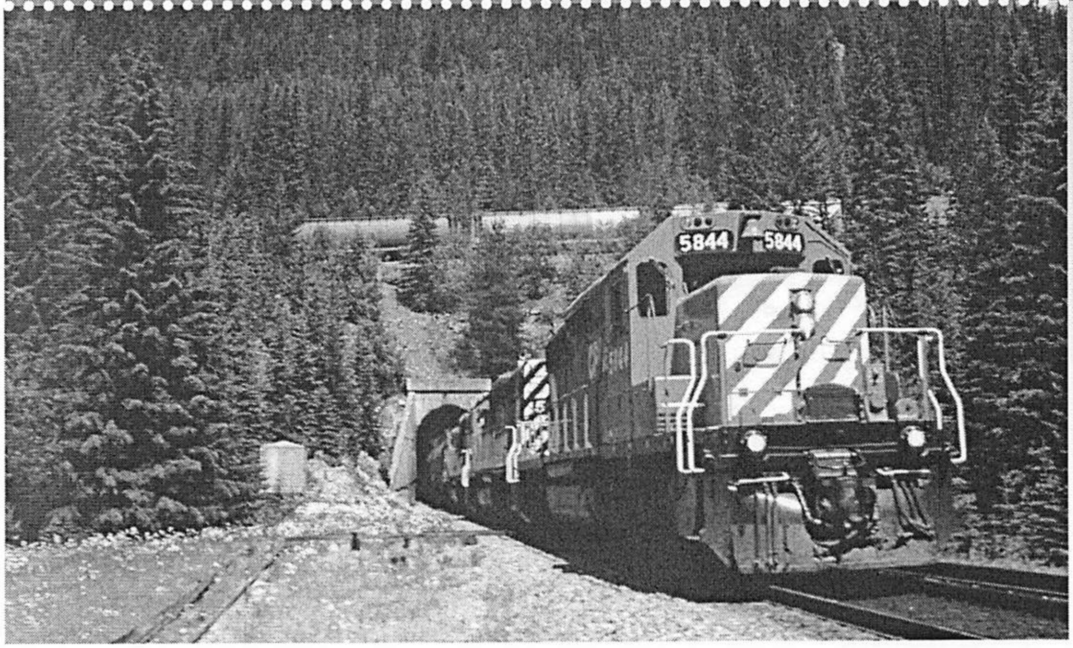
2.4 The Sine and Cosine Ratios



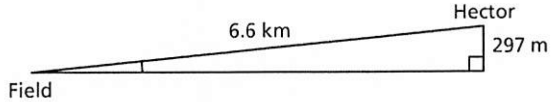
LESSON FOCUS

Develop and apply the sine and cosine ratios to determine angle measures.

Make Connections

The railroad track through the mountains between Field, B.C., and Hector, B.C., includes spiral tunnels. They were built in the early 1900s to reduce the angle of inclination of the track between the two towns. You can see a long train passing under itself after it comes out of a tunnel before it has finished going in.

Visualize the track straightened out to form the hypotenuse of a right triangle. Here is a diagram of the track before the tunnels were constructed. The diagram is *not* drawn to scale.



How could you determine the angle of inclination of the track?

Construct Understanding

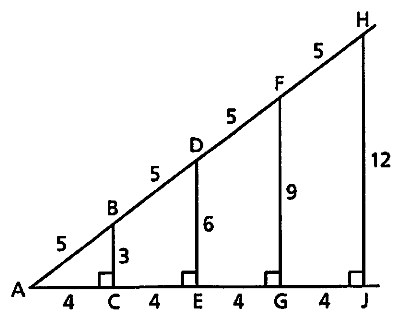
We defined the tangent ratio for an acute angle in a right triangle. There are two other ratios we can form to compare the sides of the triangle; each ratio involves the hypotenuse.

TRY THIS

Work with a partner.

You will need grid paper, a ruler, and a protractor.

A. Examine the nested right triangles below.



$\angle A$ is common to each triangle. How are the other acute angles in each triangle related? How do you know? How are the triangles related?

B. Copy and complete this table.

Triangle	Measures of Sides			Ratios	
	Hypotenuse	Side opposite $\angle A$	Side adjacent to $\angle A$	$\frac{\text{Side opposite } \angle A}{\text{Hypotenuse}}$	$\frac{\text{Side adjacent to } \angle A}{\text{Hypotenuse}}$
$\triangle ABC$					
$\triangle ADE$					
$\triangle AFG$					
$\triangle AHJ$					

C. Draw another set of nested right triangles that are not similar to those in Step A.

D. Measure the sides and angles of each triangle. Label your diagram with the measures, as in the diagram above.

E. Complete a table like the one in Step B for your triangles.

F. For each set of triangles, how do the ratios compare?

G. What do you think the value of each ratio depends on?

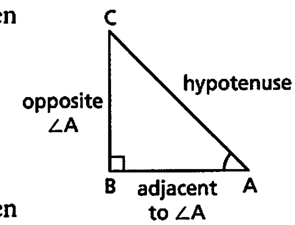
In a right triangle, the ratios that relate each leg to the hypotenuse depend only on the measure of the acute angle, and not on the size of the triangle. These ratios are called the **sine ratio** and the **cosine ratio**.

The sine ratio for ∠A is written as sin A and the cosine ratio for ∠A is written as cos A.

The Sine Ratio

If ∠A is an acute angle in a right triangle, then

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}}$$



The Cosine Ratio

If ∠A is an acute angle in a right triangle, then

$$\cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}}$$

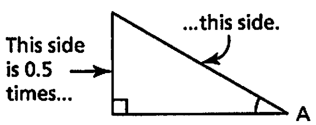
The tangent, sine, and cosine are called the **primary trigonometric ratios**. The word **trigonometry** comes from three Greek words "tri + gonia + metron" that together mean "three angle measure."

The branch of math that deals with the relations between the sides and angles of triangles is called **trigonometry**.

The values of the sine and cosine that compare the lengths of the sides are often expressed as decimals. For example, in right △ABC,

If sin A = 0.5, then in any similar right triangle, the length of the side opposite ∠A is 0.5 times the length of the hypotenuse.

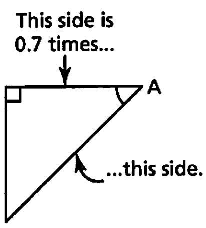
What happens to sin A as ∠A gets closer to 0°?



sin A = 0.5

If cos A = 0.7, then in any similar right triangle, the length of the side adjacent to ∠A is 0.7 times the length of the hypotenuse.

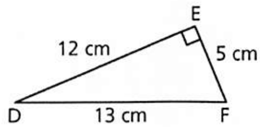
What happens to cos A as ∠A gets closer to 0°?



cos A = 0.7

Example 1 Determining the Sine and Cosine of an Angle

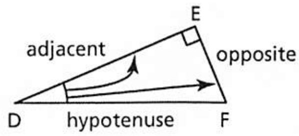
a) In $\triangle DEF$, identify the side opposite $\angle D$ and the side adjacent to $\angle D$.



b) Determine sin D and cos D to the nearest hundredth.

SOLUTION

a) In right $\triangle DEF$,
DF is the hypotenuse.
EF is opposite $\angle D$ and
DE is adjacent to $\angle D$.



b) $\sin D = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\sin D = \dfrac{EF}{DF}$ EF is opposite $\angle D$, DF is the hypotenuse.

$\sin D = \dfrac{5}{13}$

$\sin D = 0.3846...$

$\sin D \doteq 0.38$

$\cos D = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\cos D = \dfrac{DE}{DF}$ DE is adjacent to $\angle D$, DF is the hypotenuse.

$\cos D = \dfrac{12}{13}$

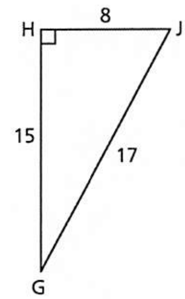
$\cos D = 0.9230...$

$\cos D \doteq 0.92$

CHECK YOUR UNDERSTANDING

1. a) In $\triangle GHJ$, identify the side opposite $\angle G$ and the side adjacent to $\angle G$.

b) Determine sin G and cos G to the nearest hundredth.



[Answers: a) HJ, HG
b) $\sin G \doteq 0.47$; $\cos G \doteq 0.88$]

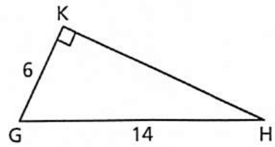
Determine sin F and cos F. How are these values related to sin D and cos D?

You can use a scientific calculator to determine the measure of an angle:

- When you know its sine, use $\sin^{-1}$ or InvSin
- When you know its cosine, use $\cos^{-1}$ or InvCos

Example 2 Using Sine or Cosine to Determine the Measure of an Angle

Determine the measures of ∠G and ∠H to the nearest tenth of a degree.



SOLUTIONS

Method 1

Determine the measure of ∠H first.

In right △GHK:

$$\sin H = \frac{\text{opposite}}{\text{hypotenuse}}$$

GK is opposite ∠H, GH is the hypotenuse.

$$\sin H = \frac{GK}{GH}$$

$$\sin H = \frac{6}{14}$$

$$\angle H = 25.3769\ldots^\circ$$

$$\angle G + \angle H = 90^\circ$$

$$\angle G = 90^\circ - \angle H$$

The angle sum of any triangle is 180°, so the two acute angles in a right triangle have a sum of 90°.

So, $\angle G = 90^\circ - 25.3769\ldots^\circ$

$$\angle G = 64.6230\ldots^\circ$$

Method 2

Determine the measure of ∠G first.

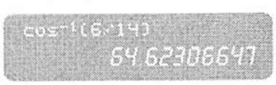
In right △GHK:

$$\cos G = \frac{\text{adjacent}}{\text{hypotenuse}}$$

GK is adjacent to ∠G, GH is the hypotenuse.

$$\cos G = \frac{GK}{GH}$$

$$\cos G = \frac{6}{14}$$



$$\angle G = 64.6230\ldots^\circ$$

$$\angle G + \angle H = 90^\circ$$

The two acute angles have a sum of 90°.

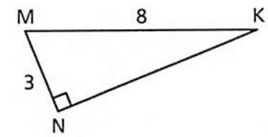
So, $\angle H = 90^\circ - 64.6230\ldots^\circ$

$$\angle H = 25.3769\ldots^\circ$$

∠G is approximately 64.6° and ∠H is approximately 25.4°.

CHECK YOUR UNDERSTANDING

2. Determine the measures of ∠K and ∠M to the nearest tenth of a degree.



[Answer: ∠K ≐ 22.0°, ∠M ≐ 68.0°]

How are cos G and sin H related? Explain why this relationship occurs.

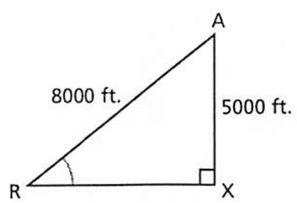
We can use the sine or cosine ratio to solve problems that can be modelled by a right triangle when we know the length of the hypotenuse, and the length of a leg or the measure of an acute angle.

Example 3 Using Sine or Cosine to Solve a Problem

A water bomber is flying at an altitude of 5000 ft. The plane's radar shows that it is 8000 ft. from the target site. What is the **angle of elevation** of the plane measured from the target site, to the nearest degree?

SOLUTION

Draw a diagram to represent the situation.
Altitude is measured vertically.
Assume the ground is horizontal.



$\angle R$ is the angle of elevation of the plane.

AX is the altitude of the plane.

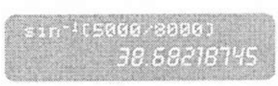
RA is the distance from the target site to the plane.

In right $\triangle ARX$:

$$\sin R = \frac{AX}{RA}$$

AX is opposite ∠R, RA is the hypotenuse.

$$\sin R = \frac{5000}{8000}$$



$$\angle R \doteq 39°$$

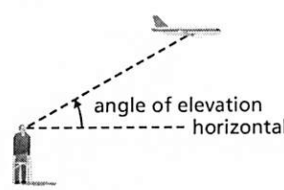
The angle of elevation of the plane is approximately 39°.

CHECK YOUR UNDERSTANDING

3. An observer is sitting on a dock watching a float plane in Vancouver harbour. At a certain time, the plane is 300 m above the water and 430 m from the observer. Determine the angle of elevation of the plane measured from the observer, to the nearest degree.

[Answer: approximately 44°]

The **angle of elevation** of an object above the horizontal is the angle between the horizontal and the line of sight from an observer.



Discuss the Ideas

1. When can you use the sine ratio to determine the measure of an acute angle in a right triangle? When can you use the cosine ratio?
2. Why is it important to draw a sketch before you start to solve a problem?
3. Why are the values of the sine of an acute angle and the cosine of an acute angle less than 1?

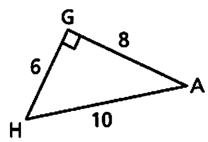
Exercises

A

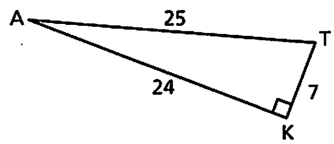
4. a) In each triangle below:

- Name the side opposite ∠A.
- Name the side adjacent to ∠A.
- Name the hypotenuse.

i)



ii)



b) For each triangle in part a, determine sin A and cos A to the nearest hundredth.

5. Determine the sine and cosine of each angle to the nearest hundredth.

a) 57° b) 5° c) 19° d) 81°

6. To the nearest degree, determine the measure of each ∠X.

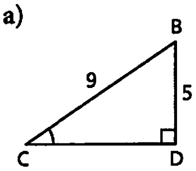
a) $\sin X = 0.25$ b) $\cos X = 0.64$

c) $\sin X = \frac{6}{11}$ d) $\cos X = \frac{7}{9}$

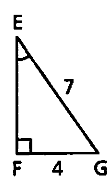
B

7. Determine the measure of each indicated angle to the nearest degree.

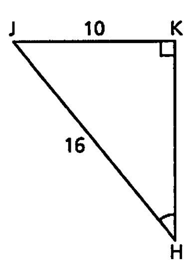
a)



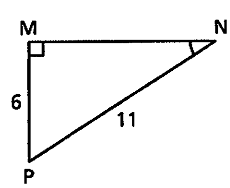
b)



c)

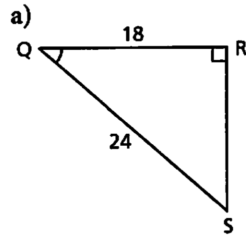


d)

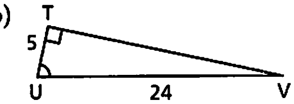


8. Determine the measure of each indicated angle to the nearest degree.

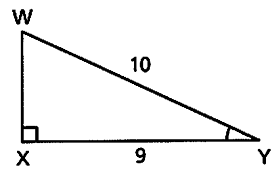
a)



b)



c)



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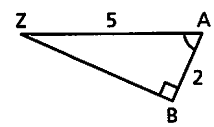


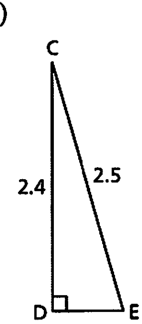
9. For each ratio below, sketch two different right triangles and label their sides.

a) $\sin B = \frac{3}{5}$ b) $\cos B = \frac{5}{8}$

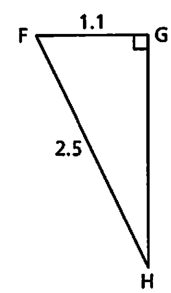
c) $\sin B = \frac{1}{4}$ d) $\cos B = \frac{4}{9}$

10. Use the sine or cosine ratio to determine the measure of each acute angle to the nearest tenth of a degree. Describe your strategy.

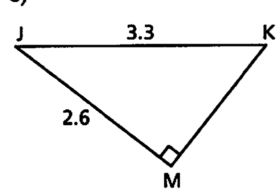
a)



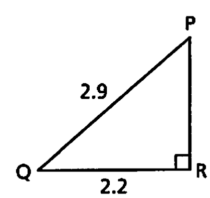
b)



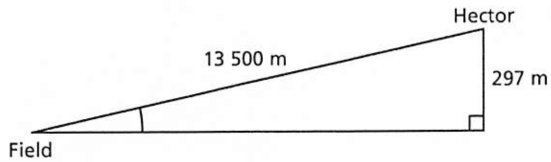
c)



d)



11. Suppose the railroad track through the spiral tunnels from Field to Hector were straightened out. It would look like the diagram below. The diagram is *not* drawn to scale. What is the angle of inclination of the track to the nearest tenth of a degree?



12. A ladder is 6.5 m long. It leans against a wall. The base of the ladder is 1.2 m from the wall. What is the angle of inclination of the ladder to the nearest tenth of a degree?

13. A rope that supports a tent is 2.4 m long. The rope is attached to the tent at a point that is 2.1 m above the ground. What is the angle of inclination of the rope to the nearest degree?

14. A rectangle is 4.8 cm long and each diagonal is 5.6 cm long. What is the measure of the angle between a diagonal and the longest side of the rectangle? Give the answer to the nearest degree.

15. **a)** Calculate:

i) sin 10° **ii)** sin 20° **iii)** sin 40°
iv) sin 50° **v)** sin 60° **vi)** sin 80°

b) Why does the sine of an angle increase as the angle increases?

16. Sketch a right isosceles triangle. Explain why the cosine of each acute angle is equal to the sine of the angle.

C

17. A cylindrical silo is 37 ft. high and has a diameter of 14 ft. The top of the silo can be reached by a spiral staircase that circles the silo once. What is the angle of inclination of the staircase to the nearest degree?

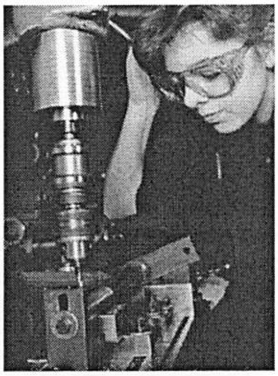
18. **a)** We have defined the sine and cosine ratios for acute angles. Use a calculator to determine:

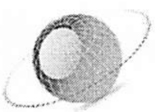
i) sin 90° **ii)** sin 0° **iii)** cos 90° **iv)** cos 0°

b) Sketch a right triangle. Use the sketch to explain the results in part a.

Reflect

Explain when you might use the sine or cosine ratio instead of the tangent ratio to determine the measure of an acute angle in a right triangle. Include examples in your explanation.



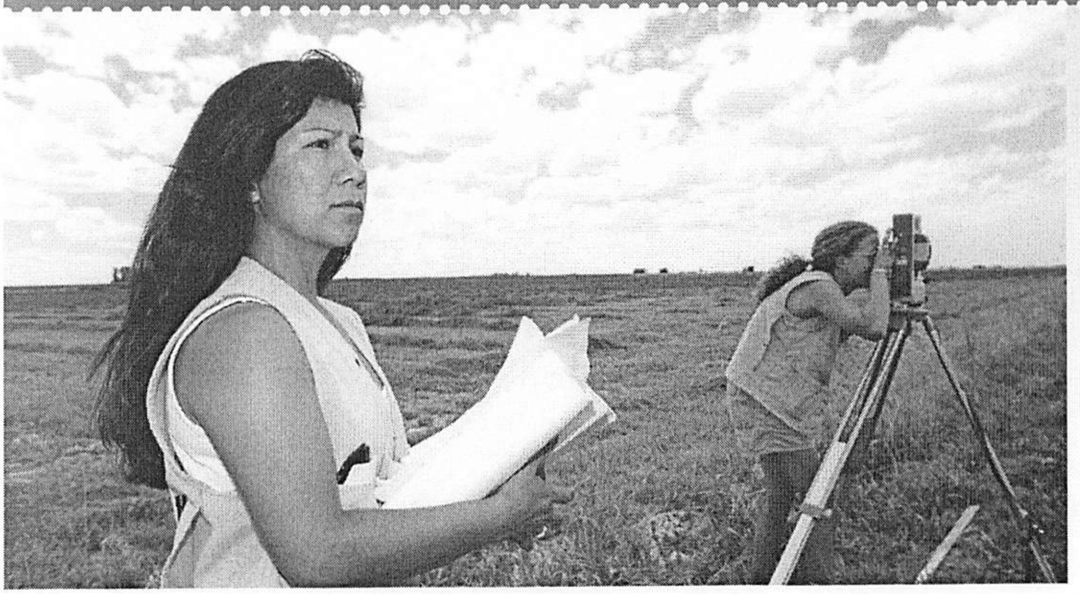


THE WORLD OF MATH

Careers: Tool and Die Maker

A tool and die maker constructs tools and prepares dies for manufacturing common objects, such as bottle caps. A *die* is made up of two plates that stamp together. A tool and die maker uses trigonometry to construct a die. She works from blueprints that show the dimensions of the design. To cut the material for a die, a tool and die maker must set the milling machine at the precise angle.

2.5 Using the Sine and Cosine Ratios to Calculate Lengths



LESSON FOCUS

Use the sine and cosine ratios to determine lengths indirectly.

Make Connections

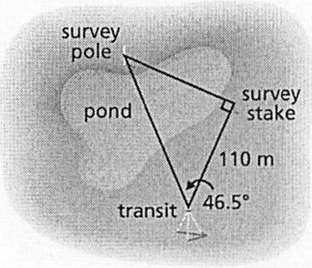
A surveyor can measure an angle precisely using an instrument called a *transit*. A measuring tape is used to measure distances. How can the surveyor use these measures and her knowledge of trigonometry in a right triangle to calculate the lengths that cannot be measured directly?

Construct Understanding

THINK ABOUT IT

Work with a partner.

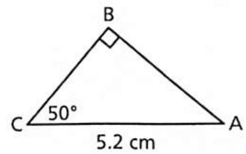
The diagram shows measurements taken by surveyors. How could you determine the distance between the transit and the survey pole?



We can use the sine ratio or cosine ratio to write an equation that we can solve to calculate the length of a leg in a right triangle when the measure of one acute angle and the length of the hypotenuse are known.

Example 1 Using the Sine or Cosine Ratio to Determine the Length of a Leg

Determine the length of BC to the nearest tenth of a centimetre.

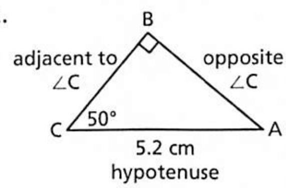


SOLUTION

In right $\triangle ABC$, AC is the hypotenuse and BC is adjacent to the known $\angle C$.

Choose the ratio that compares the adjacent side to the hypotenuse.

Use the cosine ratio to write an equation.



$$\cos C = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos C = \frac{BC}{AC}$$

$$\cos 50° = \frac{BC}{5.2}$$

Solve this equation for BC. Multiply both sides by 5.2.

$$5.2 \cos 50° = \frac{(5.2)(BC)}{5.2}$$

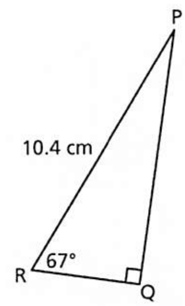
$$5.2 \cos 50° = BC$$

$$BC = 3.3424\ldots$$

BC is approximately 3.3 cm long.

CHECK YOUR UNDERSTANDING

1. Determine the length of PQ to the nearest tenth of a centimetre.



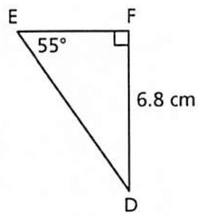
[Answer: PQ ≐ 9.6 cm]

How could you have used the sine ratio to solve this problem?

The sine and cosine ratios can be used to calculate the length of the hypotenuse when the measure of one acute angle and the length of one leg are known.

Example 2 Using Sine or Cosine to Determine the Length of the Hypotenuse

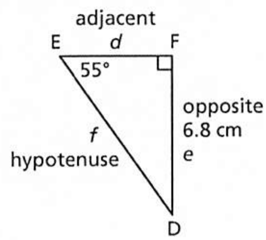
Determine the length of DE to the nearest tenth of a centimetre.



SOLUTION

In right △DEF, DE is the hypotenuse and DF is opposite the known ∠E.

The sine ratio compares the opposite side to the hypotenuse. Use lower case letters to label the lengths of the sides.



Use the sine ratio to write an equation.

$$\sin E = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin E = \frac{e}{f}$$

$$\sin 55° = \frac{6.8}{f}$$

Solve for f.

Multiply both sides by f.

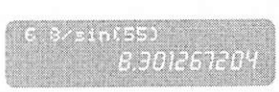
$$f \sin 55° = \frac{6.8f}{f}$$

$$f \sin 55° = 6.8$$

Divide both sides by sin 55°.

$$\frac{f \sin 55°}{\sin 55°} = \frac{6.8}{\sin 55°}$$

$$f = \frac{6.8}{\sin 55°}$$

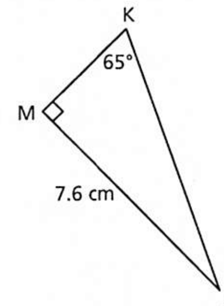


$$f = 8.3012\ldots$$

DE is approximately 8.3 cm long.

CHECK YOUR UNDERSTANDING

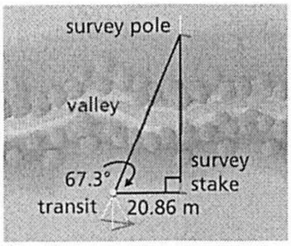
2. Determine the length of JK to the nearest tenth of a centimetre.



[Answer: JK ≐ 8.4 cm]

Example 3 Solving an Indirect Measurement Problem

A surveyor made the measurements shown in the diagram. How could the surveyor determine the distance from the transit to the survey pole to the nearest hundredth of a metre?



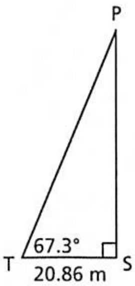
SOLUTION

Label a diagram to represent the problem.

The required distance is the hypotenuse of right △PST.

In right △PST, TP is the hypotenuse and TS is adjacent to ∠T.

Use the cosine ratio to write an equation.



$$\cos T = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos T = \frac{TS}{TP}$$

$$\cos 67.3° = \frac{20.86}{TP}$$

Solve this equation for TP.

Multiply both sides by TP.

$$TP \cos 67.3° = \frac{(TP)(20.86)}{TP}$$

$$TP \cos 67.3° = 20.86$$

Divide both sides by cos 67.3°.

$$\frac{TP \cos 67.3°}{\cos 67.3°} = \frac{20.86}{\cos 67.3°}$$

$$TP = \frac{20.86}{\cos 67.3°}$$

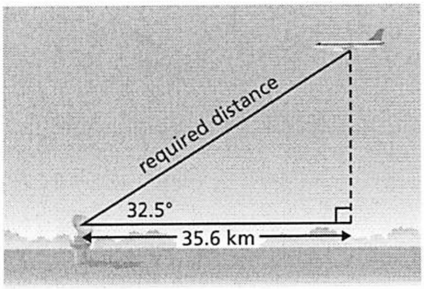
$$TP = 54.0546\ldots$$

20.86/cos(67.3)
54.05460841

The distance from the transit to the survey pole is approximately 54.05 m.

CHECK YOUR UNDERSTANDING

3. From a radar station, the angle of elevation of an approaching airplane is 32.5°. The horizontal distance between the plane and the radar station is 35.6 km. How far is the plane from the radar station to the nearest tenth of a kilometre?



[Answer: 42.2 km]

How could you use the sine ratio instead of the cosine ratio to solve Example 3?
How could you use the tangent ratio?

Discuss the Ideas

1. What are the advantages of using a trigonometric ratio instead of an accurate drawing to solve a measurement problem?

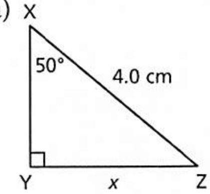
2. When would you use the sine ratio to determine the length of a side of a right triangle? When would you use the cosine ratio?

Exercises

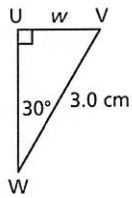
A

3. Determine the length of each indicated side to the nearest tenth of a centimetre.

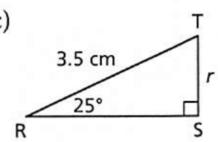
a)



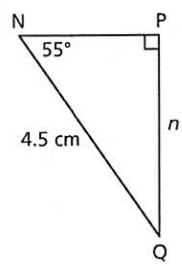
b)



c)

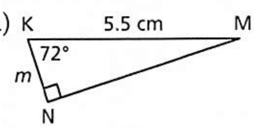


d)

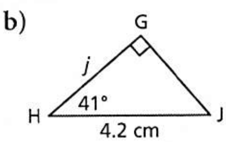


4. Determine the length of each indicated side to the nearest tenth of a centimetre.

a)



b)



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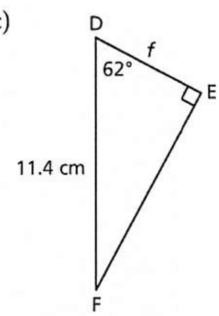
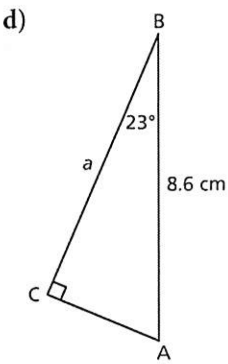


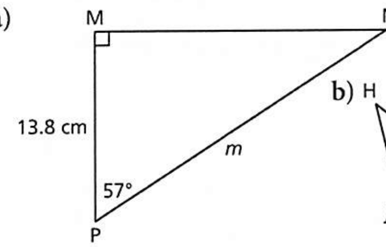
d)



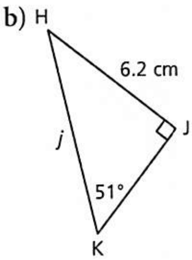
B

5. Determine the length of each indicated side to the nearest tenth of a centimetre.

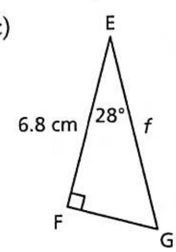
a)



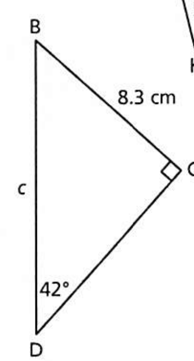
b)



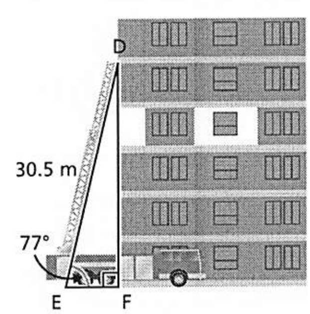
c)



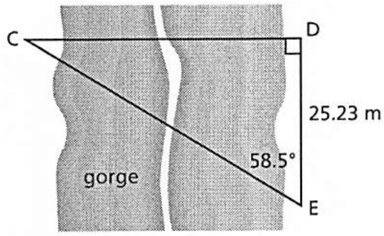
d)



6. A fire truck has an aerial ladder that extends 30.5 m measured from the ground. The angle of inclination of the ladder is 77°. To the nearest tenth of a metre, how far up the wall of an apartment building can the ladder reach?



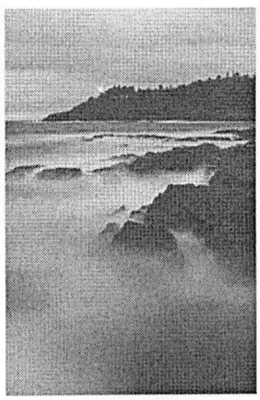
7. A surveyor makes the measurements shown in the diagram to determine the distance from C to E across a gorge.



a) To the nearest tenth of a metre, what is the distance from C to E?

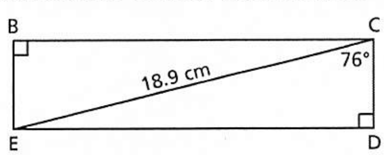
b) How could the surveyor calculate the distance from C to D?

8. A ship is sailing off the north coast of the Queen Charlotte Islands. At a certain point, the navigator sees the lighthouse at Langara Point, due south of the ship. The ship then sails 3.5 km due east. The angle between the ship's path and the line of sight to the lighthouse is then 28.5°. To the nearest tenth of a kilometre, how far is the ship from the lighthouse?

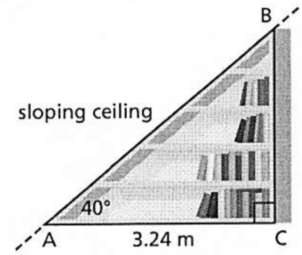


9. An airplane approaches an airport. At a certain time, it is 939 m high. Its angle of elevation measured from the airport is 19.5°. To the nearest metre, how far is the plane from the airport?

10. Calculate the dimensions of this rectangle to the nearest tenth of a centimetre.



11. A bookcase is built against the sloping ceiling of an attic. The base of the bookcase is 3.24 m long. The angle of inclination of the attic ceiling is 40°.



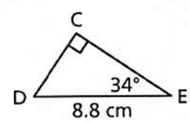
a) What is the length of the top of the bookcase, measured along the attic ceiling?

b) What is the greatest height of the bookcase?

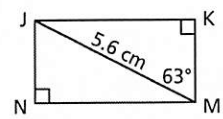
Give the answers to the nearest centimetre.

12. a) Determine the perimeter of each shape to the nearest tenth of a centimetre.

i)



ii)



b) What strategies did you use to complete part a? What other strategies could you have used instead?

C

13. In trapezoid CDEF, $\angle D = \angle E = 90°$, $\angle C = 60°$, EF = 4.5 cm, and DE = 3.5 cm. What is the perimeter of the trapezoid to the nearest millimetre? Describe your strategy.

14. A survey of a building lot that has the shape of an acute triangle shows these data:

- Two intersecting sides are 250 ft. and 170 ft. long.
- The angle between these sides is 55°.

a) Use the 250-ft. side as the base of the triangle. What is the height of the triangle?

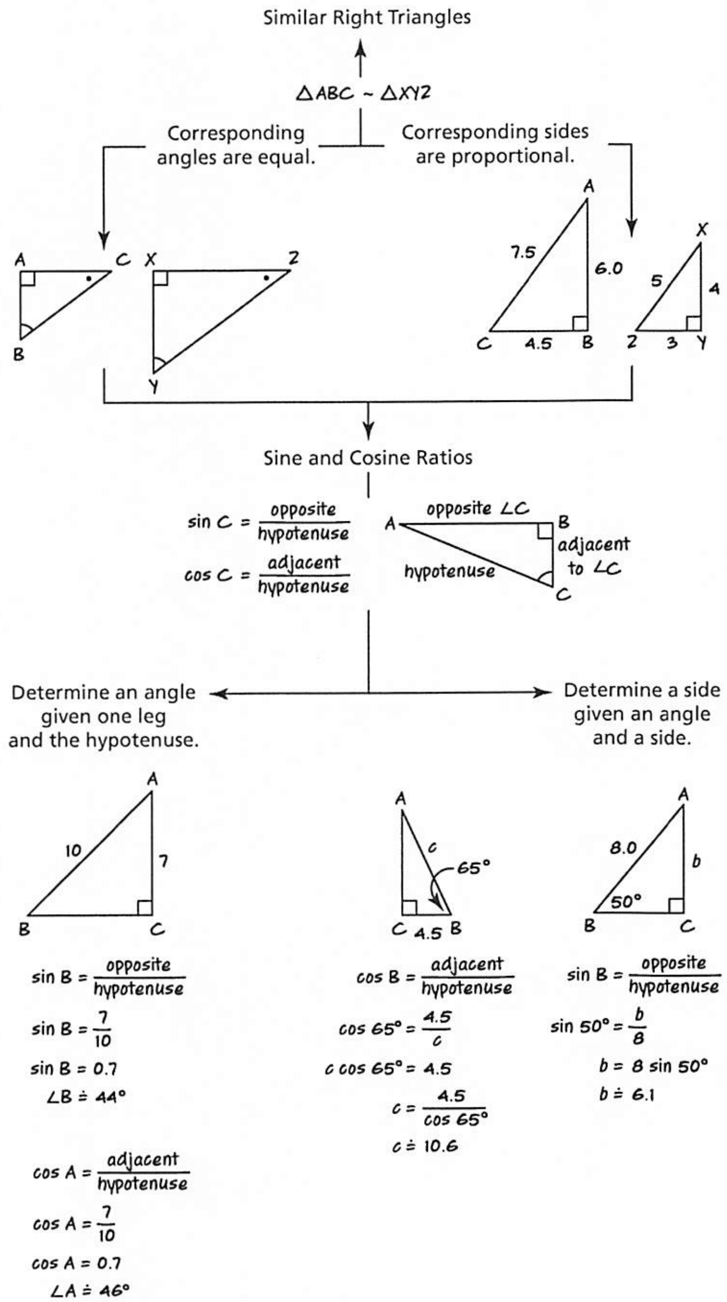
b) Determine the area of the lot to the nearest square foot.

Reflect

Explain when you might use the sine or cosine ratio instead of the tangent ratio to determine the length of a side in a right triangle. Include examples.

CHECKPOINT 2

Connections



Concept Development

In Lesson 2.4

- You applied what you know about similar right triangles to develop the **sine and cosine ratios.**
- You used the sine or cosine ratio **to determine an acute angle** in a right triangle when you know the lengths of one leg and the hypotenuse.

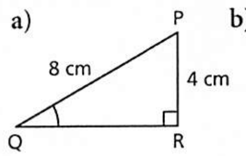
In Lesson 2.5

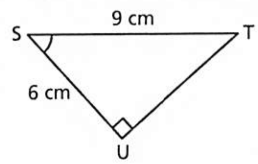
- You showed how to determine the **length of a leg** in a right triangle when you know the measure of an acute angle and the length of the hypotenuse.
- You showed how to determine the **length of the hypotenuse** when you know the measure of an acute angle and the length of one leg.

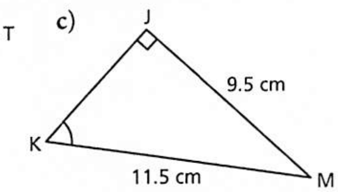
Assess Your Understanding

2.4

1. Determine the measure of each indicated angle to the nearest degree.

a)


b)


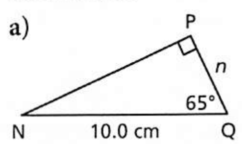
c)


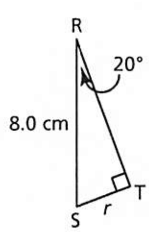
2. A factory manager plans to install a 30-ft. long conveyor that rises 7 ft. from the road to a loading dock. What is the angle of inclination of the conveyor to the nearest degree?

3. a) Calculate the cosine of each angle:
i) 10° ii) 20° iii) 30° iv) 40° v) 50° vi) 60° vii) 70° viii) 80°
b) Explain why the cosine of an angle decreases as the angle increases.

2.5

4. Determine the length of each indicated side to the nearest tenth of a centimetre.

a)


b)


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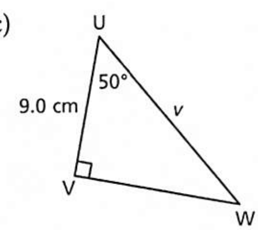

5. A ship is sailing off the south coast of the Queen Charlotte Islands. At a certain point, the navigator sees the beacon at Cape St. James, due north of the ship. The ship then sails 2.4 km due west. The angle between the ship's path and the line of sight to the beacon is 41.5°. How far is the ship from the beacon?