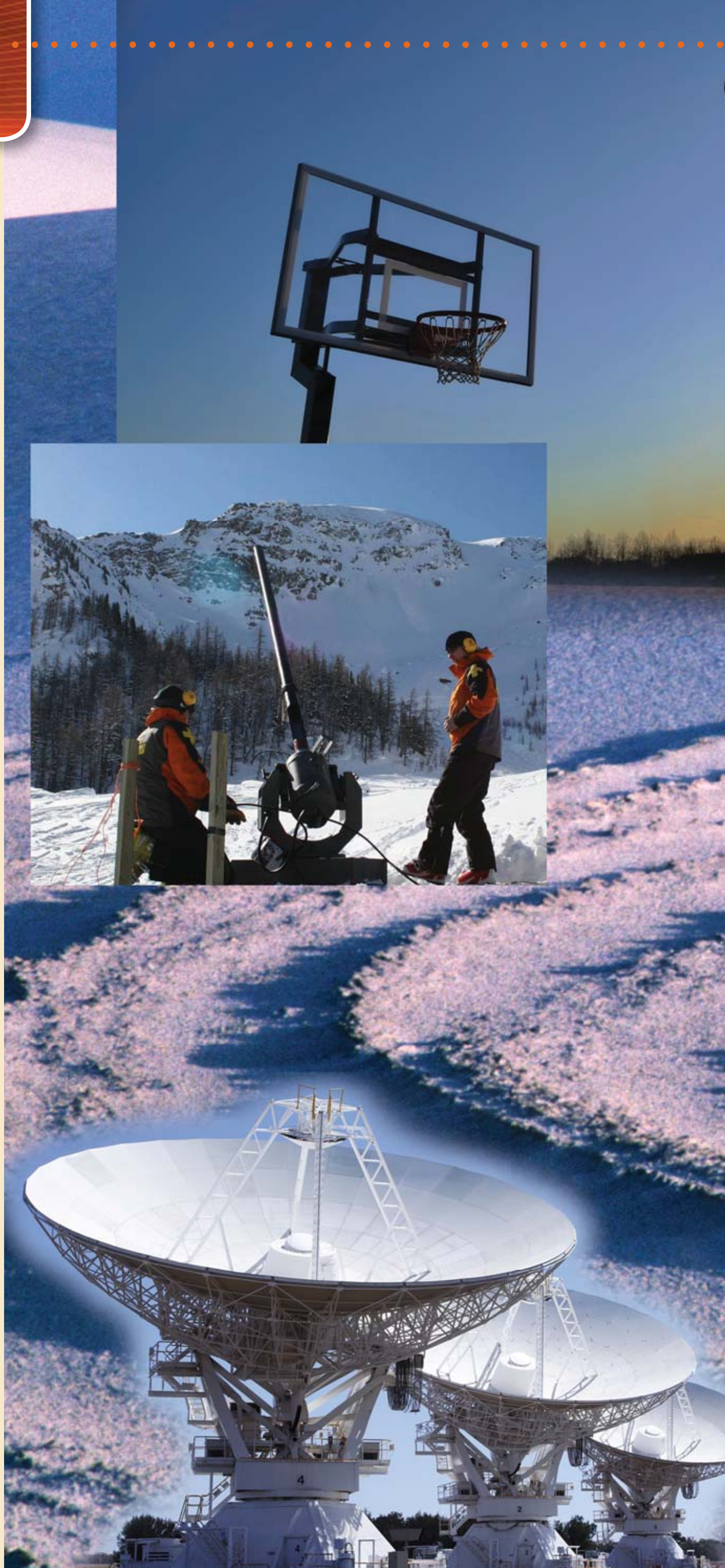
Unit 2

Quadratics

Quadratic functions and their applications can model a large part of the world around us. Consider the path of a basketball after it leaves the shooter's hand. Think about how experts determine when and where the explosive shells used in avalanche control will land, as they attempt to make snowy areas safe for everyone. Why do satellite dishes and suspension bridges have the particular shapes that they do? You can model these and many other everyday situations mathematically with quadratic functions. In this unit, you will investigate the nature of quadratic equations and quadratic functions. You will also apply them to model real-world situations and solve problems.



Looking Ahead

In this unit, you will solve problems involving...

- equations and graphs of quadratic functions
- quadratic equations

Unit 2 Project Quadratic Functions in Everyday Life

In this project, you will explore quadratic functions that occur in everyday life such as sports, science, art, architecture, and nature.

In Chapter 3, you will find information and make notes about quadratic functions in familiar situations. In Chapter 4, you will focus specifically on the subject of avalanche control.

At the end of the unit, you will choose between two options:

- You may choose to examine real-world situations that you can model using quadratic functions. For this option, you will mathematically determine the accuracy of your model. You will also investigate reasons for the quadratic nature of the situation.
- You may choose to apply the skills you have learned in this unit to the subject of projectile motion and the use of mathematics in avalanche control.

For either option, you will showcase what you have learned about quadratic relationships by modelling and analysing real situations involving quadratic functions or equations. You will also prepare a written summary of your observations.

CHAPTER 3

Quadratic Functions

Digital images are everywhere—on computer screens, digital cameras, televisions, mobile phones, and more. Digital images are composed of many individual *pixels*, or *picture elements*. Each pixel is a single dot or square of colour. The total number of pixels in a two-dimensional image is related to its dimensions. The more pixels an image has, the greater the quality of the image and the higher the resolution.

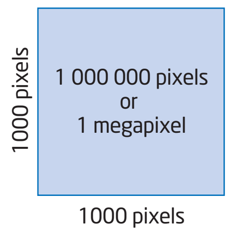
If the image is a square with a side length of x pixels, then you can represent the total number of pixels, p, by the function $p(x) = x^2$. This is the simplest example of a quadratic function. The word *quadratic* comes from the word *quadratum*, a Latin word meaning *square*. The term *quadratic* is used because a term like x^2 represents the area of a square of side length x.

Quadratic functions occur in a wide variety of real-world situations. In this chapter, you will investigate quadratic functions and use them in mathematical modelling and problem solving.

Did You Know?

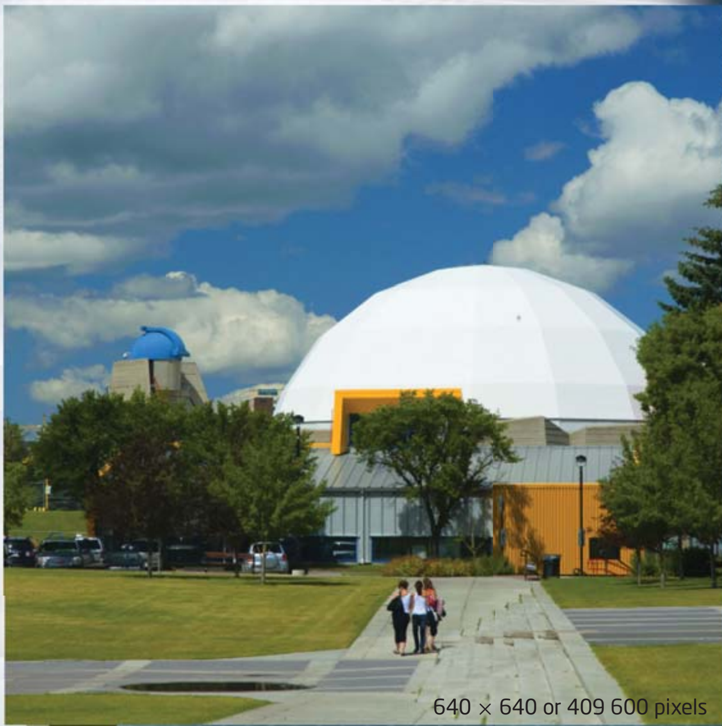
The word *pixel* comes from combining *pix* for picture and *el* for element.

The term *megapixel* is used to refer to one million pixels. Possible dimensions for a one-megapixel image could be 1000 pixels by 1000 pixels or 800 pixels by 1250 pixels—in both cases the total number of pixels is 1 million. Digital cameras often give a value in megapixels to indicate the maximum resolution of an image.

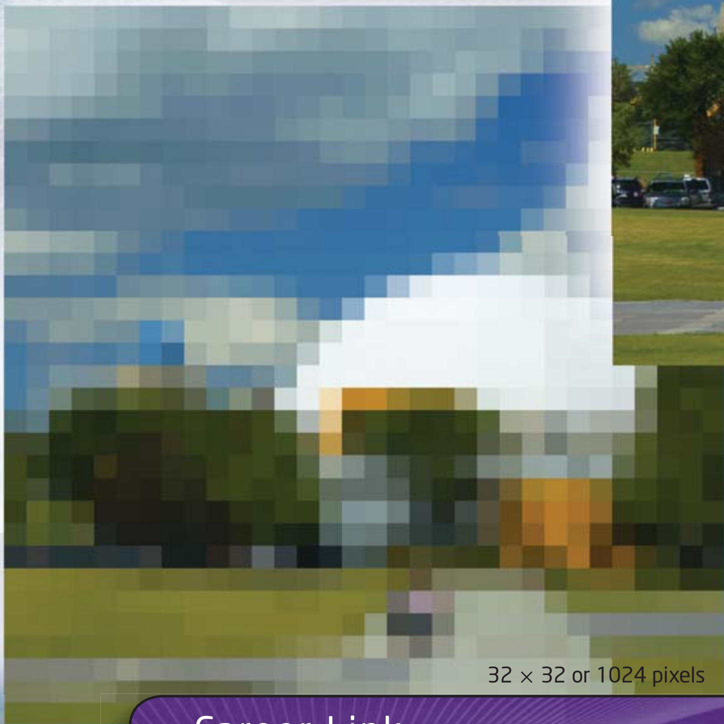


Key Terms

- quadratic function
- parabola
- vertex (of a parabola)
- minimum value
- maximum value
- axis of symmetry
- vertex form (of a quadratic function)
- standard form (of a quadratic function)
- completing the square



640 × 640 or 409 600 pixels



32 × 32 or 1024 pixels

8 × 8 or 64 pixels

Career Link

SpaceShipTwo is a sub-orbital space-plane designed to carry space tourists at a cost of hundreds of thousands of dollars per ride. Designers have developed a craft that will carry six passengers and two pilots to a height of 110 km above Earth and reach speeds of 4200 km/h. Engineers use quadratic functions to optimize the vehicle's storage capacity, create re-entry simulations, and help develop the structural design of the space-plane itself. Flights are due to begin no earlier than 2011.

Web Link

To learn more about aerospace design, go to www.mhrprecalc11.ca and follow the links.

3.1

Investigating Quadratic Functions in Vertex Form

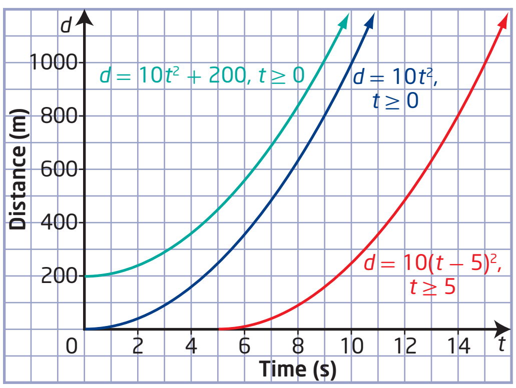
Focus on…

- identifying quadratic functions in vertex form
- determining the effect of a, p, and q on the graph of $y = a(x - p)^2 + q$
- analysing and graphing quadratic functions using transformations

The Bonneville Salt Flats is a large area in Utah, in the United States, that is a remnant of an ancient lake from glacial times. The surface is extremely flat, smooth, and hard, making it an ideal place for researchers, racing enthusiasts, and automakers to test high-speed vehicles in a safer manner than on a paved track. Recently, the salt flats have become the site of an annual time-trial event for alternative-fuel vehicles. At the 2007 event, one major automaker achieved a top speed of 335 km/h with a hydrogen-powered fuel-cell car, the highest-ever recorded land speed at the time for any fuel-cell-powered vehicle.

Suppose three vehicles are involved in speed tests. The first sits waiting at the start line in one test lane, while a second sits 200 m ahead in a second test lane. These two cars start accelerating constantly at the same time. The third car leaves 5 s later from the start line in a third lane.

The graph shows a function for the distance travelled from the start line for each of the three vehicles. How are the algebraic forms of these functions related to each other?



Did You Know?

The fuel cells used in this vehicle are manufactured by Ballard Power Systems, based in Burnaby, British Columbia. They have been developing hydrogen fuel cells for over 20 years.

Web Link

For more information about the Bonneville Salt Flats and about fuel-cell-powered vehicles, go to www.mhrprecalc11.ca and follow the links.

Investigate Graphs of Quadratic Functions in Vertex Form

Materials

- grid paper or graphing technology

Part A: Compare the Graphs of $f(x) = x^2$ and $f(x) = ax^2$, $a \neq 0$

1. a) Graph the following functions on the same set of coordinate axes, with or without technology.

$f(x) = x^2$ $\qquad$ $f(x) = -x^2$
$f(x) = 2x^2$ $\qquad$ $f(x) = -2x^2$
$f(x) = \frac{1}{2}x^2$ $\qquad$ $f(x) = -\frac{1}{2}x^2$

b) Describe how the graph of each function compares to the graph of $f(x) = x^2$, using terms such as *narrower*, *wider*, and *reflection*.

c) What relationship do you observe between the parameter, a, and the shape of the corresponding graph?

2. a) Using a variety of values of a, write several of your own functions of the form $f(x) = ax^2$. Include both positive and negative values.

b) Predict how the graphs of these functions will compare to the graph of $f(x) = x^2$. Test your prediction.

Reflect and Respond

3. Develop a rule that describes how the value of a in $f(x) = ax^2$ changes the graph of $f(x) = x^2$ when a is

a) a positive number greater than 1

b) a positive number less than 1

c) a negative number

Part B: Compare the Graphs of $f(x) = x^2$ and $f(x) = x^2 + q$

4. a) Graph the following functions on the same set of coordinate axes, with or without technology.

$f(x) = x^2$
$f(x) = x^2 + 4$
$f(x) = x^2 - 3$

b) Describe how the graph of each function compares to the graph of $f(x) = x^2$.

c) What relationship do you observe between the parameter, q, and the location of the corresponding graph?

5. a) Using a variety of values of q, write several of your own functions of the form $f(x) = x^2 + q$. Include both positive and negative values.

b) Predict how these functions will compare to $f(x) = x^2$. Test your prediction.

Reflect and Respond

6. Develop a rule that describes how the value of q in $f(x) = x^2 + q$ changes the graph of $f(x) = x^2$ when q is

a) a positive number $\qquad$ **b)** a negative number

Part C: Compare the Graphs of $f(x) = x^2$ and $f(x) = (x - p)^2$

7. a) Graph the following functions on the same set of coordinate axes, with or without technology.

$f(x) = x^2$
$f(x) = (x - 2)^2$
$f(x) = (x + 1)^2$

b) Describe how the graph of each function compares to the graph of $f(x) = x^2$.

c) What relationship do you observe between the parameter, p, and the location of the corresponding graph?

8. a) Using a variety of values of p, write several of your own functions of the form $f(x) = (x - p)^2$. Include both positive and negative values.

b) Predict how these functions will compare to $f(x) = x^2$. Test your prediction.

Reflect and Respond

9. Develop a rule that describes how the value of p in $f(x) = (x - p)^2$ changes the graph of $f(x) = x^2$ when p is

a) a positive number **b)** a negative number

Link the Ideas

quadratic function

- a function f whose value $f(x)$ at x is given by a polynomial of degree two
- for example, $f(x) = x^2$ is the simplest form of a quadratic function

parabola

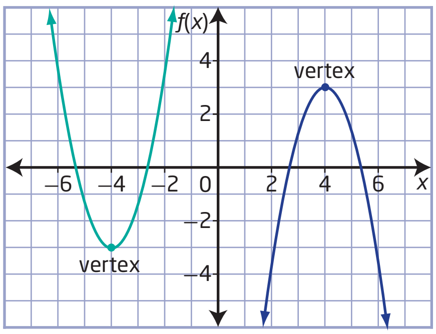
- the symmetrical curve of the graph of a quadratic function

vertex (of a parabola)

- the lowest point of the graph (if the graph opens upward) or the highest point of the graph (if the graph opens downward)

The graph of a **quadratic function** is a **parabola**.

When the graph opens upward, the **vertex** is the lowest point on the graph. When the graph opens downward, the vertex is the highest point on the graph.



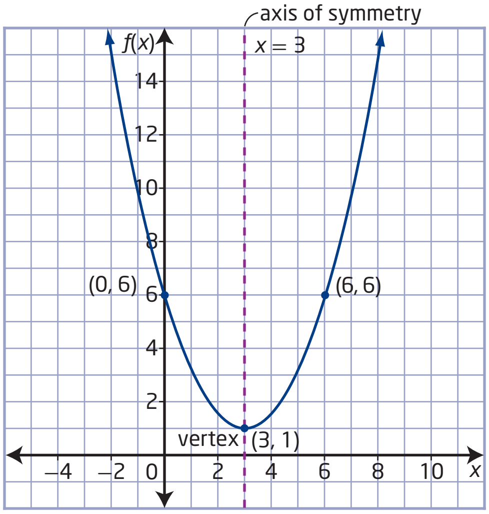
When using function notation, the values for $f(x)$ are often considered the same as the values for y.

The y-coordinate of the vertex is called the **minimum value** if the parabola opens upward or the **maximum value** if the parabola opens downward

The parabola is symmetric about a line called the **axis of symmetry**. This line divides the function graph into two parts so that the graph on one side is the mirror image of the graph on the other side. This means that if you know a point on one side of the parabola, you can determine a corresponding point on the other side based on the axis of symmetry.

The axis of symmetry intersects the parabola at the vertex.

The x-coordinate of the vertex corresponds to the equation of the axis of symmetry.



Quadratic functions written in **vertex form**, $f(x) = a(x - p)^2 + q$, are useful when graphing the function. The vertex form tells you the location of the vertex (p, q) as well as the shape of the parabola and the direction of the opening.

You can examine the parameters a, p, and q to determine information about the graph.

minimum value (of a function)

- the least value in the range of a function
- for a quadratic function that opens upward, the y-coordinate of the vertex

maximum value (of a function)

- the greatest value in the range of a function
- for a quadratic function that opens downward, the y-coordinate of the vertex

axis of symmetry

- a line through the vertex that divides the graph of a quadratic function into two congruent halves
- the x-coordinate of the vertex defines the equation of the axis of symmetry

vertex form (of a quadratic function)

- the form $y = a(x - p)^2 + q$, or $f(x) = a(x - p)^2 + q$, where a, p, and q are constants and $a \neq 0$

The Effect of Parameter a in $f(x) = ax^2$ on the Graph of $f(x) = x^2$

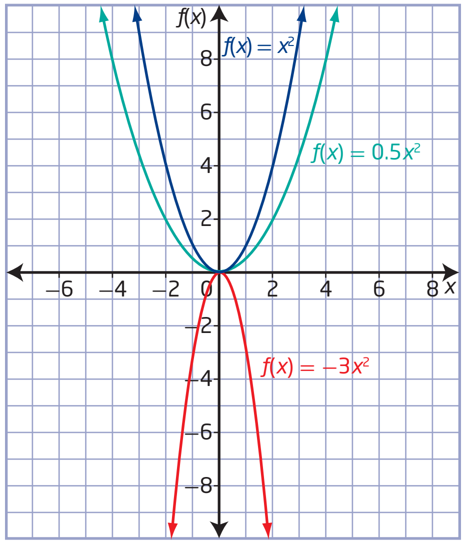
Consider the graphs of the following functions:

$f(x) = x^2$

$f(x) = 0.5x^2$ — The parabola is wider in relation to the y-axis than $f(x) = x^2$ and opens upward.

$f(x) = -3x^2$ — The parabola is narrower in relation to the y-axis than $f(x) = x^2$ and opens downward.

- Parameter a determines the orientation and shape of the parabola.
- The graph opens upward if $a > 0$ and downward if $a < 0$.
- If $-1 < a < 1$, the parabola is wider compared to the graph of $f(x) = x^2$.
- If $a > 1$ or $a < -1$, the parabola is narrower compared to the graph of $f(x) = x^2$.



The Effect of Parameter q in $f(x) = x^2 + q$ on the Graph of $f(x) = x^2$

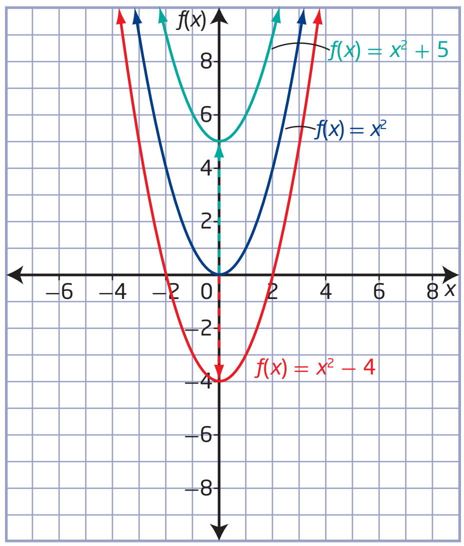
Consider the graphs of the following functions:

$f(x) = x^2$

$f(x) = x^2 + 5$ — The graph is translated 5 units up.

$f(x) = x^2 - 4$ — The graph is translated 4 units down.

- Parameter q translates the parabola vertically q units relative to the graph of $f(x) = x^2$.
- The y-coordinate of the parabola's vertex is q.



The Effect of Parameter p in $f(x) = (x - p)^2$ on the Graph of $f(x) = x^2$

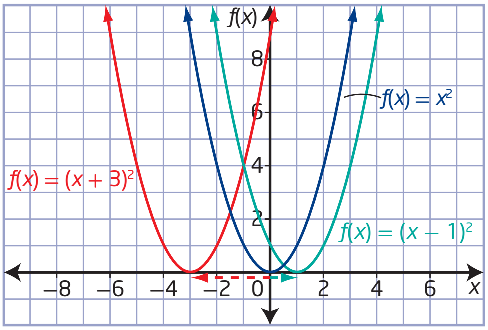
Consider the graphs of the following functions:

$f(x) = x^2$

$f(x) = (x - 1)^2$ — Since $p = +1$, the graph is translated 1 unit right.

$f(x) = (x + 3)^2$ — Since $p = -3$, the graph is translated 3 units left.

- Parameter p translates the parabola horizontally p units relative to the graph of $f(x) = x^2$.
- The x-coordinate of the parabola's vertex is p.
- The equation of the axis of symmetry is $x - p = 0$ or $x = p$.



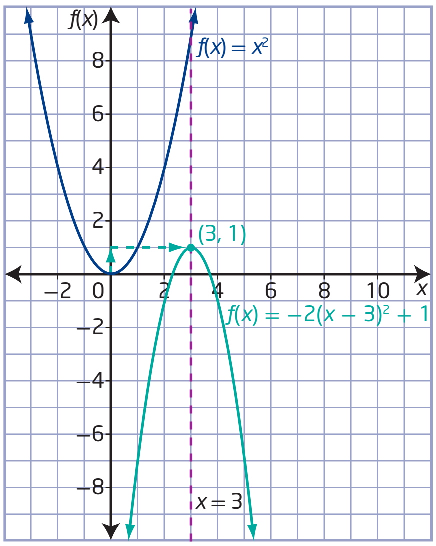
Combining Transformations

Consider the graphs of the following functions:

$f(x) = x^2$

$f(x) = -2(x - 3)^2 + 1$

- The parameter $a = -2$ determines that the parabola opens downward and is narrower than $f(x) = x^2$.
- The vertex of the parabola is located at (3, 1) and represents a horizontal translation of 3 units right and a vertical translation of 1 unit up relative to the graph of $f(x) = x^2$.
- The equation of the axis of symmetry is $x - 3 = 0$ or $x = 3$.



In general:

- The sign of a defines the direction of opening of the parabola. When $a > 0$, the graph opens upward, and when $a < 0$, the graph opens downward.
- The parameter a also determines how wide or narrow the graph is compared to the graph of $f(x) = x^2$.
- The point (p, q) defines the vertex of the parabola.
- The equation $x = p$ defines the axis of symmetry.

Example 1

Sketch Graphs of Quadratic Functions in Vertex Form

Determine the following characteristics for each function.
- the vertex
- the domain and range
- the direction of opening
- the equation of the axis of symmetry

Then, sketch each graph.

a) $y = 2(x + 1)^2 - 3$ **b)** $y = -\frac{1}{4}(x - 4)^2 + 1$

Solution

a) Use the values of a, p, and q to determine some characteristics of $y = 2(x + 1)^2 - 3$ and sketch the graph.

$$y = 2(x + 1)^2 - 3$$

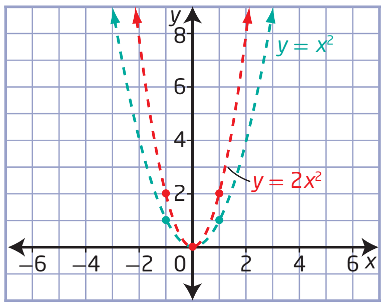
$a = 2$ $p = -1$ $q = -3$

Since $p = -1$ and $q = -3$, the vertex is located at $(-1, -3)$.
Since $a > 0$, the graph opens upward. Since $a > 1$, the parabola is narrower compared to the graph of $y = x^2$.
Since $q = -3$, the range is $\{y \mid y \geq -3, y \in \mathrm{R}\}$.
The domain is $\{x \mid x \in \mathrm{R}\}$.
Since $p = -1$, the equation of the axis of symmetry is $x = -1$.

Method 1: Sketch Using Transformations

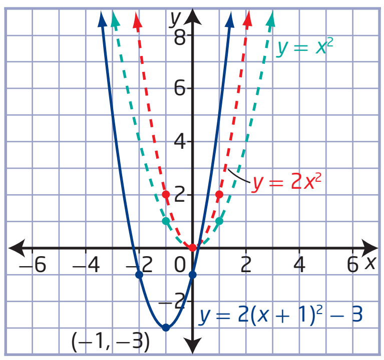
Sketch the graph of $y = 2(x + 1)^2 - 3$ by transforming the graph of $y = x^2$.
- Use the points (0, 0), (1, 1), and (−1, 1) to sketch the graph of $y = x^2$.
- Apply the change in width.

When using transformations to sketch the graph, you should deal with parameter a first, since its reference for wider or narrower is relative to the y-axis.



- Translate the graph.

How are p and q related to the direction of the translations and the location of the vertex?



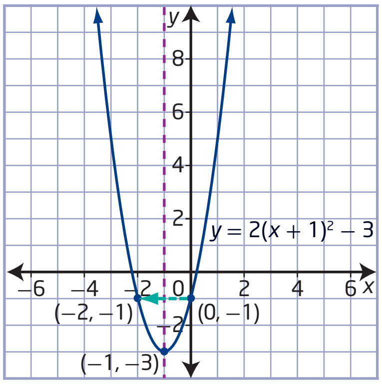
Method 2: Sketch Using Points and Symmetry

- Plot the coordinates of the vertex, $(-1, -3)$, and draw the axis of symmetry, $x = -1$.
- Determine the coordinates of one other point on the parabola.

The y-intercept is a good choice for another point.
Let $x = 0$.
$y = 2(0 + 1)^2 - 3$
$y = 2(1)^2 - 3$
$y = -1$
The point is $(0, -1)$.

For any point other than the vertex, there is a corresponding point that is equidistant from the axis of symmetry. In this case, the corresponding point for $(0, -1)$ is $(-2, -1)$.

Plot these two additional points and complete the sketch of the parabola.



b) For the quadratic function $y = -\frac{1}{4}(x - 4)^2 + 1$, $a = -\frac{1}{4}$, $p = 4$, and $q = 1$.
The vertex is located at $(4, 1)$.
The graph opens downward and is wider than the graph $y = x^2$.
The range is $\{y \mid y \leq 1, y \in R\}$.
The domain is $\{x \mid x \in R\}$.
The equation of the axis of symmetry is $x = 4$.

Sketch the graph of $y = -\frac{1}{4}(x - 4)^2 + 1$ by using the information from the vertex form of the function.

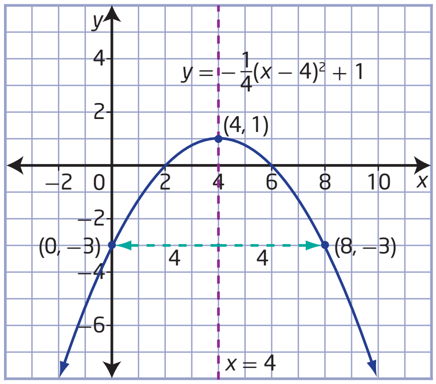
- Plot the vertex at (4, 1).
- Determine a point on the graph. For example, determine the y-intercept by substituting $x = 0$ into the function.

$$y = -\frac{1}{4}(0 - 4)^2 + 1$$
$$y = -\frac{1}{4}(-4)^2 + 1$$
$$y = -4 + 1$$
$$y = -3$$

The point (0, −3) is on the graph.

For any point other than the vertex, there is a corresponding point that is equidistant from the axis of symmetry. In this case, the corresponding point of (0, −3) is (7, −3).

Plot these two additional points and complete the sketch of the parabola.



How are the values of y affected when a is $-\frac{1}{4}$?

How are p and q related to the direction of the translations and location of the vertex?

How is the shape of the curve related to the value of a?

Your Turn

Determine the following characteristics for each function.
- the vertex
- the domain and range
- the direction of opening
- the equations of the axis of symmetry

Then, sketch each graph.

a) $y = \frac{1}{2}(x - 2)^2 - 4$

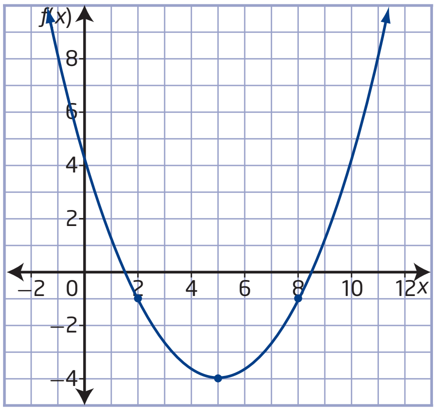
b) $y = -3(x + 1)^2 + 3$

Example 2

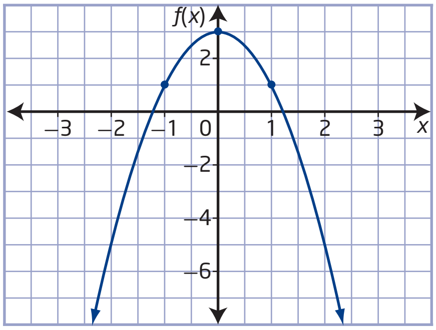
Determine a Quadratic Function in Vertex Form Given Its Graph

Determine a quadratic function in vertex form for each graph.

a)



b)



Solution

a) Method 1: Use Points and Substitution

You can determine the equation of the function using the coordinates of the vertex and one other point.

The vertex is located at (5, −4), so $p = 5$ and $q = -4$. The graph opens upward, so the value of a is greater than 0.

Express the function as

$f(x) = a(x - p)^2 + q$

$f(x) = a(x - 5)^2 + (-4)$

$f(x) = a(x - 5)^2 - 4$

Choose one other point on the graph, such as (2, −1). Substitute the values of x and y into the function and solve for a.

$f(x) = a(x - 5)^2 - 4$

$-1 = a(2 - 5)^2 - 4$

$-1 = a(-3)^2 - 4$

$-1 = a(9) - 4$

$-1 = 9a - 4$

$3 = 9a$

$\frac{1}{3} = a$

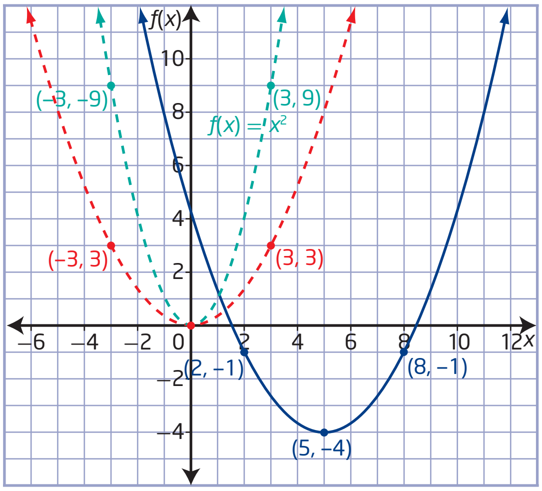
The quadratic function in vertex form is $f(x) = \frac{1}{3}(x - 5)^2 - 4$.

Method 2: Compare With the Graph of $f(x) = x^2$

The vertex is located at (5, −4), so $p = 5$ and $q = -4$. The graph involves a translation of 5 units to the right and 4 units down.

The graph opens upward, so the value of a is greater than 0.

To determine the value of a, undo the translations and compare the vertical distances of points on the non-translated parabola relative to those on the graph of $f(x) = x^2$.



How are the y-coordinates of the corresponding points on the two parabolas with a vertex at (0, 0) related?

Since the vertical distances are one third as much, the value of a is $\frac{1}{3}$. The red graph of $f(x) = \frac{1}{3}x^2$ has been stretched vertically by a factor of $\frac{1}{3}$ compared to the graph of $f(x) = x^2$.

Substitute the values $a = \frac{1}{3}$, $p = 5$, and $q = -4$ into the vertex form, $f(x) = a(x + p)^2 + q$.

The quadratic function in vertex form is $f(x) = \frac{1}{3}(x - 5)^2 - 4$.

b) You can determine the equation of the function using the coordinates of the vertex and one other point.

The vertex is located at (0, 3), so $p = 0$ and $q = 3$. The graph opens downward, so the value of a is less than 0.

Express the function as

$f(x) = a(x - p)^2 + q$
$f(x) = a(x - 0)^2 + 3$
$f(x) = ax^2 + 3$

Choose one other point on the graph, such as (1, 1). Substitute the values of x and y into the function and solve for a.

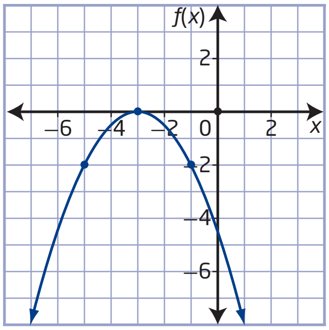
$f(x) = ax^2 + 3$
$1 = a(1)^2 + 3$
$1 = a + 3$
$-2 = a$

The quadratic function in vertex form is $f(x) = -2x^2 + 3$.

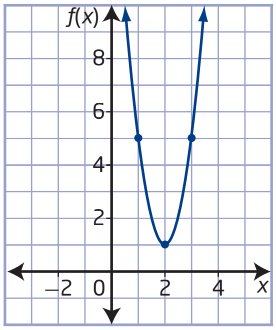
Your Turn

Determine a quadratic function in vertex form for each graph.

a)



b)



Example 3

Determine the Number of *x*-Intercepts Using *a* and *q*

Determine the number of *x*-intercepts for each quadratic function.

a) $f(x) = 0.8x^2 - 3$ **b)** $f(x) = 2(x - 1)^2$ **c)** $f(x) = -3(x + 2)^2 - 1$

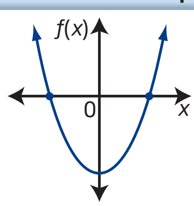
Solution

You can determine the number of *x*-intercepts if you know the location of the vertex and direction of opening. Visualize the general position and shape of the graph based on the values of *a* and *q*.

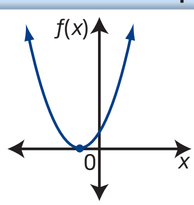
Determine the number of *x*-intercepts a quadratic function has by examining

- the value of *a* to determine if the graph opens upward or downward
- the value of *q* to determine if the vertex is above, below, or on the *x*-axis

a) $f(x) = 0.8x^2 - 3$

Value of *a*	Value of *q*	Visualize the Graph	Number of *x*-Intercepts
$a > 0$ the graph opens upward	$q < 0$ the vertex is below the *x*-axis		2 crosses the *x*-axis *twice*, since it opens *upward* from a vertex *below* the *x*-axis

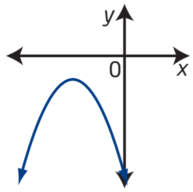
b) $f(x) = 2(x - 1)^2$

Value of *a*	Value of *q*	Visualize the Graph	Number of *x*-Intercepts
$a > 0$ the graph opens upward	$q = 0$ the vertex is on the *x*-axis		1 touches the *x*-axis *once*, since the vertex is *on* the *x*-axis

If you know that *q* is 0, does it matter what the value of *a* is?

Where on the parabola is the *x*-intercept in this case?

c) $f(x) = -3(x + 2)^2 - 1$

Value of a	Value of q	Visualize the Graph	Number of x-Intercepts
$a < 0$ the graph opens downward	$q < 0$ the vertex is below the x-axis		0 does not cross the x-axis, since it opens *down* from a vertex *below* the x-axis

Why does the value of p not affect the number of x-intercepts?

Your Turn

Determine the number of x-intercepts for each quadratic function without graphing.

a) $f(x) = 0.5x^2 - 7$ **b)** $f(x) = -2(x + 1)^2$ **c)** $f(x) = -\frac{1}{6}(x - 5)^2 - 11$

Example 4

Model Problems Using Quadratic Functions in Vertex Form

Did You Know?

The Lions' Gate Bridge carries over 60 000 vehicles per day on average. In 2009, the lights on the Lions' Gate Bridge were replaced with a new LED lighting system. The change is expected to reduce the power consumption on the bridge by 90% and significantly cut down on maintenance.

The deck of the Lions' Gate Bridge in Vancouver is suspended from two main cables attached to the tops of two supporting towers. Between the towers, the main cables take the shape of a parabola as they support the weight of the deck. The towers are 111 m tall relative to the water's surface and are 472 m apart. The lowest point of the cables is approximately 67 m above the water's surface.

a) Model the shape of the cables with a quadratic function in vertex form.

b) Determine the height above the surface of the water of a point on the cables that is 90 m horizontally from one of the towers. Express your answer to the nearest tenth of a metre.

Lion's Gate Bridge, Vancouver

Solution

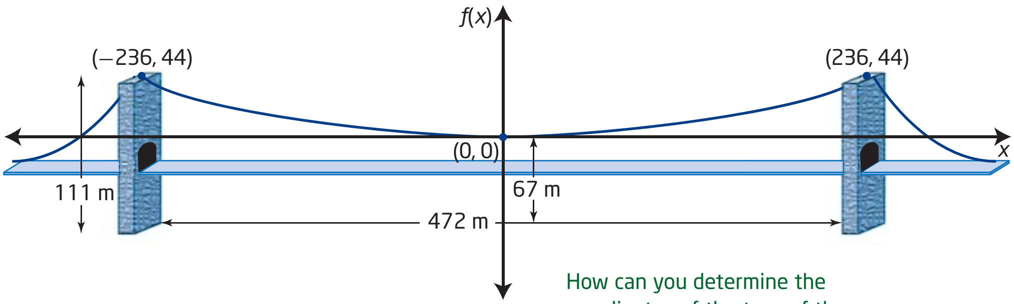
a) Draw a diagram and label it with the given information.

Let the vertex of the parabolic shape be at the low point of the cables. Consider this point to be the origin.

Why is this point the simplest to use as the origin?

Draw a set of axes. Let x and y represent the horizontal and vertical distances from the low point of the cables, respectively.

You can write a quadratic function if you know the coordinates of the vertex and one other point. The vertex is (0, 0), since it is the origin. Determine the coordinates of the point at the top of each tower from the given distances.



How can you determine the coordinates of the tops of the towers from the given information?

Since the vertex is located at the origin, (0, 0), no horizontal or vertical translation is necessary, and p and q are both zero. Therefore, the quadratic function is of the form $f(x) = ax^2$.

Substitute the coordinates of the top of one of the towers, (236, 44), into the equation $f(x) = ax^2$ and solve for a.

What other point could you use?

$$f(x) = ax^2$$
$$44 = a(236)^2$$
$$44 = a(55\ 696)$$
$$44 = 55\ 696a$$
$$\frac{44}{55\ 696} = a$$

a is $\frac{11}{13\ 924}$ in lowest terms.

Represent the shape of the cables with the following quadratic function.

$$f(x) = \frac{11}{13\ 924}x^2$$

What would the quadratic function be if the origin were placed at the water's surface directly below the lowest point of the cables?

What would it be if the origin were at water level at the base of one of the towers?

b) A point 90 m from one tower is 236 − 90, or 146 m horizontally from the vertex. Substitute 146 for x and determine the value of $f(146)$.

$$f(x) = \frac{11}{13\ 924}x^2$$
$$f(146) = \frac{11}{13\ 924}(146)^2$$
$$= \frac{11}{13\ 924}(21\ 316)$$
$$= 16.839\ldots$$

This is approximately 16.8 m above the low point in the cables, which are approximately 67 m above the water.

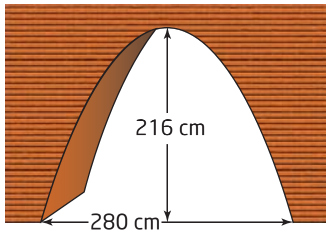
The height above the water is approximately 67 + 16.8, or 83.8 m.

Your Turn

Suppose a parabolic archway has a width of 280 cm and a height of 216 cm at its highest point above the floor.

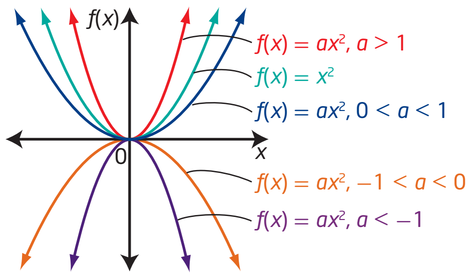
a) Write a quadratic function in vertex form that models the shape of this archway.

b) Determine the height of the archway at a point that is 50 cm from its outer edge.

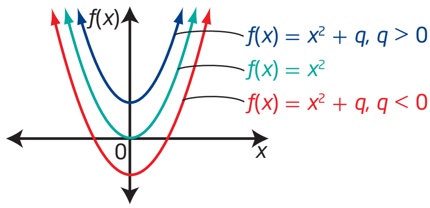


Key Ideas

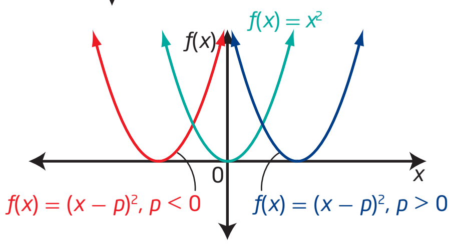
- For a quadratic function in vertex form, $f(x) = a(x - p)^2 + q$, $a \neq 0$, the graph:
  - has the shape of a parabola
  - has its vertex at (p, q)
  - has an axis of symmetry with equation $x = p$
  - is congruent to $f(x) = ax^2$ translated horizontally by p units and vertically by q units
- Sketch the graph of $f(x) = a(x - p)^2 + q$ by transforming the graph of $f(x) = x^2$.
  - The graph opens upward if $a > 0$.
  - If $a < 0$, the parabola is reflected in the x-axis; it opens downward.
  - If $-1 < a < 1$, the parabola is wider compared to the graph of $f(x) = x^2$.
  - If $a > 1$ or $a < -1$, the parabola is narrower compared to the graph of $f(x) = x^2$.



  - The parameter q determines the vertical position of the parabola.
  - If $q > 0$, then the graph is translated q units up.
  - If $q < 0$, then the graph is translated q units down.



  - The parameter p determines the horizontal position of the parabola.
  - If $p > 0$, then the graph is translated p units to the right.
  - If $p < 0$, then the graph is translated p units to the left.



- You can determine a quadratic function in vertex form if you know the coordinates of the vertex and at least one other point.
- You can determine the number of x-intercepts of the graph of a quadratic function using the value of a to determine if the graph opens upward or downward and the value of q to determine if the vertex is above, below, or on the x-axis.

Check Your Understanding

Practise

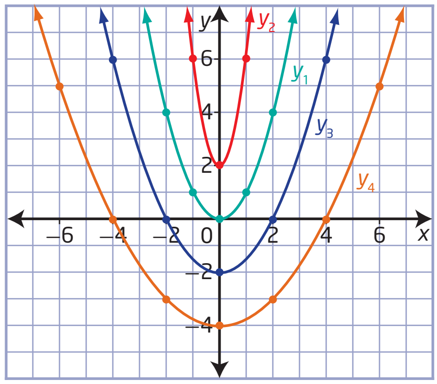
1. Describe how you can obtain the graph of each function from the graph of $f(x) = x^2$. State the direction of opening, whether it has a maximum or a minimum value, and the range for each.
   - **a)** $f(x) = 7x^2$
   - **b)** $f(x) = \frac{1}{6}x^2$
   - **c)** $f(x) = -4x^2$
   - **d)** $f(x) = -0.2x^2$

2. Describe how the graphs of the functions in each pair are related. Then, sketch the graph of the second function in each pair, and determine the vertex, the equation of the axis of symmetry, the domain and range, and any intercepts.
   - **a)** $y = x^2$ and $y = x^2 + 1$
   - **b)** $y = x^2$ and $y = (x - 2)^2$
   - **c)** $y = x^2$ and $y = x^2 - 4$
   - **d)** $y = x^2$ and $y = (x + 3)^2$

3. Describe how to sketch the graph of each function using transformations.
   - **a)** $f(x) = (x + 5)^2 + 11$
   - **b)** $f(x) = -3x^2 - 10$
   - **c)** $f(x) = 5(x + 20)^2 - 21$
   - **d)** $f(x) = -\frac{1}{8}(x - 5.6)^2 + 13.8$

4. Sketch the graph of each function. Identify the vertex, the axis of symmetry, the direction of opening, the maximum or minimum value, the domain and range, and any intercepts.
   - **a)** $y = -(x - 3)^2 + 9$
   - **b)** $y = 0.25(x + 4)^2 + 1$
   - **c)** $y = -3(x - 1)^2 + 12$
   - **d)** $y = \frac{1}{2}(x - 2)^2 - 2$

5. **a)** Write a quadratic function in vertex form for each parabola in the graph.



   b) Suppose four new parabolas open downward instead of upward but have the same shape and vertex as each parabola in the graph. Write a quadratic function in vertex form for each new parabola.

   c) Write the quadratic functions in vertex form of four parabolas that are identical to the four in the graph but translated 4 units to the left.

   d) Suppose the four parabolas in the graph are translated 2 units down. Write a quadratic function in vertex form for each new parabola.

6. For the function $f(x) = 5(x - 15)^2 - 100$, explain how you can identify each of the following without graphing.
   - **a)** the coordinates of the vertex
   - **b)** the equation of the axis of symmetry
   - **c)** the direction of opening
   - **d)** whether the function has a maximum or minimum value, and what that value is
   - **e)** the domain and range
   - **f)** the number of x-intercepts

7. Without graphing, identify the location of the vertex and the axis of symmetry, the direction of opening and the maximum or minimum value, the domain and range, and the number of x-intercepts for each function.

a) $y = -4x^2 + 14$

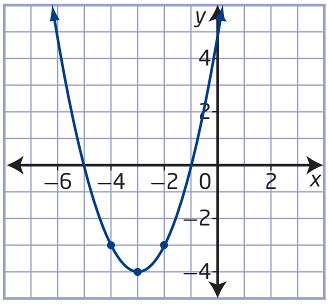
b) $y = (x + 18)^2 - 8$

c) $y = 6(x - 7)^2$

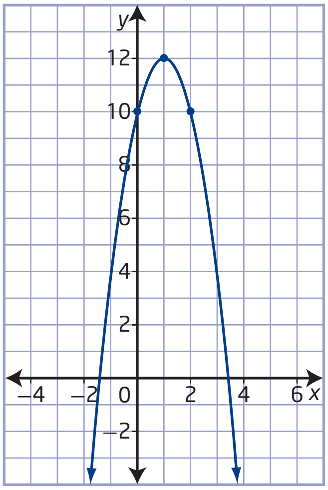
d) $y = -\frac{1}{9}(x + 4)^2 - 36$

8. Determine the quadratic function in vertex form for each parabola.

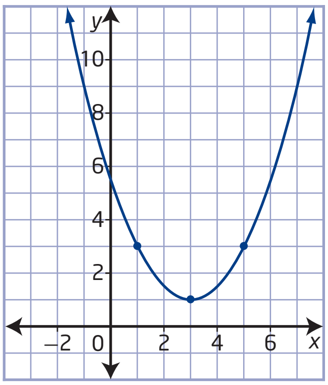
a)



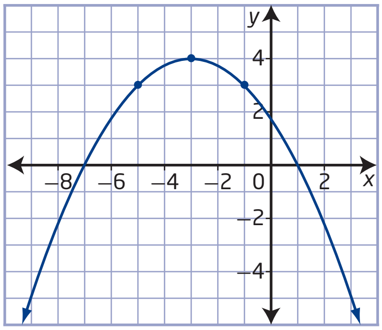
b)



c)



d)



9. Determine a quadratic function in vertex form that has the given characteristics.

a) vertex at (0, 0), passing through the point (6, −9)

b) vertex at (0, −6), passing through the point (3, 21)

c) vertex at (2, 5), passing through the point (4, −11)

d) vertex at (−3, −10), passing through the point (2, −5)

Apply

10. The point (4, 16) is on the graph of $f(x) = x^2$. Describe what happens to the point when each of the following sets of transformations is performed in the order listed. Identify the corresponding point on the transformed graph.

a) a horizontal translation of 5 units to the left and then a vertical translation of 8 units up

b) a multiplication of the y-values by a factor of $\frac{1}{4}$ and then a reflection in the x-axis

c) a reflection in the x-axis and then a horizontal translation of 10 units to the right

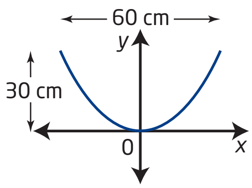
d) a multiplication of the y-values by a factor of 3 and then a vertical translation of 8 units down

11. Describe how to obtain the graph of $y = 20 - 5x^2$ using transformations on the graph of $y = x^2$.

12. Quadratic functions do not all have the same number of x-intercepts. Is the same true about y-intercepts? Explain.

13. A parabolic mirror was used to ignite the Olympic torch for the 2010 Winter Olympics in Vancouver and Whistler, British Columbia. Suppose its diameter is 60 cm and its depth is 30 cm.

a) Determine the quadratic function that represents its cross-sectional shape if the lowest point in the centre of the mirror is considered to be the origin, as shown.



b) How would the quadratic function be different if the outer edge of the mirror were considered the origin? Explain why there is a difference.

Did You Know?

Before the 2010 Winter Olympics began in Vancouver and Whistler, the Olympic torch was carried over 45 000 km for 106 days through every province and territory in Canada. The torch was initially lit in Olympia, Greece, the site of the ancient Olympic Games, before beginning its journey in Canada. The flame was lit using a special bowl-shaped reflector called a parabolic mirror that focuses the Sun's rays to a single point, concentrating enough heat to ignite the torch.

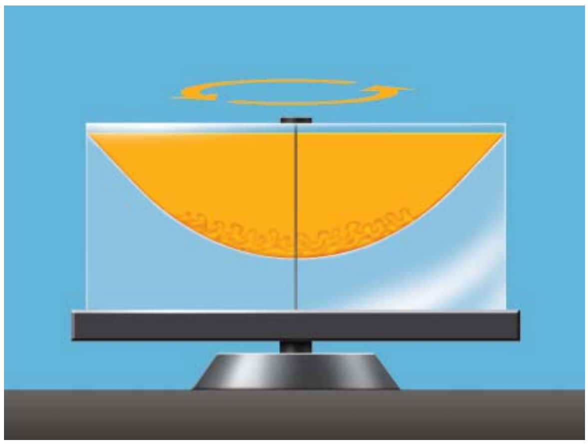
14. The finance team at an advertising company is using the quadratic function $N(x) = -2.5(x - 36)^2 + 20\ 000$ to predict the effectiveness of a TV commercial for a certain product, where N is the predicted number of people who buy the product if the commercial is aired x times per week.

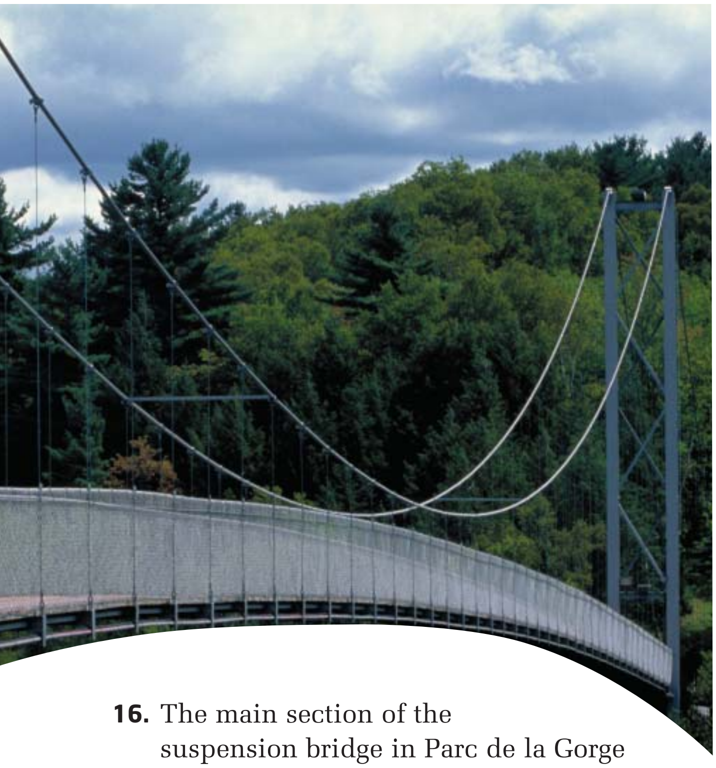
a) Explain how you could sketch the graph of the function, and identify its characteristics.

b) According to this model, what is the optimum number of times the commercial should be aired?

c) What is the maximum number of people that this model predicts will buy the product?

15. When two liquids that do not mix are put together in a container and rotated around a central axis, the surface created between them takes on a parabolic shape as they rotate. Suppose the diameter at the top of such a surface is 40 cm, and the maximum depth of the surface is 12 cm. Choose a location for the origin and write the function that models the cross-sectional shape of the surface.





16. The main section of the suspension bridge in Parc de la Gorge de Coaticook, Québec, has cables in the shape of a parabola. Suppose that the points on the tops of the towers where the cables are attached are 168 m apart and 24 m vertically above the minimum height of the cables.

a) Determine the quadratic function in vertex form that represents the shape of the cables. Identify the origin you used.

b) Choose two other locations for the origin. Write the corresponding quadratic function for the shape of the cables for each.

c) Use each quadratic function to determine the vertical height of the cables above the minimum at a point that is 35 m horizontally from one of the towers. Are your answers the same using each of your functions? Explain.

Did You Know?

The suspension bridge in Parc de la Gorge de Coaticook in Québec claims to be the longest pedestrian suspension bridge in the world.

17. During a game of tennis, Natalie hits the tennis ball into the air along a parabolic trajectory. Her initial point of contact with the tennis ball is 1 m above the ground. The ball reaches a maximum height of 10 m before falling toward the ground. The ball is again 1 m above the ground when it is 22 m away from where she hit it. Write a quadratic function to represent the trajectory of the tennis ball if the origin is on the ground directly below the spot from which the ball was hit.

Did You Know?

Tennis originated from a twelfth-century French game called *jeu de paume*, meaning game of palm (of the hand). It was a court game where players hit the ball with their hands. Over time, gloves covered bare hands and, finally, racquets became the standard equipment. In 1873, Major Walter Wingfield invented a game called *sphairistike* (Greek for *playing ball*), from which modern outdoor tennis evolved.

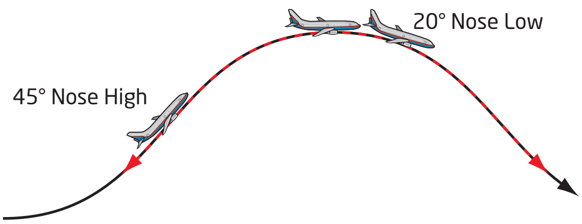
18. Water is spraying from a nozzle in a fountain, forming a parabolic path as it travels through the air. The nozzle is 10 cm above the surface of the water. The water achieves a maximum height of 100 cm above the water's surface and lands in the pool. The water spray is again 10 cm above the surface of the water when it is 120 cm horizontally from the nozzle. Write the quadratic function in vertex form to represent the path of the water if the origin is at the surface of the water directly below the nozzle.

19. The function $y = x^2 + 4$ represents a translation of 4 units up, which is in the positive direction. The function $y = (x + 4)^2$ represents a translation of 4 units to the left, which is in the negative direction. How can you explain this difference?

20. In the movie, *Apollo 13*, starring Tom Hanks, scenes were filmed involving weightlessness. Weightlessness can be simulated using a plane to fly a special manoeuvre. The plane follows a specific inverted parabolic arc followed by an upward-facing recovery arc. Suppose the parabolic arc starts when the plane is at 7200 m and takes it up to 10 000 m and then back down to 7200 m again. It covers approximately 16 000 m of horizontal distance in total.

a) Determine the quadratic function that represents the shape of the parabolic path followed by the plane if the origin is at ground level directly below where the plane starts the parabolic arc.

b) Identify the domain and range in this situation.



Did You Know?

Passengers can experience the feeling of *zero-g*, or weightlessness, for approximately 30 s during each inverted parabolic manoeuvre made. During the recovery arc, passengers feel almost *two-g*, or almost twice the sensation of gravity. In addition to achieving weightlessness, planes such as these are also able to fly parabolic arcs designed to simulate the gravity on the moon (one sixth of Earth's) or on Mars (one third of Earth's).

21. Determine a quadratic function in vertex form given each set of characteristics. Explain your reasoning.

a) vertex (6, 30) and a y-intercept of -24

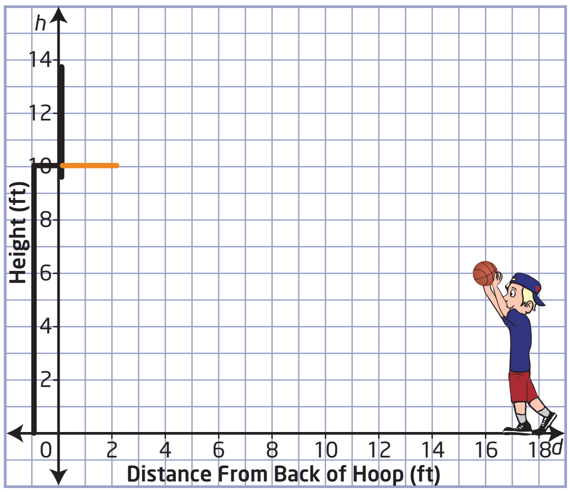
b) minimum value of -24 and x-intercepts at -21 and -5

Extend

22. a) Write quadratic functions in vertex form that represent three different trajectories the basketball shown can follow and pass directly through the hoop without hitting the backboard.

b) Which of your three quadratic functions do you think represents the most realistic trajectory for an actual shot? Explain your thoughts.

c) What do you think are a reasonable domain and range in this situation?



23. If the point (m, n) is on the graph of $f(x) = x^2$, determine expressions for the coordinates of the corresponding point on the graph of $f(x) = a(x - p)^2 + q$.

Create Connections

24. a) Write a quadratic function that is related to $f(x) = x^2$ by a change in width, a reflection, a horizontal translation, and a vertical translation.

b) Explain your personal strategy for accurately sketching the function.

25. Create your own specific examples of functions to explain how to determine the number of x-intercepts for quadratic functions of the form $f(x) = a(x - p)^2 + q$ without graphing.

26. **MINI LAB** Graphing a function like $y = -x^2 + 9$ will produce a curve that extends indefinitely. If only a portion of the curve is desired, you can state the function with a restriction on the domain. For example, to draw only the portion of the graph of $y = -x^2 + 9$ between the points where $x = -2$ and $x = 3$, write $y = -x^2 + 9$, $\{x \mid -2 \le x \le 3, x \in \mathrm{R}\}$.

Materials

- 0.5-cm grid paper

Create a line-art illustration of an object or design using quadratic and/or linear functions with restricted domains.

Step 1 Use a piece of 0.5-cm grid paper. Draw axes vertically and horizontally through the centre of the grid. Label the axes with a scale.

Step 2 Plan out a line-art drawing that you can draw using portions of the graphs of quadratic and linear functions. As you create your illustration, keep a record of the functions you use. Add appropriate restrictions to the domain to indicate the portion of the graph you want.

Step 3 Use your records to make a detailed and accurate list of instructions/functions (including restrictions) that someone else could use to recreate your illustration.

Step 4 Trade your functions/instructions list with a partner. See if you can recreate each other's illustration using only the list as a guide.

Project Corner — Parabolic Shape

- Many suspension bridge cables, the arches of bridges, satellite dishes, reflectors in headlights and spotlights, and other physical objects often appear to have parabolic shape.
- You can try to model a possible quadratic relationship by drawing a set of axes on an image of a physical object that appears to be quadratic in nature, and using one or more points on the curve.
- What images or objects can you find that might be quadratic?

3.2

Investigating Quadratic Functions in Standard Form

Focus on...

- identifying quadratic functions in standard form
- determining the vertex, domain and range, axis of symmetry, maximum or minimum value, and x-intercepts and y-intercept for quadratic functions in standard form
- graphing and analysing quadratic functions in applied situations

When a player kicks or punts a football into the air, it reaches a maximum height before falling back to the ground. The moment it leaves the punter's foot to the moment it is caught or hits the ground is called the *hang time* of the punt. A punter attempts to kick the football so there is a longer hang time to allow teammates to run downfield to tackle an opponent who catches the ball. The punter may think about exactly where or how far downfield the football will land. How can you mathematically model the path of a football through the air after it is punted?

Did You Know?

The Grey Cup has been the championship trophy for the Canadian Football League (CFL) since 1954. Earl Grey, the Governor General of Canada at the time, donated the trophy in 1909 for the Rugby Football Championship of Canada. Two Grey Cups won have been on the last play of the game: Saskatchewan in 1989 and Montreal in 2009.

Investigate Quadratic Functions in Standard Form

Materials

- grid paper

Did You Know?

In the National Football League, the field length, not including the end zones, is 100 yd. The longest regular season punt record for the NFL was 98 yd, by Steve O'Neal of the New York Jets, against the Denver Broncos in 1969.

In the Canadian Football League, the field length, not including the end zones, is 110 yd. The longest regular season punt record for the CFL was 108 yd, by Zenon Andrusyshyn of the Toronto Argonauts, against the Edmonton Eskimos in 1977.

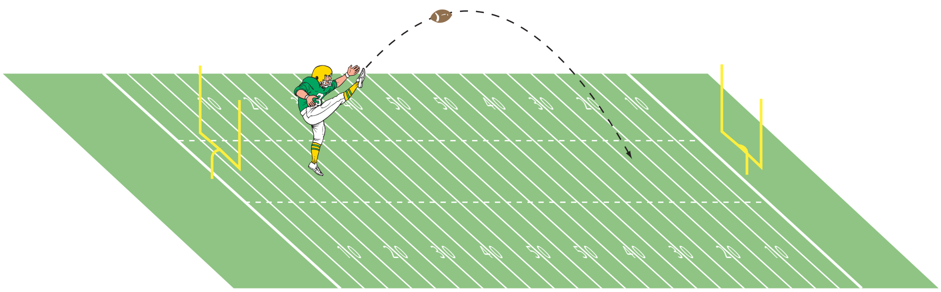
Part A: Model the Path of a Football

Depending upon the situation, the punter may kick the football so that it will follow a specific path.

1. Work with a partner. Draw a coordinate grid on a sheet of grid paper. Label the x-axis as horizontal distance downfield and the y-axis as height. How do the horizontal distance and height relate to the kicking of a football?
2. On the same grid, sketch out three possible flight paths of the football.
3. Describe the shape of your graphs. Are these shapes similar to other students' graphs?
4. Describe the common characteristics of your graphs.

Reflect and Respond

5. How would you describe the maximum or minimum heights of each of your graphs?
6. Describe any type of symmetry that you see in your graphs.
7. State the domain and range for each of your graphs.
8. How do the domain and range relate to the punting of the football?



Part B: Investigate a Quadratic Function of the Form $f(x) = ax^2 + bx + c$

The path of a football through the air is just one of many real-life phenomena that can be represented by a quadratic function. A quadratic function of the form $f(x) = ax^2 + bx + c$ is written in **standard form**.

standard form (of a quadratic function)

- the form $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$

9. Using technology, graph the quadratic function $f(x) = -x^2 + 4x + 5$.
10. Describe any symmetry that the graph has.
11. Does the function have a maximum y-value? Does it have a minimum y-value? Explain.

12. Using technology, graph on a Cartesian plane the functions that result from substituting the following c-values into the function $f(x) = -x^2 + 4x + c$.

10
0
−5

13. Using technology, graph on a Cartesian plane the functions that result from substituting the following a-values into the function $f(x) = ax^2 + 4x + 5$.

−4
−2
1
2

14. Using technology, graph on a Cartesian plane the functions that result from substituting the following b-values into the function $f(x) = -x^2 + bx + 5$.

2
0
−2
−4

Reflect and Respond

15. What do your graphs show about how the function $f(x) = ax^2 + bx + c$ is affected by changing the parameter c?

16. How is the function affected when the value of a is changed? How is the graph different when a is a positive number?

17. What effect does changing the value of b have on the graph of the function?

Do any of the parameters affect the *position* of the graph?

Do any affect the *shape* of the graph?

Link the Ideas

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$, where a, b, and c are real numbers with $a \neq 0$.

- a determines the shape and whether the graph opens upward (positive a) or downward (negative a)
- b influences the position of the graph
- c determines the y-intercept of the graph

You can expand $f(x) = a(x - p)^2 + q$ and compare the resulting coefficients with the standard form $f(x) = ax^2 + bx + c$, to see the relationship between the parameters of the two forms of a quadratic function.

$f(x) = a(x - p)^2 + q$
$f(x) = a(x^2 - 2xp + p^2) + q$
$f(x) = ax^2 - 2axp + ap^2 + q$
$f(x) = ax^2 + (-2ap)x + (ap^2 + q)$
$f(x) = ax^2 + bx + c$

By comparing the two forms, you can see that
$b = -2ap$ or $p = \frac{-b}{2a}$ and $c = ap^2 + q$ or $q = c - ap^2$.

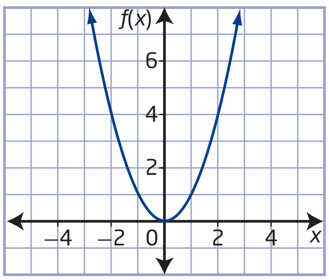
Recall that to determine the x-coordinate of the vertex, you can use the equation $x = p$. So, the x-coordinate of the vertex is $x = -\frac{b}{2a}$.

Example 1

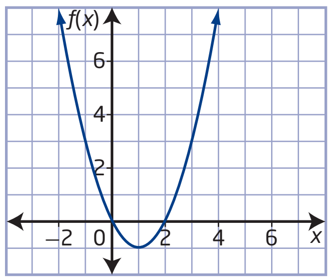
Identify Characteristics of a Quadratic Function in Standard Form

For each graph of a quadratic function, identify the following:
- the direction of opening
- the coordinates of the vertex
- the maximum or minimum value
- the equation of the axis of symmetry
- the x-intercepts and y-intercept
- the domain and range

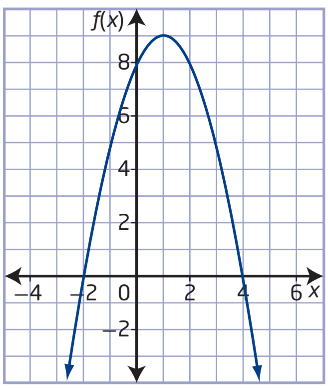
a) $f(x) = x^2$



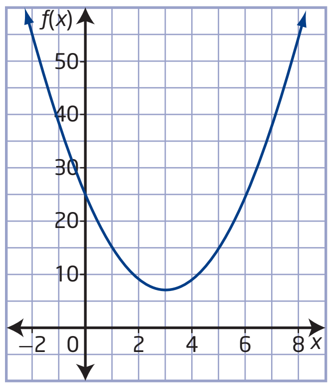
b) $f(x) = x^2 - 2x$



c) $f(x) = -x^2 + 2x + 8$

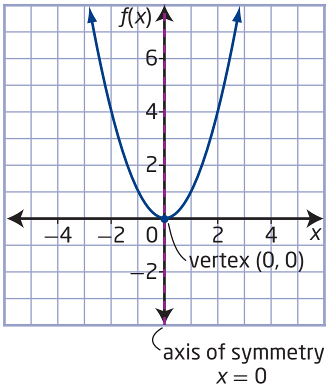


d) $f(x) = 2x^2 - 12x + 25$



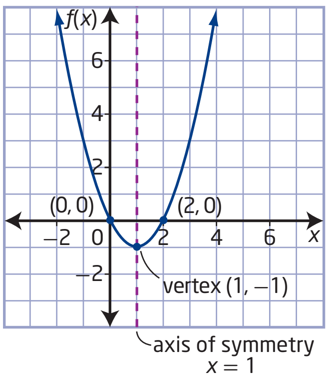
Solution

a) $f(x) = x^2$



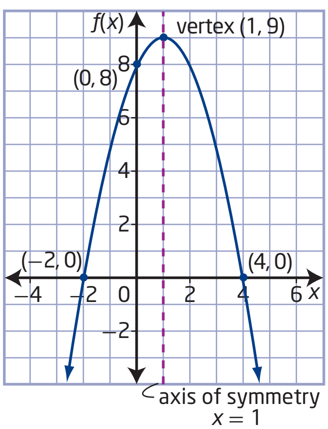
- opens upward
- vertex: (0, 0)
- minimum value of y of 0 when $x = 0$
- axis of symmetry: $x = 0$
- y-intercept occurs at (0, 0) and has a value of 0
- x-intercept occurs at (0, 0) and has a value of 0
- domain: all real numbers, or $\{x \mid x \in \mathrm{R}\}$
- range: all real numbers greater than or equal to 0, or $\{y \mid y \geq 0, y \in \mathrm{R}\}$

b) $f(x) = x^2 - 2x$



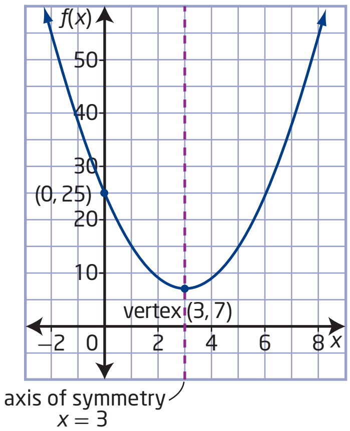
- opens upward
- vertex: (1, −1)
- minimum value of y of −1 when $x = 1$
- axis of symmetry: $x = 1$
- y-intercept occurs at (0, 0) and has a value of 0
- x-intercepts occur at (0, 0) and (2, 0) and have values of 0 and 2
- domain: all real numbers, or $\{x \mid x \in \mathrm{R}\}$
- range: all real numbers greater than or equal to −1, or $\{y \mid y \geq -1, y \in \mathrm{R}\}$

c) $f(x) = -x^2 + 2x + 8$



- opens downward
- vertex: (1, 9)
- maximum value of y of 9 when $x = 1$
- axis of symmetry: $x = 1$
- y-intercept occurs at (0, 8) and has a value of 8
- x-intercepts occur at (−2, 0) and (4, 0) and have values of −2 and 4
- domain: all real numbers, or $\{x \mid x \in \mathrm{R}\}$
- range: all real numbers less than or equal to 9, or $\{y \mid y \leq 9, y \in \mathrm{R}\}$

d) $f(x) = 2x^2 - 12x + 25$



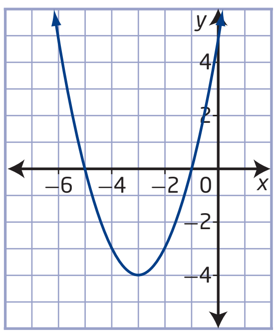
- opens upward
- vertex: (3, 7)
- minimum value of y of 7 when $x = 3$
- axis of symmetry: $x = 3$
- y-intercept occurs at (0, 25) and has a value of 25
- no x-intercepts
- domain: all real numbers, or $\{x \mid x \in R\}$
- range: all real numbers greater than or equal to 7, or $\{y \mid y \geq 7, y \in R\}$

Your Turn

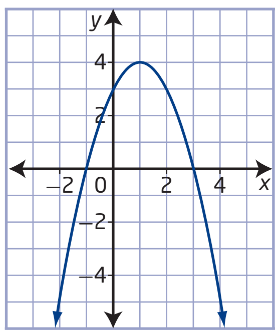
For each quadratic function, identify the following:

- the direction of opening
- the coordinates of the vertex
- the maximum or minimum value
- the equation of the axis of symmetry
- the x-intercepts and y-intercept
- the domain and range

a) $y = x^2 + 6x + 5$



b) $y = -x^2 + 2x + 3$



Example 2

Analysing a Quadratic Function

A frog sitting on a rock jumps into a pond. The height, h, in centimetres, of the frog above the surface of the water as a function of time, t, in seconds, since it jumped can be modelled by the function $h(t) = -490t^2 + 150t + 25$. Where appropriate, answer the following questions to the nearest tenth.

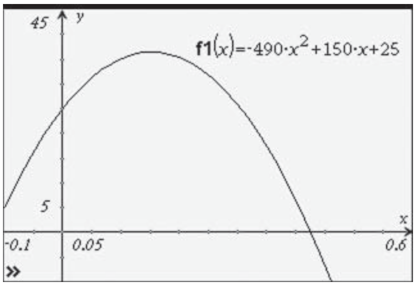
a) Graph the function.

b) What is the y-intercept? What does it represent in this situation?

c) What maximum height does the frog reach? When does it reach that height?
d) When does the frog hit the surface of the water?
e) What are the domain and range in this situation?
f) How high is the frog 0.25 s after it jumps?

Solution

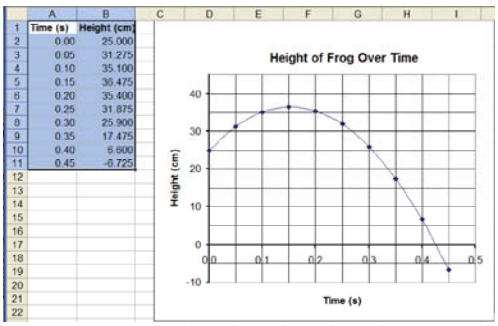
a) **Method 1: Use a Graphing Calculator**
Enter the function and adjust the dimensions of the graph until the vertex and intercepts are visible.



Why is it not necessary to show the negative *x*-intercept?

The shape of the graph might appear to resemble the path the frog follows through the air, but it is important to realize that the graph compares height to time rather than height to horizontal distance.

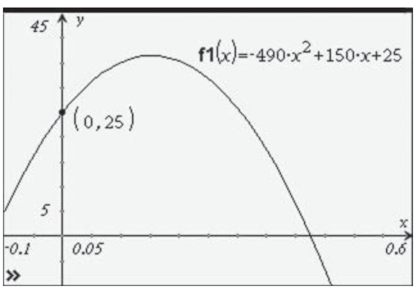
Method 2: Use a Spreadsheet
You can generate a table of values using a spreadsheet. From these values, you can create a graph.



Time (s)	Height (cm)
0.00	25.000
0.05	31.275
0.10	35.100
0.15	36.475
0.20	35.400
0.25	31.875
0.30	25.900
0.35	17.475
0.40	6.600
0.45	-6.725

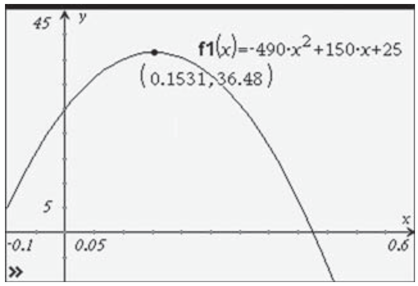
How is the pattern in the heights connected to shape of the graph?

b) The graph shows that the *y*-intercept is 25. This is the value of *h* at $t = 0$. It represents the initial height, 25 cm, from which the frog jumped.

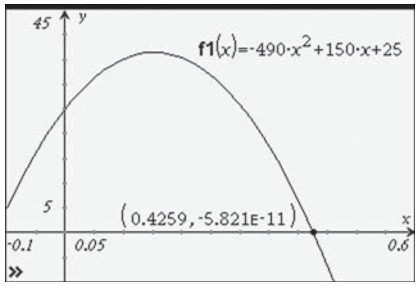


The *y*-intercept of the graph of $h(t) = -490t^2 + 150t + 25$ is equal to the value of the constant term, 25.

c) The coordinates of the vertex represent the time and height of the frog at its maximum point during the jump. The graph shows that after approximately 0.2 s, the frog achieves a maximum height of approximately 36.5 cm.



d) The positive x-intercept represents the time at which the height is 0 cm, or when the frog hits the water. The graph shows that the frog hits the water after approximately 0.4 s.



e) The domain is the set of all possible values of the independent variable, or time.

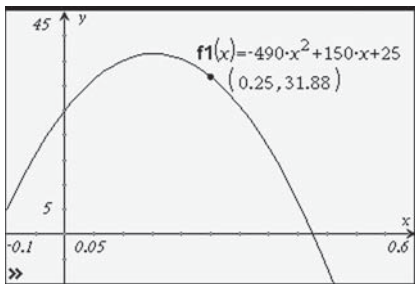
The range is the set of all possible values of the dependent variable, or height.

The values of time and height cannot be negative in this situation.

The domain is the set of all real numbers from 0 to approximately 0.4, or $\{t \mid 0 \leq t \leq 0.4, t \in \mathrm{R}\}$.

The range is the set of all real numbers from 0 to approximately 36.5, or $\{h \mid 0 \leq h \leq 36.5, h \in \mathrm{R}\}$.

f) The height of the frog after 0.25 s is the h-coordinate when t is 0.25. The graph shows that after 0.25 s, the height of the frog is approximately 31.9 cm.



You can also determine the height after 0.25 s by substituting 0.25 for t in $h(t) = -490t^2 + 150t + 25$.

$$h(t) = -490t^2 + 150t + 25$$
$$h(0.25) = -490(0.25)^2 + 150(0.25) + 25$$
$$h(0.25) = -30.625 + 37.5 + 25$$
$$h(0.25) = 31.875$$

The height of the frog after 0.25 s is approximately 31.9 cm.

Your Turn

A diver jumps from a 3-m springboard with an initial vertical velocity of 6.8 m/s. Her height, h, in metres, above the water t seconds after leaving the diving board can be modelled by the function $h(t) = -4.9t^2 + 6.8t + 3$.

a) Graph the function.
b) What does the y-intercept represent?
c) What maximum height does the diver reach? When does she reach that height?
d) How long does it take before the diver hits the water?
e) What domain and range are appropriate in this situation?
f) What is the height of the diver 0.6 s after leaving the board?

Example 3

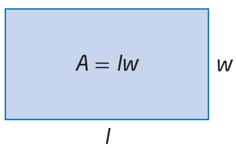
Write a Quadratic Function to Model a Situation

A rancher has 100 m of fencing available to build a rectangular corral.

a) Write a quadratic function in standard form to represent the area of the corral.
b) What are the coordinates of the vertex? What does the vertex represent in this situation?
c) Sketch the graph for the function you determined in part a).
d) Determine the domain and range for this situation.
e) Identify any assumptions you made in modelling this situation mathematically.

Solution

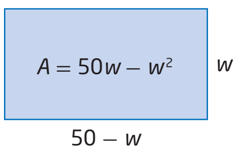
a) Let l represent the length, w represent the width, and A represent the area of the corral.



The formula $A = lw$ has three variables. To create a function for the area in terms of the width alone, you can use an expression for the length in terms of the width to eliminate the length. The formula for the perimeter of the corral is $P = 2l + 2w$, which gives the equation $2l + 2w = 100$. Solving for l gives $l = 50 - w$.

$A = lw$
$A = (50 - w)(w)$
$A = 50w - w^2$

How could you write a similar function using the length instead of the width?



b) Use the equation $x = p$ to determine the x-coordinate of the vertex.

$x = \frac{-b}{2a}$

$x = \frac{-50}{2(-1)}$

$x = 25$

Substitute the x-coordinate of the vertex into the function to determine the y-coordinate.

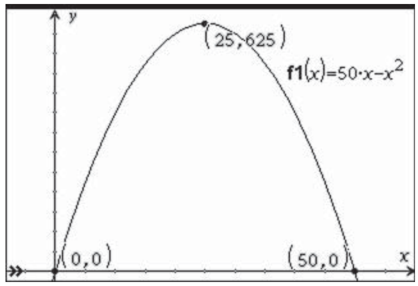
$y = 50x - x^2$

$y = 50(25) - (25)^2$

$y = 625$

The vertex is located at (25, 625). The y-coordinate of the vertex represents the maximum area of the rectangle. The x-coordinate represents the width when this occurs.

c) For the function $f(x) = 50x - x^2$, the y-intercept is the point (0, 0). Using the axis of symmetry, a point symmetric to the y-intercept is (50, 0). Sketch the parabola through these points and the vertex (25, 625).



d) Negative widths, lengths, and areas do not have any meaning in this situation, so the domain and range are restricted.

The width is any real number from 0 to 50.
The domain is $\{w \mid 0 \le w \le 50, w \in R\}$.

The area is any real number from 0 to 625.
The range is $\{A \mid 0 \le A \le 625, A \in R\}$.

Although 0 and 50 are theoretically possible, can they really be used as dimensions?

e) The quadratic function written in part a) assumes that the rancher will use all of the fencing to make the corral. It also assumes that any width or length from 0 m to 50 m is possible. In reality, there may be other limitations on the dimensions of the corral, such as the available area and landscape of the location on the rancher's property.

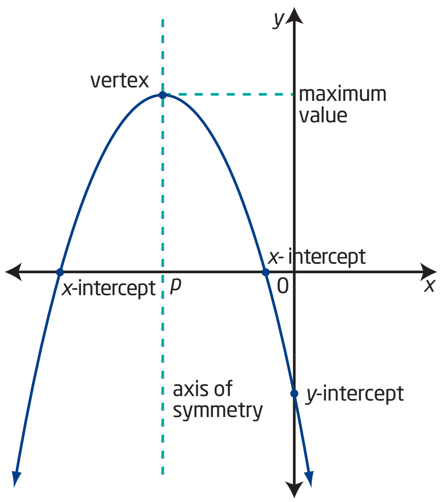
Your Turn

At a children's music festival, the organizers are roping off a rectangular area for stroller parking. There is 160 m of rope available to create the perimeter.

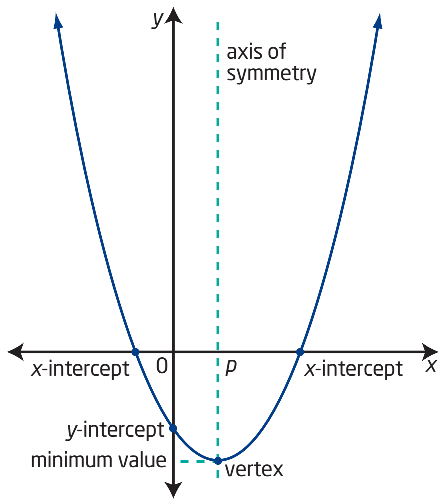
a) Write a quadratic function in standard form to represent the area for the stroller parking.
b) What are the coordinates of the vertex? What does the vertex represent in this situation?
c) Sketch the graph for the function you determined in part a).
d) Determine the domain and range for this situation.
e) Identify any assumptions you made.

Key Ideas

- The standard form of a quadratic function is $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$, where $a \neq 0$.
- The graph of a quadratic function is a parabola that
  - is symmetric about a vertical line, called the axis of symmetry, that passes through the vertex
  - opens upward and has a minimum value if $a > 0$
  - opens downward and has a maximum value if $a < 0$
  - has a y-intercept at $(0, c)$ that has a value of c
- You can determine the vertex, domain and range, direction of opening, axis of symmetry, x-intercepts and y-intercept, and maximum or minimum value from the graph of a quadratic function.



Domain: all real numbers
Range: all real numbers less than or equal to the maximum value of y



Domain: all real numbers
Range: all real numbers greater than or equal to the minimum value of y

- For any quadratic function in standard form, the x-coordinate of the vertex is given by $x = -\frac{b}{2a}$.
- For quadratic functions in applied situations,
  - the y-intercept represents the value of the function when the independent variable is 0
  - the x-intercept(s) represent(s) the value(s) of the independent variable for which the function has a value of 0
  - the vertex represents the point at which the function reaches its maximum or minimum
  - the domain and range may need to be restricted based on the values that are actually possible in the situation

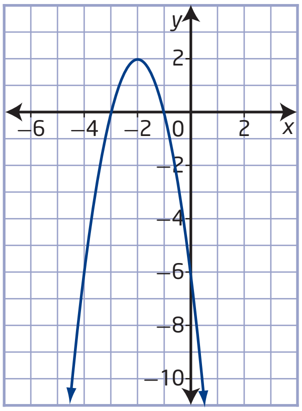
Check Your Understanding

Practise

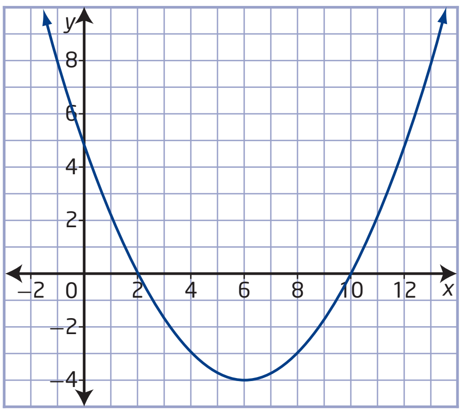
1. Which functions are quadratic? Explain.
   - **a)** $f(x) = 2x^2 + 3x$
   - **b)** $f(x) = 5 - 3x$
   - **c)** $f(x) = x(x + 2)(4x - 1)$
   - **d)** $f(x) = (2x - 5)(3x - 2)$

2. For each graph, identify the following:
   - the coordinates of the vertex
   - the equation of the axis of symmetry
   - the x-intercepts and y-intercept
   - the maximum or minimum value and how it is related to the direction of opening
   - the domain and range

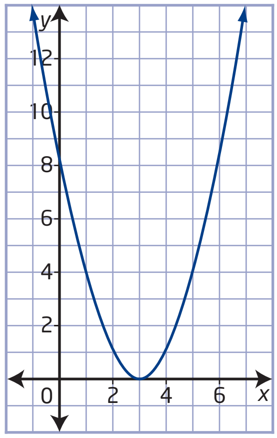
   a)



   b)



   c)



3. Show that each function fits the definition of a quadratic function by writing it in standard form.
   - **a)** $f(x) = 5x(10 - 2x)$
   - **b)** $f(x) = (10 - 3x)(4 - 5x)$

4. Create a table of values and then sketch the graph of each function. Determine the vertex, the axis of symmetry, the direction of opening, the maximum or minimum value, the domain and range, and any intercepts.
   - **a)** $f(x) = x^2 - 2x - 3$
   - **b)** $f(x) = -x^2 + 16$
   - **c)** $p(x) = x^2 + 6x$
   - **d)** $g(x) = -2x^2 + 8x - 10$

5. Use technology to graph each function. Identify the vertex, the axis of symmetry, the direction of opening, the maximum or minimum value, the domain and range, and any intercepts. Round values to the nearest tenth, if necessary.
   - **a)** $y = 3x^2 + 7x - 6$
   - **b)** $y = -2x^2 + 5x + 3$
   - **c)** $y = 50x - 4x^2$
   - **d)** $y = 1.2x^2 + 7.7x + 24.3$

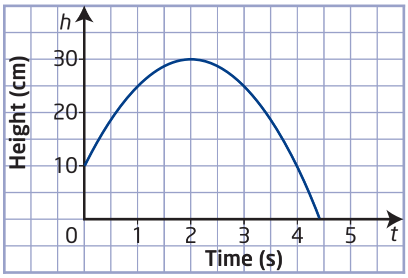
6. The x-coordinate of the vertex is given by $x = \frac{-b}{2a}$. Use this information to determine the vertex of each quadratic function.

a) $y = x^2 + 6x + 2$

b) $y = 3x^2 - 12x + 5$

c) $y = -x^2 + 8x - 11$

7. A siksik, an Arctic ground squirrel, jumps from a rock, travels through the air, and then lands on the tundra. The graph shows the height of its jump as a function of time. Use the graph to answer each of the following, and identify which characteristic(s) of the graph you used in each case.



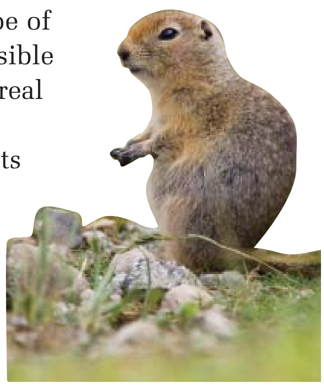
a) What is the height of the rock that the siksik jumped from?

b) What is the maximum height of the siksik? When did it reach that height?

c) How long was the siksik in the air?

d) What are the domain and range in this situation?

e) Would this type of motion be possible for a siksik in real life? Use your answers to parts a) to d) to explain why or why not.



Did You Know?

The *siksik* is named because of the sound it makes.

8. How many x-intercepts does each function have? Explain how you know. Then, determine whether each intercept is positive, negative, or zero.

a) a quadratic function with an axis of symmetry of $x = 0$ and a maximum value of 8

b) a quadratic function with a vertex at (3, 1), passing through the point (1, −3)

c) a quadratic function with a range of $y \geq 1$

d) a quadratic function with a y-intercept of 0 and an axis of symmetry of $x = -1$

9. Consider the function $f(x) = -16x^2 + 64x + 4$.

a) Determine the domain and range of the function.

b) Suppose this function represents the height, in feet, of a football kicked into the air as a function of time, in seconds. What are the domain and range in this case?

c) Explain why the domain and range are different in parts a) and b).

Apply

10. Sketch the graph of a quadratic function that has the characteristics described in each part. Label the coordinates of three points that you know are on the curve.

a) x-intercepts at −1 and 3 and a range of $y \geq -4$

b) one of its x-intercepts at −5 and vertex at (−3, −4)

c) axis of symmetry of $x = 1$, minimum value of 2, and passing through (−1, 6)

d) vertex at (2, 5) and y-intercept of 1

11. Satellite dish antennas have the shape of a parabola. Consider a satellite dish that is 80 cm across. Its cross-sectional shape can be described by the function $d(x) = 0.0125x^2 - x$, where d is the depth, in centimetres, of the dish at a horizontal distance of x centimetres from one edge of the dish.

a) What is the domain of this function?

b) Graph the function to show the cross-sectional shape of the satellite dish.

c) What is the maximum depth of the dish? Does this correspond to the maximum value of the function? Explain.

d) What is the range of the function?

e) How deep is the dish at a point 25 cm from the edge of the dish?

12. A jumping spider jumps from a log onto the ground below. Its height, h, in centimetres, as a function of time, t, in seconds, since it jumped can be modelled by the function $h(t) = -490t^2 + 75t + 12$. Where appropriate, answer the following questions to the nearest tenth.

a) Graph the function.

b) What does the h-intercept represent?

c) When does the spider reach its maximum height? What is its maximum height?

d) When does the spider land on the ground?

e) What domain and range are appropriate in this situation?

f) What is the height of the spider 0.05 s after it jumps?

Did You Know?

There are an estimated 1400 spider species in Canada. About 110 of these are jumping spiders. British Columbia has the greatest diversity of jumping spiders. Although jumping spiders are relatively small (3 mm to 10 mm in length), they can jump horizontal distances of up to 16 cm.

13. A quadratic function can model the relationship between the speed of a moving object and the wind resistance, or drag force, it experiences. For a typical car travelling on a highway, the relationship between speed and drag can be approximated with the function $f(v) = 0.002v^2$, where f is the drag force, in newtons, and v is the speed of the vehicle, in kilometres per hour.

a) What domain do you think is appropriate in this situation?

b) Considering your answer to part a), create a table of values and a graph to represent the function.

c) How can you tell from your graph that the function is not a linear function? How can you tell from your table?

d) What happens to the values of the drag force when the speed of the vehicle doubles? Does the drag force also double?

e) Why do you think a driver might be interested in understanding the relationship between the drag force and the speed of the vehicle?

Did You Know?

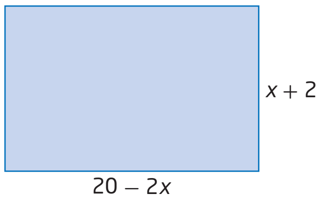
A *newton* (abbreviated N) is a unit of measure of force. One newton is equal to the force required to accelerate a mass of one kilogram at a rate of one metre per second squared.

14. Assume that you are an advisor to business owners who want to analyse their production costs. The production costs, C, to produce n thousand units of their product can be modelled by the function $C(n) = 0.3n^2 - 48.6n + 13\ 500$.

a) Graph the function and identify the important characteristics of the graph.

b) Write a short explanation of what the graph and each piece of information you determined in part a) shows about the production costs.

15. a) Write a function to represent the area of the rectangle. Show that the function fits the definition of a quadratic function.



b) Graph the function.

c) What do the x-intercepts represent in this situation? How do they relate to the dimensions of the rectangle?

d) What information does the vertex give about this situation?

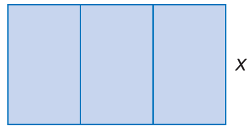
e) What are the domain and range? What do they represent in this situation?

f) Does the function have a maximum value in this situation? Does it have a minimum value?

g) If you graph the function you wrote in part a) with the domain being the set of all real numbers, does it have a minimum value? Explain.

16. Does the function $f(x) = 4x^2 - 3x + 2x(3 - 2x) + 1$ represent a quadratic function? Show why or why not using several different methods.

17. Maria lives on a farm. She is planning to build an enclosure for her animals that is divided into three equal-sized sections, as shown in the diagram. She has 280 m of fencing to use.



a) Write a function that represents the area of the entire enclosure in terms of its width. How do you know that it fits the definition of a quadratic function?

b) Graph the function.

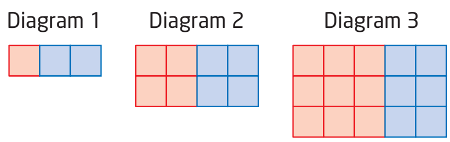
c) What are the coordinates of the vertex? What do they represent?

d) What are the domain and range in this situation?

e) Does the function have a maximum value? Does it have a minimum value? Explain.

f) What assumptions did you make in your analysis of this situation?

18. a) Consider the pattern shown in the sequence of diagrams. The area of each small square is 1 square unit.



a) Draw the next three diagrams in the sequence. What is the total area of each diagram?

b) Write a function to model the total area, A, of each diagram in terms of the diagram number, n.

c) Is the function linear or quadratic? Explain why in terms of the diagrams as well as the function.

d) If the sequence of diagrams continues, what is the domain? Are the values in the domain continuous or discrete? Explain.

e) Considering your answer to part b), graph the function to show the relationship between A and n.

19. a) Write a function for the area, A, of a circle, in terms of its radius, r.

b) What domain and range are appropriate for this function?

c) Considering your answer to part b), graph the function.

d) What are the intercepts of the graph? What meaning do they have in this situation?

e) Does this graph have an axis of symmetry? Explain.

20. The stopping distance of a vehicle is the distance travelled between the time a driver notices a need to stop and the time when the vehicle actually stops. This includes the reaction time before applying the brakes and the time it takes to stop once the brakes are applied. The stopping distance, d, in metres, for a certain vehicle can be approximated using the formula $d(t) = \frac{vt}{3.6} + \frac{v^2}{130}$, where v is the speed of the vehicle, in kilometres per hour, before braking, and t is the reaction time, in seconds, before the driver applies the brakes.

a) Suppose the driver of this vehicle has a reaction time of 1.5 s. Write a function to model the stopping distance, d, for the vehicle and driver as a function of the pre-braking speed, v.

b) Create a table of values and graph the function using a domain of $0 \leq v \leq 200$.

c) When the speed of the vehicle doubles, does the stopping distance also double? Use your table and graph to explain.

d) Assume that you are writing a newspaper or magazine article about safe driving. Write an argument aimed at convincing drivers to slow down. Use your graphs and other results to support your case.

Extend

21. A *family* of functions is a set of functions that are related to each other in some way.

a) Write a set of functions for part of the family defined by $f(x) = k(x^2 + 4x + 3)$ if $k = 1, 2, 3$. Simplify each equation so that it is in standard form.

b) Graph the functions on the same grid using the restricted domain of $\{x \mid 0 \leq x \leq 4, x \in \mathrm{R}\}$.

c) Describe how the graphs are related to each other. How are the values of y related for points on each graph for the same value of x?

d) Predict what the graph would look like if $k = 4$ and if $k = 0.5$. Sketch the graph in each case to test your prediction.

e) Predict what the graphs would look like for negative values of k. Test your prediction.

f) What does the graph look like if $k = 0$?

g) Explain how the members of the family of functions defined by $f(x) = k(x^2 + 4x + 3)$ for all values of k are related.

22. Milos said, "The a in the quadratic function $f(x) = ax^2 + bx + c$ is like the 'steepness' of the graph, just like it is for the a in $f(x) = ax + b$." In what ways might this statement be a reasonable comparison? In what ways is it not completely accurate? Explain, using examples.

23. a) The point $(-2, 1)$ is on the graph of the quadratic function $f(x) = -x^2 + bx + 11$. Determine the value of b.

b) If the points $(-1, 6)$ and $(2, 3)$ are on the graph of the quadratic function $f(x) = 2x^2 + bx + c$, determine the values of b and c.

24. How would projectile motion be different on the moon? Consider the following situations:

- an object launched from an initial height of 35 m above ground with an initial vertical velocity of 20 m/s
- a flare that is shot into the air with an initial velocity of 800 ft/s from ground level
- a rock that breaks loose from the top of a 100-m-high cliff and starts to fall straight down

a) Write a pair of functions for each situation, one representing the motion if the situation occurred on Earth and one if on the moon.

b) Graph each pair of functions.

c) Identify the similarities and differences in the various characteristics for each pair of graphs.

d) What do your graphs show about the differences between projectile motion on Earth and the moon? Explain.

Did You Know?

You can create a function representing the height of any projectile over time using the formula $h(t) = -0.5gt^2 + v_0t + h_0$, where g is the acceleration due to gravity, v_0 is the initial vertical velocity, and h_0 is the initial height.

The *acceleration due to gravity* is a measure of how much gravity slows down an object fired upward or speeds up an object dropped or thrown downward. On the surface of Earth, the acceleration due to gravity is 9.81 m/s^2 in metric units or 32 ft/s^2 in imperial units. On the surface of the moon, the acceleration due to gravity is much less than on Earth, only 1.63 m/s^2 or 5.38 ft/s^2.

25. Determine an expression for the coordinates of each missing point described below. Explain your reasoning.

a) A quadratic function has a vertex at (m, n) and a y-intercept of r. Identify one other point on the graph.

b) A quadratic function has an axis of symmetry of $x = j$ and passes through the point $(4j, k)$. Identify one other point on the graph.

c) A quadratic function has a range of $y \geq d$ and x-intercepts of s and t. What are the coordinates of the vertex?

Create Connections

26. For the graph of a given quadratic function, how are the range, direction of opening, and location of the vertex, axis of symmetry, and x-intercepts connected?

27. MINI LAB Use computer or graphing calculator technology to investigate how the values of a, b, and c affect the graph of the function that corresponds to $y = ax^2 + bx + c$. If you are using graphing/geometry software, you may be able to use it to make sliders to change the values of a, b, and c. The will allow you to dynamically see the effect each parameter has.

For each step below, sketch one or more graphs to illustrate your findings.

Step 1 Change the values of a, b, and c, one at a time. Observe any changes that occur in the graph as a, b, or c is

- increased or decreased
- made positive or negative

Step 2 How is the y-intercept affected by the values of a, b, and c? Do all the values affect its location?

Step 3 Explore how the location of the axis of symmetry is affected by the values. Which values are involved, and how?

Step 4 Explore how the values affect the steepness of the curve as it crosses the y-axis. Do they all have an effect on this aspect?

Step 5 Observe whether any other aspects of the graph are affected by changes in a, b, and c. Explain your findings.

3.3

Completing the Square

Focus on...

- converting quadratic functions from standard to vertex form
- analysing quadratic functions of the form $y = ax^2 + bx + c$
- writing quadratic functions to model situations

Every year, staff and students hold a craft fair as a fundraiser. Sellers are charged a fee for a table at the fair. As sellers prepare for the fair, they consider what price to set for the items they sell. If items are priced too high, few people may buy them. If the prices are set too low, sellers may not take in much revenue even though many items sell. The key is to find the optimum price. How can you determine the price at which to sell items that will give the maximum revenue?

Investigate Completing the Square

Materials

- grid paper or graphing technology

Part A: Comparing Different Forms of a Quadratic Function

1. Suppose that Adine is considering pricing for the mukluks she sells at a craft fair. Last year, she sold mukluks for \$400 per pair, and she sold 14 pairs. She predicts that for every \$40 increase in price, she will sell one fewer pair. The revenue from the mukluk sales, $R(x)$, is (Number of Mukluks Sold)(Cost Per Mukluk).

Copy and complete the table to model how Adine's total revenue this year might change for each price increase or decrease of \$40. Continue the table to see what will happen to the total revenue if the price continues to increase or decrease.

Number of Mukluks Sold	Cost Per Mukluk (\$)	Revenue, $R(x)$ (\$)
14	400	5600
13	440	5720

2. What pattern do you notice in the revenue as the price changes? Why do you think that this pattern occurs?

3. Let x represent the number of $40 increases. Develop an algebraic function to model Adine's total revenue.

   a) Determine an expression to represent the cost of the mukluks.

   b) Determine an expression to represent the number of mukluks sold.

   c) Determine the revenue function, $R(x)$, where $R(x) =$ (Number of Mukluks Sold)(Cost Per Mukluk).

   d) Expand $R(x)$ to give a quadratic function in standard form.

4. a) Graph the revenue as a function of the number of price changes.

   b) What maximum possible revenue can Adine expect?

   c) What price would give her the maximum possible revenue?

5. A friend of Adine's determined a function in the form $R(x) = -40(x - 2)^2 + 5760$ where x represents the number of price decreases.

   a) Expand this function and compare it to Adine's function. What do you notice?

   b) Which quadratic function allows you to determine the best price and maximum revenue without graphing or creating a table of values? Explain.

Reflect and Respond

6. a) Consider the shape of your graph in step 4. Why is a quadratic function a good model to use in this situation? Why is a linear function not appropriate to relate revenue to price change?

   b) What assumptions did you make in using this model to predict Adine's sales? Why might her actual sales at the fair not exactly follow the predictions made by this model?

Did You Know?

Mukluks are soft winter boots traditionally made from animal fur and hide by Arctic Aboriginal peoples. The Inuit have long worn and continue to wear this type of boot and refer to them as *kamik*.

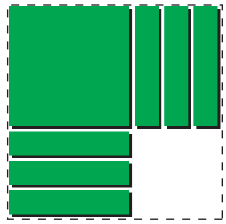
Materials

- algebra tiles

Part B: Completing the Square

The quadratic function developed in step 3 in Part A is in standard form. The function used in step 5 is in vertex form. These quadratic functions are equivalent and can provide different information. You can convert from vertex to standard form by expanding the vertex form. How can you convert from standard to vertex form?

7. a) Select algebra tiles to represent the expression $x^2 + 6x$. Arrange them into an incomplete square as shown.



b) What tiles must you add to complete the square?

c) What trinomial represents the new completed square?

d) How can you rewrite this trinomial in factored form as the square of a binomial?

8. a) Repeat the activity in step 7 using each expression in the list. Record your results in an organized fashion. Include a diagram of the tiles for each expression.

$x^2 + 2x$
$x^2 + 4x$
$x^2 + 8x$
$x^2 + 10x$

What tiles must you add to each expression to make a complete square?

b) Continue to model expressions until you can clearly describe the pattern that emerges. What relationship is there between the original expression and the tiles necessary to complete the square? Explain.

9. Repeat the activity, but this time model expressions that have a negative x-term, such as $x^2 - 2x$, $x^2 - 4x$, $x^2 - 6x$, and so on.

10. a) Without using algebra tiles, predict what value you need to add to the expression $x^2 + 32x$ to represent it as a completed square. What trinomial represents this completed square?

b) How can you rewrite the trinomial in factored form as the square of a binomial?

Reflect and Respond

11. a) How are the tiles you need to complete each square related to the original expression?

b) Does it matter whether the x-term in the original expression is positive or negative? Explain.

c) Is it possible to complete the square for an expression with an x-term with an odd coefficient? Explain your thinking.

12. The expressions $x^2 + ■x + ▲$ and $(x + ●)^2$ both represent the same perfect square. Describe how the missing values are related to each other.

Link the Ideas

You can express a quadratic function in standard form, $f(x) = ax^2 + bx + c$, or in vertex form, $f(x) = a(x - p)^2 + q$. You can determine the shape of the graph and direction of opening from the value of a in either form. The vertex form has the advantage that you can identify the coordinates of the vertex as (p, q) directly from the algebraic form. It is useful to be able to determine the coordinates of the vertex algebraically when using quadratic functions to model problem situations involving maximum and minimum values.

How can you use the values of a, p, and q to determine whether a function has a maximum or minimum value, what that value is, and where it occurs?

You can convert a quadratic function in standard form to vertex form using an algebraic process called **completing the square**. Completing the square involves adding a value to and subtracting a value from a quadratic polynomial so that it contains a perfect square trinomial. You can then rewrite this trinomial as the square of a binomial.

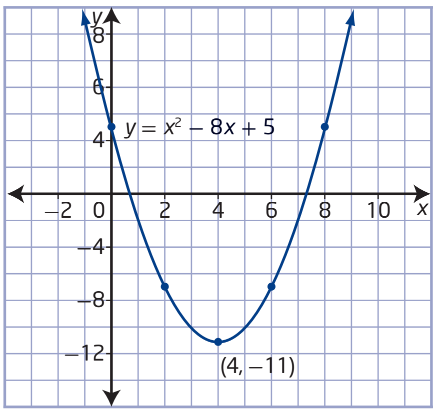
completing the square

- an algebraic process used to write a quadratic polynomial in the form $a(x - p)^2 + q$.

$y = x^2 - 8x + 5$	
$y = (x^2 - 8x) + 5$	Group the first two terms.
$y = (x^2 - 8x + 16 - 16) + 5$	Add and subtract the square of half the coefficient of the x-term.
$y = (x^2 - 8x + 16) - 16 + 5$	Group the perfect square trinomial.
$y = (x - 4)^2 - 16 + 5$	Rewrite as the square of a binomial.
$y = (x - 4)^2 - 11$	Simplify.

In the above example, both the standard form, $y = x^2 - 8x + 5$, and the vertex form, $y = (x - 4)^2 - 11$, represent the same quadratic function. You can use both forms to determine that the graph of the function will open up, since $a = 1$. However, the vertex form also reveals without graphing that the vertex is at $(4, -11)$, so this function has a minimum value of -11 when $x = 4$.

x	y
0	5
2	–7
4	–11
6	–7
8	5



Example 1

Convert From Standard Form to Vertex Form

Rewrite each function in vertex form by completing the square.

a) $f(x) = x^2 + 6x + 5$

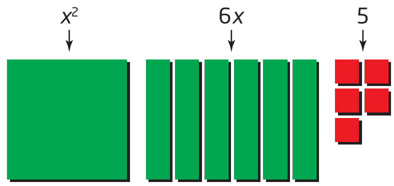
b) $f(x) = 3x^2 - 12x - 9$

c) $f(x) = -5x^2 - 70x$

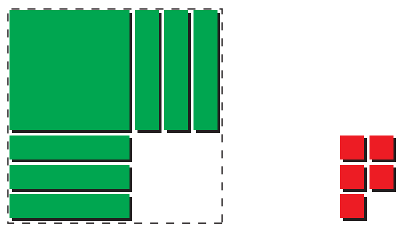
Solution

a) Method 1: Model with Algebra Tiles

Select algebra tiles to represent the quadratic polynomial $x^2 + 6x + 5$.



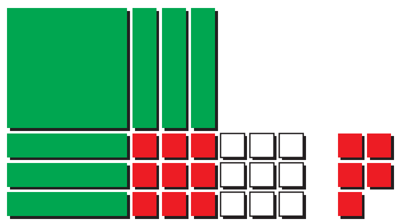
Using the x^2-tile and x-tiles, create an incomplete square to represent the first two terms. Leave the unit tiles aside for now.



How is the side length of the incomplete square related to the number of x-tiles in the original expression?

How is the number of unit tiles needed to complete the square related to the number of x-tiles in the original expression?

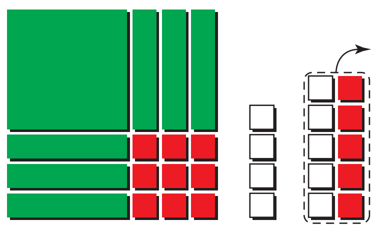
To complete the square, add nine zero pairs. The nine positive unit tiles complete the square and the nine negative unit tiles are necessary to maintain an expression equivalent to the original.



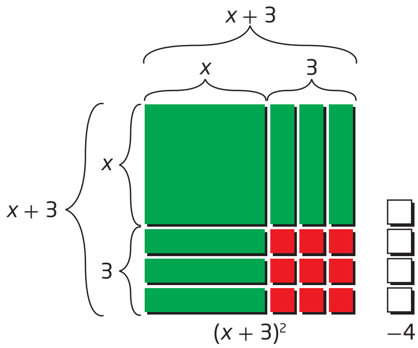
Why is it necessary to add the same number of red and white tiles?

Why are the positive unit tiles used to complete the square rather than the negative ones?

Simplify the expression by removing zero pairs.



You can express the completed square in expanded form as $x^2 + 6x + 9$, but also as the square of a binomial as $(x + 3)^2$. The vertex form of the function is $y = (x + 3)^2 - 4$.



How are the tiles in this arrangement equivalent to the original group of tiles?

Method 2: Use an Algebraic Method

For the function $y = x^2 + 6x + 5$, the value of a is 1. To complete the square,

- group the first two terms
- inside the brackets, add and subtract the square of half the coefficient of the x-term
- group the perfect square trinomial
- rewrite the perfect square trinomial as the square of a binomial
- simplify

$y = x^2 + 6x + 5$

$y = (x^2 + 6x) + 5$

$y = (x^2 + 6x + 9 - 9) + 5$ — Why is the value 9 used here? Why is 9 also subtracted?

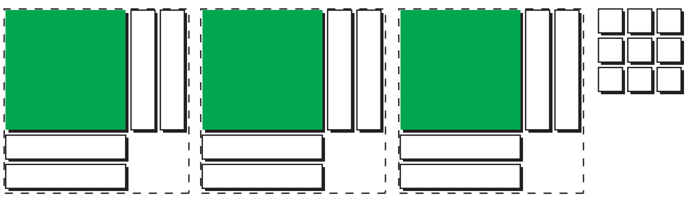
$y = (x^2 + 6x + 9) - 9 + 5$ — Why are the first three terms grouped together?

$y = (x + 3)^2 - 9 + 5$ — How is the 3 inside the brackets related to the original function? How is the 3 related to the 9 that was used earlier?

$y = (x + 3)^2 - 4$ — How could you check that this is equivalent to the original expression?

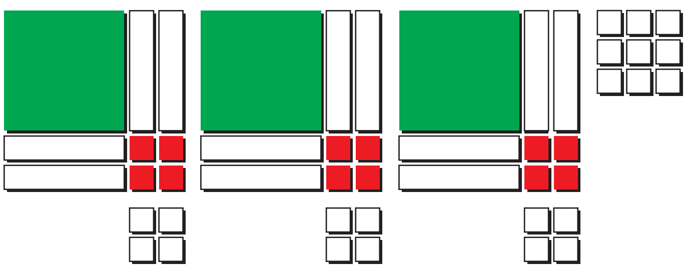
b) Method 1: Use Algebra Tiles

Select algebra tiles to represent the quadratic expression $3x^2 - 12x - 9$. Use the x^2-tiles and x-tiles to create three incomplete squares as shown. Leave the unit tiles aside for now.



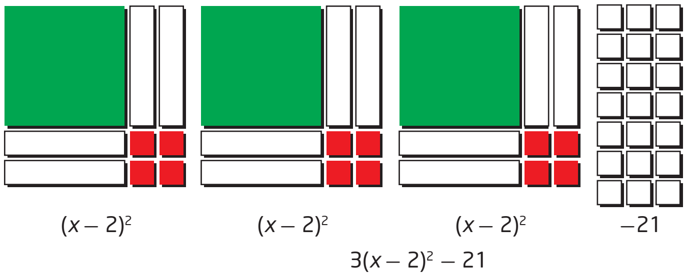
Why are three incomplete squares created?

Add enough positive unit tiles to complete each square, as well as an equal number of negative unit tiles.



Why do positive tiles complete each square even though the x-tiles are negative?

Simplify by combining the negative unit tiles.



How are the tiles in this arrangement equivalent to the original group of tiles?

You can express each completed square as $x^2 - 4x + 4$, but also as $(x - 2)^2$. Since there are three of these squares and 21 extra negative unit tiles, the vertex form of the function is $y = 3(x - 2)^2 - 21$.

Method 2: Use an Algebraic Method

To complete the square when the leading coefficient, a, is not 1,

- group the first two terms and factor out the leading coefficient
- inside the brackets, add and subtract the square of half of the coefficient of the x-term
- group the perfect square trinomial
- rewrite the perfect square trinomial as the square of a binomial
- expand the square brackets and simplify

$y = 3x^2 - 12x - 9$

$y = 3(x^2 - 4x) - 9$ — Why does 3 need to be factored from the first two terms?

$y = 3(x^2 - 4x + 4 - 4) - 9$ — Why is the value 4 used inside the brackets?

$y = 3[(x^2 - 4x + 4) - 4] - 9$

$y = 3[(x - 2)^2 - 4] - 9$

$y = 3(x - 2)^2 - 12 - 9$ — What happens to the square brackets? Why are the brackets still needed?

$y = 3(x - 2)^2 - 21$ — Why is the constant term, –21, 12 less than at the start, when only 4 was added inside the brackets?

c) Use the process of completing the square to convert to vertex form.

$y = -5x^2 - 70x$ — What happens to the x-term when a negative number is factored?

$y = -5(x^2 + 14x)$

$y = -5(x^2 + 14x + 49 - 49)$ — How does a leading coefficient that is negative affect the process? How would the result be different if it had been positive?

$y = -5[(x^2 + 14x + 49) - 49]$

$y = -5[(x + 7)^2 - 49]$

$y = -5(x + 7)^2 + 245$ — Why would algebra tiles not be suitable to use for this function?

Your Turn

Rewrite each function in vertex form by completing the square.

a) $y = x^2 + 8x - 7$

b) $y = 2x^2 - 20x$

c) $y = -3x^2 - 18x - 24$

Example 2

Convert to Vertex Form and Verify

a) Convert the function $y = 4x^2 - 28x - 23$ to vertex form.
b) Verify that the two forms are equivalent.

Solution

a) Complete the square to convert to vertex form.

Method 1: Use Fractions

$$y = 4x^2 - 28x - 23$$
$$y = 4(x^2 - 7x) - 23$$
$$y = 4\left[x^2 - 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2\right] - 23$$
$$y = 4\left(x^2 - 7x + \frac{49}{4} - \frac{49}{4}\right) - 23$$
$$y = 4\left[\left(x^2 - 7x + \frac{49}{4}\right) - \frac{49}{4}\right] - 23$$
$$y = 4\left[\left(x - \frac{7}{2}\right)^2 - \frac{49}{4}\right] - 23$$
$$y = 4\left(x - \frac{7}{2}\right)^2 - 4\left(\frac{49}{4}\right) - 23$$
$$y = 4\left(x - \frac{7}{2}\right)^2 - 49 - 23$$
$$y = 4\left(x - \frac{7}{2}\right)^2 - 72$$

Why is the number being added and subtracted inside the brackets not a whole number in this case?

Method 2: Use Decimals

$$y = 4x^2 - 28x - 23$$
$$y = 4(x^2 - 7x) - 23$$
$$y = 4[x^2 - 7x + (3.5)^2 - (3.5)^2] - 23$$
$$y = 4(x^2 - 7x + 12.25 - 12.25) - 23$$
$$y = 4[(x^2 - 7x + 12.25) - 12.25] - 23$$
$$y = 4[(x - 3.5)^2 - 12.25] - 23$$
$$y = 4(x - 3.5)^2 - 4(12.25) - 23$$
$$y = 4(x - 3.5)^2 - 49 - 23$$
$$y = 4(x - 3.5)^2 - 72$$

Do you find it easier to complete the square using fractions or decimals? Why?

b) Method 1: Work Backward

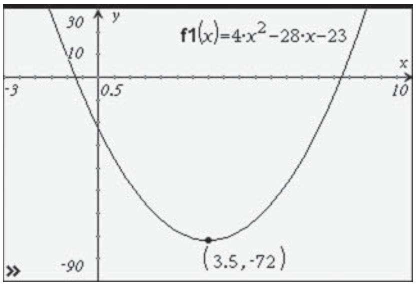
$$y = 4(x - 3.5)^2 - 72$$
$$y = 4(x^2 - 7x + 12.25) - 72$$ Expand the binomial square expression.
$$y = 4x^2 - 28x + 49 - 72$$ Eliminate the brackets by distributing.
$$y = 4x^2 - 28x - 23$$ Combine like terms to simplify.

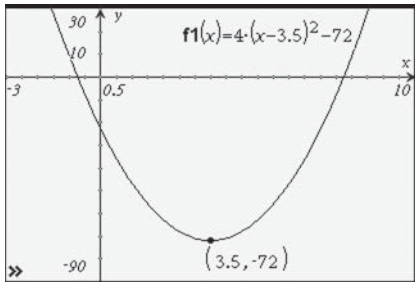
Since the result is the original function, the two forms are equivalent.

How are these steps related to the steps used to complete the square in part a)?

Method 2: Use Technology

Use graphing technology to graph both functions together or separately using identical window settings.





Since the graphs are identical, the two forms are equivalent.

Your Turn

a) Convert the function $y = -3x^2 - 27x + 13$ to vertex form.

b) Verify that the two forms are equivalent.

Example 3

Determine the Vertex of a Quadratic Function by Completing the Square

Consider the function $y = 5x^2 + 30x + 41$.

a) Complete the square to determine the vertex and the maximum or minimum value of the function.

b) Use the process of completing the square to verify the relationship between the value of p in vertex form and the values of a and b in standard form.

c) Use the relationship from part b) to determine the vertex of the function. Compare with your answer from part a).

Solution

a)
$$y = 5x^2 + 30x + 41$$
$$y = 5(x^2 + 6x) + 41$$
$$y = 5(x^2 + 6x + 9 - 9) + 41$$
$$y = 5[(x^2 + 6x + 9) - 9] + 41$$
$$y = 5[(x + 3)^2 - 9] + 41$$
$$y = 5(x + 3)^2 - 45 + 41$$
$$y = 5(x + 3)^2 - 4$$

The vertex form of the function, $y = a(x - p)^2 + q$, reveals characteristics of the graph.

The vertex is located at the point (p, q). For the function $y = 5(x + 3)^2 - 4$, $p = -3$ and $q = -4$. So, the vertex is located at $(-3, -4)$. The graph opens upward since a is positive. Since the graph opens upward from the vertex, the function has a minimum value of -4 when $x = -3$.

b) Look back at the steps in completing the square.

$y = ax^2 + bx + 41$
$y = 5x^2 + 30x + 41$
$y = 5(x^2 + 6x) + 41$
$\vdots$
$y = 5(x + 3)^2 - 4$
$y = 5(x - p)^2 - 4$

b divided by *a* gives the coefficient of *x* inside the brackets.

6 is $\frac{30}{5}$, or $\frac{b}{a}$.

Half the coefficient of *x* inside the brackets gives the value of *p* in the vertex form.

3 is half of 6, or half of $\frac{b}{a}$, or $\frac{b}{2a}$.

Considering the steps in completing the square, the value of p in vertex form is equal to $-\frac{b}{2a}$. For any quadratic function in standard form, the equation of the axis of symmetry is $x = -\frac{b}{2a}$.

c) Determine the x-coordinate of the vertex using $x = -\frac{b}{2a}$.

$x = -\frac{30}{2(5)}$
$x = -\frac{30}{10}$
$x = -3$

Determine the y-coordinate by substituting the x-coordinate into the function.
$y = 5(-3)2 + 30(-3) + 41$
$y = 5(9) - 90 + 41$
$y = 45 - 90 + 41$
$y = -4$

The vertex is $(-3, -4)$.
This is the same as the coordinates for the vertex determined in part a).

Your Turn

Consider the function $y = 3x^2 + 30x + 41$.

a) Complete the square to determine the vertex of the graph of the function.

b) Use $x = -\frac{b}{2a}$ and the standard form of the quadratic function to determine the vertex. Compare with your answer from part a).

Example 4

Write a Quadratic Model Function

The student council at a high school is planning a fundraising event with a professional photographer taking portraits of individuals or groups. The student council gets to charge and keep a session fee for each individual or group photo session. Last year, they charged a $10 session fee and 400 sessions were booked. In considering what price they should charge this year, student council members estimate that for every $1 increase in the price, they expect to have 20 fewer sessions booked.

a) Write a function to model this situation.
b) What is the maximum revenue they can expect based on these estimates. What session fee will give that maximum?
c) How can you verify the solution?
d) What assumptions did you make in creating and using this model function?

Solution

a) The starting price is $10/session and the price increases are in $1 increments.

Let n represent the number of price increases. The new price is $10 plus the number of price increases times $1, or $10 + 1n$ or, more simply, $10 + n$.

The original number of sessions booked is 400. The new number of sessions is 400 minus the number of price increases times 20, or $400 - 20n$.

Let R represent the expected revenue, in dollars. The revenue is calculated as the product of the price per session and the number of sessions.

Revenue = (price)(number of sessions)
$R = (10 + n)(400 - 20n)$
$R = 4000 + 200n - 20n^2$
$R = -20n^2 + 200n + 4000$

b) Complete the square to determine the maximum revenue and the price that gives that revenue.

$R = -20n^2 + 200n + 4000$
$R = -20(n^2 - 10n) + 4000$
$R = -20(n^2 - 10n + 25 - 25) + 4000$
$R = -20[(n^2 - 10n + 25) - 25] + 4000$
$R = -20[(n - 5)^2 - 25] + 4000$
$R = -20(n - 5)^2 + 500 + 4000$
$R = -20(n - 5)^2 + 4500$

Why does changing to vertex form help solve the problem?

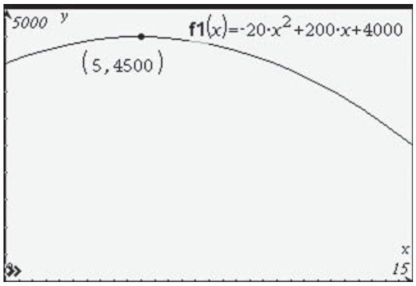
What other methods could you use to find the maximum revenue and the price that gives that revenue?

The vertex form of the function shows that the vertex is at (5, 4500).

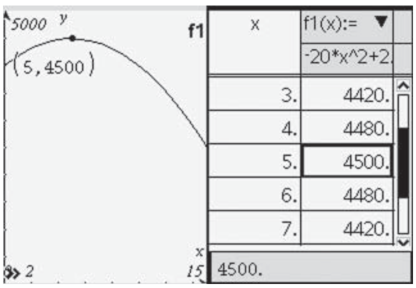
The revenue, R, will be at its maximum value of $4500 when $n = 5$, or when there are five price increases of $1. So, the price per session, or session fee, should be 10 + 5, or $15.

c) You can verify the solution using technology by graphing the function expressed in standard form.

The vertex of the graph is located at (5, 4500). This verifies that the maximum revenue is $4500 with five price increases, or a session fee of $15.



You can also verify the solution numerically by examining the function table. The table shows that a maximum revenue of $4500 occurs with five price increases, or a session fee of $15.



A function table is a table of values generated using a given function.

d) The price the student council sets will affect their revenue from this fundraiser, as they have predicted in using this model.

This model assumes that the price affects the revenue. The revenue function in this situation was based on information about the number of sessions booked last year and predictions on how price changes might affect revenue. However, other factors might affect revenue this year, such as

- how happy people were with their photos last year and whether they tell others or not
- whether the student council advertises the event more this year
- whether the photographer is the same or different from last year
- the date, time, and duration chosen for the event

Your Turn

A sporting goods store sells reusable sports water bottles for \$8. At this price their weekly sales are approximately 100 items. Research says that for every \$2 increase in price, the manager can expect the store to sell five fewer water bottles.

a) Represent this situation with a quadratic function.
b) Determine the maximum revenue the manager can expect based on these estimates. What selling price will give that maximum revenue?
c) Verify your solution.
d) Explain any assumptions you made in using a quadratic function in this situation.

Key Ideas

- You can convert a quadratic function from standard form to vertex form by completing the square.

$y = 5x^2 - 30x + 7$	← standard form
$y = 5(x^2 - 6x) + 7$	Group the first two terms. Factor out the leading coefficient if $a \neq 1$.
$y = 5(x^2 - 6x + 9 - 9) + 7$	Add and then subtract the square of half the coefficient of the x-term.
$y = 5[(x^2 - 6x + 9) - 9] + 7$	Group the perfect square trinomial.
$y = 5[(x - 3)^2 - 9] + 7$	Rewrite using the square of a binomial.
$y = 5(x - 3)^2 - 45 + 7$	Simplify.
$y = 5(x - 3)^2 - 38$	← vertex form

- Converting a quadratic function to vertex form, $y = a(x - p)^2 + q$, reveals the coordinates of the vertex, (p, q).
- You can use information derived from the vertex form to solve problems such as those involving maximum and minimum values.

Check Your Understanding

Practise

1. Use a model to determine the value of c that makes each trinomial expression a perfect square. What is the equivalent binomial square expression for each?

a) $x^2 + 6x + c$

b) $x^2 - 4x + c$

c) $x^2 + 14x + c$

d) $x^2 - 2x + c$

2. Write each function in vertex form by completing the square. Use your answer to identify the vertex of the function.

a) $y = x^2 + 8x$

b) $y = x^2 - 18x - 59$

c) $y = x^2 - 10x + 31$

d) $y = x^2 + 32x - 120$

3. Convert each function to the form $y = a(x - p)^2 + q$ by completing the square. Verify each answer with or without technology.

a) $y = 2x^2 - 12x$

b) $y = 6x^2 + 24x + 17$

c) $y = 10x^2 - 160x + 80$

d) $y = 3x^2 + 42x - 96$

4. Convert each function to vertex form algebraically, and verify your answer.

a) $f(x) = -4x^2 + 16x$

b) $f(x) = -20x^2 - 400x - 243$

c) $f(x) = -x^2 - 42x + 500$

d) $f(x) = -7x^2 + 182x - 70$

5. Verify, in at least two different ways, that the two algebraic forms in each pair represent the same function.

a) $y = x^2 - 22x + 13$
and
$y = (x - 11)^2 - 108$

b) $y = 4x^2 + 120x$
and
$y = 4(x + 15)^2 - 900$

c) $y = 9x^2 - 54x - 10$
and
$y = 9(x - 3)^2 - 91$

d) $y = -4x^2 - 8x + 2$
and
$y = -4(x + 1)^2 + 6$

6. Determine the maximum or minimum value of each function and the value of x at which it occurs.

a) $y = x^2 + 6x - 2$

b) $y = 3x^2 - 12x + 1$

c) $y = -x^2 - 10x$

d) $y = -2x^2 + 8x - 3$

7. For each quadratic function, determine the maximum or minimum value.

a) $f(x) = x^2 + 5x + 3$

b) $f(x) = 2x^2 - 2x + 1$

c) $f(x) = -0.5x^2 + 10x - 3$

d) $f(x) = 3x^2 - 4.8x$

e) $f(x) = -0.2x^2 + 3.4x + 4.5$

f) $f(x) = -2x^2 + 5.8x - 3$

8. Convert each function to vertex form.

a) $y = x^2 + \frac{3}{2}x - 7$

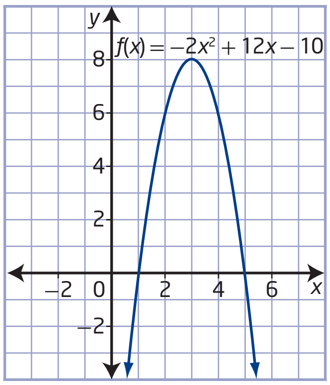
b) $y = -x^2 - \frac{3}{8}x$

c) $y = 2x^2 - \frac{5}{6}x + 1$

Apply

9. a) Convert the quadratic function $f(x) = -2x^2 + 12x - 10$ to vertex form by completing the square.

b) The graph of $f(x) = -2x^2 + 12x - 10$ is shown. Explain how you can use the graph to verify your answer.



10. a) For the quadratic function $y = -4x^2 + 20x + 37$, determine the maximum or minimum value and domain and range without making a table of values or graphing.

b) Explain the strategy you used in part a).

11. Determine the vertex of the graph of $f(x) = 12x^2 - 78x + 126$. Explain the method you used.

12. Identify, explain, and correct the error(s) in the following examples of completing the square.

a) $y = x^2 + 8x + 30$
$y = (x^2 + 4x + 4) + 30$
$y = (x + 2)^2 + 30$

b) $f(x) = 2x^2 - 9x - 55$
$f(x) = 2(x^2 - 4.5x + 20.25 - 20.25) - 55$
$f(x) = 2[(x^2 - 4.5x + 20.25) - 20.25] - 55$
$f(x) = 2[(x - 4.5)^2 - 20.25] - 55$
$f(x) = 2(x - 4.5)^2 - 40.5 - 55$
$f(x) = (x - 4.5)^2 - 95.5$

c) $y = 8x^2 + 16x - 13$
$y = 8(x^2 + 2x) - 13$
$y = 8(x^2 + 2x + 4 - 4) - 13$
$y = 8[(x^2 + 2x + 4) - 4] - 13$
$y = 8[(x + 2)^2 - 4] - 13$
$y = 8(x + 2)^2 - 32 - 13$
$y = 8(x + 2)^2 - 45$

d) $f(x) = -3x^2 - 6x$
$f(x) = -3(x^2 - 6x - 9 + 9)$
$f(x) = -3[(x^2 - 6x - 9) + 9]$
$f(x) = -3[(x - 3)^2 + 9]$
$f(x) = -3(x - 3)^2 + 27$

13. The managers of a business are examining costs. It is more cost-effective for them to produce more items. However, if too many items are produced, their costs will rise because of factors such as storage and overstock. Suppose that they model the cost, C, of producing n thousand items with the function $C(n) = 75n^2 - 1800n + 60\ 000$. Determine the number of items produced that will minimize their costs.

14. A gymnast is jumping on a trampoline. His height, h, in metres, above the floor on each jump is roughly approximated by the function $h(t) = -5t^2 + 10t + 4$, where t represents the time, in seconds, since he left the trampoline. Determine algebraically his maximum height on each jump.

15. Sandra is practising at an archery club. The height, h, in feet, of the arrow on one of her shots can be modelled as a function of time, t, in seconds, since it was fired using the function $h(t) = -16t^2 + 10t + 4$.

a) What is the maximum height of the arrow, in feet, and when does it reach that height?

b) Verify your solution in two different ways.

Did You Know?

The use of the bow and arrow dates back before recorded history and appears to have connections with most cultures worldwide. Archaeologists can learn great deal about the history of the ancestors of today's First Nations and Inuit populations in Canada through the study of various forms of spearheads and arrowheads, also referred to as *projectile points*.

16. Austin and Yuri were asked to convert the function $y = -6x^2 + 72x - 20$ to vertex form. Their solutions are shown.

Austin's solution:
$y = -6x^2 + 72x - 20$
$y = -6(x^2 + 12x) - 20$
$y = -6(x^2 + 12x + 36 - 36) - 20$
$y = -6[(x^2 + 12x + 36) - 36] - 20$
$y = -6[(x + 6) - 36] - 20$
$y = -6(x + 6) + 216 - 20$
$y = -6(x + 6) + 196$

Yuri's solution:

$y = -6x^2 + 72x - 20$
$y = -6(x^2 - 12x) - 20$
$y = -6(x^2 - 12x + 36 - 36) - 20$
$y = -6[(x^2 - 12x + 36) - 36] - 20$
$y = -6[(x - 6)^2 - 36] - 20$
$y = -6(x - 6)^2 - 216 - 20$
$y = -6(x - 6)^2 + 236$

a) Identify, explain, and correct any errors in their solutions.

b) Neither Austin nor Yuri verified their answers. Show several methods that they could have used to verify their solutions. Identify how each method would have pointed out if their solutions were incorrect.

17. A parabolic microphone collects and focuses sound waves to detect sounds from a distance. This type of microphone is useful in situations such as nature audio recording and sports broadcasting. Suppose a particular parabolic microphone has a cross-sectional shape that can be described by the function $d(x) = 0.03125x^2 - 1.5x$, where d is the depth, in centimetres, of the microphone's dish at a horizontal distance of x centimetres from one edge of the dish. Use an algebraic method to determine the depth of the dish, in centimetres, at its centre.

18. A concert promoter is planning the ticket price for an upcoming concert for a certain band. At the last concert, she charged \$70 per ticket and sold 2000 tickets. After conducting a survey, the promoter has determined that for every \$1 decrease in ticket price, she might expect to sell 50 more tickets.

a) What maximum revenue can the promoter expect? What ticket price will give that revenue?

b) How many tickets can the promoter expect to sell at that price?

c) Explain any assumptions the concert promoter is making in using this quadratic function to predict revenues.

Digging Roots, a First Nations band, from Barriere, British Columbia

19. The manager of a bike store is setting the price for a new model. Based on past sales history, he predicts that if he sets the price at \$360, he can expect to sell 280 bikes this season. He also predicts that for every \$10 increase in the price, he expects to sell five fewer bikes.

a) Write a function to model this situation.

b) What maximum revenue can the manager expect? What price will give that maximum?

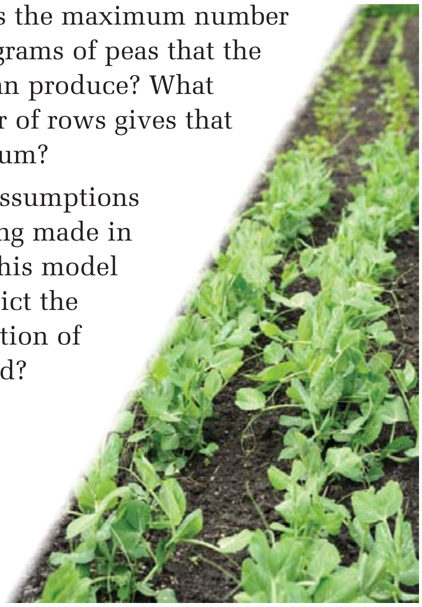
c) Explain any assumptions involved in using this model.

20. A gardener is planting peas in a field. He knows that if he spaces the rows of pea plants closer together, he will have more rows in the field, but fewer peas will be produced by the plants in each row. Last year he planted the field with 30 rows of plants. At this spacing, he got an average of 4000 g of peas per row. He estimates that for every additional row, he will get 100 g less per row.

a) Write a quadratic function to model this situation.

b) What is the maximum number of kilograms of peas that the field can produce? What number of rows gives that maximum?

c) What assumptions are being made in using this model to predict the production of the field?



21. A holding pen is being built alongside a long building. The pen requires only three fenced sides, with the building forming the fourth side. There is enough material for 90 m of fencing.

a) Predict what dimensions will give the maximum area of the pen.

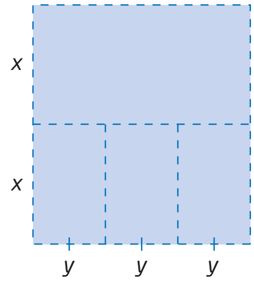
b) Write a function to model the area.

c) Determine the maximum possible area.

d) Verify your solution in several ways, with or without technology. How does the solution compare to your prediction?

e) Identify any assumptions you made in using the model function that you wrote.

22. A set of fenced-in areas, as shown in the diagram, is being planned on an open field. A total of 900 m of fencing is available. What measurements will maximize the overall area of the entire enclosure?

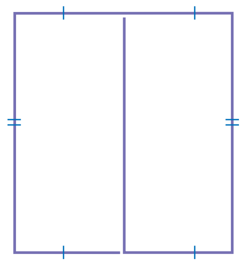


23. Use a quadratic function model to solve each problem.

a) Two numbers have a sum of 29 and a product that is a maximum. Determine the two numbers and the maximum product.

b) Two numbers have a difference of 13 and a product that is a minimum. Determine the two numbers and the minimum product.

24. What is the maximum total area that 450 cm of string can enclose if it is used to form the perimeters of two adjoining rectangles as shown?



Extend

25. Write $f(x) = -\frac{3}{4}x^2 + \frac{9}{8}x + \frac{5}{16}$ in vertex form.

26. a) Show the process of completing the square for the function $y = ax^2 + bx + c$.

b) Express the coordinates of the vertex in terms of a, b, and c.

c) How can you use this information to solve problems involving quadratic functions in standard form?

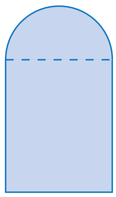
27. The vertex of a quadratic function in standard form is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

a) Given the function $f(x) = 2x^2 - 12x + 22$ in standard form, determine the vertex.

b) Determine the vertex by converting the function to vertex form.

c) Show the relationship between the parameters a, b, and c in standard form and the parameters a, p, and q in vertex form.

28. A Norman window has the shape of a rectangle with a semicircle on the top. Consider a Norman window with a perimeter of 6 m.



a) Write a function to approximate the area of the window as a function of its width.

b) Complete the square to approximate the maximum possible area of the window and the width that gives that area.

c) Verify your answer to part b) using technology.

d) Determine the other dimensions and draw a scale diagram of the window. Does its appearance match your expectations?

Create Connections

29. a) Is the quadratic function $f(x) = 4x^2 + 24$ written in vertex or in standard form? Discuss with a partner.

b) Could you complete the square for this function? Explain.

30. Martine's teacher asks her to complete the square for the function $y = -4x^2 + 24x + 5$. After looking at her solution, the teacher says that she made four errors in her work. Identify, explain, and correct her errors.

Martine's solution:

$y = -4x^2 + 24x + 5$
$y = -4(x^2 + 6x) + 5$
$y = -4(x^2 + 6x + 36 - 36) + 5$
$y = -4[(x^2 + 6x + 36) - 36] + 5$
$y = -4[(x + 6)^2 - 36] + 5$
$y = -4(x + 6)^2 - 216 + 5$
$y = -4(x + 6)^2 - 211$

31. A local store sells T-shirts for \$10. At this price, the store sells an average of 100 shirts each month. Market research says that for every \$1 increase in the price, the manager of the store can expect to sell five fewer shirts each month.

a) Write a quadratic function to model the revenue in terms of the increase in price.

b) What information can you determine about this situation by completing the square?

c) What assumptions have you made in using this quadratic function to predict revenue?

Project Corner — Quadratic Functions in Motion

- Quadratic functions appear in the shapes of various types of stationary objects, along with situations involving moving ones. You can use a video clip to show the motion of a person, animal, or object that appears to create a quadratic model function using a suitably placed set of coordinate axes.
- What situations involving motion could you model using quadratic functions?

Chapter 3 Review

3.1 Investigating Quadratic Functions in Vertex Form, pages 142–162

1. Use transformations to explain how the graph of each quadratic function compares to the graph of $f(x) = x^2$. Identify the vertex, the axis of symmetry, the direction of opening, the maximum or minimum value, and the domain and range without graphing.

a) $f(x) = (x + 6)^2 - 14$

b) $f(x) = -2x^2 + 19$

c) $f(x) = \frac{1}{5}(x - 10)^2 + 100$

d) $f(x) = -6(x - 4)^2$

2. Sketch the graph of each quadratic function using transformations. Identify the vertex, the axis of symmetry, the maximum or minimum value, the domain and range, and any intercepts.

a) $f(x) = 2(x + 1)^2 - 8$

b) $f(x) = -0.5(x - 2)^2 + 2$

3. Is it possible to determine the number of x-intercepts in each case without graphing? Explain why or why not.

a) $y = -3(x - 5)^2 + 20$

b) a parabola with a domain of all real numbers and a range of $\{y \mid y \geq 0, y \in R\}$

c) $y = 9 + 3x^2$

d) a parabola with a vertex at $(-4, -6)$

4. Determine a quadratic function with each set of characteristics.

a) vertex at $(0, 0)$, passing through the point $(20, -150)$

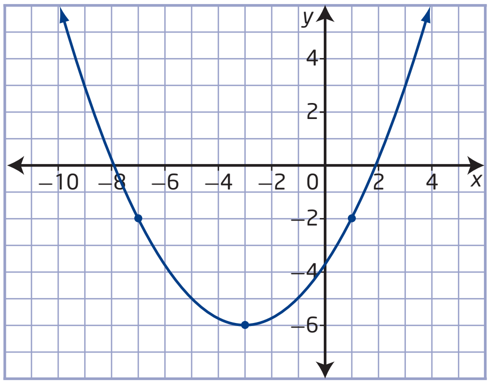
b) vertex at $(8, 0)$, passing through the point $(2, 54)$

c) minimum value of 12 at $x = -4$ and y-intercept of 60

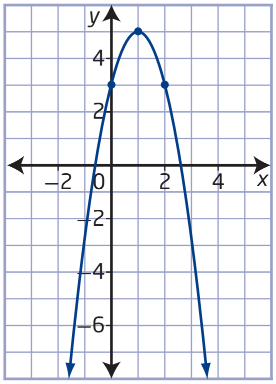
d) x-intercepts of 2 and 7 and maximum value of 25

5. Write a quadratic function in vertex form for each graph.

a)



b)



6. A parabolic trough is a solar-energy collector. It consists of a long mirror with a cross-section in the shape of a parabola. It works by focusing the Sun's rays onto a central axis running down the length of the trough. Suppose a particular solar trough has width 180 cm and depth 56 cm. Determine the quadratic function that represents the cross-sectional shape of the mirror.

7. The main span of the suspension bridge over the Peace River in Dunvegan, Alberta, has supporting cables in the shape of a parabola. The distance between the towers is 274 m. Suppose that the ends of the cables are attached to the tops of the two supporting towers at a height of 52 m above the surface of the water, and the lowest point of the cables is 30 m above the water's surface.

a) Determine a quadratic function that represents the shape of the cables if the origin is at

i) the minimum point on the cables

ii) a point on the water's surface directly below the minimum point of the cables

iii) the base of the tower on the left

b) Would the quadratic function change over the course of the year as the seasons change? Explain.

8. A flea jumps from the ground to a height of 30 cm and travels 15 cm horizontally from where it started. Suppose the origin is located at the point from which the flea jumped. Determine a quadratic function in vertex form to model the height of the flea compared to the horizontal distance travelled.

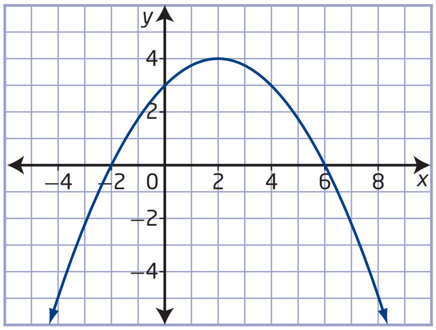
Did You Know?

The average flea can pull 160 000 times its own mass and can jump 200 times its own length. This is equivalent to a human being pulling 24 million pounds and jumping nearly 1000 ft!

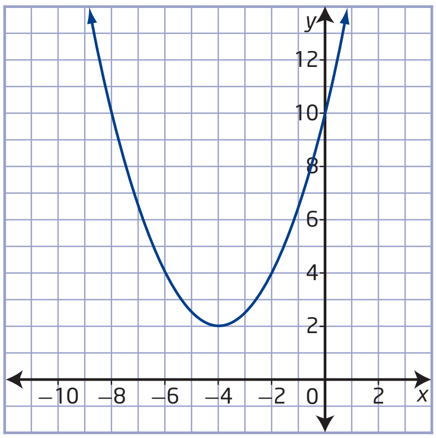
3.2 Investigating Quadratic Functions in Standard Form, pages 163–179

9. For each graph, identify the vertex, axis of symmetry, maximum or minimum value, direction of opening, domain and range, and any intercepts.

a)



b)



10. Show why each function fits the definition of a quadratic function.

a) $y = 7(x + 3)^2 - 41$

b) $y = (2x + 7)(10 - 3x)$

11. a) Sketch the graph of the function $f(x) = -2x^2 + 3x + 5$. Identify the vertex, the axis of symmetry, the direction of opening, the maximum or minimum value, the domain and range, and any intercepts.

b) Explain how each feature can be identified from the graph.

12. A goaltender kicks a soccer ball through the air to players downfield. The trajectory of the ball can be modelled by the function $h(d) = -0.032d^2 + 1.6d$, where d is the horizontal distance, in metres, from the person kicking the ball and h is the height at that distance, in metres.

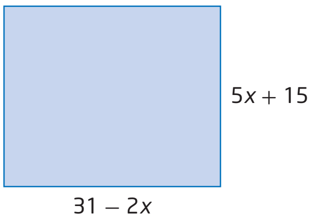
a) Represent the function with a graph, showing all important characteristics.

b) What is the maximum height of the ball? How far downfield is the ball when it reaches that height?

c) How far downfield does the ball hit the ground?

d) What are the domain and range in this situation?

13. a) Write a function to represent the area of the rectangle.



b) Graph the function.

c) What do the x-intercepts represent in this situation?

d) Does the function have a maximum value in this situation? Does it have a minimum value?

e) What information does the vertex give about this situation?

f) What are the domain and range?

3.3 Completing the Square, pages 180–197

14. Write each function in vertex form, and verify your answer.

a) $y = x^2 - 24x + 10$

b) $y = 5x^2 + 40x - 27$

c) $y = -2x^2 + 8x$

d) $y = -30x^2 - 60x + 105$

15. Without graphing, state the vertex, the axis of symmetry, the maximum or minimum value, and the domain and range of the function $f(x) = 4x^2 - 10x + 3$.

16. Amy tried to convert the function $y = -22x^2 - 77x + 132$ to vertex form.

Amy's solution:

$y = -22x^2 - 77x + 132$
$y = -22(x^2 - 3.5x) + 132$
$y = -22(x^2 - 3.5x - 12.25 + 12.25) + 132$
$y = -22(x^2 - 3.5x - 12.25) - 269.5 + 132$
$y = -22(x - 3.5)^2 - 137.5$

a) Identify, explain, and correct the errors.

b) Verify your correct solution in several different ways, both with and without technology.

17. The manager of a clothing company is analysing its costs, revenues, and profits to plan for the upcoming year. Last year, a certain type of children's winter coat was priced at \$40, and the company sold 10 000 of them. Market research says that for every \$2 decrease in the price, the manager can expect the company to sell 500 more coats.

a) Model the expected revenue as a function of the number of price decreases.

b) Without graphing, determine the maximum revenue and the price that will achieve that revenue.

c) Graph the function to confirm your answer.

d) What does the y-intercept represent in this situation? What do the x-intercepts represent?

e) What are the domain and range in this situation?

f) Explain some of the assumptions that the manager is making in using this function to model the expected revenue.

Chapter 3 Practice Test

Multiple Choice

For #1 to #6, choose the best answer.

1. Which function is NOT a quadratic function?

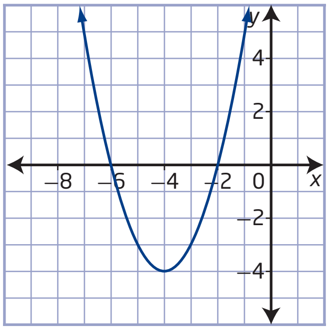
A $f(x) = 2(x + 1)^2 - 7$

B $f(x) = (x - 3)(2x + 5)$

C $f(x) = 5x^2 - 20$

D $f(x) = 3(x - 9) + 6$

2. Which quadratic function represents the parabola shown?



A $y = (x + 4)^2 + 4$

B $y = (x - 4)^2 + 4$

C $y = (x + 4)^2 - 4$

D $y = (x - 4)^2 - 4$

3. Identify the range for the function $y = -6(x - 6)^2 + 6$.

A $\{y \mid y \le 6, y \in R\}$

B $\{y \mid y \ge 6, y \in R\}$

C $\{y \mid y \le -6, y \in R\}$

D $\{y \mid y \ge -6, y \in R\}$

4. Which quadratic function in vertex form is equivalent to $y = x^2 - 2x - 5$?

A $y = (x - 2)^2 - 1$

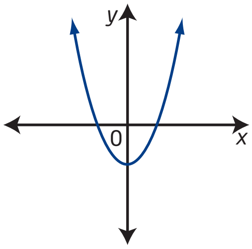
B $y = (x - 2)^2 - 9$

C $y = (x - 1)^2 - 4$

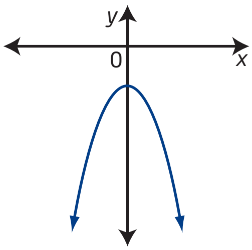
D $y = (x - 1)^2 - 6$

5. Which graph shows the function $y = 1 + ax^2$ if $a < 0$?

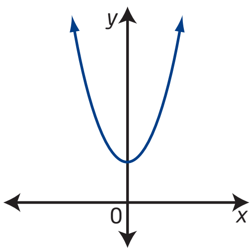
A



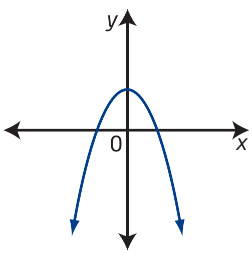
B



C



D



6. What conditions on a and q will give the function $f(x) = a(x - p)^2 + q$ no x-intercepts?

A $a > 0$ and $q > 0$

B $a < 0$ and $q > 0$

C $a > 0$ and $q = 0$

D $a < 0$ and $q = 0$

Short Answer

7. Write each quadratic function in vertex form by completing the square.

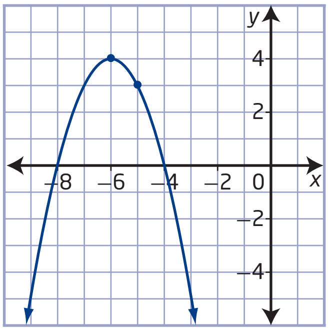
a) $y = x^2 - 18x - 27$

b) $y = 3x^2 + 36x + 13$

c) $y = -10x^2 - 40x$

8. a) For the graph shown, give the coordinates of the vertex, the equation of the axis of symmetry, the minimum or maximum value, the domain and range, and the x-intercepts.

b) Determine a quadratic function in vertex form for the graph.



9. a) Identify the transformation(s) on the graph of $f(x) = x^2$ that could be used to graph each function.

i) $f(x) = 5x^2$

ii) $f(x) = x^2 - 20$

iii) $f(x) = (x + 11)^2$

iv) $f(x) = -\frac{1}{7}x^2$

b) For each function in part a), state which of the following would be different as compared to $f(x) = x^2$ as a result of the transformation(s) involved, and explain why.

i) vertex

ii) axis of symmetry

iii) range

10. Sketch the graph of the function $y = 2(x - 1)^2 - 8$ using transformations. Then, copy and complete the table.

Vertex	
Axis of Symmetry	
Direction of Opening	
Domain	
Range	
x-Intercepts	
y-Intercept	

11. The first three steps in completing the square below contain one or more errors.

$y = 2x^2 - 8x + 9$
$y = 2(x^2 - 8x) + 9$
$y = 2(x^2 - 8x - 64 + 64) + 9$

a) Identify and correct the errors.

b) Complete the process to determine the vertex form of the function.

c) Verify your correct solution in several different ways.

12. The fuel consumption for a vehicle is related to the speed that it is driven and is usually given in litres per one hundred kilometres. Engines are generally more efficient at higher speeds than at lower speeds. For a particular type of car driving at a constant speed, the fuel consumption, C, in litres per one hundred kilometres, is related to the average driving speed, v, in kilometres per hour, by the function $C(v) = 0.004v^2 - 0.62v + 30$.

a) Without graphing, determine the most efficient speed at which this car should be driven. Explain/show the strategy you use.

b) Describe any characteristics of the graph that you can identify without actually graphing, and explain how you know.

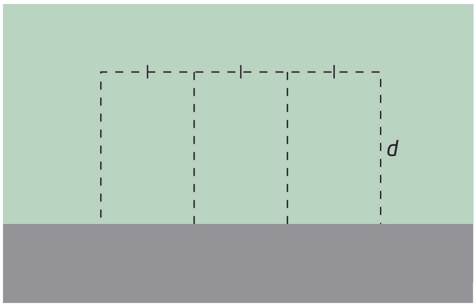
13. The height, h, in metres, of a flare t seconds after it is fired into the air can be modelled by the function $h(t) = -4.9t^2 + 61.25t$.

a) At what height is the flare at its maximum? How many seconds after being shot does this occur?

b) Verify your solution both with and without technology.

Extended Response

14. Three rectangular areas are being enclosed along the side of a building, as shown. There is enough material to make 24 m of fencing.



a) Write the function that represents the total area in terms of the distance from the wall.

b) Show that the function fits the definition of a quadratic function.

c) Graph the function. Explain the strategy you used.

d) What are the coordinates of the vertex? What do they represent?

e) What domain and range does the function have in this situation? Explain.

f) Does the function have a maximum value? Does it have a minimum value? Explain.

g) What assumptions are made in using this quadratic function model?

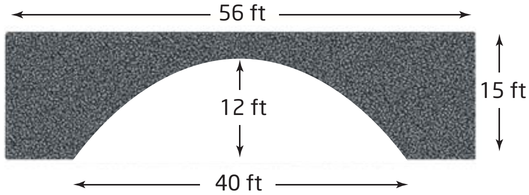
15. A stone bridge has the shape of a parabolic arch, as shown. Determine a quadratic function to represent the shape of the arch if the origin

a) is at the top of the opening under the bridge

b) is on the ground at the midpoint of the opening

c) is at the base of the bridge on the right side of the opening

d) is on the left side at the top surface of the bridge



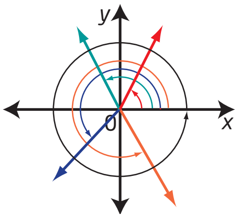
16. A store sells energy bars for \$2.25. At this price, the store sold an average of 120 bars per month last year. The manager has been told that for every 5¢ decrease in price, he can expect the store to sell eight more bars monthly.

a) What quadratic function can you use to model this situation?

b) Use an algebraic method to determine the maximum revenue the manager can expect the store to achieve. What price will give that maximum?

c) What assumptions are made in this situation?

4. C
5. D
6. $0.15 per cup
7. 45°
8. 300°
9. 2775
10. a) 5 b) −6
c) $t_n = 5n - 11$ d) $S_{10} = 165$
11. $14 880.35
12. 4 km
13. a) 64, 32, 16, 8, … b) $t_n = 64\left(\frac{1}{2}\right)^{n-1}$
c) 63 games
14. a)

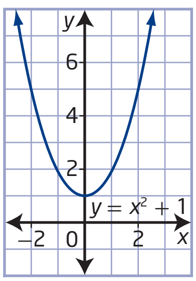


b) 60, 120, 180, 240, 300, 360
c) $t_n = 60n$
15. a) 58° b) 5.3 m
16. 38°

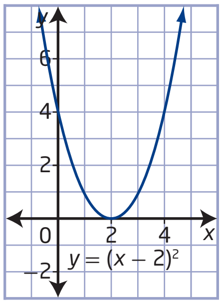
Chapter 3 Quadratic Functions

3.1 Investigating Quadratic Functions in Vertex Form, pages 157 to 162

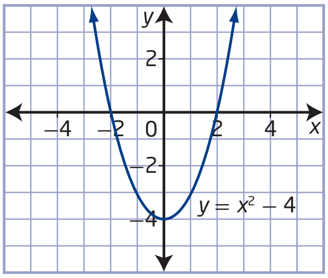
1. a) Since $a > 0$ in $f(x) = 7x^2$, the graph opens upward, has a minimum value, and has a range of $\{y \mid y \geq 0, y \in \mathrm{R}\}$.
b) Since $a > 0$ in $f(x) = \frac{1}{6}x^2$, the graph opens upward, has a minimum value, and has a range of $\{y \mid y \geq 0, y \in \mathrm{R}\}$.
c) Since $a < 0$ in $f(x) = -4x^2$, the graph opens downward, has a maximum value, and has a range of $\{y \mid y \leq 0, y \in \mathrm{R}\}$.
d) Since $a < 0$ in $f(x) = -0.2x^2$, the graph opens downward, has a maximum value, and has a range of $\{y \mid y \leq 0, y \in \mathrm{R}\}$.
2. a) The shapes of the graphs are the same with the parabola of $y = x^2 + 1$ being one unit higher. vertex: (0, 1), axis of symmetry: $x = 0$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq 1, y \in \mathrm{R}\}$, no x-intercepts, y-intercept occurs at (0, 1)



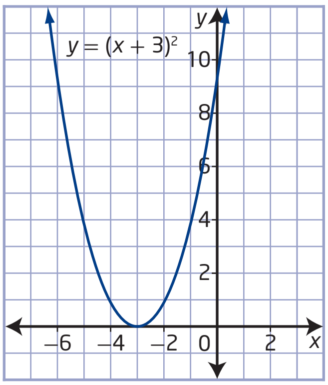
b) The shapes of the graphs are the same with the parabola of $y = (x - 2)^2$ being two units to the right. vertex: (2, 0), axis of symmetry: $x = 2$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq 0, y \in \mathrm{R}\}$, x-intercept occurs at (2, 0), y-intercept occurs at (0, 4)



c) The shapes of the graphs are the same with the parabola of $y = x^2 - 4$ being four units lower.



vertex: (0, −4), axis of symmetry: $x = 0$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq -4, y \in \mathrm{R}\}$, x-intercepts occur at (−2, 0) and (2, 0), y-intercept occurs at (0, −4)
d) The shapes of the graphs are the same with the parabola of $y = (x + 3)^2$ being three units to the left.



vertex: (−3, 0), axis of symmetry: $x = -3$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq 0, y \in \mathrm{R}\}$, x-intercept occurs at (−3, 0), y-intercept occurs at (0, 9)

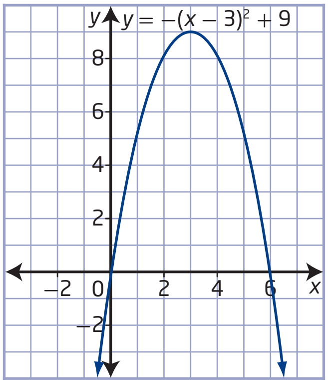
3. a) Given the graph of $y = x^2$, move the entire graph 5 units to the left and 11 units up.

b) Given the graph of $y = x^2$, apply the change in width, which is a multiplication of the y-values by a factor of 3, making it narrower, reflect it in the x-axis so it opens downward, and move the entire new graph down 10 units.

c) Given the graph of $y = x^2$, apply the change in width, which is a multiplication of the y-values by a factor of 5, making it narrower. Move the entire new graph 20 units to the left and 21 units down.

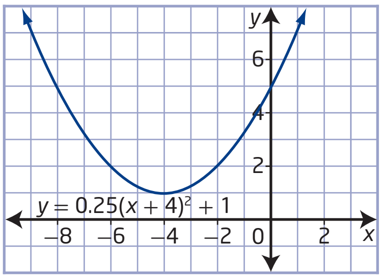
d) Given the graph of $y = x^2$, apply the change in width, which is a multiplication of the y-values by a factor of $\frac{1}{8}$, making it wider, reflect it in the x-axis so it opens downward, and move the entire new graph 5.6 units to the right and 13.8 units up.

4. a)



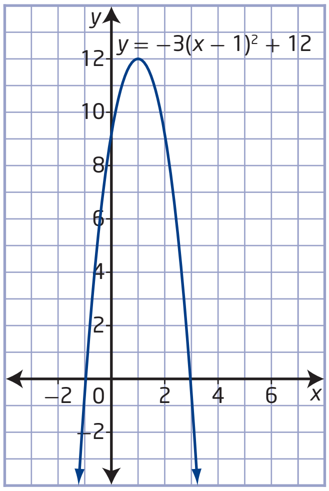
vertex: (3, 9), axis of symmetry: $x = 3$, opens downward, maximum value of 9, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq 9, y \in \mathrm{R}\}$, x-intercepts occur at (0, 0) and (6, 0), y-intercept occurs at (0, 0)

b)



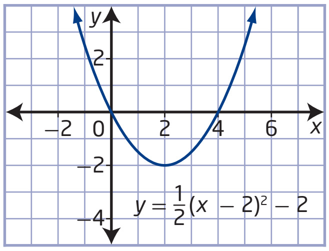
vertex: (−4, 1), axis of symmetry: $x = -4$, opens upward, minimum value of 1, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq 1, y \in \mathrm{R}\}$, no x-intercepts, y-intercept occurs at (0, 5)

c)



vertex: (1, 12), axis of symmetry: $x = 1$, opens downward, maximum value of 12, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq 12, y \in \mathrm{R}\}$, x-intercepts occur at (−1, 0) and (3, 0), y-intercept occurs at (0, 9)

d)



vertex: (2, −2), axis of symmetry: $x = 2$, opens upward, minimum value of −2, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq -2, y \in \mathrm{R}\}$, x-intercepts occur at (0, 0) and (4, 0), y-intercept occurs at (0, 0)

5. a) $y_1 = x^2$, $y_2 = 4x^2 + 2$, $y_3 = \frac{1}{2}x^2 - 2$, $y_4 = \frac{1}{4}x^2 - 4$

b) $y_1 = -x^2$, $y_2 = -4x^2 + 2$, $y_3 = -\frac{1}{2}x^2 - 2$, $y_4 = -\frac{1}{4}x^2 - 4$

c) $y_1 = (x + 4)^2$, $y_2 = 4(x + 4)^2 + 2$, $y_3 = \frac{1}{2}(x + 4)^2 - 2$, $y_4 = \frac{1}{4}(x + 4)^2 - 4$

d) $y_1 = x^2 - 2$, $y_2 = 4x^2$, $y_3 = \frac{1}{2}x^2 - 4$, $y_4 = \frac{1}{4}x^2 - 6$

6. For the function $f(x) = 5(x - 15)^2 - 100$, $a = 5$, $p = 15$, and $q = -100$.

a) The vertex is located at (p, q), or (15, −100).

b) The equation of the axis of symmetry is $x = p$, or $x = 15$.

c) Since $a > 0$, the graph opens upward.

d) Since $a > 0$, the graph has a minimum value of q, or -100.

e) The domain is $\{x \mid x \in R\}$. Since the function has a minimum value of -100, the range is $\{y \mid y \geq -100, y \in R\}$.

f) Since the graph has a minimum value of -100 and opens upward, there are two x-intercepts.

7. a) vertex: (0, 14), axis of symmetry: $x = 0$, opens downward, maximum value of 14, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \leq 14, y \in R\}$, two x-intercepts

b) vertex: $(-18, -8)$, axis of symmetry: $x = -18$, opens upward, minimum value of -8, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \geq -8, y \in R\}$, two x-intercepts

c) vertex: (7, 0), axis of symmetry: $x = 7$, opens upward, minimum value of 0, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \geq 0, y \in R\}$, one x-intercept

d) vertex: $(-4, -36)$, axis of symmetry: $x = -4$, opens downward, maximum value of -36, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \leq -36, y \in R\}$, no x-intercepts

8. a) $y = (x + 3)^2 - 4$ **b)** $y = -2(x - 1)^2 + 12$

c) $y = \frac{1}{2}(x - 3)^2 + 1$ **d)** $y = -\frac{1}{4}(x + 3)^2 + 4$

9. a) $y = -\frac{1}{4}x^2$ **b)** $y = 3x^2 - 6$

c) $y = -4(x - 2)^2 + 5$ **d)** $y = \frac{1}{5}(x + 3)^2 - 10$

10. a) $(4, 16) \rightarrow (-1, 16) \rightarrow (-1, 24)$

b) $(4, 16) \rightarrow (4, 4) \rightarrow (4, -4)$

c) $(4, 16) \rightarrow (4, -16) \rightarrow (14, -16)$

d) $(4, 16) \rightarrow (4, 48) \rightarrow (4, 40)$

11. Starting with the graph of $y = x^2$, apply the change in width, which is a multiplication of the y-values by a factor of 5, reflect the graph in the x-axis, and then move the entire graph up 20 units.

12. Example: Quadratic functions will always have one y-intercept. Since the graphs always open upward or downward and have a domain of $\{x \mid x \in R\}$, the parabola will always cross the y-axis. The graphs must always have a value at $x = 0$ and therefore have one y-intercept.

13. a) $y = \frac{1}{30}x^2$

b) The new function could be $y = \frac{1}{30}(x - 30)^2 - 30$ or $y = \frac{1}{30}(x + 30)^2 - 30$. Both graphs have the same size and shape, but the new function has been transformed by a horizontal translation of 30 units to the right or to the left and a vertical translation of 30 units down to represent a point on the edge as the origin.

14. a) The vertex is located at (36, 20 000), it opens downward, and it has a change in width by a multiplication of the y-values by a factor of 2.5 of the graph $y = x^2$. The equation of the axis of symmetry is $x = 36$, and the graph has a maximum value of 20 000.

b) 36 times

c) 20 000 people

15. Examples: If the vertex is at the origin, the quadratic function will be $y = 0.03x^2$. If the edge of the rim is at the origin, the quadratic function will be $y = 0.03(x - 20)^2 - 12$.

16. a) Example: Placing the vertex at the origin, the quadratic function is $y = \frac{1}{294}x^2$ or $y \approx 0.0034x^2$.

b) Example: If the origin is at the top of the left tower, the quadratic function is $y = \frac{1}{294}(x - 84)^2 - 24$ or $y \approx 0.0034(x - 84)^2 - 24$. If the origin is at the top of the right tower, the quadratic function is $y = \frac{1}{294}(x + 84)^2 - 24$ or $y \approx 0.0034(x + 84)^2 - 24$.

c) 8.17 m; this is the same no matter which function is used.

17. $y = -\frac{9}{121}(x - 11)^2 + 9$

18. $y = -\frac{1}{40}(x - 60)^2 + 90$

19. Example: Adding q is done after squaring the x-value, so the transformation applies directly to the parabola $y = x^2$. The value of p is added or subtracted before squaring, so the shift is opposite to the sign in the bracket to get back to the original y-value for the graph of $y = x^2$.

20. a) $y = -\frac{7}{160\,000}(x - 8000)^2 + 10\,000$

b) domain: $\{x \mid 0 \leq x \leq 16\,000, x \in R\}$, range: $\{y \mid 7200 \leq y \leq 10\,000, y \in R\}$

21. a) Since the vertex is located at (6, 30), $p = 6$ and $q = 30$. Substituting these values into the vertex form of a quadratic function and using the coordinates of the given point, the function is $y = -1.5(x - 6)^2 + 30$.

b) Knowing that the x-intercepts are -21 and -5, the equation of the axis of symmetry must be $x = -13$. Then, the vertex is located at $(-13, -24)$. Substituting the coordinates of the vertex and one of the x-intercepts into the vertex form, the quadratic function is $y = 0.375(x + 13)^2 - 24$.

22. a) Examples: I chose $x = 8$ as the axis of symmetry, I choose the position of the hoop to be (1, 10), and I allowed the basketball to be released at various heights (6 ft, 7 ft, and 8 ft) from a distance of 16 ft from the hoop. For each scenario, substitute the coordinates of the release point into the function $y = a(x - 8)^2 + q$ to get an expression for q. Then, substitute the expression for q and the coordinates of the hoop into the function. My three functions are

$y = -\frac{4}{15}(x - 8)^2 + \frac{346}{15}$,
$y = -\frac{3}{15}(x - 8)^2 + \frac{297}{15}$, and
$y = -\frac{2}{15}(x - 8)^2 + \frac{248}{15}$.

b) Example: $y = -\frac{4}{15}(x - 8)^2 + \frac{346}{15}$ ensures that the ball passes easily through the hoop.

c) domain: $\{x \mid 0 \le x \le 16, x \in R\}$, range: $\left\{y \mid 0 \le y \le \frac{346}{15}, y \in R\right\}$

23. $(m + p, an + q)$

24. Examples:

a) $f(x) = -2(x - 1)^2 + 3$

b) Plot the vertex (1, 3). Determine a point on the curve, say the y-intercept, which occurs at (0, 1). Determine that the corresponding point of (0, 1) is (2, 1). Plot these two additional points and complete the sketch of the parabola.

25. Example: You can determine the number of x-intercepts if you know the location of the vertex and the direction of opening. Visualize the general position and shape of the graph based on the values of a and q. Consider $f(x) = 0.5(x + 1)^2 - 3$, $g(x) = 2(x - 3)^2$, and $h(x) = -2(x + 3)^2 - 4$. For $f(x)$, the parabola opens upward and the vertex is below the x-axis, so the graph has two x-intercepts. For $g(x)$, the parabola opens upward and the vertex is on the x-axis, so the graph has one x-intercept. For $h(x)$, the parabola opens downward and the vertex is below the x-axis, so the graph has no x-intercepts.

26. Answers may vary.

3.2 Investigating Quadratic Functions in Standard Form, pages 174 to 179

1. a) This is a quadratic function, since it is a polynomial of degree two.

b) This is not a quadratic function, since it is a polynomial of degree one.

c) This is not a quadratic function. Once the expression is expanded, it is a polynomial of degree three.

d) This is a quadratic function. Once the expression is expanded, it is a polynomial of degree two.

2. a) The coordinates of the vertex are (−2, 2). The equation of the axis of symmetry is $x = -2$. The x-intercepts occur at (−3, 0) and (−1, 0), and the y-intercept occurs at (0, −6). The graph opens downward, so the graph has a maximum of 2 of when $x = -2$. The domain is $\{x \mid x \in R\}$ and the range is $\{y \mid y \le 2, y \in R\}$.

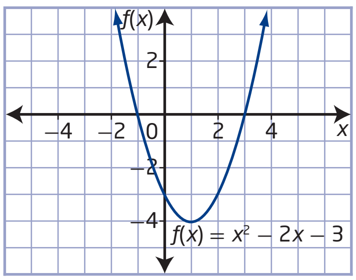
b) The coordinates of the vertex are (6, −4). The equation of the axis of symmetry is $x = 6$. The x-intercepts occur at (2, 0) and (10, 0), and the y-intercept occurs at (0, 5). The graph opens upward, so the graph has a minimum of −4 when $x = 6$. The domain is $\{x \mid x \in R\}$ and the range is $\{y \mid y \ge -4, y \in R\}$.

c) The coordinates of the vertex are (3, 0). The equation of the axis of symmetry is $x = 3$. The x-intercept occurs at (3, 0), and the y-intercept occurs at (0, 8). The graph opens upward, so the graph has a minimum of 0 when $x = 3$. The domain is $\{x \mid x \in R\}$ and the range is $\{y \mid y \ge 0, y \in R\}$.

3. a) $f(x) = -10x^2 + 50x$

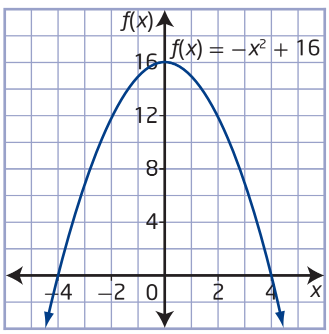
b) $f(x) = 15x^2 - 62x + 40$

4. a)

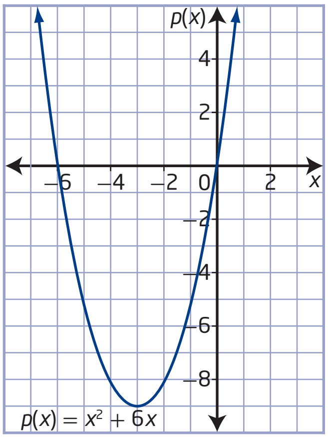


vertex is (1, −4); axis of symmetry is $x = 1$; opens upward; minimum value of −4 when $x = 1$; domain is $\{x \mid x \in R\}$, range is $\{y \mid y \ge -4, y \in R\}$; x-intercepts occur at (−1, 0) and (3, 0), y-intercept occurs at (0, −3)

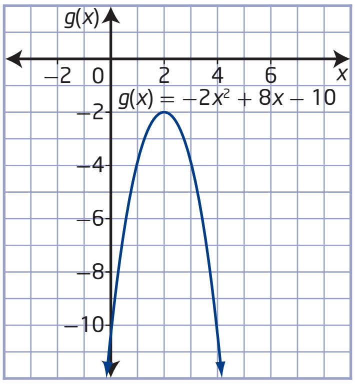
b)



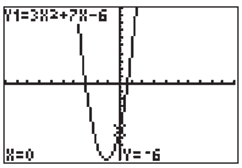
vertex is (0, 16); axis of symmetry is $x = 0$; opens downward; maximum value of 16 when $x = 0$; domain is $\{x \mid x \in \mathrm{R}\}$, range is $\{y \mid y \le 16, y \in \mathrm{R}\}$; x-intercepts occur at $(-4, 0)$ and $(4, 0)$, y-intercept occurs at $(0, 16)$

c) 

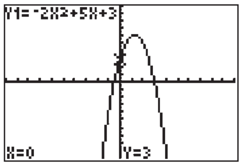
vertex is $(-3, -9)$; axis of symmetry is $x = -3$; opens upward; minimum value of -9 when $x = -3$; domain is $\{x \mid x \in \mathrm{R}\}$, range is $\{y \mid y \ge -9, y \in \mathrm{R}\}$; x-intercepts occur at $(-6, 0)$ and $(0, 0)$, y-intercept occurs at $(0, 0)$

d) 

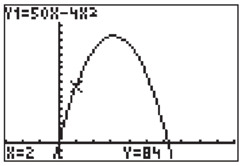
vertex is $(2, -2)$; axis of symmetry is $x = 2$; opens downward; maximum value of -2 when $x = 2$; domain is $\{x \mid x \in \mathrm{R}\}$, range is $\{y \mid y \le -2, y \in \mathrm{R}\}$; no x-intercepts, y-intercept occurs at $(0, -10)$

5. a) 

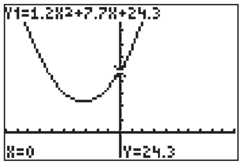
vertex is $(-1.2, -10.1)$; axis of symmetry is $x = -1.2$; opens upward; minimum value of -10.1 when $x = -1.2$; domain is $\{x \mid x \in \mathrm{R}\}$, range is $\{y \mid y \ge -10.1, y \in \mathrm{R}\}$; x-intercepts occur at $(-3, 0)$ and $(0.7, 0)$, y-intercept occurs at $(0, -6)$

b) 

vertex is (1.3, 6.1); axis of symmetry is $x = 1.3$; opens downward; maximum value of 6.1 when $x = 1.3$; domain is $\{x \mid x \in \mathrm{R}\}$, range is $\{y \mid y \le 6.1, y \in \mathrm{R}\}$; x-intercepts occur at $(-0.5, 0)$ and $(3, 0)$, y-intercept occurs at $(0, 3)$

c) 

vertex is (6.3, 156.3); axis of symmetry is $x = 6.3$; opens downward; maximum value of 156.3 when $x = 6.3$; domain is $\{x \mid x \in \mathrm{R}\}$, range is $\{y \mid y \le 156.3, y \in \mathrm{R}\}$; x-intercepts occur at (0, 0) and (12.5, 0), y-intercept occurs at (0, 0)

d) 

vertex is $(-3.2, 11.9)$; axis of symmetry is $x = -3.2$; opens upward; minimum value of 11.9 when $x = -3.2$; domain is $\{x \mid x \in \mathrm{R}\}$, range is $\{y \mid y \ge 11.9, y \in \mathrm{R}\}$; no x-intercepts, y-intercept occurs at (0, 24.3)

6. a) $(-3, -7)$ **b)** $(2, -7)$ **c)** (4, 5)

7. a) 10 cm, h-intercept of the graph
b) 30 cm after 2 s, vertex of the parabola
c) approximately 4.4 s, t-intercept of the graph
d) domain: $\{t \mid 0 \le t \le 4.4, t \in \mathrm{R}\}$, range: $\{h \mid 0 \le h \le 30, h \in \mathrm{R}\}$
e) Example: No, siksik cannot stay in the air for 4.4 s in real life.

8. Examples:
a) Two; since the graph has a maximum value, it opens downward and would cross the x-axis at two different points. One x-intercept is negative and the other is positive.
b) Two; since the vertex is at (3, 1) and the graph passes through the point $(1, -3)$, it opens downward and crosses the x-axis at two different points. Both x-intercepts are positive.

c) Zero; since the graph has a minimum of 1 and opens upward, it will not cross the x-axis.

d) Two; since the graph has an axis of symmetry of $x = -1$ and passes through the x- and y-axes at (0, 0), the graph could open upward or downward and has another x-intercept at $(-2, 0)$. One x-intercept is zero and the other is negative.

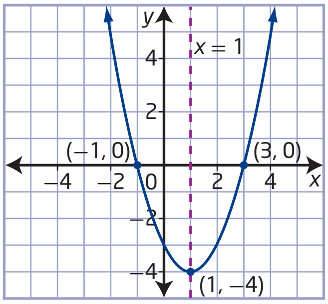
9. a) domain: $\{x \mid x \in R\}$, range: $\{y \mid y \leq 68, y \in R\}$

b) domain: $\{x \mid 0 \leq x \leq 4.06, x \in R\}$, range: $\{y \mid 0 \leq y \leq 68, y \in R\}$

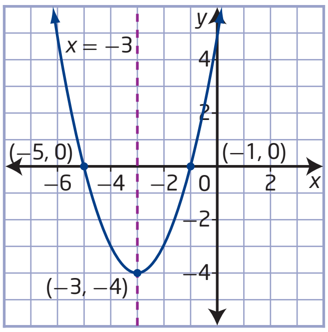
c) Example: The domain and range of algebraic functions may include all real values. For given real-world situations, the domain and range are determined by physical constraints such as time must be greater than or equal to zero and the height must be above ground, or greater than or equal to zero.

10. Examples:

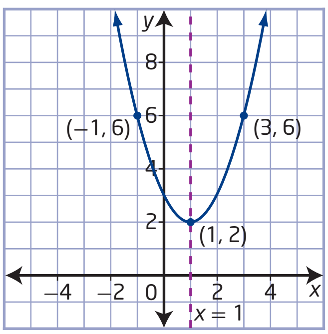
a)



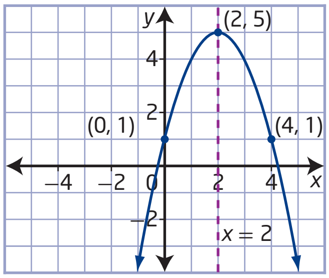
b)



c)

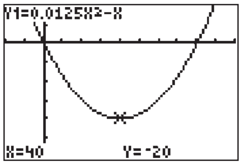


d)



11. a) $\{x \mid 0 \leq x \leq 80, x \in R\}$

b)

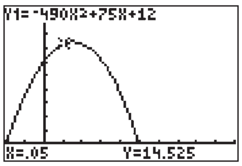


c) The maximum depth of the dish is 20 cm, which is the y-coordinate of the vertex $(40, -20)$. This is not the maximum value of the function. Since the parabola opens upward, this the minimum value of the function.

d) $\{d \mid -20 \leq d \leq 0, d \in R\}$

e) The depth is approximately 17.19 cm, 25 cm from the edge of the dish.

12. a)



b) The h-intercept represents the height of the log.

c) 0.1 s; 14.9 cm

d) 0.3 s

e) domain: $\{t \mid 0 \leq t \leq 0.3, t \in R\}$, range: $\{h \mid 0 \leq h \leq 14.9, h \in R\}$

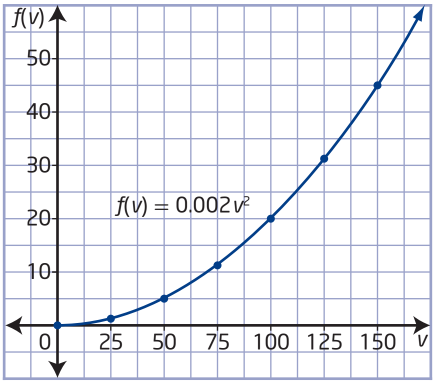
f) 14.5 cm

13. Examples:

a) $\{v \mid 0 \leq v \leq 150, v \in R\}$

b)

v	f
0	0
25	1.25
50	5
75	11.25
100	20
125	31.25
150	45

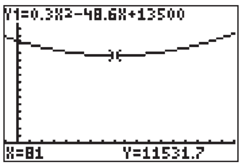


c) The graph is a smooth curve instead of a straight line. The table of values shows that the values of f are not increasing at a constant rate for equal increments in the value of v.

d) The values of the drag force increase by a value other than 2. When the speed of the vehicle doubles, the drag force quadruples.

e) The driver can use this information to improve gas consumption and fuel economy.

14. a)

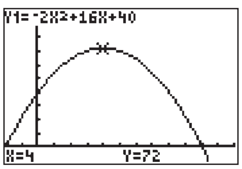


The coordinates of the vertex are (81, 11 532). The equation of the axis of symmetry is $x = 81$. There are no x-intercepts. The y-intercept occurs at (0, 13 500). The graph opens upward, so the graph has a minimum value of 11 532 when $x = 81$. The domain is $\{n \mid n \geq 0, n \in \mathrm{R}\}$. The range is $\{C \mid C \geq 11\,532, C \in \mathrm{R}\}$.

b) Example: The vertex represents the minimum cost of \$11 532 to produce 81 000 units. Since the vertex is above the n-axis, there are no n-intercepts, which means the cost of production is always greater than zero. The C-intercept represents the base production cost. The domain represents thousands of units produced, and the range represents the cost to produce those units.

15. a) $A = -2x^2 + 16x + 40$

b)



c) The values between the x-intercepts will produce a rectangle. The rectangle will have a width that is 2 greater than the value of x and a length that is 20 less 2 times the value of x.

d) The vertex indicates the maximum area of the rectangle.

e) domain: $\{x \mid -2 \leq x \leq 10, x \in \mathrm{R}\}$, range: $\{A \mid 0 \leq A \leq 72, A \in \mathrm{R}\}$; the domain represents the values for x that will produce dimensions of a rectangle. The range represents the possible values of the area of the rectangle.

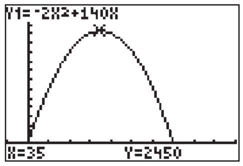
f) The function has both a maximum value and a minimum value for the area of the rectangle.

g) Example: No; the function will open downward and therefore will not have a minimum value for a domain of real numbers.

16. Example: No; the simplified version of the function is $f(x) = 3x + 1$. Since this is not a polynomial of degree two, it does not represent a quadratic function. The graph of the function $f(x) = 4x^2 - 3x + 2x(3 - 2x) + 1$ is a straight line.

17. a) $A = -2x^2 + 140x$; this is a quadratic function since it is a polynomial of degree two.

b)



c) (35, 2450); The vertex represents the maximum area of 2450 m² when the width is 35 m.

d) domain: $\{x \mid 0 \leq x \leq 70, x \in \mathrm{R}\}$, range: $\{A \mid 0 \leq A \leq 2450, A \in \mathrm{R}\}$
The domain represents the possible values of the width, and the range represents the possible values of the area.

e) The function has a maximum area (value) of 2450 m² and a minimum value of 0 m². Areas cannot have negative values.

f) Example: The quadratic function assumes that Maria will use all of the fencing to make the enclosure. It also assumes that any width from 0 m to 70 m is possible.

18. a)

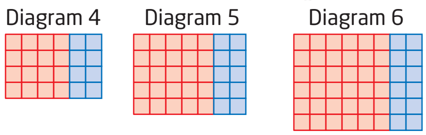


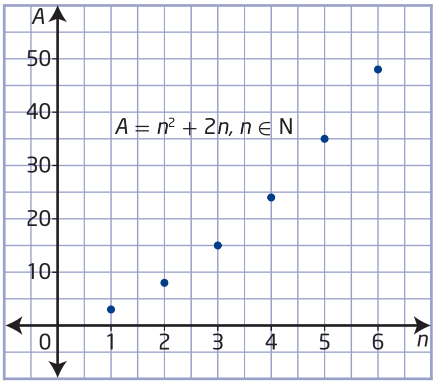
Diagram 4: 24 square units
Diagram 5: 35 square units
Diagram 6: 48 square units

b) $A = n^2 + 2n$

c) Quadratic; the function is a polynomial of degree two.

d) $\{n \mid n \geq 1, n \in \mathrm{N}\}$; The values of n are natural numbers. So, the function is discrete. Since the numbers of both diagrams and small squares are countable, the function is discrete.

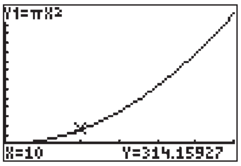
e)



19. a) $A = \pi r^2$

b) domain: $\{r \mid r \geq 0, r \in \mathrm{R}\}$,
range: $\{A \mid A \geq 0, A \in \mathrm{R}\}$

c)



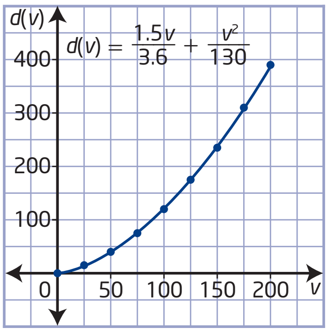
d) The x-intercept and the y-intercept occur at (0, 0). They represent the minimum values of the radius and the area.

e) Example: There is no axis of symmetry within the given domain and range.

20. a) $d(v) = \frac{1.5v}{3.6} + \frac{v^2}{130}$

b)

v	d
0	0
25	15
50	40
75	75
100	119
125	172
150	236
175	308
200	391

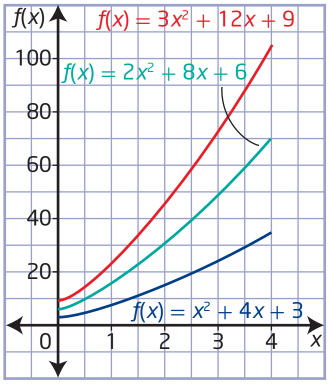


c) No; when v doubles from 25 km/h to 50 km/h, the stopping distance increases by a factor of $\frac{40}{15} = 2.67$, and when the velocity doubles from 50 km/h to 100 km/h, the stopping distance increases by a factor of $\frac{119}{40} = 2.98$. Therefore, the stopping distance increases by a factor greater than two.

d) Example: Using the graph or table, notice that as the speed increases the stopping distances increase by a factor greater than the increase in speed. Therefore, it is important for drivers to maintain greater distances between vehicles as the speed increases to allow for increasing stopping distances.

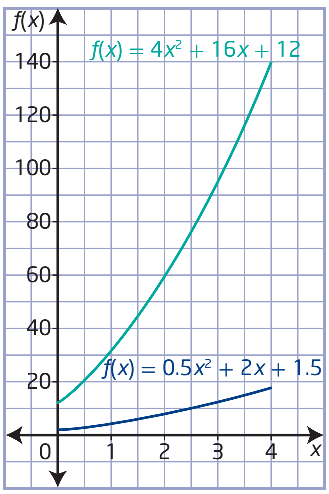
21. a) $f(x) = x^2 + 4x + 3$, $f(x) = 2x^2 + 8x + 6$, and $f(x) = 3x^2 + 12x + 9$

b)

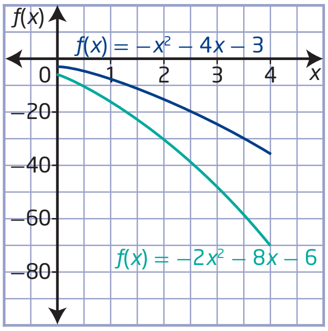


c) Example: The graphs have similar shapes, curving upward at a rate that is a multiple of the first graph. The values of y for each value of x are multiples of each other.

d) Example: If $k = 4$, the graph would start with a y-intercept 4 times as great as the first graph and increase with values of y that are 4 times as great as the values of y of the first function. If $k = 0.5$, the graph would start with a y-intercept $\frac{1}{2}$ of the original y-intercept and increase with values of y that are $\frac{1}{2}$ of the original values of y for each value of x.



e) Example: For negative values of k, the graph would be reflected in the x-axis, with a smooth decreasing curve. Each value of y would be a negative multiple of the original value of y for each value of x.



f) The graph is a line on the x-axis.

g) Example: Each member of the family of functions for $f(x) = k(x^2 + 4x + 3)$ has values of y that are multiples of the original function for each value of x.

22. Example: The value of a in the function $f(x) = ax^2 + bx + c$ indicates the steepness of the curved section of a function in that when $a > 0$, the curve will move up more steeply as a increases and when $-1 < a < 1$, the curve will move up more slowly the closer a is to 0. The sign of a is also similar in that if $a > 0$, then the graph curves up and when $a < 0$, the graph will curve down from the vertex. The value of a in the function $f(x) = ax + b$ indicates the exact steepness or slope of the line determined by the function, whereas the slope of the function $f(x) = ax^2 + bx + c$ changes as the value of x changes and is not a direct relationship for the entire graph.

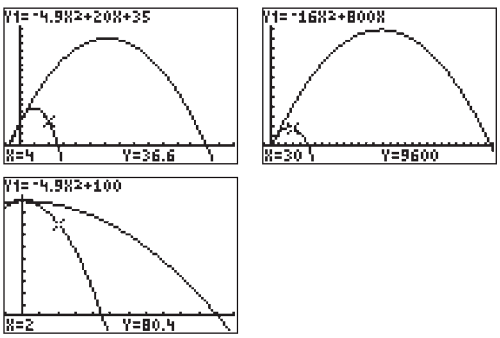
23. a) $b = 3$

b) $b = -3$ and $c = 1$

24. a)

Earth	Moon
$h(t) = -4.9t^2 + 20t + 35$	$h(t) = -0.815t^2 + 20t + 35$
$h(t) = -16t^2 + 800t$	$h(t) = -2.69t^2 + 800t$
$h(t) = -4.9t^2 + 100$	$h(t) = -0.815t^2 + 100$

b)



c) Example: The first two graphs have the same y-intercept at (0, 35). The second two graphs pass through the origin (0, 0). The last two graphs share the same y-intercept at (0, 100). Each pair of graphs share the same y-intercept and share the same constant term.

d) Example: Every projectile on the moon had a higher trajectory and stayed in the air for a longer period of time.

25. Examples:

a) $(2m, r)$; apply the definition of the axis of symmetry. The horizontal distance from the y-intercept to the x-coordinate of the vertex is $m - 0$, or m. So, one other point on the graph is $(m + m, r)$, or $(2m, r)$.

b) $(-2j, k)$; apply the definition of the axis of symmetry. The horizontal distance from the given point to the axis of symmetry is $4j - j$, or $3j$. So, one other point on the graph is $(j - 3j, k)$, or $(-2j, k)$.

c) $\left(\frac{s + t}{2}, d\right)$; apply the definitions of the axis of symmetry and the minimum value of a function. The x-coordinate of the vertex is halfway between the x-intercepts, or $\frac{s + t}{2}$. The y-coordinate of the vertex is the least value of the range, or d.

26. Example: The range and direction of opening are connected and help determine the location of the vertex. If $y \geq q$, then the graph will open upward. If $y \leq q$, then the graph will open downward. The range also determines the maximum or minimum value of the function and the y-coordinate of the vertex. The equation of the axis of symmetry determines the x-coordinate of the vertex. If the vertex is above the x-axis and the graph opens upward, there will be no x-intercepts. However, if it opens downward, there will be two x-intercepts. If the vertex is on the x-axis, there will be only one x-intercept.

27. Step 2 The y-intercept is determined by the value of c. The values of a and b do not affect its location.

Step 3 The axis of symmetry is affected by the values of a and b. As the value of a increases, the value of the axis of symmetry decreases. As the value of b increases, the value of the axis of symmetry increases.

Step 4 Increasing the value of a increases the steepness of the graph.

Step 5 Changing the values of a, b, and c affects the position of the vertex, the steepness of the graph, and whether the graph opens upward ($a > 0$) or downward ($a < 0$). a affects the steepness and determines the direction of opening. b and a affect the value of the axis of symmetry, with b having a greater effect. c determines the value of the y-intercept.

3.3 Completing the Square, pages 192 to 197

1. a) $x^2 + 6x + 9$; $(x + 3)^2$
b) $x^2 - 4x + 4$; $(x - 2)^2$
c) $x^2 + 14x + 49$; $(x + 7)^2$
d) $x^2 - 2x + 1$; $(x - 1)^2$

2. a) $y = (x + 4)^2 - 16$; $(-4, -16)$
b) $y = (x - 9)^2 - 140$; $(9, -140)$
c) $y = (x - 5)^2 + 6$; $(5, 6)$
d) $y = (x + 16)^2 - 376$; $(-16, -376)$

3. a) $y = 2(x - 3)^2 - 18$; working backward, $y = 2(x - 3)^2 - 18$ results in the original function, $y = 2x^2 - 12x$.
b) $y = 6(x + 2)^2 - 7$; working backward, $y = 6(x + 2)^2 - 7$ results in the original function, $y = 6x^2 + 24x + 17$.
c) $y = 10(x - 8)^2 - 560$; working backward, $y = 10(x - 8)^2 - 560$ results in the original function, $y = 10x^2 - 160x + 80$.
d) $y = 3(x + 7)^2 - 243$; working backward, $y = 3(x + 7)^2 - 243$ results in the original function, $y = 3x^2 + 42x - 96$.

4. a) $f(x) = -4(x - 2)^2 + 16$; working backward, $f(x) = -4(x - 2)^2 + 16$ results in the original function, $f(x) = -4x^2 + 16x$.
b) $f(x) = -20(x + 10)^2 + 1757$; working backward, $f(x) = -20(x + 10)^2 + 1757$ results in the original function, $f(x) = -20x^2 - 400x - 243$.
c) $f(x) = -(x + 21)^2 + 941$; working backward, $f(x) = -(x + 21)^2 + 941$ results in the original function, $f(x) = -x^2 - 42x + 500$.
d) $f(x) = -7(x - 13)^2 + 1113$; working backward, $f(x) = -7(x - 13)^2 + 1113$ results in the original function, $f(x) = -7x^2 + 182x - 70$.

5. Verify each part by expanding the vertex form of the function and comparing with the standard form and by graphing both forms of the function.

6. a) minimum value of -11 when $x = -3$
b) minimum value of -11 when $x = 2$
c) maximum value of 25 when $x = -5$
d) maximum value of 5 when $x = 2$

7. a) minimum value of $-\frac{13}{4}$
b) minimum value of $\frac{1}{2}$
c) maximum value of 47
d) minimum value of -1.92
e) maximum value of 18.95
f) maximum value of 1.205

8. a) $y = \left(x + \frac{3}{4}\right)^2 - \frac{121}{16}$
b) $y = -\left(x + \frac{3}{16}\right)^2 + \frac{9}{256}$
c) $y = 2\left(x - \frac{5}{24}\right)^2 + \frac{263}{288}$

9. a) $f(x) = -2(x - 3)^2 + 8$
b) Example: The vertex of the graph is $(3, 8)$. From the function $f(x) = -2(x - 3)^2 + 8$, $p = 3$ and $q = 8$. So, the vertex is $(3, 8)$.

10. a) maximum value of 62; domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq 62, y \in \mathrm{R}\}$
b) Example: By changing the function to vertex form, it is possible to find the maximum value since the function opens down and $p = 62$. This also helps to determine the range of the function. The domain is all real numbers for non-restricted quadratic functions.

11. Example: By changing the function to vertex form, the vertex is $\left(\frac{13}{4}, -\frac{3}{4}\right)$ or $(3.25, -0.75)$.

12. a) There is an error in the second line of the solution. You need to add and subtract the square of half the coefficient of the x-term.
$y = x^2 + 8x + 30$
$y = (x^2 + 8x + 16 - 16) + 30$
$y = (x + 4)^2 + 14$
b) There is an error in the second line of the solution. You need to add and subtract the square of half the coefficient of the x-term. There is also an error in the last line. The factor of 2 disappeared.
$f(x) = 2x^2 - 9x - 55$
$f(x) = 2[x^2 - 4.5x + 5.0625 - 5.0625] - 55$
$f(x) = 2[(x^2 - 4.5x + 5.0625) - 5.0625] - 55$
$f(x) = 2[(x - 2.25)^2 - 5.0625] - 55$
$f(x) = 2(x - 2.25)^2 - 10.125 - 55$
$f(x) = 2(x - 2.25)^2 - 65.125$
c) There is an error in the third line of the solution. You need to add and subtract the square of half the coefficient of the x-term.
$y = 8x^2 + 16x - 13$
$y = 8[x^2 + 2x] - 13$
$y = 8[x^2 + 2x + 1 - 1] - 13$
$y = 8[(x^2 + 2x + 1) - 1] - 13$
$y = 8[(x + 1)^2 - 1] - 13$
$y = 8(x + 1)^2 - 8 - 13$
$y = 8(x + 1)^2 - 21$

d) There are two errors in the second line of the solution. You need to factor the leading coefficient from the first two terms and add and subtract the square of half the coefficient of the x-term. There is also an error in the last line. The -3 factor was not distributed correctly.

$f(x) = -3x^2 - 6x$
$f(x) = -3[x^2 + 2x + 1 - 1]$
$f(x) = -3[(x^2 + 2x + 1) - 1]$
$f(x) = -3[(x + 1)^2 - 1]$
$f(x) = -3(x + 1)^2 + 3$

13. 12 000 items

14. 9 m

15. a) 5.56 ft; 0.31 s after being shot

b) Example: Verify by graphing and finding the vertex or by changing the function to vertex form and using the values of p and q to find the maximum value and when it occurs.

16. a) Austin got $+12x$ when dividing $72x$ by -6 and should have gotten $-12x$. He also forgot to square the quantity $(x + 6)$. Otherwise his work was correct and his answer should be $y = -6(x - 6)^2 + 196$. Yuri got an answer of -216 when he multiplied -6 by -36. He should have gotten 216 to get the correct answer of $y = -6(x - 6)^2 + 196$.

b) Example: To verify an answer, either work backward to show the functions are equivalent or use technology to show the graphs of the functions are identical.

17. 18 cm

18. a) The maximum revenue is \$151 250 when the ticket price is \$55.

b) 2750 tickets

c) Example: Assume that the decrease in ticket prices determines the same increase in ticket sales as indicated by the survey.

19. a) $R(n) = -50n^2 + 1000n + 100\,800$, where R is the revenue of the sales and n is the number of \$10 increases in price.

b) The maximum revenue is \$105 800 when the bikes are sold for \$460.

c) Example: Assume that the predictions of a decrease in sales for every increase in price holds true.

20. a) $P(n) = -0.1n^2 + n + 120$, where P is the production of peas, in kilograms, and n is the increase in plant rows.

b) The maximum production is 122.5 kg of peas when the farmer plants 35 rows of peas.

c) Example: Assume that the prediction holds true.

21. a) Answers may vary.

b) $A = -2w^2 + 90w$, where A is the area and w is the width.

c) 1012.5 m^2

d) Example: Verify the solution by graphing or changing the function to vertex form, where the vertex is (22.5, 1012.5).

e) Example: Assume that the measurements can be any real number.

22. The dimensions of the large field are 75 m by 150 m, and the dimensions of the small fields are 75 m by 50 m.

23. a) The two numbers are 14.5 and 14.5, and the maximum product is 210.25.

b) The two numbers are 6.5 and -6.5, and the minimum product is -42.25.

24. 8437.5 cm^2

25. $f(x) = -\frac{3}{4}\left(x - \frac{3}{4}\right)^2 + \frac{47}{64}$

26. a)

$y = ax^2 + bx + c$
$y = a\left(x^2 + \frac{b}{a}x\right) + c$
$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2}\right)\right) + c$
$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + c$
$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4a^2c - ab^2}{4a^2}$
$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{a(4ac - b^2)}{4a^2}$
$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$

b) $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

c) Example: This formula can be used to find the vertex of any quadratic function without using an algebraic method to change the function to vertex form.

27. a) (3, 4)

b) $f(x) = 2(x - 3)^2 + 4$, so the vertex is (3, 4).

c) $a = a$, $p = -\frac{b}{2a}$, and $q = \frac{4ac - b^2}{4a}$

28. a) $A = -\left(\frac{4 + \pi}{8}\right)w^2 + 3w$

b) maximum area of $\frac{18}{4 + \pi}$, or approximately 2.52 m^2, when the width is $\frac{12}{4 + \pi}$, or approximately 1.68 m

c) Verify by graphing and comparing the vertex values, $\left(\frac{12}{4 + \pi}, \frac{18}{4 + \pi}\right)$, or approximately (1.68, 2.52).

d) width: $\frac{12}{4 + \pi}$ or approximately 1.68 m, length: $\frac{6}{4 + \pi}$ or approximately 0.84 m, radius: $\frac{6}{4 + \pi}$ or approximately 0.84 m; Answers may vary.

29. Examples:

a) The function is written in both forms; standard form is $f(x) = 4x^2 + 24$ and vertex form is $f(x) = 4(x + 0)^2 + 24$.

b) No, since it is already in completed square form.

30. Martine's first error was that she did not correctly factor -4 from $-4x^2 + 24x$. Instead of $y = -4(x^2 + 6x) + 5$, it should have been $y = -4(x^2 - 6x) + 5$. Her second error occurred when she completed the square. Instead of $y = -4(x^2 + 6x + 36 - 36) + 5$, it should have been $y = -4(x^2 - 6x + 9 - 9) + 5$. Her third error occurred when she factored $(x^2 + 6x + 36)$. This is not a perfect square trinomial and is not factorable. Her last error occurred when she expanded the expression $-4[(x + 6)^2 - 36] + 5$. It should be $-4(x - 3)^2 + 36 + 5$ not $-4(x + 6)^2 - 216 + 5$. The final answer is $y = -4(x - 3)^2 + 41$.

31. a) $R = -5x^2 + 50x + 1000$

b) By completing the square, you can determine the maximum revenue and price to charge to produce the maximum revenue, as well as predict the number of T-shirts that will sell.

c) Example: Assume that the market research holds true for all sales of T-shirts.

Chapter 3 Review, pages 198 to 200

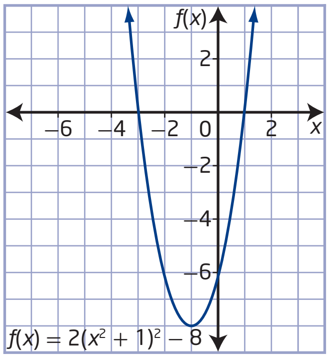
1. a) Given the graph of $f(x) = x^2$, move it 6 units to the left and 14 units down.
vertex: $(-6, -14)$, axis of symmetry: $x = -6$, opens upward, minimum value of -14, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \geq -14, y \in R\}$

b) Given the graph of $f(x) = x^2$, change the width by multiplying the y-values by a factor of 2, reflect it in the x-axis, and move the entire graph up 19 units.
vertex: $(0, 19)$, axis of symmetry: $x = 0$, opens downward, maximum value of 19, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \leq 19, y \in R\}$

c) Given the graph of $f(x) = x^2$, change the width by multiplying the y-values by a factor of $\frac{1}{5}$, move the entire graph 10 units to the right and 100 units up.
vertex: $(10, 100)$, axis of symmetry: $x = 10$, opens upward, minimum value of 100, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \geq 100, y \in R\}$

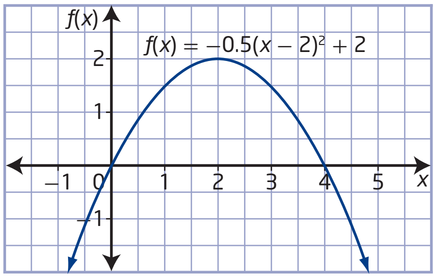
d) Given the graph of $f(x) = x^2$, change the width by multiplying the y-values by a factor of 6, reflect it in the x-axis, and move the entire graph 4 units to the right.
vertex: $(4, 0)$, axis of symmetry: $x = 4$, opens downward, maximum value of 0, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \leq 0, y \in R\}$

2. a)



vertex: $(-1, -8)$, axis of symmetry: $x = -1$, minimum value of -8, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \geq -8, y \in R\}$, x-intercepts occur at $(-3, 0)$ and $(1, 0)$, y-intercept occurs at $(0, -6)$

b)



vertex: $(2, 2)$, axis of symmetry: $x = 2$, maximum value of 2, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \leq 2, y \in R\}$, x-intercepts occur at $(0, 0)$ and $(4, 0)$, y-intercept occurs at $(0, 0)$

3. Examples:

a) Yes. The vertex is $(5, 20)$, which is above the x-axis, and the parabola opens downward to produce two x-intercepts.

b) Yes. Since $y \geq 0$, the graph touches the x-axis at only one point and has one x-intercept.

c) Yes. The vertex of $(0, 9)$ is above the x-axis and the parabola opens upward, so the graph does not cross or touch the x-axis and has no x-intercepts.

d) No. It is not possible to determine if the graph opens upward to produce two x-intercepts or downward to produce no x-intercepts.

4. a) $y = -0.375x^2$ **b)** $y = 1.5(x - 8)^2$

c) $y = 3(x + 4)^2 + 12$

d) $y = -4(x - 4.5)^2 + 25$

5. a) $y = \frac{1}{4}(x + 3)^2 - 6$ **b)** $y = -2(x - 1)^2 + 5$

6. Example: Two possible functions for the mirror are $y = 0.0069(x - 90)^2 - 56$ and $y = 0.0069x^2$.

7. a) **i)** $y = \frac{22}{18\ 769}x^2$ **ii)** $y = \frac{22}{18\ 769}x^2 + 30$

iii) $y = \frac{22}{18\ 769}(x - 137)^2 + 30$

b) Example: The function will change as the seasons change with the heat or cold changing the length of the cable and therefore the function.

8. $y = -\frac{8}{15}(x - 7.5)^2 + 30$ or $y \approx -0.53(x - 7.5)^2 + 30$

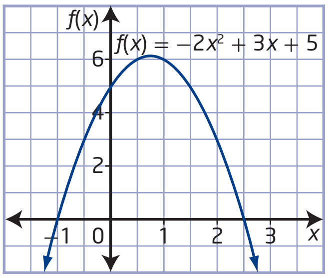
9. a) vertex: (2, 4), axis of symmetry: $x = 2$, maximum value of 4, opens downward, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq 4, y \in \mathrm{R}\}$, x-intercepts occur at (−2, 0) and (6, 0), y-intercept occurs at (0, 3)

b) vertex: (−4, 2), axis of symmetry: $x = -4$, maximum value of 2, opens upward, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq 2, y \in \mathrm{R}\}$, no x-intercepts, y-intercept occurs at (0, 10)

10. a) Expanding $y = 7(x + 3)^2 - 41$ gives $y = 7x^2 + 42x + 22$, which is a polynomial of degree two.

b) Expanding $y = (2x + 7)(10 - 3x)$ gives $y = -6x^2 - x + 70$, which is a polynomial of degree two.

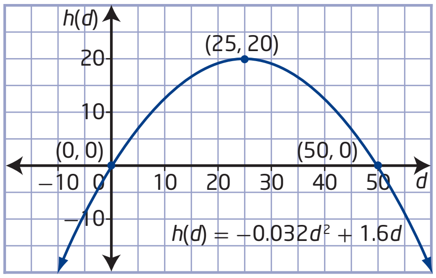
11. a)



vertex: (0.75, 6.125), axis of symmetry: $x = 0.75$, opens downward, maximum value of 6.125, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq 6.125, y \in \mathrm{R}\}$, x-intercepts occur at (−1, 0) and (2.5, 0), y-intercept occurs at (0, 5)

b) Example: The vertex is the highest point on the curve. The axis of symmetry divides the graph in half and is defined by the x-coordinate of the vertex. Since $a < 0$, the graph opens downward. The maximum value is the y-coordinate of the vertex. The domain is all real numbers. The range is less than or equal to the maximum value, since the graph opens downward. The x-intercepts are where the graph crosses the x-axis, and the y-intercept is where the graph crosses the y-axis.

12. a)



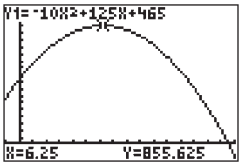
b) The maximum height of the ball is 20 m. The ball is 25 m downfield when it reaches its maximum height.

c) The ball lands downfield 50 m.

d) domain: $\{x \mid 0 \leq x \leq 50, x \in \mathrm{R}\}$, range: $\{y \mid 0 \leq y \leq 20, y \in \mathrm{R}\}$

13. a) $y = (5x + 15)(31 - 2x)$ or $y = -10x^2 + 125x + 465$

b)



c) The values between the x-intercepts will produce a rectangle.

d) Yes; the maximum value is 855.625; the minimum value is 0.

e) The vertex represents the maximum area and the value of x that produces the maximum area.

f) domain: $\{x \mid 0 \leq x \leq 15.5, x \in \mathrm{R}\}$, range: $\{y \mid 0 \leq y \leq 855.625, y \in \mathrm{R}\}$

14. a) $y = (x - 12)^2 - 134$

b) $y = 5(x + 4)^2 - 107$

c) $y = -2(x - 2)^2 + 8$

d) $y = -30(x + 1)^2 + 135$

15. vertex: $\left(\frac{5}{4}, -\frac{13}{4}\right)$, axis of symmetry: $x = \frac{5}{4}$, minimum value of $-\frac{13}{4}$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\left\{y \mid y \geq -\frac{13}{4}, y \in \mathrm{R}\right\}$

16. a) In the second line, the second term should have been $+3.5x$. In the third line, Amy found the square of half of 3.5 to be 12.25; it should have been 3.0625 and this term should be added and then subtracted. The solution should be

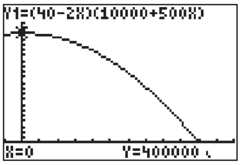
$y = -22x^2 - 77x + 132$
$y = -22(x^2 + 3.5x) + 132$
$y = -22(x^2 + 3.5x + 3.0625 - 3.0625) + 132$
$y = -22(x^2 + 3.5x + 3.0625) + 67.375 + 132$
$y = -22(x + 1.75)^2 + 199.375$

b) Verify by expanding the vertex form to standard form and by graphing both forms to see if they produce the same graph.

17. a) $R = (40 - 2x)(10\ 000 + 500x)$ or $R = -1000x^2 + 400\ 000$ where R is the revenue and x is the number of price decreases.

b) The maximum revenue is \$400 000 and the price is \$40 per coat.

c)



d) The y-intercept represents the sales before changing the price. The x-intercepts indicate the number of price increases or decreases that will produce revenue.

e) domain: $\{x \mid -20 \le x \le 20, x \in R\}$, range: $\{y \mid 0 \le y \le 400\ 000, y \in R\}$

f) Example: Assume that a whole number of price increases can be used.

Chapter 3 Practice Test, pages 201 to 203

1. D

2. C

3. A

4. D

5. D

6. A

7. a) $y = (x - 9)^2 - 108$

b) $y = 3(x + 6)^2 - 95$

c) $y = -10(x + 2)^2 + 40$

8. a) vertex: $(-6, 4)$, axis of symmetry: $x = -6$, maximum value of 4, domain: $\{x \mid x \in R\}$, range: $\{y \mid y \le 4, y \in R\}$, x-intercepts occur at $(-8, 0)$ and $(-4, 0)$

b) $y = -(x + 6)^2 + 4$

9. a) **i)** change in width by a multiplication of the y-values by a factor of 5

ii) vertical translation of 20 units down

iii) horizontal translation of 11 units to the left

iv) change in width by a multiplication of the y-values by a factor of $\frac{1}{7}$ and a reflection in the x-axis

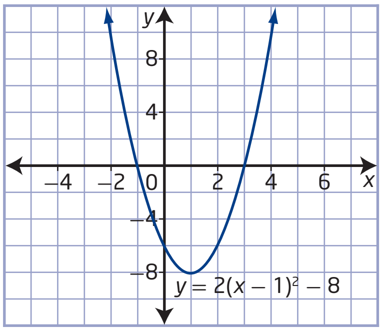
b) Examples:

i) The vertex of the functions in part a) ii) and iii) will be different as compared to $f(x) = x^2$ because the entire graph is translated. Instead of a vertex of (0, 0), the graph of the function in part a) ii) will be located at (0, −20) and the vertex of the graph of the function in part a) iii) will be located at (−11, 0).

ii) The axis of symmetry of the function in part a) iii) will be different as compared to $f(x) = x^2$ because the entire graph is translated horizontally. Instead of an axis of symmetry of $x = 0$, the graph of the function in part a) iii) will have an axis of symmetry of $x = -11$.

iii) The range of the functions in part a) ii) and iv) will be different as compared to $f(x) = x^2$ because the entire graph is either translated vertically or reflected in the x-axis. Instead of a range of $\{y \mid y \ge 0, y \in R\}$, the function in part a) ii) will have a range of $\{y \mid y \ge -20, y \in R\}$ and the function in part a) iv) will have a range of $\{y \mid y \le 0, y \in R\}$.

10.



Vertex	$(1, -8)$
Axis of Symmetry	$x = 1$
Direction of Opening	upward
Domain	$\{x \mid x \in R\}$
Range	$\{y \mid y \ge -8, y \in R\}$
x-Intercepts	−1 and 3
y-Intercept	−6

11. a) In the second line, the 2 was not factored out of the second term. In the third line, you need to add and subtract the square of half the coefficient of the x-term. The first three steps should be

$y = 2x^2 - 8x + 9$

$y = 2(x^2 - 4x) + 9$

$y = 2(x^2 - 4x + 4 - 4) + 9$

b) The rest of the process is shown.

$y = 2[(x^2 - 4x + 4) - 4] + 9$

$y = 2(x - 2)^2 - 8 + 9$

$y = 2(x - 2)^2 + 1$

c) The solution can be verified by expanding the vertex form to standard form or by graphing both functions to see that they coincide.

12. Examples:

a) The vertex form of the function $C(v) = 0.004v^2 - 0.62v + 30$ is $C(v) = 0.004(v - 77.5)^2 + 5.975$. The most efficient speed would be 77.5 km/h and will produce a fuel consumption of 5.975 L/100 km.

b) By completing the square and determining the vertex of the function, you can determine the most efficient fuel consumption and at what speed it occurs.

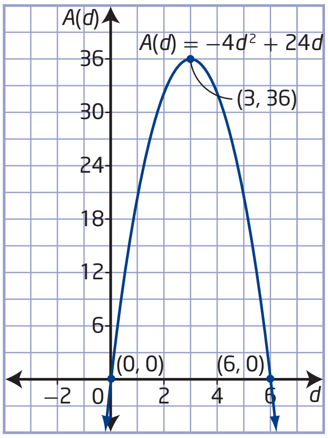
13. a) The maximum height of the flare is 191.406 25 m, 6.25 s after being shot.

b) Example: Complete the square to produce the vertex form and use the value of q to determine the maximum height and the value of p to determine when it occurs, or use the fact that the x-coordinate of the vertex of a quadratic function in standard form is $x = -\frac{b}{2a}$ and substitute this value into the function to find the corresponding y-coordinate, or graph the function to find the vertex.

14. a) $A(d) = -4d^2 + 24d$

b) Since the function is a polynomial of degree two, it satisfies the definition of a quadratic function.

c)



Example: By completing the square, determine the vertex, find the y-intercept and its corresponding point, plot the three points, and join them with a smooth curve.

d) (3, 36); the maximum area of 36 m^2 happens when the fence is extended to 3 m from the building.

e) domain: $\{d \mid 0 \le d \le 6, d \in R\}$, range: $\{A \mid 0 \le A \le 36, A \in R\}$; negative distance and area do not have meaning in this situation.

f) Yes; the maximum value is 36 when d is 3, and the minimum value is 0 when d is 0 or 6.

g) Example: Assume that any real-number distance can be used to build the fence.

15. a) $f(x) = -0.03x^2$

b) $f(x) = -0.03x^2 + 12$

c) $f(x) = -0.03(x + 20)^2 + 12$

d) $f(x) = -0.03(x - 28)^2 - 3$

16. a) $R = (2.25 - 0.05x)(120 + 8x)$

b) Expand and complete the square to get the vertex form of the function. A price of $1.50 gives the maximum revenue of $360.

c) Example: Assume that any whole number of price decreases can occur.

Chapter 4 Quadratic Equations

4.1 Graphical Solutions of Quadratic Equations, pages 215 to 217

1. a) 1 **b)** 2 **c)** 0 **d)** 2

2. a) 0 **b)** −1 and −4

c) none **d)** −3 and 8

3. a) $x = -3, x = 8$ **b)** $r = -3, r = 0$

c) no real solutions **d)** $x = 3, x = -2$

e) $z = 2$ **f)** no real solutions

4. a) $n \approx -3.2, n \approx 3.2$ **b)** $x = -4, x = 1$

c) $w = 1, w = 3$ **d)** $d = -8, d = -2$

e) $v \approx -4.7, v \approx -1.3$ **f)** $m = 3, m = 7$

5. 60 yd

6. a) $-x^2 + 9x - 20 = 0$ or $x^2 - 9x + 20 = 0$

b) 4 and 5

7. a) $x^2 + 2x - 168 = 0$

b) $x = 12$ and $x = 14$ or $x = -12$ and $x = -14$

8. a) Example: Solving the equation leads to the distance from the firefighter that the water hits the ground. The negative solution is not part of this situation.

b) 12.2 m

c) Example: Assume that aiming the hose higher would not reach farther. Assume that wind does not affect the path of the water.

9. a) Example: Solving the equation leads to the time that the fireworks hit the ground. The negative solution is not part of the situation.

b) 6.1 s

10. a) $-0.75d^2 + 0.9d + 1.5 = 0$ **b)** 2.1 m

11. a) $-2d^2 + 3d + 10 = 0$ **b)** 3.1 m

12. a) first arch: $x = 0$ and $x = 84$, second arch: $x = 84$ and $x = 168$, third arch: $x = 168$ and $x = 252$

b) The zeros represent where the arches reach down to the bridge deck.

c) 252 m