2 Trigonometry

BUILDING ON

- applying the Pythagorean Theorem
- solving problems using properties of similar polygons
- solving problems involving ratios

BIG IDEAS

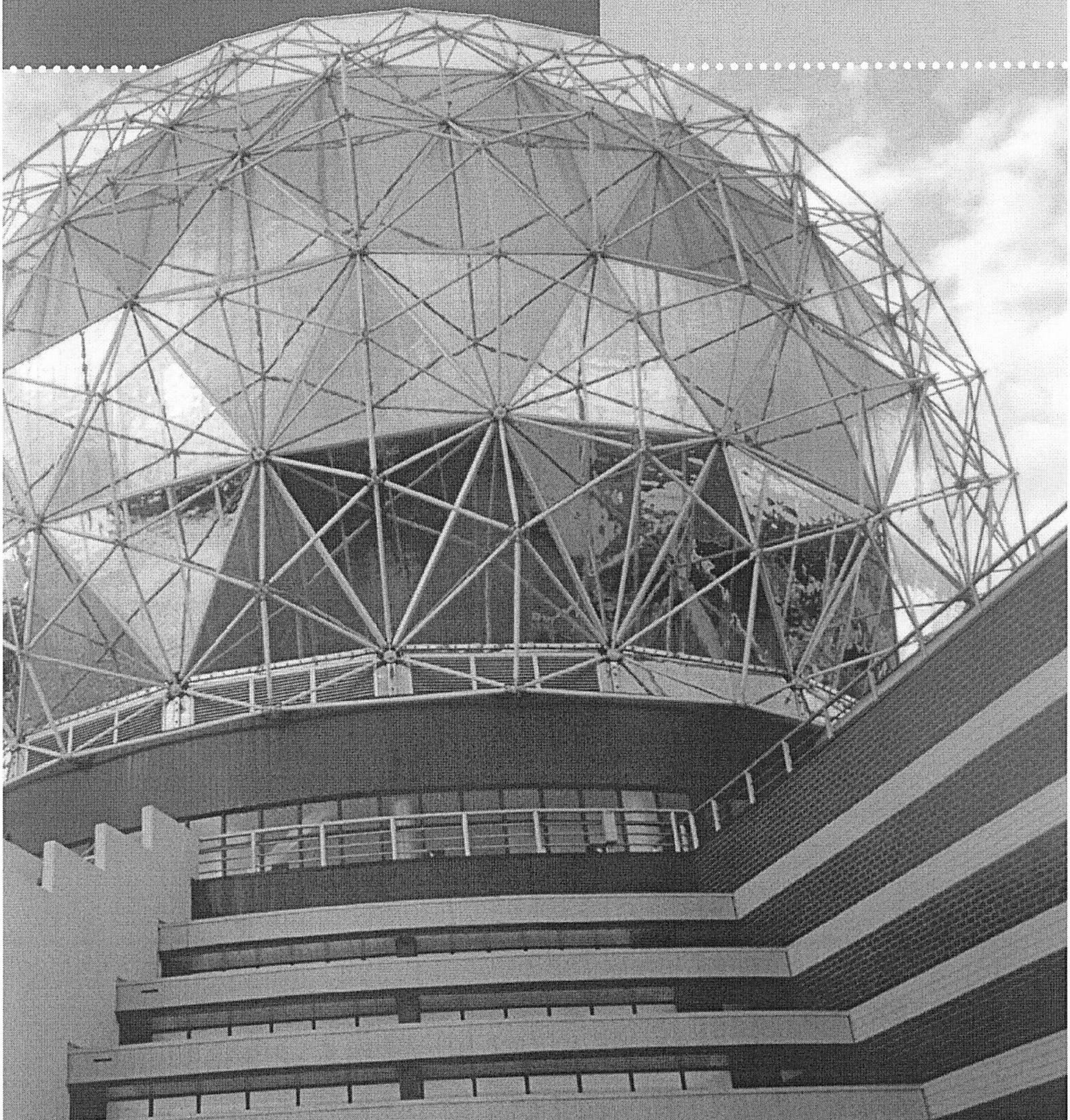
In a right triangle,

- The ratio of any two sides remains constant even if the triangle is enlarged or reduced.
- You can use the ratio of the lengths of two sides to determine the measure of one of the acute angles.
- You can use the length of one side and the measure of an acute angle to determine the length of another side of the triangle.

NEW VOCABULARY

angle of inclination
tangent ratio
indirect measurement
sine ratio
cosine ratio
angle of elevation
angle of depression

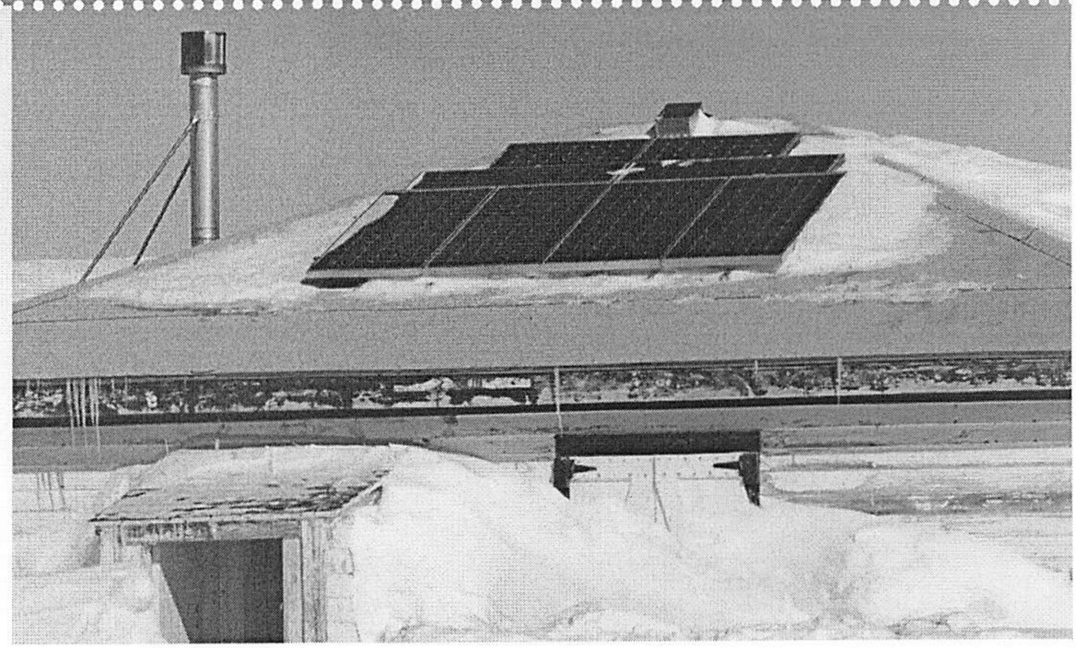
SCIENCE WORLD *This building was constructed for the Expo '86 World Fair held in Vancouver, British Columbia. The structure is a geodesic dome containing 766 triangles.*



2.1 The Tangent Ratio

LESSON FOCUS

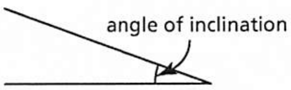
Develop the tangent ratio and relate it to the angle of inclination of a line segment.



This ranger's cabin on Herschel Island, Yukon, has solar panels on its roof.

Make Connections

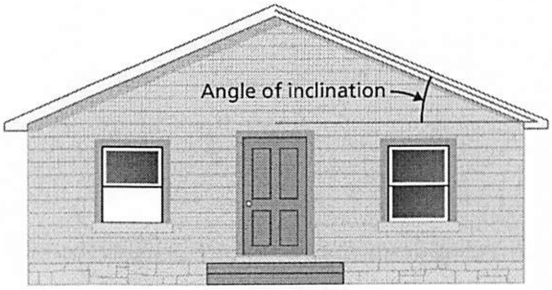
The **angle of inclination** of a line or line segment is the acute angle it makes with the horizontal.



South-facing solar panels on a roof work best when the **angle of inclination** of the roof, that is, the angle between the roof and the horizontal, is approximately equal to the latitude of the house.

When an architect designs a house that will have solar panels on its roof, she has to determine the width and height of the roof so that the panels work efficiently.

What happens to the angle of inclination if the diagram of the house is drawn using a different scale?



You will investigate the relationship between one acute angle in a right triangle and two sides of that triangle.

Construct Understanding

Recall that two triangles are similar if one triangle is an enlargement or a reduction of the other.

TRY THIS

Work with a partner.

You will need grid paper, a ruler, and a protractor.

A. On grid paper, draw a right $\triangle ABC$ with $\angle B = 90°$.

B. Each of you draws a different right triangle that is similar to $\triangle ABC$.

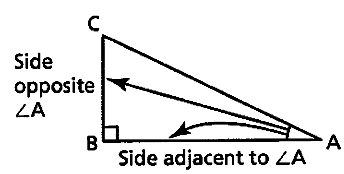
C. Measure the sides and angles of each triangle. Label your diagrams with the measures.

D. The two shorter sides of a right triangle are its legs. Calculate the ratio of the legs $\frac{CB}{BA}$ as a decimal, then the corresponding ratio for each of the similar triangles.

E. How do the ratios compare?

F. What do you think the value of each ratio depends on?

We name the sides of a right triangle in relation to one of its acute angles.



The ratio

Length of side opposite $\angle A$: Length of side adjacent to $\angle A$

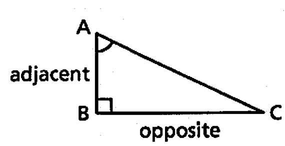
depends only on the measure of the angle, not on how large or small the triangle is.

This ratio is called the **tangent ratio** of $\angle A$.
The tangent ratio for $\angle A$ is written as tan A.
We usually write the tangent ratio as a fraction.

The Tangent Ratio

If $\angle A$ is an acute angle in a right triangle, then

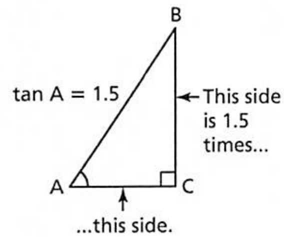
$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A}$$



As the size of ∠A increases, what happens to tan A?

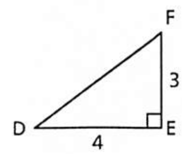
The value of the tangent ratio is usually expressed as a decimal that compares the lengths of the sides.

For example, if tan A = 1.5; then, in any similar right triangle with ∠A, the length of the side opposite ∠A is 1.5 times the length of the side adjacent to ∠A.



Example 1 Determining the Tangent Ratios for Angles

Determine tan D and tan F.

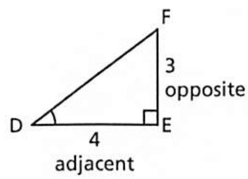


SOLUTION

$$\tan D = \frac{\text{length of side opposite } \angle D}{\text{length of side adjacent to } \angle D}$$

$$\tan D = \frac{EF}{DE}$$

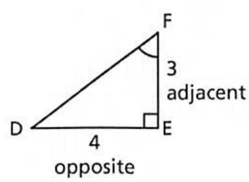
EF is opposite ∠D, DE is adjacent to ∠D.



$$\tan D = \frac{3}{4}$$

$$\tan D = 0.75$$

$$\tan F = \frac{\text{length of side opposite } \angle F}{\text{length of side adjacent to } \angle F}$$



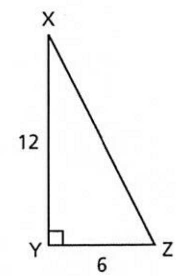
$$\tan F = \frac{DE}{EF}$$

$$\tan F = \frac{4}{3}$$

$$\tan F = 1.\bar{3}$$

CHECK YOUR UNDERSTANDING

1. Determine tan X and tan Z.



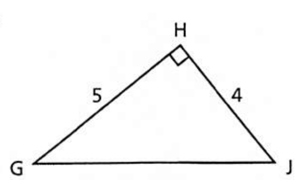
[Answer: tan X = 0.5; tan Z = 2]

How are the values of tan D and tan F related? Explain why this relation will always be true for the acute angles in a right triangle.

You can use a scientific calculator to determine the measure of an acute angle when you know the value of its tangent. The $\tan^{-1}$ or InvTan calculator operation does this.

Example 2 Using the Tangent Ratio to Determine the Measure of an Angle

Determine the measures of ∠G and ∠J to the nearest tenth of a degree.



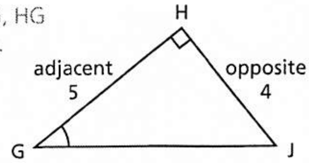
SOLUTION

In right $\triangle GHJ$:

$$\tan G = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan G = \frac{HJ}{GH}$$

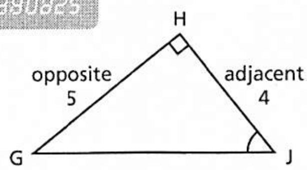
HJ is opposite $\angle G$, HG is adjacent to $\angle G$.



$$\tan G = \frac{4}{5}$$

$$\angle G \doteq 38.7°$$

$$\tan J = \frac{\text{opposite}}{\text{adjacent}}$$



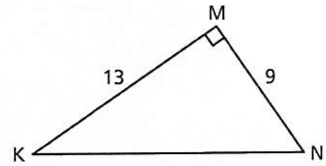
$$\tan J = \frac{GH}{HJ}$$

$$\tan J = \frac{5}{4}$$

$$\angle J \doteq 51.3°$$

CHECK YOUR UNDERSTANDING

2. Determine the measures of $\angle K$ and $\angle N$ to the nearest tenth of a degree.

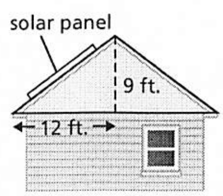


[Answer: $\angle K \doteq 34.7°$; $\angle N \doteq 55.3°$]

What other strategy could you use to determine $\angle J$?

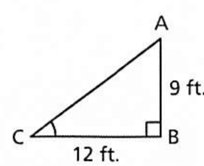
Example 3 Using the Tangent Ratio to Determine an Angle of Inclination

The latitude of Fort Smith, NWT, is approximately 60°. Determine whether this design for a solar panel is the best for Fort Smith. Justify your answer.



SOLUTION

The best angle of inclination for the solar panel is the same as the latitude, 60°. Draw a right triangle to represent the cross-section of the roof and solar panel. $\angle C$ is the angle of inclination. In $\triangle ABC$:



$$\tan C = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan C = \frac{AB}{BC}$$

AB is opposite $\angle C$, BC is adjacent to $\angle C$.

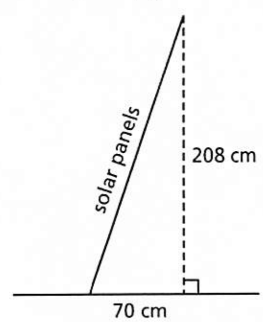
$$\tan C = \frac{9}{12}$$

$$\angle C \doteq 37°$$

The angle of inclination of the solar panel is about 37°, which is not equal to the latitude of Fort Smith. So, this is not the best design.

CHECK YOUR UNDERSTANDING

3. Clyde River on Baffin Island, Nunavut, has a latitude of approximately 70°. The diagram shows the side view of some solar panels. Determine whether this design for solar panels is the best for Clyde River. Justify your answer.



[Answer: The angle of inclination is approximately 71°. So, the design is the best.]

Example 4 Using the Tangent Ratio to Solve a Problem

A 10-ft. ladder leans against the side of a building with its base 4 ft. from the wall.

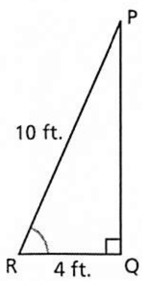
What angle, to the nearest degree, does the ladder make with the ground?

SOLUTION

Draw a diagram.

Label the vertices of the triangle PQR.

Assume the ground is horizontal and the building vertical.



To use the tangent ratio to determine ∠R, we first need to know the length of PQ.

Use the Pythagorean Theorem in right △PQR.

$PR^2 = PQ^2 + QR^2$ Isolate the unknown.

$PQ^2 = PR^2 - QR^2$

$PQ^2 = 10^2 - 4^2$

$= 84$

$PQ = \sqrt{84}$

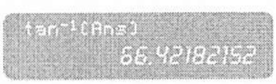
Use the tangent ratio in right △PQR.

$\tan R = \frac{PQ}{QR}$ PQ is opposite ∠R, QR is adjacent to ∠R.

$\tan R = \frac{\sqrt{84}}{4}$

$\tan R = 2.2913...$

$\angle R \doteq 66°$



The angle between the ladder and the ground is approximately 66°.

CHECK YOUR UNDERSTANDING

4. A support cable is anchored to the ground 5 m from the base of a telephone pole. The cable is 19 m long. It is attached near the top of the pole. What angle, to the nearest degree, does the cable make with the ground?

[Answer: The angle is approximately 75°.]

Suppose you used PQ ≐ 9.2, instead of PQ = $\sqrt{84}$. How could this affect the calculated measure of ∠R?

Discuss the Ideas

1. Why does the value of the tangent ratio of a given angle not depend on the right triangle you use to calculate it?

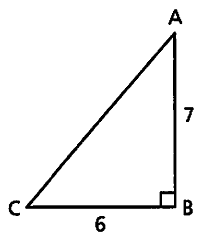
2. How can you use the tangent ratio to determine the measures of the acute angles of a right triangle when you know the lengths of its legs?

Exercises

A

3. In each triangle, write the tangent ratio for each acute angle.

a)



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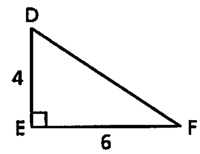
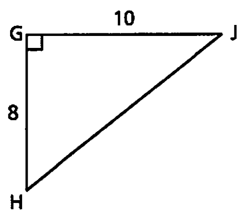
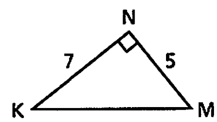


c)



d)

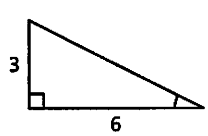


4. To the nearest degree, determine the measure of ∠X for each value of tan X.

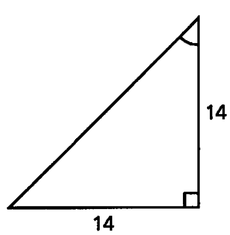
a) tan X = 0.25 b) tan X = 1.25
c) tan X = 2.50 d) tan X = 20

5. Determine the measure of each indicated angle to the nearest degree.

a)



b)



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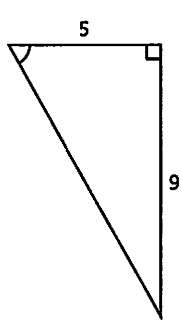

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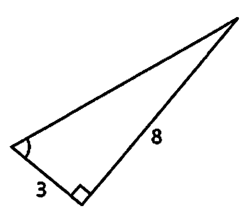


B

6. Use grid paper. Illustrate each tangent ratio by sketching a right triangle, then labelling the measures of its legs.

a) $\tan B = \frac{3}{5}$ b) $\tan E = \frac{5}{3}$ c) $\tan F = \frac{1}{4}$

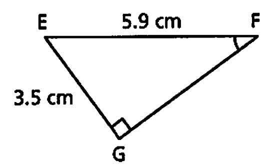
d) $\tan G = 4$ e) $\tan H = 1$ f) $\tan J = 25$

7. a) Is tan 60° greater than or less than 1? How do you know without using a calculator?

b) Is tan 30° greater than or less than 1? How do you know without using a calculator?

8. Determine the measure of each indicated angle to the nearest tenth of a degree. Describe your solution method.

a)



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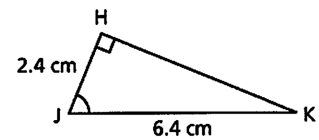
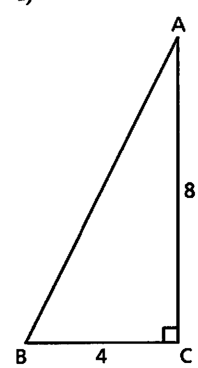
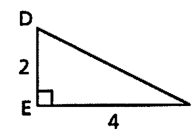


9. a) Why are these triangles similar?

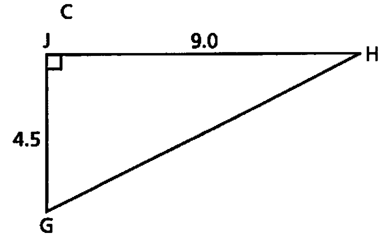
i)



ii)



iii)

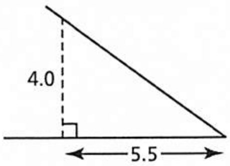


b) For each triangle in part a, determine the measures of the acute angles to the nearest tenth of a degree.

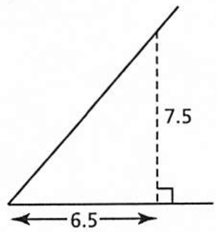
c) To complete part b, did you have to calculate the measures of all 6 acute angles? Explain.

10. Determine the angle of inclination of each line to the nearest tenth of a degree.

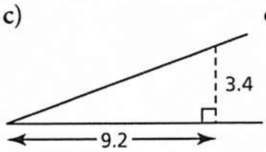
a)



b)



c)



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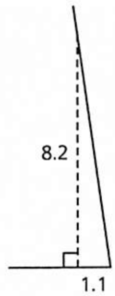
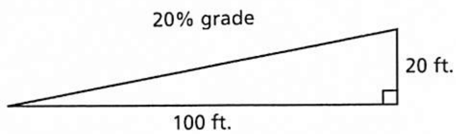


11. The grade or inclination of a road is often expressed as a percent. When a road has a grade of 20%, it increases 20 ft. in altitude for every 100 ft. of horizontal distance.

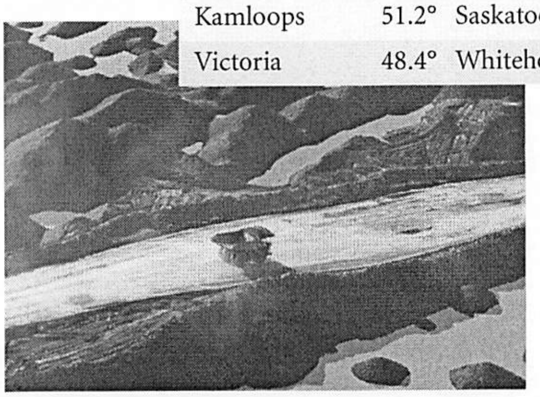


Calculate the angle of inclination, to the nearest degree, of a road with each grade.

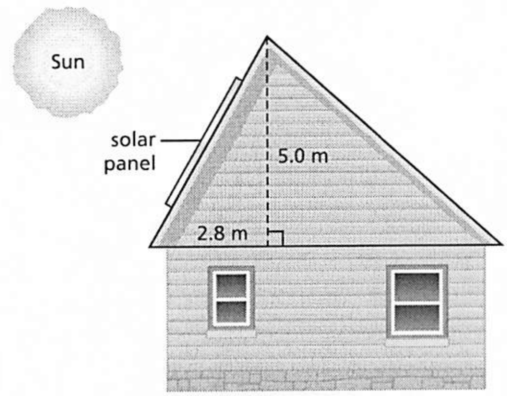
a) 20% b) 25% c) 10% d) 15%

12. The approximate latitudes for several cities in western and northern Canada are shown.

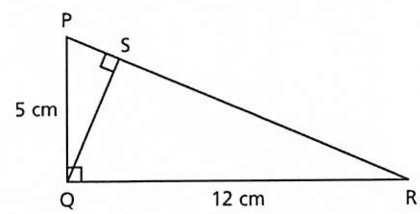
Calgary	51.1°	Edmonton	53.5°
Fort McMurray	56.5°	Inuvik	68.4°
Kamloops	51.2°	Saskatoon	52.2°
Victoria	48.4°	Whitehorse	60.7°



For which locations might the following roof design be within 1° of the recommended angle for solar panels? Justify your answer.



13. Determine the measures of all the acute angles in this diagram, to the nearest tenth of a degree.



14. A birdwatcher sights an eagle at the top of a 20-m tree. The birdwatcher is lying on the ground 50 m from the tree. At what angle must he incline his camera to take a photograph of the eagle? Give the answer to the nearest degree.

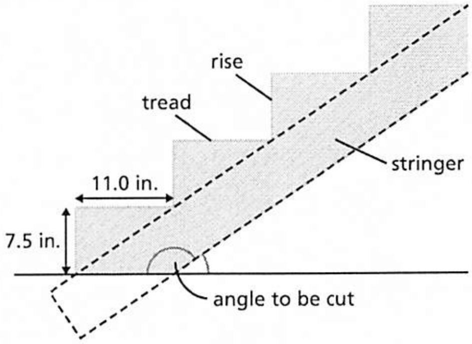
15. A rectangle has dimensions 3 cm by 8 cm. What angles does a diagonal of the rectangle make with the sides of the rectangle? Give the measures to the nearest tenth of a degree.

16. In a right isosceles triangle, why is the tangent of an acute angle equal to 1?

17. A playground slide starts 107 cm above the ground and is 250 cm long. What angle does the slide make with the ground? Give the answer to the nearest degree.

18. The Pioneer ski lift at Golden, B.C., is 1366 m long. It rises 522 m vertically. What is the angle of inclination of the ski lift? Give the answer to the nearest degree.

19. From a rectangular board, a carpenter cuts a stringer to support some stairs. Each stair rises 7.5 in. and has a tread of 11.0 in. To the nearest degree, at what angle should the carpenter cut the board?



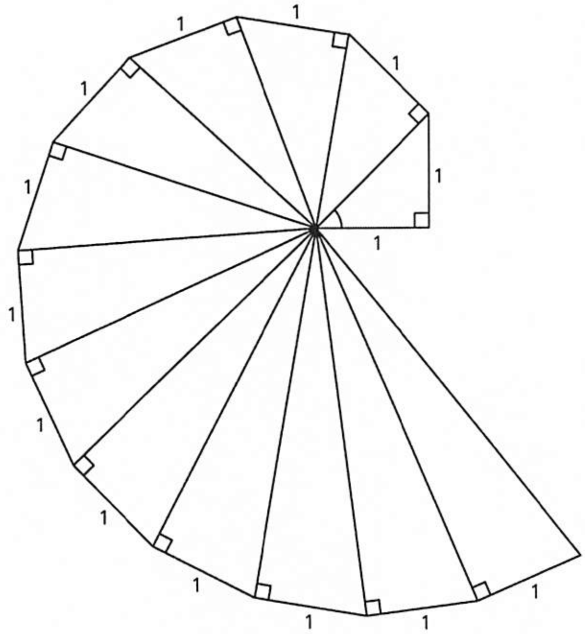
20. For safety reasons, a ladder is positioned so that the distance between its base and the wall is no greater than $\frac{1}{4}$ the length of the ladder. To the nearest degree, what is the greatest angle of inclination allowed for a ladder?

C

21. In isosceles $\triangle XYZ$, $XY = XZ = 5.9$ cm and $YZ = 5.0$ cm. Determine the measures of the angles of the triangle to the nearest tenth of a degree.

22. For the tangent of an acute angle in a right triangle:
a) What is the least possible value?
b) What is the greatest possible value?
Justify your answers.

23. A Pythagorean spiral is constructed by drawing right triangles on the hypotenuse of other right triangles. Start with a right triangle in which each leg is 1 unit long. Use the hypotenuse of that triangle as one leg of a new triangle and draw the other leg 1 unit long. Continue the process. A spiral is formed.

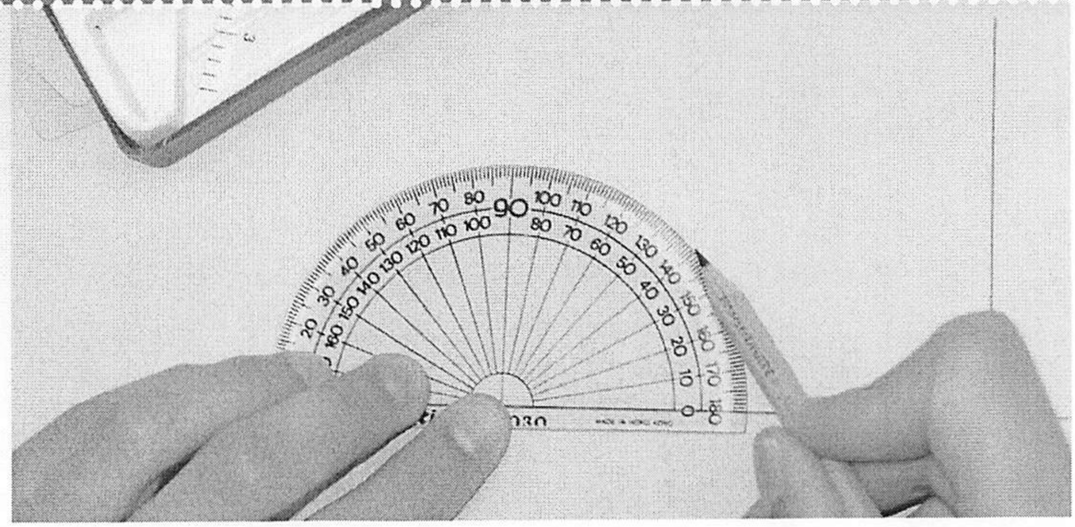


a) Determine the tangent of the angle at the centre of the spiral in each of the first 5 triangles.
b) Use the pattern in part a to predict the tangent of the angle at the centre of the spiral for the 100th triangle. Justify your answer.

Reflect

Summarize what you have learned about the tangent ratio and its relationship to the sides and angles of a right triangle.

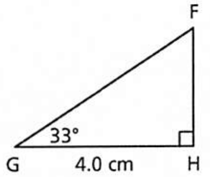
2.2 Using the Tangent Ratio to Calculate Lengths



LESSON FOCUS

Apply the tangent ratio to calculate lengths.

Make Connections



In Lesson 2.1, you used the measures of two legs of a right triangle to calculate the measures of the acute angles of the triangle. When you know the length of one leg of a right triangle and the measure of one acute angle, you can draw the triangle.

What other measures in the triangle can you calculate?

Construct Understanding

THINK ABOUT IT

Work with a partner.

In right $\triangle PQR$, $\angle Q = 90°$, $\angle P = 34.5°$, and PQ = 46.1 cm.
Determine the length of RQ to the nearest tenth of a centimetre.

We use **direct measurement** when we use a measuring instrument to determine a length or an angle in a polygon. We use **indirect measurement** when we use mathematical reasoning to calculate a length or an angle.

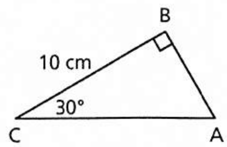
The tangent ratio is a powerful tool we can use to calculate the length of a leg of a right triangle. We are then measuring the length of a side of a triangle **indirectly**.

In a right triangle, we can use the tangent ratio, $\frac{\text{opposite}}{\text{adjacent}}$, to write an equation.

When we know the measure of an acute angle and the length of a leg, we solve the equation to determine the length of the other leg.

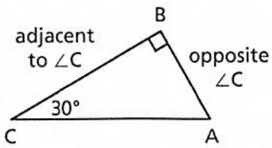
Example 1 Determining the Length of a Side Opposite a Given Angle

Determine the length of AB to the nearest tenth of a centimetre.



SOLUTION

In right △ABC, AB is the side opposite ∠C and BC is the side adjacent to ∠C.



Use the tangent ratio to write an equation.

$$\tan C = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan C = \frac{AB}{BC}$$

$$\tan 30° = \frac{AB}{10}$$

Solve this equation for AB. Multiply both sides by 10.

$$10 \times \tan 30° = \frac{AB}{10} \times 10$$

We write: $10 \times \tan 30°$ as $10 \tan 30°$. When an operation sign is omitted, it is understood to be multiplication.

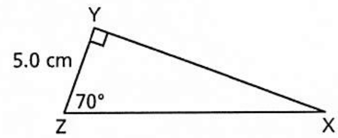
$$10 \tan 30° = AB$$

$$AB = 5.7735\ldots$$

AB is approximately 5.8 cm long.

CHECK YOUR UNDERSTANDING

1. Determine the length of XY to the nearest tenth of a centimetre.

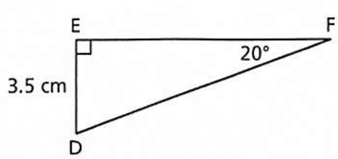


[Answer: XY ≐ 13.7 cm]

How can you determine the length of the hypotenuse in △ABC?

Example 2 Determining the Length of a Side Adjacent to a Given Angle

Determine the length of EF to the nearest tenth of a centimetre.



SOLUTIONS

Method 1

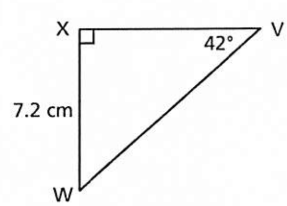
In right △DEF, DE is opposite ∠F and EF is adjacent to ∠F.

$$\tan F = \frac{\text{opposite}}{\text{adjacent}}$$

(Solution continues.)

CHECK YOUR UNDERSTANDING

2. Determine the length of VX to the nearest tenth of a centimetre.



[Answer: VX ≐ 8.0 cm]

$$\tan F = \frac{DE}{EF}$$

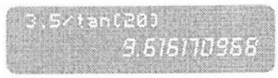
$$\tan 20° = \frac{3.5}{EF}$$

Solve the equation for EF. Multiply both sides by EF.

$$EF \tan 20° = EF\left(\frac{3.5}{EF}\right)$$

$$EF \tan 20° = 3.5$$ Divide both sides by tan 20°.

$$\frac{EF \tan 20°}{\tan 20°} = \frac{3.5}{\tan 20°}$$



$$EF = \frac{3.5}{\tan 20°}$$

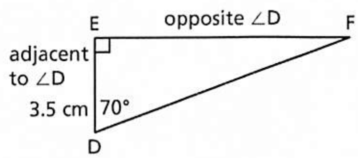
$$EF = 9.6161\ldots$$

EF is approximately 9.6 cm long.

What is the advantage of solving the equation for EF before calculating tan 20°?

Method 2

In right △DEF:



$$\angle D + \angle F = 90°$$

$$\angle D + 20° = 90°$$

$$\angle D = 90° - 20°$$

$$\angle D = 70°$$

EF is opposite ∠D and DE is adjacent to ∠D.

$$\tan D = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan D = \frac{EF}{DE}$$

$$\tan 70° = \frac{EF}{3.5}$$

Solve the equation for EF. Multiply both sides by 3.5.

$$3.5 \tan 70° = \frac{(EF)(3.5)}{3.5}$$

$$3.5 \tan 70° = EF$$

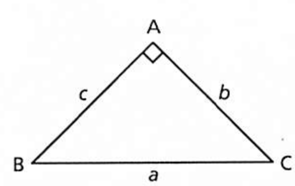
$$EF = 9.6161\ldots$$

EF is approximately 9.6 cm long.

Which method to determine EF do you think is easier? Why?

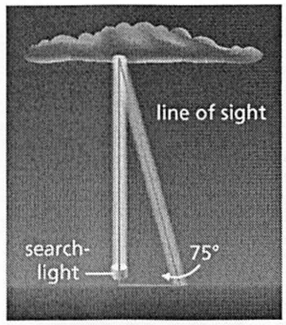
How could you determine the length of DF?

It is often convenient to use the lower case letter to name the side opposite a vertex of a triangle.



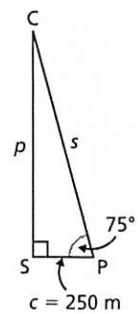
Example 3 Using Tangent to Solve an Indirect Measurement Problem

A searchlight beam shines vertically on a cloud. At a horizontal distance of 250 m from the searchlight, the angle between the ground and the line of sight to the cloud is 75°. Determine the height of the cloud to the nearest metre.



SOLUTION

Sketch and label a diagram to represent the information in the problem.



Assume the ground is horizontal.

In right △CSP, side CS is opposite ∠P and SP is adjacent to ∠P.

$$\tan P = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan P = \frac{p}{c}$$

$$\tan 75° = \frac{p}{250}$$

Solve the equation for p. Multiply both sides by 250.

$$250 \tan 75° = \left(\frac{p}{250}\right)250$$

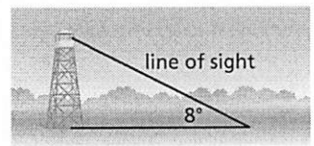
$$250 \tan 75° = p$$

$$p = 933.0127\ldots$$

The cloud is approximately 933 m high.

CHECK YOUR UNDERSTANDING

3. At a horizontal distance of 200 m from the base of an observation tower, the angle between the ground and the line of sight to the top of the tower is 8°. How high is the tower to the nearest metre? The diagram is *not* drawn to scale.



[Answer: 28 m]

Why can we draw a right triangle to represent the problem?

Discuss the Ideas

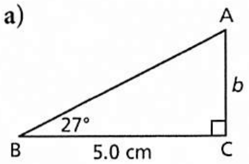
1. How can you use the tangent ratio to determine the length of a leg in a right triangle?
2. Suppose you know or can calculate the lengths of the legs in a right triangle. Why can you always calculate its hypotenuse?

Exercises

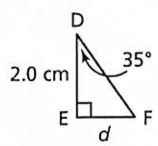
A

3. Determine the length of each indicated side to the nearest tenth of a centimetre.

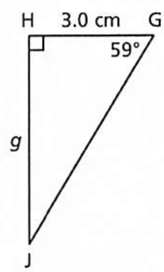
a)



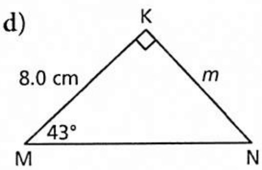
b)



c)

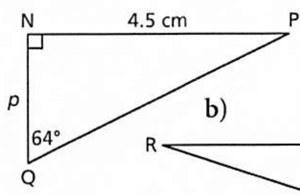


d)

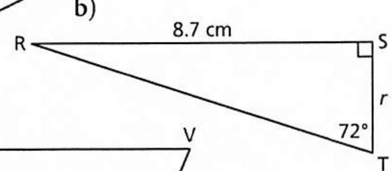


4. Determine the length of each indicated side to the nearest tenth of a centimetre.

a)



b)



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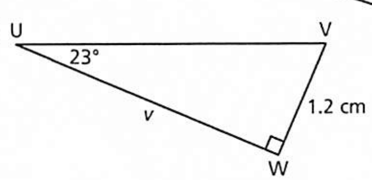
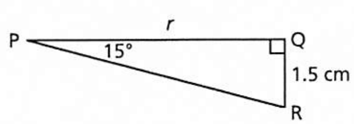


5. Determine the length of each indicated side to the nearest tenth of a centimetre.

a)



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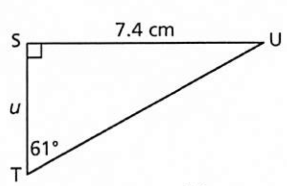

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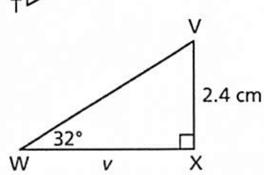
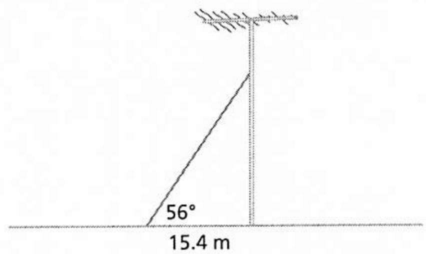
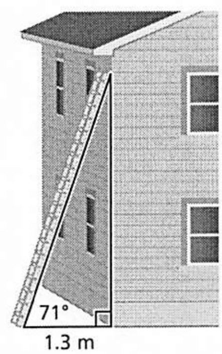


B

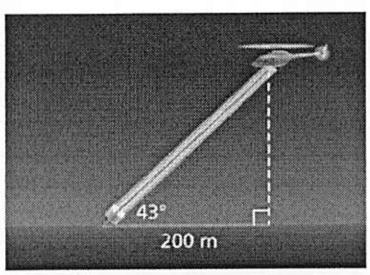
6. A guy wire helps to support a tower. The angle between the wire and the level ground is 56°. One end of the wire is 15.4 m from the base of the tower. How high up the tower does the wire reach to the nearest tenth of a metre?



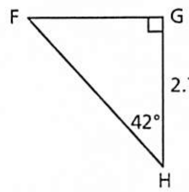
7. The base of a ladder is on level ground 1.3 m from a wall. The ladder leans against the wall. The angle between the ladder and the ground is 71°. How far up the wall does the ladder reach to the nearest tenth of a metre?

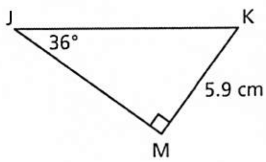


8. A helicopter is descending vertically. On the ground, a searchlight is 200 m from the point where the helicopter will land. It shines on the helicopter and the angle the beam makes with the ground is 43°. How high is the helicopter at this point to the nearest metre?

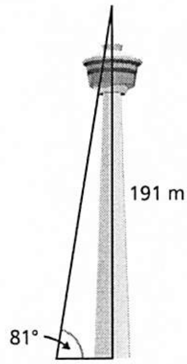


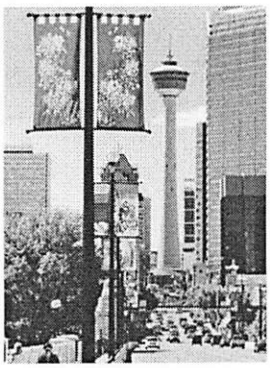
9. Determine the length of the hypotenuse of each right triangle to the nearest tenth of a centimetre. Describe your strategy.

a) 

b) 

10. Claire knows that the Calgary Tower is 191 m high. At a certain point, the angle between the ground and Claire's line of sight to the top of the tower was 81°. To the nearest metre, about how far was Claire from the tower? Why is this distance approximate?





11. The angle between one longer side of a rectangle and a diagonal is 34°. One shorter side of the rectangle is 2.3 cm.
   a) Sketch and label the rectangle.
   b) What is the length of the rectangle to the nearest tenth of a centimetre?

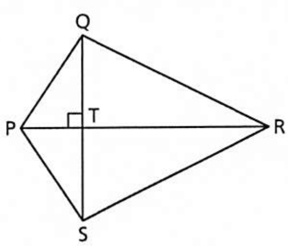
12. In △PQR, ∠R = 90°, ∠P = 58°, and PR = 7.1 cm. Determine the area of △PQR to the nearest tenth of a square centimetre. Describe your strategy.

13. The height of the Manitoba Legislature Building, from the ground to the top of the Golden Boy statue, is about 77 m. Liam is lying on the ground near the building. The angle between the ground and his line of sight to the top of the building is 52°. About how far is Liam from a point on the ground vertically below the statue? How do you know?

14. Janelle sees a large helium-filled balloon anchored to the roof of a store. When she is 100 m from the store, the angle between the ground and her line of sight to the balloon is 30°. About how high is the balloon? What assumptions are you making?

C

15. In kite PQRS, the shorter diagonal, QS, is 6.0 cm long, ∠QRT is 26.5°, and ∠QPT is 56.3°. Determine the measures of all the angles and the lengths of the sides of the kite to the nearest tenth.



16. On a coordinate grid:
   a) Draw a line through the points A(4, 5) and B(−4, −5). Determine the measure of the acute angle between AB and the y-axis.
   b) Draw a line through the points C(1, 4) and D(4, −2). Determine the measure of the acute angle between CD and the x-axis.

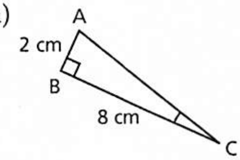
Reflect

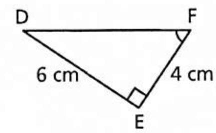
Summarize what you have learned about using the tangent ratio to determine the length of a side of a right triangle.

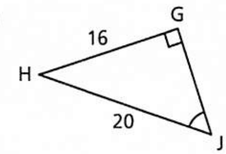
Assess Your Understanding

2.1

1. Determine the measure of each indicated angle to the nearest degree.

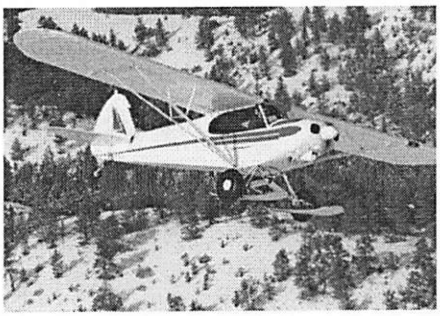
a) 

b) 

c) 

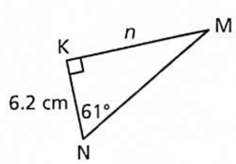
2. Why does the tangent of an angle increase as the angle increases?

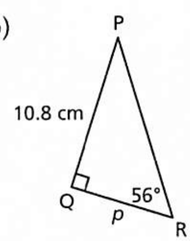
3. A small plane is flying at an altitude of 1000 m and is 5000 m from the beginning of the landing strip. What is the angle between the ground and the line of sight from an observer at the beginning of the landing strip? Give the measure to the nearest tenth of a degree.

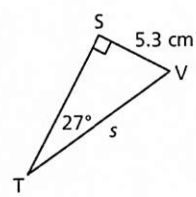


2.2

4. Determine the length of each indicated side to the nearest tenth of a centimetre.

a) 

b) 

c) 

5. A hiker saw a hoodoo on a cliff at Willow Creek in Alberta's badlands. The hiker was 9.1 m from the base of the cliff. From that point, the angle between the level ground and the line of sight to the top of the hoodoo was 69°. About how high was the top of the hoodoo above the level ground?