1.5 Investments Involving Regular Payments

YOU WILL NEED

- financial application on a graphing calculator or spreadsheet
- spreadsheet software

EXPLORE...

- Indu has been depositing $200 into a savings account at the end of every month. The interest rate on her minimum monthly balance is 6%, compounded monthly. She now has between $1000 and $1500. How long has she been depositing money into the account?

GOAL

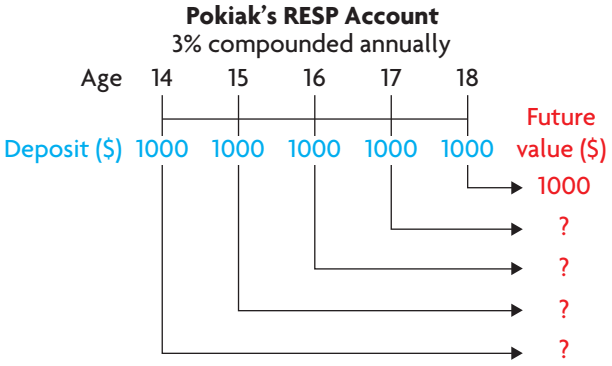
Determine the future value of an investment that earns compound interest involving regular payments.

INVESTIGATE the Math

Pokiak is now 18 years old, and he needs money for his post-secondary education. On his 14th birthday, his family deposited $1000 into a Registered Education Savings Plan (RESP) at 3% interest, compounded annually. Since then, Pokiak has deposited $1000 of his own money, earned by working part-time, into the account each year.

? How much money is in Pokiak's RESP account, and how much interest has it earned altogether?

A. Each of the five $1000 deposits could be thought of as a separate investment, as shown on the timeline below. Each deposit earns interest at the same interest rate. Will each deposit earn the same amount of interest? Explain.



B. For how many compounding periods does each deposit earn interest?

C. Determine the future value of each deposit.

D. Determine the current value of Pokiak's RESP account. What strategy did you use?

E. How much interest did Pokiak's RESP account earn altogether?

Reflecting

F. How is determining the future value of an investment involving regular deposits like determining the future value of a single deposit? How is it different?

G. How is determining the total interest earned on an investment involving regular deposits different from determining the total interest earned on a single deposit?

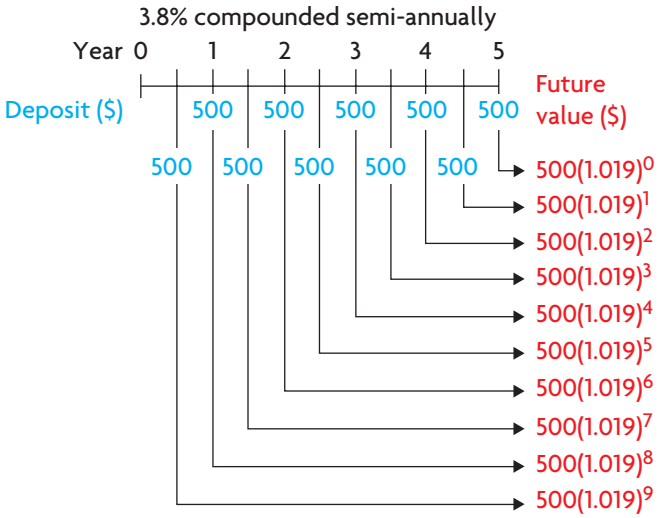
APPLY the Math

EXAMPLE 1 Determining the future value of an investment involving regular deposits

Darva is saving for a trip to Australia in 5 years. She plans to work on a student visa while she is there, so she needs only enough money for a return flight and her expenses until she finds a job. She deposits $500 into her savings account at the end of each 6-month period from what she earns as a server. The account earns 3.8%, compounded semi-annually. How much money will be in the account at the end of 5 years? How much of this money will be earned interest?

Darva's Solution: Using a spreadsheet

I drew a timeline to show the future value of each of the $500 deposits that I made at the end of each 6-month period for 5 years.



I could see that I needed to do 10 calculations and then determine the sum.

$A = P(1 + i)^n$

I represented each deposit's future value, A, using $P = \$500$, $i = \frac{0.038}{2}$, or 0.019, and the number of compounding periods for each deposit, n.

	A	B	C	D
1	**Year**	**Number of Compounding Periods (*n*)**	**Deposit ($)**	**Future Value ($)**
2	0.5	9	500	592.2944
3	1	8	500	581.2507
4	1.5	7	500	570.4128
5	2	6	500	559.7771
6	2.5	5	500	549.3396
7	3	4	500	539.0968
8	3.5	3	500	529.0449
9	4	2	500	519.1805
10	4.5	1	500	509.5
11	5	0	500	500
12		Totals	5000	5449.897

It made sense to use a spreadsheet because there are a lot of calculations.

In column D, I used the expression $500(1.019)^n$ to create the spreadsheet formula for the future value of each deposit.

In column B, I entered the value of n for each deposit.

In cell D12, I entered a formula to determine the future value of the investment (the sum of the future values of the 10 deposits).

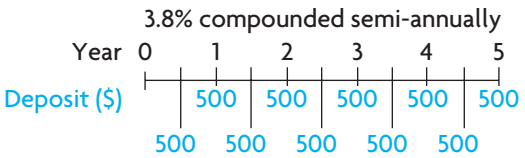
In cell C12, I entered a formula to determine the sum of the 10 deposits.

Interest earned $= 5449.90 - 5000$
Interest earned $= 449.90$

To determine the interest earned, I subtracted the total of the deposits from the future value of the investment.

There will be $5449.90 in the account at the end of 5 years, and $449.90 of this will be interest.

Len's Solution: Using a financial application



I drew a timeline to represent the investment. I could see that I needed to determine the future value of the ten $500 deposits, each earning interest at the same rate but for a different number of compounding periods.

The regular payment amount is $500.
The payment frequency is semi-annual, or 2 times per year.
The number of payments is 10.
The payments are made at the end of each payment period.
The annual interest rate is 3.8%.
The compounding frequency is semi-annual, or 2 times per year.
The future value is unknown.

I decided to use the financial application on my calculator.

Future value $= 5449.896...$

Interest earned $= 5449.896... - 10(500)$
Interest earned $= 449.896...$

I determined the total interest earned by subtracting the total payments from the future value.

There will be $5449.90 in the account at the end of 5 years, and $449.90 of this will be interest.

$5449.90 seems reasonable because deposits of $500 made twice a year for 5 years would total $5000, without counting the interest.

Your Turn

Suppose that Darva's deposits were only $400 every 6 months instead of $500 every 6 months, and that the interest rate on her account remains 3.8%, compounded semi-annually. At the end of 5 years, how much less would the future value of the account be? How much interest would Darva earn?

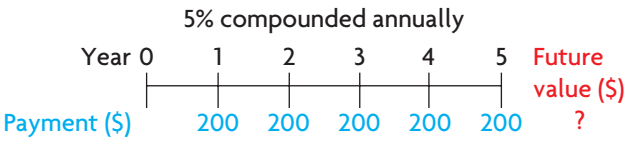
EXAMPLE 2 Comparing a regular payment investment with a single payment investment

Adam made a $200 payment at the end of each year into an investment that earned 5%, compounded annually. Blake made a single investment at 5%, compounded annually. At the end of 5 years, their future values were equal.

a) What was their future value?
b) What principal amount did Blake invest 5 years ago?
c) Who earned more interest? Why?

Eva's Solution

a) Adam's investment:



I drew a timeline to help me visualize the problem.

I could see that there would be a lot of calculations if I determined the future value of each payment and then determined the sum, so I decided to use technology.

The regular payment amount is $200.
The payment frequency is annual, or 1 time per year.
The number of payments is 5.
The payments are made at the end of each payment period.
The annual interest rate is 5%.
The compounding frequency is annual, or 1 time per year.
The future value is unknown.

I used the financial application in my spreadsheet software to determine the future value of the investment.

The future value of both investments was $1105.126....

b) Blake's investment:

The present value is unknown.
The annual interest rate is 5%.
The compounding frequency is annual, or 1 time per year.
The term (in years) is 5.
The future value is \$1105.126....

Present value = 865.895...
Five years ago, Blake invested \$865.90.

I predicted that Blake's investment would be less than the total amount Adam invested (\$1000), because Blake's entire investment had 5 years to earn interest.

I used the financial application again, but this time I used it to determine the present value of Blake's single payment investment.

c) Adam's investment:

Interest earned = Future value − Present value
Interest earned = 1105.126... − 5(200)
Interest earned = 105.126...

Blake's investment:

Interest earned = Future value − Present value
Interest earned = 1105.126... − 865.895...
Interest earned = 239.230...

Difference in interest earned = 239.230... − 105.126...

Blake earned \$134.10 more interest.

Blake invested less but earned more interest, even though the interest rate, compounding frequency, and term were the same.

Blake's principal of \$865.90 earned 5% interest, compounded annually, for 5 years. In contrast, only \$200 of Adam's investment earned 5% interest, compounded annually, for 5 years. The second \$200 payment earned interest for only 4 years, the third \$200 payment earned interest for only 3 years, and so on.

Adam's investment was like a series of individual investments, each with a term that was 1 year shorter than the term before it.

Your Turn

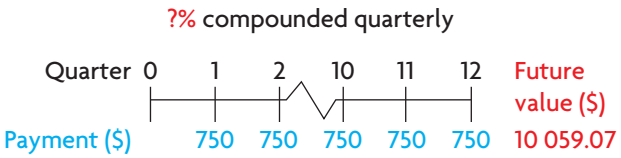
a) Suppose that Blake had invested the same total amount as Adam, but as a single investment. Predict how their future values would compare. Explain and then verify your prediction.

b) What rate of interest would Blake's \$1000 investment need to earn for it to have the same future value as Adam's investment?

EXAMPLE 3 Determining the interest rate of a regular payment investment

Jeremiah deposits $750 into an investment account at the end of every 3 months. Interest is compounded quarterly, the term is 3 years, and the future value is $10 059.07. What annual rate of interest does Jeremiah's investment earn?

Jeremiah's Solution



I drew a timeline to organize the given information and the information I needed so that I could determine the annual interest rate.

The regular payment amount is $750.
The payment frequency is 4 times per year.
The number of payments is 3(4) or 12.
The payments are made at the end of each payment period.
The annual interest rate is unknown.
The compounding frequency is 4 times per year.

I entered these values into the financial application on my graphing calculator and then solved for the annual interest rate.

The future value is $10 059.07.

Annual interest rate $= 0.080...$

The annual interest rate on Jeremiah's investment is 8.00%.

Your Turn

Predict whether the interest rate would be greater or less than 8% in each situation below, assuming the term remains 3 years, and the future value remains $10 059.07. Explain and then verify your prediction.

a) Jeremiah made payments of $800 every 3 months.
b) Jeremiah made payments of $750 every 6 months, and interest was compounded semi-annually.

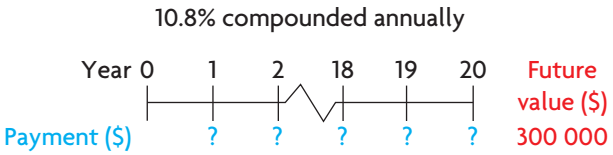
EXAMPLE 4 Determining the regular payment amount of an investment

Celia wants to have $300 000 in 20 years so that she can retire. Celia has found a trust account that earns a fixed rate of 10.8%, compounded annually.

a) What regular payments must Celia make at the end of each year to meet her goal of $300 000?
b) How much interest will she earn over the 20 years?

Liv's Solution

a)



I drew a timeline to organize the given information and the information I needed so that I could determine the payment amount.

The regular payment amount is unknown.
The payment frequency is 1 time per year.
The number of payments is 20.
The payments are made at the end of each payment period.
The annual interest rate is 10.8%.
The compounding frequency is 1 time per year.
The future value is $300 000.

I entered these values into the financial application on my graphing calculator and then solved for the payment amount.

Regular payment amount $= 4781.089\ldots$

Celia must make annual payments of $4781.09.

b) $I = 300\,000 - (20)(4781.089...)$
$I = 204\,378.20$

To determine the total interest earned, I subtracted the total of the 20 payments from the future value.

Celia will earn $204 378.20 in interest on 20 regular payments of $4781.09.

Your Turn

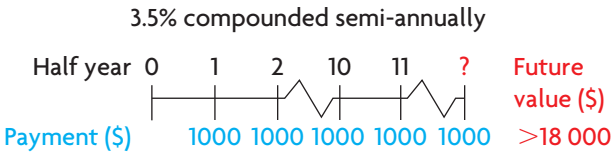
Predict whether Celia's payment amount would be greater than or less than $4781.09 in each situation. Explain and then verify your prediction.

a) Celia wants a future value of $500 000.
b) The payment frequency is every 6 months for 20 years (assume compounding is also every 6 months).
c) The interest rate is 7.8%, compounded annually.
d) Celia wants to have $300 000 in 10 years.

EXAMPLE 5 Determining the term of a regular payment investment

On Luis's 20th birthday, he started making regular $1000 payments into an investment account at the end of every 6 months. He wants to save for a down payment on a home. His investment earns 3.5%, compounded semi-annually. At what age will he have more than $18 000?

Greg's Solution



I drew a timeline to organize the given information and the information I needed. I could see that I had to figure out the number of semi-annual compounding periods in order to determine the number of years.

The regular payment amount is $1000.
The payment frequency is 2 times per year.
The number of payments is unknown.
The payments are made at the end of each payment period.
The annual interest rate is 3.5%.
The compounding frequency is 2 times per year.
The future value is $18 000.

I entered these values into the financial application on my graphing calculator and then solved for the number of payments.

Number of payments = 15.784...

The number of payments must be a whole number. Since Luis needs more than $18 000, I rounded up to 16 payments.

16 payments will result in more than $18 000.
16 payments made semi-annually is 8 years.

Luis will be 28 by the time his investment is worth more than $18 000.

Your Turn

What payments would Luis have to make if he wanted exactly $18 000 in 8 years?

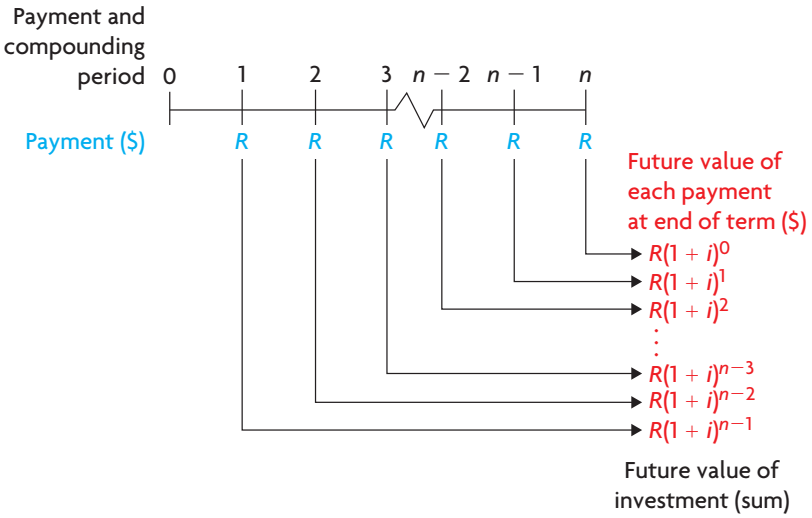
In Summary

Key Ideas

- For an investment that involves a series of equal deposits or payments made at regular intervals, the future value is the sum of all the regular payments plus the accumulated interest.
- The future value of an investment involving regular payments can be found by determining the sum of all the future values of each regular payment:

  $A = R(1 + i)^0 + R(1 + i)^1 + R(1 + i)^2 + R(1 + i)^3 + R(1 + i)^{n-1}$

  where A is the amount, or future value of the investment;
  R is the regular payment;
  i is the interest rate per compounding period, expressed as a decimal; and
  n is the number of compounding periods.



- Problems that involve the future value of an investment with regular payments can be solved using spreadsheet software or using the financial application on a graphing calculator or spreadsheet.

Need to Know

- The future value of a single deposit has a greater future value than a series of regular payments of the same total amount.
- Small deposits over a long term can have a greater future value than large deposits over a short term because there is more time for compound interest to be earned.

CHECK Your Understanding

1. Determine the future value of each investment.

	Regular Payment ($)	Interest Rate (%)	Compounding and Payment Frequency	Term (years)
a)	200	4.8	monthly	50
b)	1750	5.6	semi-annually	20
c)	50	8.4	quarterly	40
d)	5500	6.5	semi-annually	12

2. Determine the unknown values.

	Regular Payment ($)	Interest Rate (%)	Compounding and Payment Frequency	Term (years)	Future Value ($)
a)	100	?	monthly	6	7800.61
b)	?	3.50	semi-annually	7	3927.38
c)	20 000	4.75	quarterly	?	1 080 978.04

3. Darlene has invested $350 at the end of each month, at 7.2% compounded monthly, for 18 years. What is the investment's future value? How much interest has she earned?

4. Predict which investment will earn more interest. Explain and then verify your prediction.

A. $5000 invested at 6%, compounded annually, for 5 years

B. $1000 invested every year at 6%, compounded annually, for 5 years

PRACTISING

5. Fraser, who is 16 years old, wants to buy a car when he is 21. He deposits $600 every 3 months, from his part-time job, in a savings account that earns 6.8%, compounded quarterly. How much money will he have to buy his car when he is 21? How much interest will he have earned?

6. Zoey deposited the same amount of money at the end of each month for 2 years in a savings account that earned 6% interest, compounded monthly. She ended up with $5000. How much did Zoey deposit each month?

7. a) Jayne plans to retire in 35 years, when she is 55, and hopes to have $1 000 000 saved. For each investment option below, how much does she need to invest at the end of each month to reach her goal?

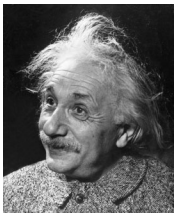
i) 14.6% compounded monthly

ii) 6.9% compounded monthly

b) Compare the rates of return for options i) and ii). Which option should she choose?

Math in Action

Wonder of the World?



Einstein allegedly claimed that "the most powerful force in the universe is compound interest." Certainly, banks and other financial institutions use compound interest as a cornerstone for their success in the business of making money. How do they do this? In simple terms, they earn the difference between what they charge their customers to borrow money and what they pay in interest to investors.

- Research the interest rate for a 5-year GIC and the interest rate for a 5-year loan offered by one or more financial institutions.
- Determine the difference between the amount of interest each institution would receive on a $50 000 loan and the amount of interest it would pay on a $50 000 investment.
- Make a conjecture about the difference between loan interest rates and investment interest rates, and explain your thinking.

8. Aaron and Casey started investing at the same time. Aaron makes payments of $25 at the end of each month into an investment that earns 4.2%, compounded monthly. Casey made a single payment into an investment that earns 4.2%, compounded annually.

a) At the end of 5 years, what is the future value of Aaron's investment?

b) Casey's investment has the same future value as Aaron's in 5 years. How much principal did Casey invest?

c) Predict whose investment will be worth more at the end of 10 years. Explain and then verify your prediction.

9. What interest rate, compounded monthly, is required to make monthly payments of $500 grow to $35 000 in 5 years?

10. How long will it take for $1000 payments every 6 months to grow to more than $10 000 if the interest rate is 7.5%, compounded semi-annually?

11. Dee deposited $1000 at the end of each month into a 5-year investment that earned 4%, compounded monthly. Pete deposited half as much each month for twice as long, but at the same interest rate.

a) Determine the future value of each investment.

b) Explain why the future values are different even though they invested the same amount.

12. For 2 years, Trey deposited $600 at the end of every 3 months into an investment that earned 3.6%, compounded quarterly. Over the same 2 years, Sam deposited $2400 annually into an investment that earned 3.8%, compounded annually. Whose investment is worth more, and by how much?

13. Miguel wants to buy an entertainment system as a wedding gift for his sister. He estimates that when she marries a year from now, the system will cost $2499, plus 13% tax. Will Miguel have enough money to buy the system if he deposits $225 at the end of each month into an account that earns 3.6%, compounded monthly? Explain.

14. Both Jill and Vaughn set up a 30-year investment and want to have $250 000 at the end of the term. Jill's bank pays a rate of 7.4%, compounded monthly. Vaughn is investing through the company he works for, at a rate of 11.6%, compounded monthly.

a) How much more does Jill need to invest than Vaughn over the 30 years?

b) Vaughn decides to make the same payments at the end of each month as Jill. How much will he have at the end of the 30 years?

15. Tim has found his dream sailboat in Victoria. It is selling for $120 000. He intends to sell his current sailboat in 2 years for $50 000. During those 2 years, Tim is going to put $300 at the end of each week into an investment account that earns 10.5%, compounded weekly. Will he have enough to buy his dream sailboat? Explain.

16. Andrew had two investment options:

A. A one-time deposit of $1200, which will earn 6%, compounded monthly, for 10 years

B. Deposits of $10 at the end of each month, which will earn 6%, compounded monthly, for 10 years

a) How are the investments the same? How are they different?

b) On the same grid, graph both investments. Compare their values over time. What do you notice?

Closing

17. Quinn is a server at a restaurant. He plans to deposit what he collects in tips each month for 5 years into an account that earns 5%, compounded monthly. He wonders how much he will be able to save altogether. Could you solve this as an investment involving regular payments? Explain.

Extending

18. Farah is a filmmaker. She paid $5000 for a computer that she can use to edit videos. She plans to sell this computer and buy a new one, with upgraded hardware, for $5000 in 2 years. In 2 years, her current computer will be worth about 25% of its current value. Farah started saving for her next computer by investing at the end of each month in an account that earns 3.6%/a, compounded monthly. How much should each payment be, so that she can pay cash for her next computer in 2 years?

19. When Blythe's adopted son was 6 months old, Blythe started depositing $50 at the end of each 6 months into an account that earned 5%, compounded semi-annually. When her son turned 20, she stopped making deposits but kept the money in the account. On her son's 30th birthday, she signed the account over to her son to help cover the start-up costs of his new business. How much did Blythe's son receive?

20. Pat wants his next vehicle to be environmentally friendly. He predicts that the vehicle will cost $46 000, plus 5% tax. He plans to invest $550 at the end of each month for the next 5 years into an investment that earns 9.8%, compounded monthly. If he sells his old car for $4000, will he have the purchase price of the new car? Explain.