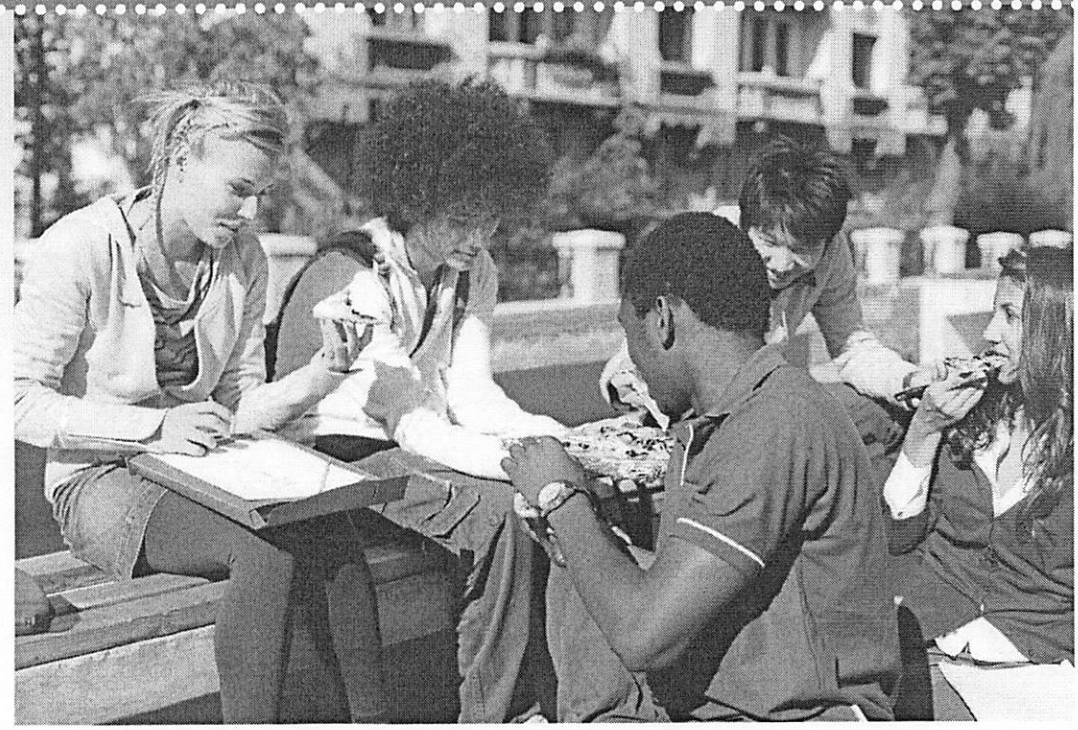
5.6 Properties of Linear Relations

LESSON FOCUS

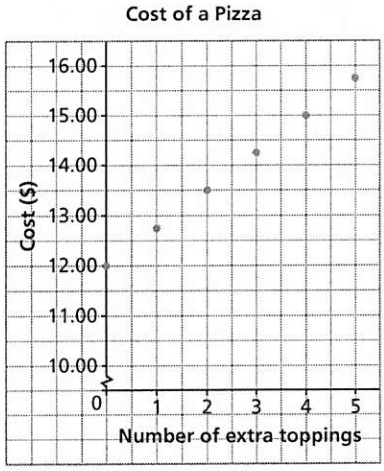
Identify and represent linear relations in different ways.



Make Connections

The table of values and graph show the cost of a pizza with up to 5 extra toppings.

Number of Extra Toppings	Cost ($)
0	12.00
1	12.75
2	13.50
3	14.25
4	15.00
5	15.75



What patterns do you see in the table?

Write a rule for the pattern that relates the cost of a pizza to the number of its toppings.

How are the patterns in the table shown in the graph?

How can you tell from the table that the graph represents a linear relation?

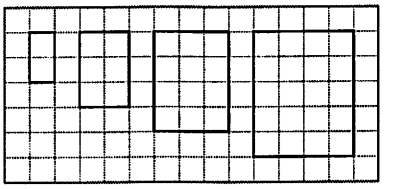
Construct Understanding

TRY THIS

Work with a partner.

You will need 1-cm grid paper.

Use this pattern of rectangles.
This pattern continues.



A. Draw the next two rectangles in the pattern.
Copy and complete each table of values for the 6 rectangles.

Width of Rectangle (cm)	Area (cm^2)
1	
2	

Width of Rectangle (cm)	Perimeter (cm)
1	
2	

B. Which table of values represents a linear relation? How can you tell?

C. Graph the data in each table of values.
Does each graph represent a linear relation?
How do you know?

The cost for a car rental is \$60, plus \$20 for every 100 km driven.
The independent variable is the distance driven and the dependent variable is the cost.
We can identify that this is a linear relation in different ways.

- a table of values

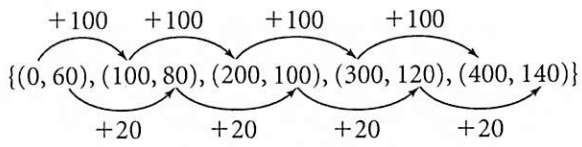
Independent variable → Distance (km) | Cost (\$) ← Dependent variable

	Distance (km)	Cost (\$)	
	0	60	
+100			+20
	100	80	
+100			+20
	200	100	
+100			+20
	300	120	
+100			+20
	400	140	

For a linear relation, a constant change in the independent variable results in a constant change in the dependent variable.

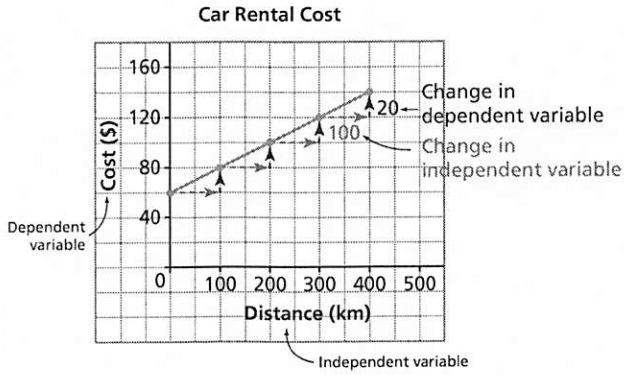
- a set of ordered pairs

Why is it important that the ordered pairs are listed so their first elements are in numerical order?



- a graph

The graph of a linear relation is a straight line.



We can use each representation above to calculate the **rate of change**.
The rate of change can be expressed as a fraction:

$$\frac{\text{change in dependent variable}}{\text{change in independent variable}} = \frac{\$20}{100 \text{ km}}$$

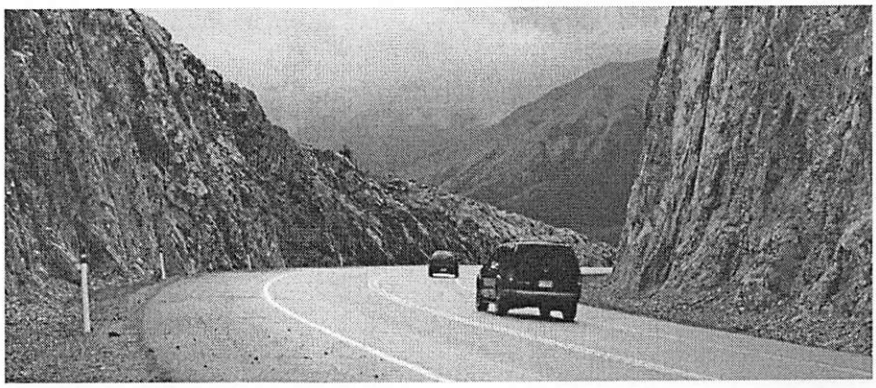
$$= \$0.20/\text{km}$$

The rate of change is \$0.20/km; that is, for each additional 1 km driven, the rental cost increases by 20¢. The rate of change is constant for a linear relation.

We can determine the rate of change from the equation that represents the linear function.
Let the cost be C dollars and the distance driven be d kilometres.
An equation for this linear function is:

$C = 0.20d + 60$

- 60: initial amount
- d: independent variable
- 0.20: rate of change
- C: dependent variable



Example 1 Determining whether a Table of Values Represents a Linear Relation

Which table of values represents a linear relation? Justify the answer.

a) The relation between temperature in degrees Celsius, C, and temperature in degrees Fahrenheit, F

C	F
0	32
5	41
10	50
15	59
20	68

b) The relation between the current, I amps, and power, P watts, in an electrical circuit

I	P
0	0
5	75
10	300
15	675
20	1200

SOLUTION

The terms in the first column are in numerical order. So, calculate the change in each variable.

a)

C	Change in C	F	Change in F
0		32	
5	$5 - 0 = 5$	41	$41 - 32 = 9$
10	$10 - 5 = 5$	50	$50 - 41 = 9$
15	$15 - 10 = 5$	59	$59 - 50 = 9$
20	$20 - 15 = 5$	68	$68 - 59 = 9$

Since the changes in both variables are constant, the table of values represents a linear relation.

b)

I	Change in I	P	Change in P
0		0	
5	$5 - 0 = 5$	75	$75 - 0 = 75$
10	$10 - 5 = 5$	300	$300 - 75 = 225$
15	$15 - 10 = 5$	675	$675 - 300 = 375$
20	$20 - 15 = 5$	1200	$1200 - 675 = 525$

The changes in I are constant, but the changes in P are not constant. So, the table of values does not represent a linear relation.

CHECK YOUR UNDERSTANDING

1. Which table of values represents a linear relation? Justify your answer.

a) The relation between the number of bacteria in a culture, n, and time, t minutes.

t	n
0	1
20	2
40	4
60	8
80	16
100	32

b) The relation between the amount of goods and services tax charged, T dollars, and the amount of the purchase, A dollars

A	T
60	3
120	6
180	9
240	12
300	15

[Answers: a) not linear b) linear]

What other strategies could you use to check whether each table of values represents a linear relation?

When an equation is written using the variables x and y, x represents the independent variable and y represents the dependent variable.

Example 2 Determining whether an Equation Represents a Linear Relation

a) Graph each equation.

i) $y = -3x + 25$ ii) $y = 2x^2 + 5$

iii) $y = 5$ iv) $x = 1$

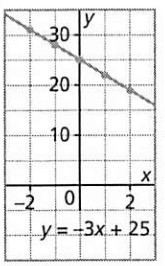
b) Which equations in part a represent linear relations? How do you know?

SOLUTION

a) Create a table of values, then graph the relation.

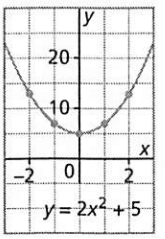
i) $y = -3x + 25$

x	y
-2	31
-1	28
0	25
1	22
2	19



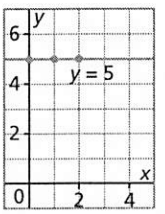
ii) $y = 2x^2 + 5$

x	y
-2	13
-1	7
0	5
1	7
2	13



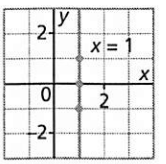
iii) $y = 5$

x	y
0	5
1	5
2	5



iv) $x = 1$

x	y
1	-1
1	0
1	1



b) The graphs in parts i, iii, and iv are straight lines, so their equations represent linear relations; that is, $y = -3x + 25$, $y = 5$, and $x = 1$.
The graph in part ii is not a straight line, so its equation does not represent a linear relation.

CHECK YOUR UNDERSTANDING

2. a) Graph each equation.
i) $x = -2$
ii) $y = x + 25$
iii) $y = 25$
iv) $y = x^2 + 25$

b) Which equations in part a represent linear relations? How do you know?

[Answers: b) $x = -2$; $y = x + 25$; $y = 25$]

Example 3 Identifying a Linear Relation

Which relation is linear? Justify the answer.

a) A new car is purchased for $24 000. Every year, the value of the car decreases by 15%. The value is related to time.

b) For a service call, an electrician charges a $75 flat rate, plus $50 for each hour he works. The total cost for service is related to time.

SOLUTION

Create a table of values, then check to see if the relation is linear.

a) Every year, the value decreases by 15%.
The value of the car is:
100% − 15% = 85% of its previous value
So, multiply each value by 0.85.

	Time (years)	Value ($)	
	0	24 000	
+1			−3600
	1	20 400	
+1			−3060
	2	17 340	
+1			−2601
	3	14 739	

There is a constant change of 1 in the 1st column, but the differences in the 2nd column are not constant. So, the relation is not linear.

b) After the first hour, the cost increases by $50 per hour.

	Time (h)	Cost ($)	
	0	75	
+1			+50
	1	125	
+1			+50
	2	175	
+1			+50
	3	225	
+1			+50
	4	275	

There is a constant change of 1 in the 1st column and a constant change of 50 in the 2nd column, so the relation is linear.

CHECK YOUR UNDERSTANDING

3. Which relation is linear? Justify your answer.

a) A dogsled moves at an average speed of 10 km/h along a frozen river. The distance travelled is related to time.

b) The area of a square is related to the side length of the square.

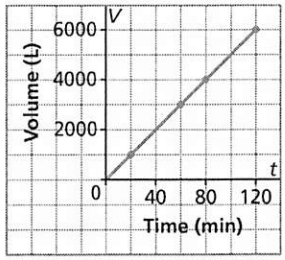
[Answers: a) linear
b) not linear]

What equation could you write for the linear relation in part b?

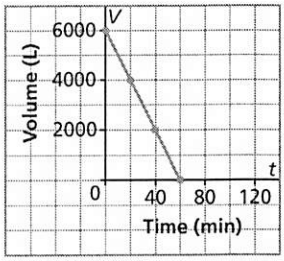
Example 4 Determining the Rate of Change of a Linear Relation from Its Graph

A water tank on a farm near Swift Current, Saskatchewan, holds 6000 L.
Graph A represents the tank being filled at a constant rate.
Graph B represents the tank being emptied at a constant rate.

Graph A
Filling a Water Tank



Graph B
Emptying a Water Tank



a) Identify the independent and dependent variables.

b) Determine the rate of change of each relation, then describe what it represents.

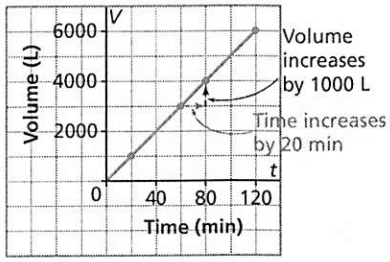
SOLUTION

For Graph A

a) The independent variable is the time, t.
The dependent variable is the volume, V.

b) Choose two points on the line. Calculate the change in each variable from one point to the other.

Graph A
Filling a Water Tank



Change in volume:
4000 L − 3000 L = 1000 L

Change in time:
80 min − 60 min = 20 min

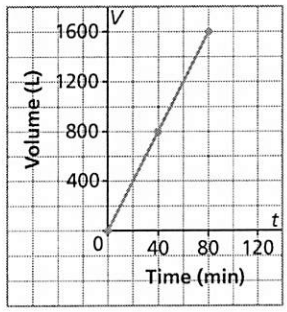
Rate of change: $\frac{1000 \text{ L}}{20 \text{ min}} = 50 \text{ L/min}$

The rate of change is positive so the volume is increasing with time.
Every minute, 50 L of water are added to the tank.

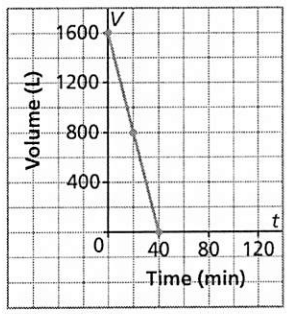
CHECK YOUR UNDERSTANDING

4. A hot tub contains 1600 L of water. Graph A represents the hot tub being filled at a constant rate. Graph B represents the hot tub being emptied at a constant rate.

Graph A
Filling a Hot Tub



Graph B
Emptying a Hot Tub



a) Identify the dependent and independent variables.

b) Determine the rate of change of each relation, then describe what it represents.

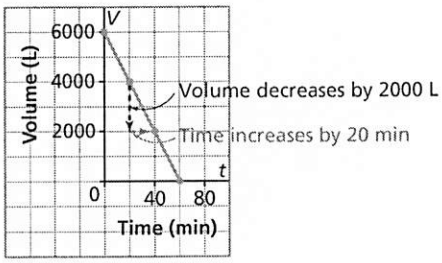
[Answers: Graph A a) V, t b) 20 L/min
Graph B a) V, t b) −40 L/min]

For Graph B

a) The independent variable is the time, t.
The dependent variable is the volume, V.

b) Choose two points on the line.
Calculate the change in each variable from one point to the other.

Graph B
Emptying a Water Tank



Change in volume: $2000 \text{ L} - 4000 \text{ L} = -2000 \text{ L}$
Change in time: $40 \text{ min} - 20 \text{ min} = 20 \text{ min}$

Rate of change: $\dfrac{-2000 \text{ L}}{20 \text{ min}} = -100 \text{ L/min}$

The rate of change is negative so the volume is decreasing with time.
Every minute, 100 L of water are removed from the tank.

Discuss the Ideas

1. How can you tell from each format whether a relation is linear?
   - a description in words
   - a set of ordered pairs
   - a table of values
   - an equation
   - a graph

2. What is "rate of change"? How can you use each format in question 1 to determine the rate of change of a linear relation?

Exercises

A

3. Which tables of values represent linear relations? Explain your answers.

a)

Time (min)	Distance (m)
0	10
2	50
4	90
6	130

b)

Time (s)	Speed (m/s)
0	10
1	20
2	40
3	80

c)

Speed (m/s)	Time (s)
15	7.5
10	5
5	2.5
0	0

d)

Distance (m)	Speed (m/s)
4	2
16	4
1	1
9	3

4. Which sets of ordered pairs represent linear relations? Explain your answers.

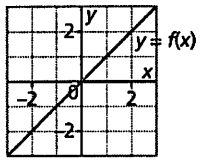
a) $\{(3, 11), (5, 9), (7, 7), (9, 5)\}$

b) $\{(-2, 3), (0, 1), (2, -3), (4, -7)\}$

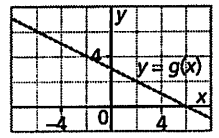
c) $\{(1, 1), (1, 3), (2, 1), (2, 3)\}$

5. Which graphs represent linear relations? How do you know?

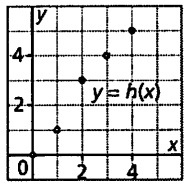
a)



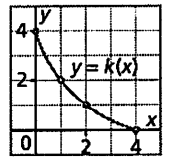
b)



c)



d)



B

6. a) Create a table of values when necessary, then graph each relation.

i) $y = 2x + 8$ ii) $y = 0.5x + 12$

iii) $y = x^2 + 8$ iv) $y = 2x$

v) $x = 7$ vi) $x + y = 6$

b) Which equations in part a represent linear relations? How do you know?

7. For each relation below:

i) Identify the dependent and independent variables.

ii) Use the table of values to determine whether the relation is linear.

iii) If the relation is linear, determine its rate of change.

a) The distance required for a car to come to a complete stop after its brakes are applied is the *braking distance*. The braking distance, d metres, is related to the speed of the car, s kilometres per hour, when the brakes are first applied.

s (km/h)	d (m)
50	13
60	20
70	27
80	35

b) The altitude of a plane, a metres, is related to the time, t minutes, that has elapsed since it started its descent.

t (min)	a (m)
0	12 000
2	11 600
4	11 200
6	10 800
8	10 400

8. In a hot-air balloon, a chart shows how the distance to the horizon, d kilometres, is related to the height of the balloon, h metres.

h (m)	d (km)
5	8
10	11
30	20
50	25
100	36

a) Graph these data.
b) Is the relation linear? What strategy did you use?

9. Earth rotates through approximately 360° every 24 h. The set of ordered pairs below describes the rotation. The first coordinate is the time in hours, and the second coordinate is the approximate angle of rotation in degrees. Describe two strategies you could use to determine if this relation is linear.
{(0, 0), (6, 90), (12, 180), (18, 270), (24, 360)}

10. Sophie and 4 of her friends plan a trip to the Edmonton Chante for one night. The hotel room is $95 for the first 2 people, plus $10 for each additional person in the room. The total cost is related to the number of people. Is the relation linear? How do you know?

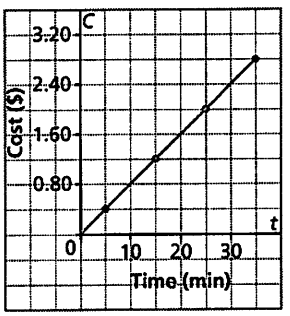
11. A skydiver jumps from an altitude of 3600 m. For the first 12 s, her height in metres above the ground is described by this set of ordered pairs:
{(0, 3600), (4, 3526), (8, 3353.5), (12, 3147.5)}
For the next 21 s, her height above the ground is described by this set of ordered pairs:
{(15, 2988.5), (21, 2670.5), (27, 2352.5), (33, 2034.5)}
Determine whether either set of ordered pairs represents a linear relation. Explain.

12. The cost, C dollars, to rent a hall for a banquet is given by the equation $C = 550 + 15n$, where n represents the number of people attending the banquet.
a) Explain why the equation represents a linear relation.
b) State the rate of change. What does it represent?

13. A safety flare is shot upward from the top of a cliff 200 m above sea level. An equation for the height of the flare, d metres, above sea level t seconds after the flare is fired, is given by the equation $d = -4.9t^2 + 153.2t + 200$. Describe two strategies you could use to determine whether this relation is linear.

14. This graph represents Jerome's long distance phone call to his pen pal in Nunavut. Jerome is charged a constant rate.

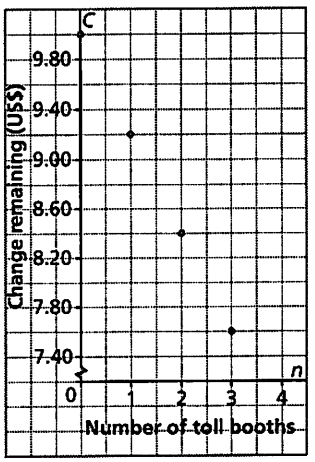
The Cost of Jerome's Phone Call



a) Identify the dependent and independent variables.
b) Determine the rate of change, then describe what it represents.

15. Kashala takes a cross-country trip from her home in Lethbridge through the United States. In Illinois, she drives on a toll highway. This graph represents the cost of Kashala's drive on the toll highway. She is charged a constant amount at each toll booth and she starts with US$10 in change. Determine the rate of change, then describe what it represents.

Kashala's Drive on the Toll Highway



16. Match each description of a linear relation with its equation and set of ordered pairs below. Explain your choices.

a) The amount a person earns is related to her hourly wage.

b) The cost of a banquet is related to a flat fee plus an amount for each person who attends.

c) The volume of gas in a car's gas tank is related to the distance driven since the time when the tank was filled.

Equation 1: $y = 500 + 40x$

Equation 2: $y = 35 - 0.06x$

Equation 3: $y = 20x$

Set A: {(100, 29), (200, 23), (300, 17), (400, 11)}

Set B: {(1, 20), (5, 100), (10, 200), (15, 300)}

Set C: {(0, 500), (40, 2100), (80, 3700), (100, 4500)}

17. a) Which situations represent linear relations? Explain how you know.

i) A hang glider starts her descent at an altitude of 2000 m. She descends at a constant speed to an altitude of 1500 m in 10 min.

ii) A population of bacteria triples every hour for 4 h.

iii) A taxi service charges a $5 flat fee plus $2 for each kilometre travelled.

iv) The cost to print each yearbook is $5. There is a start up fee of $500 to set up the printing press.

v) An investment increases in value by 12% each year.

b) For each linear relation in part a, identify:

- the dependent and independent variables
- the rate of change and explain what it represents

18. Identify the measurement formulas that represent linear relations. Explain how you know.

a) Perimeter, P, of an equilateral triangle with side length s: $P = 3s$

b) Surface area, A, of a cube with edge length s: $A = 6s^2$

c) Volume, V, of a sphere with radius r: $V = \frac{4}{3}\pi r^3$

d) Circumference, C, of a circle with diameter d: $C = \pi d$

e) Area, A, of a circle with radius r: $A = \pi r^2$

19. Here are two equations that can be used to model the value, V dollars, of a $24 000 truck as it depreciates over n years:

$V = 24\,000 - 2000n$ and $V = 24\,000(0.2^n)$

a) Which equation represents a linear relation? Justify your answer.

b) For the linear relation, state the rate of change. What does it represent?

20. You can estimate the distance in kilometres between you and a distant storm by measuring the time in seconds between seeing a lightning flash and hearing the thunder, then dividing by 3. This works because sound travels at approximately 0.3 km/s. Is this relation between distance and time linear? Justify your answer.

21. A berry patch is to be harvested. Is the relation between the time it will take to harvest the patch and the number of pickers needed linear? Justify your answer.

22. Which statements are true? Use examples to justify your answers.

a) A relation described by exactly two ordered pairs is always linear.

b) An equation of the form $Ax + By = C$ for non-zero constants, A, B, and C, always represents a linear function.

c) An equation of the form $y = Cx^2$ for a non-zero constant C, always represents a linear function.

d) An equation of the form $x = C$ for a constant C, always represents a linear relation.

e) A linear relation is always a linear function.

Reflect

List three different strategies you can use to tell whether a relation is linear. Include an example with each strategy.