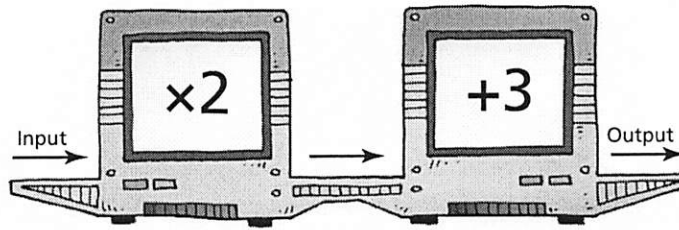


# 5.2 Properties of Functions

## LESSON FOCUS

Develop the concept of a function.



## Make Connections

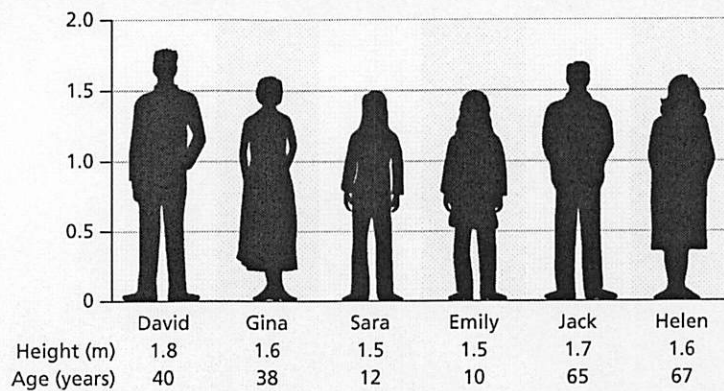
What is the rule for the Input/Output machine above?

Which numbers would complete this table for the machine?

Input	Output
1	5
2	7
	9
4	
	13

## Construct Understanding

### THINK ABOUT IT



Work in a group of 3.  
Use the picture above.

Each of you chooses one of the relations below.

- *name* related to *age*
- *name* related to *height*
- *height* related to *name*

Represent the relation you chose. Compare the relations.  
How are they alike? How are they different?

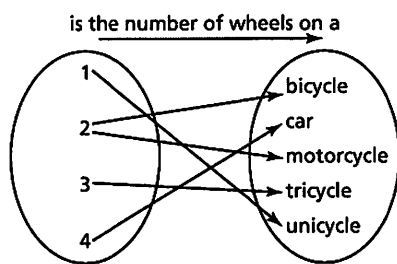
The set of first elements of a relation is called the **domain**.

The set of related second elements of a relation is called the **range**.

A **function** is a special type of relation where each element in the domain is associated with exactly one element in the range.

Here are some different ways to relate vehicles and the number of wheels each has.

This relation associates a number with a vehicle with that number of wheels.



This diagram does not represent a function because there is one element in the first set that associates with two elements in the second set; that is, there are two arrows from 2 in the first set.

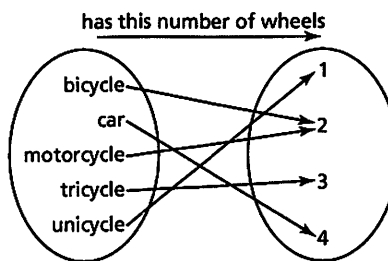
$\{(1, \text{unicycle}), (2, \text{bicycle}), (2, \text{motorcycle}), (3, \text{tricycle}), (4, \text{car})\}$

The set of ordered pairs above does not represent a function because two ordered pairs have the same first element.

The domain is the set of first elements:  $\{1, 2, 3, 4\}$

The range is the set of associated second elements:  $\{\text{unicycle, bicycle, motorcycle, tricycle, car}\}$

This relation associates a vehicle with the number of wheels it has.



This diagram represents a function because each element in the first set associates with exactly one element in the second set; that is, there is only one arrow from each element in the first set.

$\{(\text{unicycle}, 1), (\text{bicycle}, 2), (\text{motorcycle}, 2), (\text{tricycle}, 3), (\text{car}, 4)\}$

The set of ordered pairs above represents a function because the ordered pairs have different first elements.

The domain is the set of first elements:  $\{\text{unicycle, bicycle, motorcycle, tricycle, car}\}$

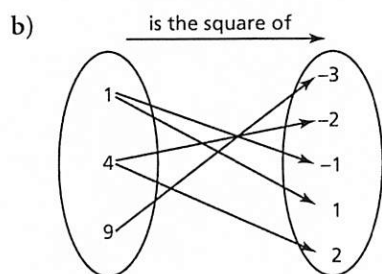
The range is the set of associated second elements:  $\{1, 2, 3, 4\}$

When we list the elements of the range, we do not repeat an element that occurs more than once.

## Example 1 Identifying Functions

For each relation below:

- Determine whether the relation is a function. Justify the answer.
  - Identify the domain and range of each relation that is a function.
- a) A relation that associates given shapes with the number of right angles in the shape: {(right triangle, 1), (acute triangle, 0), (square, 4), (rectangle, 4), (regular hexagon, 0)}



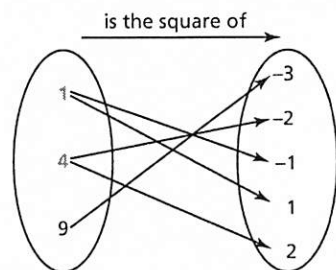
### SOLUTION

- a) Check to see if any ordered pairs have the same first element: {(right triangle, 1), (acute triangle, 0), (square, 4), (rectangle, 4), (regular hexagon, 0)}
- Each ordered pair has a different first element, so for every first element there is exactly one second element. So, the relation is a function.

The domain is the set of the first elements of the ordered pairs: {right triangle, acute triangle, square, rectangle, regular hexagon}

The range is the set of second elements of the ordered pairs: {0, 1, 4}

- b) Check to see if any element in the first set associates with more than one element in the second set.



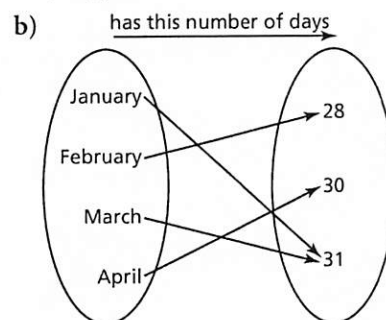
This relation is not a function because each of the numbers 1 and 4 in the first set associates with more than one number in the second set.

### CHECK YOUR UNDERSTANDING

1. For each relation below:

- Determine whether the relation is a function. Justify your answer.
- Identify the domain and range of each relation that is a function.

- a) A relation that associates a number with a prime factor of the number: {(4, 2), (6, 2), (6, 3), (8, 2), (9, 3)}



[Answers: a) no b) yes; domain: {January, February, March, April}; range: {28, 30, 31}]

What other strategies could you use to determine whether each relation is a function?

If  $(a, b)$  and  $(a, c)$  are ordered pairs in a function, what can you say about  $b$  and  $c$ ?

In the workplace, a person's gross pay,  $P$  dollars, often depends on the number of hours worked,  $h$ .

So, we say  $P$  is the *dependent variable*. Since the number of hours worked,  $h$ , does not depend on the gross pay,  $P$ , we say that  $h$  is the *independent variable*.

independent variable $\longrightarrow$	Hours Worked, $h$	Gross Pay, $P$ (\$)	$\longleftarrow$ dependent variable
	1	12	
	2	24	
	3	36	
	4	48	
	5	60	

domain  $\left\{ \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array} \right.$  range  $\left. \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array} \right\}$

A table of values usually represents a sample of the ordered pairs in a relation.

The values of the independent variable are listed in the first column of a table of values. These elements belong to the domain.

The values of the dependent variable are listed in the second column of a table of values. These elements belong to the range.

## Example 2 Describing Functions

The table shows the masses,  $m$  grams, of different numbers of identical marbles,  $n$ .

Number of Marbles, $n$	Mass of Marbles, $m$ (g)
1	1.27
2	2.54
3	3.81
4	5.08
5	6.35
6	7.62

- Why is this relation also a function?
- Identify the independent variable and the dependent variable. Justify the choices.
- Write the domain and range.

(Solution continues.)

### CHECK YOUR UNDERSTANDING

- The table shows the costs of student bus tickets,  $C$  dollars, for different numbers of tickets,  $n$ .

Number of Tickets, $n$	Cost, $C$ (\$)
1	1.75
2	3.50
3	5.25
4	7.00
5	8.75

- Why is this relation also a function?
- Identify the independent variable and the dependent variable. Justify your choices.
- Write the domain and range.

[Answers: b)  $n$ ;  $C$  c)  $\{1, 2, 3, 4, 5, \dots\}$ ;  $\{1.75, 3.50, 5.25, 7.00, 8.75, \dots\}$ ]

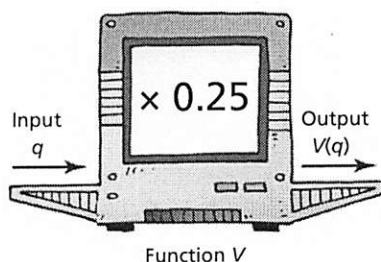
## SOLUTION

- a) For each number in the first column, there is only one number in the second column. So, the relation is a function.
- b) From an understanding of the situation, the mass of the marbles,  $m$ , depends on the number of marbles,  $n$ . So,  $m$  is the dependent variable and  $n$  is the independent variable.
- c) The first column of the table is representative of the domain.  
The domain is:  $\{1, 2, 3, 4, 5, 6, \dots\}$ ; that is, all natural numbers  
The second column of the table is representative of the range.  
The range is:  $\{1.27, 2.54, 3.81, 5.08, 6.35, 7.62, \dots\}$ ; that is, the product of each natural number and 1.27

We can think of a function as an input/output machine. The input can be any number in the domain, and the output depends on the input number. So, the input is the independent variable and the output is the dependent variable.

Consider two machines that both accept quarters. Machine A calculates the value of the quarters. Machine B weighs the quarters. Each machine performs a different operation, so the machines represent two different functions.

### ■ Machine A



When the input is  $q$  quarters, the output or value,  $V$ , in dollars is:  $0.25q$   
The equation  $V = 0.25q$  describes this function.

Since  $V$  is a function of  $q$ , we can write this equation using **function notation**:

$$V(q) = 0.25q$$

We say: “ $V$  of  $q$  is equal to  $0.25q$ .”

This notation shows that  $V$  is the dependent variable and that  $V$  depends on  $q$ .

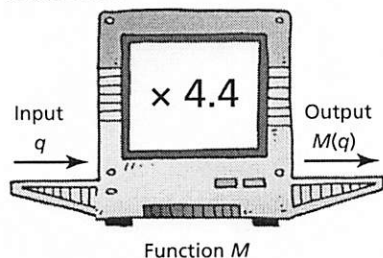
$V(3)$  represents the value of the function when  $q = 3$ .

$$V(3) = 0.25(3)$$

$$V(3) = 0.75$$

So, the value of 3 quarters is \$0.75.

Machine B



The mass of 1 quarter is 4.4 g.

When the input is  $q$  quarters, the output or mass,  $M$ , in grams is:  $4.4q$

The equation  $M = 4.4q$  describes this function.

Since  $M$  is a function of  $q$ , we can write this equation using function notation:

$$M(q) = 4.4q$$

This notation shows that  $M$  is the dependent variable and that  $M$  depends on  $q$ .

Any function that can be written as an equation in two variables can be written in function notation. For example, to write the equation  $d = 4t + 5$  in function notation, we may write  $d(t) = 4t + 5$ .  $t$  represents an element of the domain and  $d(t)$  represents an element of the range.

When we write an equation that is not related to a context, we use  $x$  as the independent variable and  $y$  as the dependent variable. Then an equation in two variables such as  $y = 3x - 2$  may be written as  $f(x) = 3x - 2$ .

Conversely, we may write an equation in function notation as an equation in two variables.

For example, for the equation  $C(n) = 300 + 25n$ , we write  $C = 300 + 25n$ .

And, for the equation  $g(x) = -2x + 5$ , we write  $y = -2x + 5$ .

We can use any other letter, such as  $g$  or  $h$  or  $k$ , to name a function.

### Example 3 Using Function Notation to Determine Values

The equation  $V = -0.08d + 50$  represents the volume,  $V$  litres, of gas remaining in a vehicle's tank after travelling  $d$  kilometres. The gas tank is not refilled until it is empty.

- Describe the function.  
Write the equation in function notation.
- Determine the value of  $V(600)$ .  
What does this number represent?
- Determine the value of  $d$  when  $V(d) = 26$ .  
What does this number represent?

(Solution continues.)

#### CHECK YOUR UNDERSTANDING

- The equation  $C = 25n + 1000$  represents the cost,  $C$  dollars, for a feast following an Arctic sports competition, where  $n$  is the number of people attending.
  - Describe the function.  
Write the equation in function notation.

(Question continues.)

## SOLUTION

a) The volume of gas remaining in a vehicle's tank is a function of the distance travelled. In function notation:

$$V(d) = -0.08d + 50$$

b) To determine  $V(600)$ , use:

$$V(d) = -0.08d + 50 \quad \text{Substitute: } d = 600$$

$$V(600) = -0.08(600) + 50$$

$$V(600) = -48 + 50$$

$$V(600) = 2$$

$V(600)$  is the value of  $V$  when  $d = 600$ .

This means that when the car has travelled 600 km, the volume of gas remaining in the vehicle's tank is 2 L.

c) To determine the value of  $d$  when  $V(d) = 26$ , use:

$$V(d) = -0.08d + 50 \quad \text{Substitute: } V(d) = 26$$

$$26 = -0.08d + 50 \quad \text{Solve for } d.$$

$$-24 = -0.08d \quad \text{Divide each side by } -0.08.$$

$$d = 300$$

$V(300) = 26$  means that when  $d = 300$ ,  $V = 26$ ; that is, after the car has travelled 300 km, 26 L of gas remains in the vehicle's tank.

b) Determine the value of  $C(100)$ . What does this number represent?

c) Determine the value of  $n$  when  $C(n) = 5000$ . What does this number represent?

[Answers: a)  $C(n) = 25n + 1000$   
b) \$3500 c) 160]

What values of  $d$  do not make sense as possible domain values?

## Discuss the Ideas

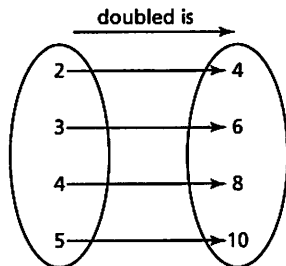
1. How can you tell whether a set of ordered pairs represents a function?
2. When a function is completely represented using a set of ordered pairs or a table of values, how can you determine the domain and range of the function?
3. Why are some relations not functions? Why are all functions also relations?

## Exercises

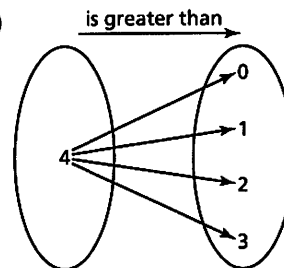


4. Which arrow diagrams represent functions?

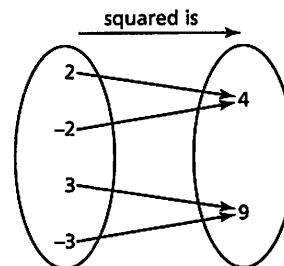
a)



b)



c)



5. Which sets of ordered pairs represent functions? Identify the domain and range of each set of ordered pairs.

- a)  $\{(1, 3), (2, 6), (3, 9), (4, 12)\}$
- b)  $\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$
- c)  $\{(2, 3), (4, 5), (6, 7), (8, 9)\}$
- d)  $\{(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3)\}$

6. Write in function notation.

- a)  $C = 20n + 8$
- b)  $P = n - 3$
- c)  $t = 5d$
- d)  $y = -x$

7. Write as an equation in two variables.

- a)  $d(t) = 3t - 5$
- b)  $f(x) = -6x + 4$
- c)  $C(n) = 5n$
- d)  $P(n) = 2n - 7$



8. For each relation below:

- ▣ Determine whether the relation is a function. Justify your answer.
  - ▣ Identify the domain and range of each relation.
- a)  $\{(1, 1), (2, 8), (3, 27), (4, 64)\}$
  - b)  $\{(3, 4), (3, 5), (3, 6), (3, 7)\}$

9. For each table of values below:

- i) Explain why the relation is a function.
- ii) Identify the independent variable and the dependent variable. Justify your choices.
- iii) Write the domain and range.

a)

Number of Cans of Juice Purchased, $n$	Cost, $C$ (\$)
1	2.39
2	4.00
3	6.39
4	8.00
5	10.39
6	12.00

b)

Altitude, $A$ (m)	Temperature, $T$ ( $^{\circ}\text{C}$ )
610	15.0
1220	11.1
1830	7.1
2440	3.1
3050	-0.8
3660	-4.8

10. This set of ordered pairs associates a number with a polygon that has that number of sides:  $\{(3, \text{isosceles triangle}), (3, \text{equilateral triangle}), (3, \text{right triangle}), (3, \text{scalene triangle}), (4, \text{square}), (4, \text{rectangle}), (4, \text{rhombus}), (4, \text{trapezoid}), (4, \text{parallelogram}), (5, \text{pentagon}), (6, \text{hexagon})\}$

- a) Does the set of ordered pairs represent a function? Explain.
- b) Suppose the elements in the ordered pairs were reversed. Use the association "has this number of sides." Would the new relation be a function? Explain.
- c) Identify the domain and range of each relation in parts a and b.

11. The Rassemblement jeunesse francophone in Alberta brings together French language high school students from all over the province for a day of activities. Use two columns in this table to represent a relation.

- a) Name two relations that are functions.
- b) Name two relations that are not functions. Justify your answers.

Name	From	Age	Gender
Marie	Edmonton	13	F
Gabriel	Falher	16	M
Élise	Bonnyville	14	F
Christophe	Calgary	13	M
Jean	Edmonton	15	M
Mélanie	Edmonton	15	F
Nicole	Red Deer	17	F
Marc	Légal	13	M



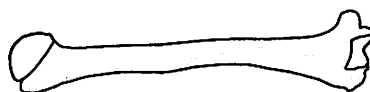
12. Which statement is true? Give an example to justify your choice.
- All functions are relations, but not all relations are functions.
  - All relations are functions, but not all functions are relations.
13. In a crossword game, each letter is worth a certain number of points. Here are some letters and their points.

A <sub>1</sub>	D <sub>2</sub>	F <sub>4</sub>
G <sub>2</sub>	M <sub>3</sub>	Q <sub>10</sub>
T <sub>1</sub>	X <sub>8</sub>	Z <sub>10</sub>

- Create two different tables to represent relations that associate these letters and their points.
  - Which table in part a represents a function? Justify your choice.
14. For the function  $f(x) = -5x + 11$ , determine:
- $f(1)$
  - $f(-3)$
  - $f(0)$
  - $f(1.2)$
15. a) For the function  $f(n) = 2n - 7$ , determine  $n$  when:
- $f(n) = 11$
  - $f(n) = -6$
- b) For the function  $g(x) = -5x + 1$ , determine  $x$  when:
- $g(x) = 41$
  - $g(x) = -16$
16. The function  $C(i) = 2.54i$  converts a measurement of  $i$  inches to a measurement of  $C$  centimetres.
- Write the function as an equation in 2 variables.
  - Determine the value of  $C(12)$ . What does this number represent?
  - Determine the value of  $i$  when  $C(i) = 100$ . What does this number represent?

17. A car is travelling toward Meadow Lake Park, Saskatchewan. The equation  $D = -80t + 300$  represents the distance,  $D$  kilometres, to Meadow Lake after  $t$  hours of driving.
- Describe the function.  
Write this equation in function notation.
  - How far away from Meadow Lake Park was the car at the start of its journey?  
How do you know?

18. Anthropologists who study human remains have developed equations for estimating the height of a person from a measure of her or his bones. The height in centimetres is a function of the length,  $l$  centimetres, of the humerus (the upper arm bone).



For a female:  $f(l) = 2.754l + 71.475$

For a male:  $m(l) = 2.894l + 70.641$

- Determine each value. What does each number represent?
    - $f(15)$
    - $m(20)$
  - Determine each value of  $l$ . What does each number represent?
    - $f(l) = 142$
    - $m(l) = 194$
  - Measure the length of your humerus. Use an equation to estimate your height. How close was your answer to your actual height?
19. The function  $C(f) = \frac{5}{9}(f - 32)$  converts a temperature,  $f$  degrees Fahrenheit, to  $C$  degrees Celsius.
- Determine:
    - $C(50)$
    - $C(-13)$
  - Determine each value of  $f$  when:
    - $C(f) = 20$
    - $C(f) = -35$
  - Write an equation in function notation to relate the temperatures in each fact.
    - Pure water freezes at  $0^{\circ}\text{C}$  or  $32^{\circ}\text{F}$ .
    - Pure water boils at  $100^{\circ}\text{C}$  or  $212^{\circ}\text{F}$ .
    - Cookies are baked at  $180^{\circ}\text{C}$  or  $356^{\circ}\text{F}$ .

**C**

20. To convert a temperature in degrees Celsius to degrees Fahrenheit, multiply the Celsius temperature by  $\frac{9}{5}$  then add 32. Use these instructions to write an equation in function notation for this conversion.
21. The area of a rectangle with length  $l$  centimetres and width  $w$  centimetres is  $9 \text{ cm}^2$ . Express the perimeter of the rectangle as a function of its length.
22. A rectangle with length  $l$  centimetres and width  $w$  centimetres has a perimeter of 12 cm. Use function notation to express the length of the rectangle as a function of its width. What are the domain and range of the function?
23. The lengths of the sides of a triangle, in units, are  $s$ ,  $s + 5$ , and  $t$ . Its perimeter is 16 units. Use function notation to express  $t$  as a function of  $s$ . What are the domain and range of the function?

**Reflect**

Describe how you can determine if a relation is a function. Use an example to illustrate each strategy you might use.

**THE WORLD OF MATH****Careers: Forensic Anthropologist**

Forensic anthropologists study human remains to understand more about how people develop, both as individuals and as societies. They collect data on bones and teeth to identify the sex, height, mass, race, and age at death. Forensic anthropologists work in crime labs, law enforcement agencies, museums, or at archaeological sites; and may give expert testimony in court. They may identify bones and bone fragments that have been in storage for many years.

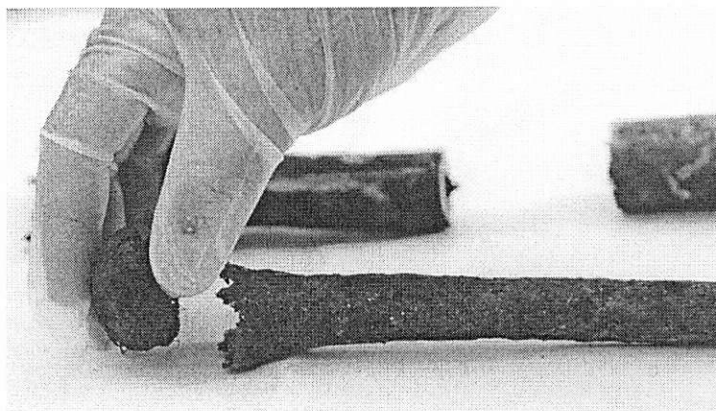
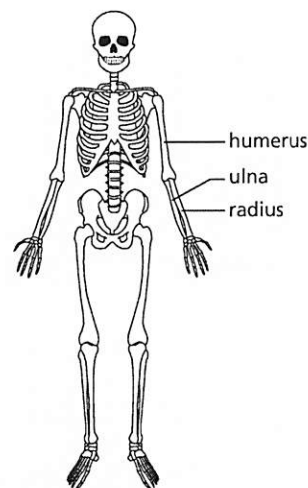
Specimens must be cleaned, accurately measured, and catalogued. Measurements of the skull and teeth can help to estimate the age of a person. When the length of the humerus, radius, or ulna is known, then a person's height can be approximated.

Here are some typical equations used to estimate the height of a person from the length of her or his radius.

$$\text{For a female: } h = 3.343r + 81.224$$

$$\text{For a male: } h = 3.271r + 89.925,$$

where  $h$  is the height of the person in centimetres and  $r$  is the length of the radius in centimetres.



# CHECKPOINT 1

## Connections

Here is a Frayer model for a function.

<p><b>Definition</b></p> <p>A function is a relation where each element in the first set is associated with exactly one element in the second set.</p>	<p><b>Essential Characteristics</b></p> <p>The domain is the set of first elements in the ordered pairs. These are the values of the independent variable.</p> <p>The range is the set of second elements in the ordered pairs. These are the values of the dependent variable.</p>																				
<p><b>Function</b></p>																					
<p><b>Example</b></p> <table style="margin-left: auto; margin-right: auto; border-collapse: collapse;"> <tr> <td style="padding: 5px;"><math>x</math></td> <td style="padding: 5px;"><math>y</math></td> </tr> <tr> <td style="padding: 5px;">0</td> <td style="padding: 5px;">4</td> </tr> <tr> <td style="padding: 5px;">1</td> <td style="padding: 5px;">5</td> </tr> <tr> <td style="padding: 5px;">2</td> <td style="padding: 5px;">6</td> </tr> <tr> <td style="padding: 5px;">3</td> <td style="padding: 5px;">7</td> </tr> </table> <p style="margin-left: 40px;">{(0, 4), (1, 5), (2, 6), (3, 7)}</p> <div style="margin-left: 40px;"> <p>plus 4 equals →</p> </div>	$x$	$y$	0	4	1	5	2	6	3	7	<p><b>Non-examples</b></p> <table style="margin-left: auto; margin-right: auto; border-collapse: collapse;"> <thead> <tr> <th style="padding: 5px;">Number of Faces</th> <th style="padding: 5px;">Object</th> </tr> </thead> <tbody> <tr> <td style="padding: 5px;">4</td> <td style="padding: 5px;">triangular pyramid</td> </tr> <tr> <td style="padding: 5px;">5</td> <td style="padding: 5px;">square pyramid</td> </tr> <tr> <td style="padding: 5px;">6</td> <td style="padding: 5px;">cube</td> </tr> <tr> <td style="padding: 5px;">6</td> <td style="padding: 5px;">rectangular prism</td> </tr> </tbody> </table> <p style="margin-left: 40px;">{(4, triangular pyramid), (5, square pyramid), (6, cube), (6, rectangular prism)}</p> <div style="margin-left: 40px;"> <p>is the number of faces on a →</p> </div>	Number of Faces	Object	4	triangular pyramid	5	square pyramid	6	cube	6	rectangular prism
$x$	$y$																				
0	4																				
1	5																				
2	6																				
3	7																				
Number of Faces	Object																				
4	triangular pyramid																				
5	square pyramid																				
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6	rectangular prism																				

## Concept Development

### In Lesson 5.1

- You described a relation in words and represented it using: a set of ordered pairs, an arrow diagram, a table, and a bar graph.

### In Lesson 5.2

- You identified a function by checking to see whether its ordered pairs had different first elements.
- You listed the elements of the domain and of the range.
- You related the elements of the domain to the independent variable and the elements of the range to the dependent variable.
- You described functions in words, and algebraically using function notation.

## Assess Your Understanding

### 5.1

1. Copy and complete this table for different representations of relations.

	Description in Words	Set of Ordered Pairs	Arrow Diagram	Table or Graph										
a)		{(skin, drum), (skin, kayak), (bark, basket), (stone, inukshuk), (stone, carving)}												
b)				<table border="1"> <thead> <tr> <th>Number</th> <th>Number of Factors</th> </tr> </thead> <tbody> <tr> <td>1</td> <td>1</td> </tr> <tr> <td>2</td> <td>2</td> </tr> <tr> <td>3</td> <td>2</td> </tr> <tr> <td>4</td> <td>3</td> </tr> </tbody> </table>	Number	Number of Factors	1	1	2	2	3	2	4	3
Number	Number of Factors													
1	1													
2	2													
3	2													
4	3													
c)			<p>is usually coloured</p>											
d)	For the numbers 1 to 4, the first number in an ordered pair is greater than the second number.													

### 5.2

- Which relations in question 1 are functions? Justify your answers.
  - State the domain and range of each function.
- Think about two sets of numbers and an association.
    - Create a relation that is not a function.
    - Create a function.
  - Represent each relation in part a in different ways.
- The temperature,  $T$  degrees Celsius, of Earth's interior is a function of the distance,  $d$  kilometres, below the surface:  $T(d) = 10d + 20$

  - Identify the dependent and independent variables.
  - Write this function as an equation in two variables.
  - Determine the value of  $T(5)$ . Describe what this number represents.
  - Determine the value of  $d$  when  $T(d) = 50$ . Describe what this number represents.