln|u| + C,\ C\in\mathbb{R}$

$= \frac{2}{(2-x)^2} + \frac{4}{2-x} + \ln|2-x| + C$

h) $\int \frac{y\,dy}{(y-2)^3}$

$u = y-2 \quad y = u+2$

$dy = du$

$= \int \frac{u+2}{u^3}\,du$

$= \int (u^{-2} + 2u^{-3})\,du$

$= -\frac{1}{y-2} - \frac{1}{(y-2)^2} + C,\ C \in \mathbb{R}$

i) $\int \sqrt{\frac{1+x}{x^5}}\,dx\ , x > 0$

$= \int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx$

$= \int \frac{1}{x^2}\sqrt{\frac{1}{x}+1}\,dx$

$u = \frac{1}{x}+1$

$du = -\frac{1}{x^2}\,dx$

$= -\int \sqrt{u}\,du$

$= -\frac{2}{3}\left(\frac{1}{x}+1\right)^{3/2} + C,\ C \in \mathbb{R}$

j) $\int \frac{e^x+1}{e^x-1}\,dx$

$= \int \frac{e^{x/2}(e^{x/2}+e^{-x/2})}{e^{x/2}(e^{x/2}-e^{-x/2})}\,dx$

$u = e^{x/2} - e^{-x/2}$

$du = \left(\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right)dx$

$= \int \frac{2}{u}\,du$

$= 2\ln\left|e^{x/2} - e^{-x/2}\right| + C,\ C \in \mathbb{R}$

k) $\int \frac{dx}{\sqrt{x}(1-2\sqrt{x})}$

$u = 1 - 2\sqrt{x}$

$du = -\frac{1}{\sqrt{x}}\,dx$

$= -\int \frac{1}{u}\,du$

$= -\ln|1-2\sqrt{x}| + C\ ,\ C \in \mathbb{R}$

l) $\int e^{x+e^x}\,dx$

$= \int e^x . e^{e^x}\,dx$

$u = e^x$

$du = e^x dx$

$= \int e^u\,du$

$= e^{e^x} + C,\ C \in \mathbb{R}$

m) $\int \sqrt[3]{x^3 + 3\cos x}\,(x^2 - \sin x)\,dx$

$u = x^3 + 3\cos x$

$du = (3x^2 - 3\sin x)dx$

$= \frac{1}{3}\int u^{1/3}\,du$

$= \frac{1}{4}(x^3 + 3\cos x)^{4/3} + C,\ C \in \mathbb{R}$

n) $\int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$

$= \tan x - x + C,\ C \in \mathbb{R}$

p) $\int \cos^2(3x)\,dx$

$u = 3x$

$du = 3dx$

$= \frac{1}{3}\int \cos^2 u\,du$

$= \frac{1}{3}\int \frac{\cos 2u + 1}{2}\,du$

$= \frac{1}{6}\int \cos 2u\,du + \frac{u}{6}$

$= \frac{1}{12}\int \cos t\,dt + \frac{u}{6}$

$t = 2u$

$dt = 2du$

$= \frac{1}{12}\sin(6x) + \frac{x}{2} + C,\ C \in \mathbb{R}$

q) $\int 5x\,\tan(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$

$= \frac{5}{2}\int \tan u\,du$

$= \frac{5}{2}\int \frac{\sin u}{\cos u}\,du$

$t = \cos u$

$dt = -\sin u\,du$

$= -\frac{5}{2}\int \frac{1}{t}\,dt$

$= -\frac{5}{2}\ln|\cos x^2| + C$

$C \in \mathbb{R}$

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Definite Integrals

13. True or False: $\int_0^{\pi/4} \tan^3(x)\sec^2(x)\,dx = \int_0^{\pi/4} u^3\,du$

let $u = \tan x$
$du = \sec^2 x\,dx$

$\int_0^{\pi/4} \tan^3 x \cdot \sec^2 x\,dx = \int_0^{1} u^3 \cdot du$

While no one is going to "force" you to do a definite integral problem using substitution a specific way, the previous problem is <u>less likely to be missed</u> if you get in the habit of changing the limits <u>at the same time</u> that you make your substitution!

14. $\int_1^{\sqrt{2}} x \cdot 2^{-x^2}\,dx$

let $u = -x^2$
$du = -2x\,dx$

$= -\frac{1}{2}\int_{-1}^{-2} 2^u\,du$

$= -\frac{1}{2}\left.\frac{2^u}{\ln 2}\right]_{-1}^{-2}$

$= -\frac{1}{2}\frac{2^{-2}}{\ln 2} + \frac{1}{2}\frac{2^{-1}}{\ln 2}$

$= \frac{1}{4\ln 2} - \frac{1}{8\ln 2}$

$= \frac{2}{8\ln 2} - \frac{1}{8\ln 2}$

$= \boxed{\frac{1}{8\ln 2}}$

15. $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

let $u = \ln x$
$du = \frac{1}{x}\,dx$

$= \int_1^2 \frac{1}{u}\,du$

$= \ln|u|\Big]_1^2$

$= \ln 2 - \ln 1$

$= \boxed{\ln 2}$

16. $\int_0^2 \sqrt{4x+1}\,dx$

let $u = 4x+1$
$du = 4\,dx$

$= \frac{1}{4}\int_1^9 \sqrt{u}\,du$

$= \frac{1}{4}\cdot\frac{2}{3}\,u^{3/2}\Big]_1^9$

$= \frac{1}{6}\cdot 27 - \frac{1}{6}\cdot 1$

$= \boxed{\frac{13}{3}}$

17. $\int_1^5 \frac{(\ln x)^{1/2}}{x}\,dx$

let $u = \ln x$
$du = \frac{1}{x}\,dx$

$= \int_0^{\ln 5} \sqrt{u}\,du$

$= \frac{2}{3}\,u^{3/2}\Big]_0^{\ln 5}$

$= \boxed{\frac{2}{3}(\ln 5)^{3/2}}$

18. $\int_0^1 \frac{1}{(2x+3)^3}\,dx$

let $u = 2x+3$
$du = 2\,dx$

$= \frac{1}{2}\int_3^5 \frac{1}{u^3}\,du$

$= \frac{1}{2}\cdot\frac{1}{-2}\,u^{-2}\Big]_3^5$

$= -\frac{1}{100} + \frac{1}{36}$

$= \boxed{\frac{4}{225}}$

20. $\int_0^{\pi} \sin\left(\frac{x}{2}\right)dx$

let $u = \frac{x}{2}$
$du = \frac{1}{2}\,dx$

$= 2\int_0^{\pi/2} \sin u \cdot du$

$= 2(-\cos u)\Big]_0^{\pi/2}$

$= \boxed{2}$

21. $\int_{-\pi}^{\pi} x\sin(x^2)\,dx$

let $u = x^2$
$du = 2x\,dx$

$= \frac{1}{2}\int_{\pi^2}^{\pi^2} \sin u \cdot du$

$= 0$

22. $\int_{-1}^{0} \frac{2}{6x-1}\,dx$

let $u = 6x-1$
$du = 6\,dx$

$= \frac{1}{3}\int_{-7}^{-1} \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u|\Big]_{-7}^{-1}$

$= \frac{1}{3}\ln 1 - \frac{1}{3}\ln 7$

$= \boxed{-\frac{1}{3}\ln 7}$

5. Integrate each definite integral using any method possible. All Mixed UP!

a) $\int_0^1 xe^{-x^2}\,dx$

$u=-x^2$
$du=-2x\,dx$

$= -\frac{1}{2}\int_0^{-1} e^u\,du$

$= -\frac{1}{2}e^u\Big]_0^{-1}$

$= -\frac{1}{2}e^{-1} + \frac{1}{2}e^0$

$= -\frac{1}{2e} + \frac{1}{2}$

b) $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

$= \tan^{-1}x\Big]_{-1}^{1}$

$= \tan^{-1}1 - \tan^{-1}(-1)$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$

c) $\int_0^2 \left(2^x + x^2\right)dx$

$= \int_0^2 2^x\,dx + \int_0^2 x^2\,dx$

$= \frac{2^x}{\ln 2}\Big]_0^2 + \frac{x^3}{3}\Big]_0^2$

$= \frac{4}{\ln 2} - \frac{1}{\ln 2} + \frac{8}{3}$

$= \frac{3}{\ln 2} + \frac{8}{3}$

d) $\int_{\pi^2/4}^{4\pi^2} \frac{1}{\sqrt{x}} \sin\sqrt{x}\,dx$

$u=\sqrt{x}$
$du=\frac{1}{2\sqrt{x}}dx$

$= 2\int_{\pi/2}^{2\pi} \sin u.\,du$

$= -2\cos u\Big]_{\pi/2}^{2\pi}$

$= -2$

e) $\int_1^2 \frac{1}{t^4}\left(1-\frac{1}{t^3}\right)^3 dt$

$u=1-\frac{1}{t^3}$

$du=\frac{3}{t^4}dt$

$= \int_0^{7/8} \frac{1}{3}u^3\,du$

$= \frac{1}{3}\frac{u^4}{4}\Big]_0^{7/8}$

$= \frac{2401}{49152}$

f) $\int_{\frac{1}{5}}^{3} \frac{\ln(5x)}{x}dx$

$u=\ln 5x$
$du=\frac{1}{x}dx$

$= \int_0^{\ln 15} u\,du$

$= \frac{u^2}{2}\Big]_0^{\ln 15}$

$= \frac{1}{2}(\ln 15)^2$

g) $\int_{-1}^{1} \frac{5^{-x}}{2^x} dx$

$= \int_{-1}^{1} \frac{1}{10^x} dx$

$= \int_{-1}^{1} 10^{-x} dx$

$= \frac{-10^{-x}}{\ln 10}\Big]_{-1}^{1} = \frac{99}{10\ln 10}$

i) $\int_0^1 \frac{x\,dx}{\sqrt{2-x^2}}$ $\quad u = 2-x^2$, $du = -2x\,dx$

$= -\frac{1}{2}\int_2^1 u^{-1/2} du$

$= -\frac{1}{2} \cdot 2u^{1/2}\Big]_2^1$

$= -1 + \sqrt{2}$

j) $\int_0^1 \frac{2}{\sqrt{1-x^2}} dx$

$= 2\sin^{-1}x\Big]_0^1$

$= 2 \times \frac{\pi}{2}$

$= \pi$

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AP Calculus
6.4 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Suppose the rate of change of y is proportional to the amount of y present.

   a) Write the differential equation that this statement represents. $y' = ky$

   b) Solve the differential equation from part *a* … do not skip ANY steps.

$\frac{y'}{y} = k$

$\ln|y| = kt + c$

$|y| = e^{kt+c}$

$y = \pm e^{kt+c}$

$y = y_0 e^{kt}$ (with $y_0 = \pm e^c$)

2. Find the particular solution $y = f(x)$ to each differential equation using the given initial value.

a) $\frac{dy}{dx} = (y+5)(x+2)$ and $y = 1$ when $x = 0$.

$\frac{y'}{y+5} = x+2$

$\ln|y+5| = \frac{x^2}{2} + 2x + c$

$|y+5| = e^{\frac{x^2}{2}+2x+c}$

$y = \pm e^{\frac{x^2}{2}+2x+c} - 5$

when $x=0$, $y=1$

$1 = +e^c - 5 \quad e^c = 6 \quad c = \ln 6$

$\boxed{y = 6e^{\frac{x^2}{2}+2x} - 5}$

b) $\frac{dy}{dx} = \frac{1}{5}(8-y)$ and $y = 6$ when $x = 0$

$\frac{y'}{8-y} = \frac{1}{5}$

$-\ln|8-y| = \frac{1}{5}x + c$

$|8-y| = e^{-\frac{1}{5}x - c}$

$8 - y = \pm e^{-\frac{1}{5}x - c}$

$y = \pm e^{-\frac{1}{5}x - c} + 8$

when $x=0$, $y=6$

$6 = -e^{-c} + 8$

$e^{-c} = 2$

$c = -\ln 2$

$\Rightarrow \boxed{y = -2e^{-\frac{1}{5}x} + 8}$

c) $\frac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$.

$\frac{y'}{\cos^2 y} = 1$

$y' \sec^2 y = 1$

$\tan y = x + c$

$y = \tan^{-1}(x+c)$

when $x=0$, $y=0$

$0 = \tan^{-1}(c)$

$c = 0$

$\Rightarrow \boxed{y = \tan^{-1} x}$

d) $\frac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$.

$y' = \frac{e^x}{e^y}$

$y' e^y = e^x$

$e^y = e^x + c$

$y = \ln(e^x + c)$

when $x=0$, $y=2$

$2 = \ln(1+c)$

$e^2 = 1 + c$

$c = e^2 - 1$

$\boxed{y = \ln(e^x + e^2 - 1)}$

3. [No Calculator] The rate of change in the population of a group of elk in the local national forest is proportional to the difference between the maximum number of elk the forest can support and the number of elk currently present. At time $t = 0$, when the number of elk are first counted, there are 40 elk. If $L(t)$ is the number of elk at time t years after they are first counted, then

$$\frac{dL}{dt} = \frac{1}{4}(500 - L)$$

$L' = -\frac{1}{4}(L-500)$ $y' = -\frac{1}{4}y$ w $y = L - 500$
$y_0 = -460$

a) Are the elk increasing in number faster when there are 160 or when there are 360? Explain and use correct notation.

$k = -\frac{1}{4}$ ∴ decay for y.
$\Rightarrow$ faster when $L = 160$

$L'(160) = -\frac{1}{4}(160-500) = 85$ elks / year
$L'(360) = -\frac{1}{4}(360-500) = 35$ elks / year

b) Find an equation for $\frac{d^2L}{dt^2}$ in terms of L. What does $\frac{d^2L}{dt^2}$ tell you about the graph of L?

$L'' = -\frac{1}{4}L'$

$L'' = -\frac{1}{4}\left(-\frac{1}{4}(L-500)\right)$

$L'' = \frac{1}{16}(L-500) < 0$ (because $L \leq 500$)

The graph is concave down

c) Use separation of variables to find the particular solution to $\frac{dL}{dt} = \frac{1}{4}(500 - L)$ if $L(0) = 40$.

$L' = -\frac{1}{4}(L-500)$

$y' = -\frac{1}{4}y$

$\frac{y'}{y} = -\frac{1}{4}$

$\ln|y| = -\frac{1}{4}t + C$

$|y| = e^{-\frac{1}{4}t + C}$

$y = \pm e^{C} \cdot e^{-\frac{1}{4}t}$

• when $t=0$, $y = -460$

$y = -460e^{-\frac{1}{4}t}$

$\Rightarrow \boxed{L = -460e^{-\frac{1}{4}t} + 500}$

4. [No Calculator] If $\frac{dy}{dt} = -2y$ and if $y = 1$ when $t = 0$, what is the value of t for which $y = \frac{1}{2}$?

A) $-\frac{1}{2}\ln 2$
B) $-\frac{1}{4}$
C) $\frac{1}{2}\ln 2$
D) $\frac{\sqrt{2}}{2}$
E) $\ln 2$

$y' = -2y \Rightarrow y = y_0 e^{-2t}$
$y = 1e^{-2t}$

$\Rightarrow \frac{1}{2} = e^{-2t}$
$\ln\left(\frac{1}{2}\right) = -2t$ $t = +\frac{1}{2}\ln 2$

5. [Calculator] A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

A) 4.2 pounds
B) 4.6 pounds
C) 4.8 pounds
D) 5.6 pounds
E) 6.5 pounds

$W' = kW \Rightarrow W = W_0 e^{kt}$

• when $t = 0$: $W_0 = 2$

• when $t = 2$: $W_2 = 2e^{2k}$
$3.5 = 2e^{2k}$
$\frac{7}{4} = e^{2k}$
$k = \frac{1}{2}\ln\left(\frac{7}{4}\right)$

• when $t = 3$:
$W = 2e^{\frac{3}{2}\ln\left(\frac{7}{4}\right)}$
$= 2 \times \left(\frac{7}{4}\right)^{3/2}$