CHAPTER 4

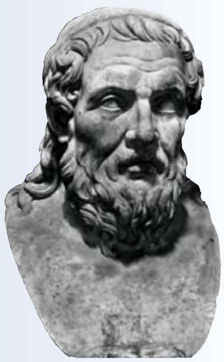
Quadratic Equations

What do water fountains, fireworks, satellite dishes, bridges, and model rockets have in common? They all involve a parabolic shape. You can develop and use quadratic equations to solve problems involving these parabolic shapes. Quadratic equations are also used in other situations such as avalanche control, setting the best ticket prices for concerts, designing roller coasters, and planning gardens.

In this chapter, you will relate quadratic equations to the graphs of quadratic functions, and solve problems by determining and analysing quadratic equations.

Did You Know?

Apollonius, also known as the "The Greek Geometer" (c. 210 B.C.E.), was the first mathematician to study parabolas in depth.



Key Terms

quadratic equation
root(s) of an equation
zero(s) of a function
extraneous root
quadratic formula
discriminant





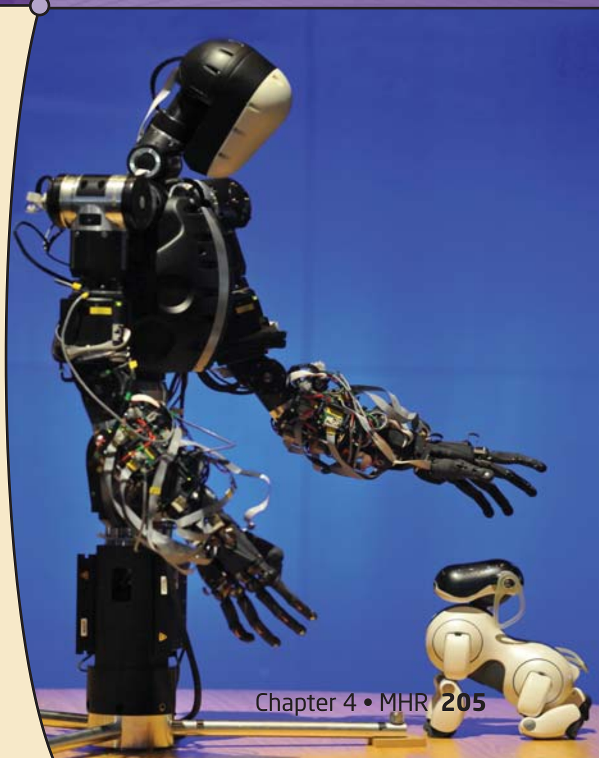

Career Link

Robotics engineering is a sub-field of mechanical engineering. A robotics engineer designs, maintains, and develops new applications for robots. These applications range from production line robots to those used in the medical and military fields, and from aerospace and mining to walking machines and tele-operators controlled by microchips.

A visionary robotics engineer could work on designing mobile robots, cars that drive themselves, and parts of space probes.

Web Link

To learn more about robotics engineering, go to www.mhrprecalc11.ca and follow the links.



4.1

Graphical Solutions of Quadratic Equations

Focus on...

- describing the relationships between the roots of a quadratic equation, the zeros of the corresponding quadratic function, and the x-intercepts of the graph of the quadratic function
- solving quadratic equations by graphing the corresponding quadratic function

Water fountains are usually designed to give a specific visual effect. For example, the water fountain shown consists of individual jets of water that each arch up in the shape of a parabola. Notice how the jets of water are designed to land precisely on the underwater spotlights.

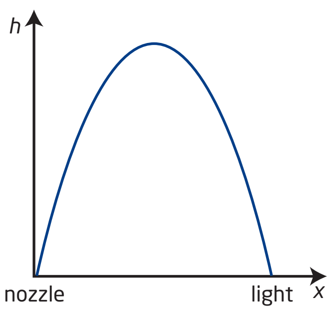
How can you design a water fountain to do this? Where must you place the underwater lights so the jets of water land on them? What are some of the factors to consider when designing a water fountain? How do these factors affect the shape of the water fountain?

Investigate Solving Quadratic Equations by Graphing

Materials

- grid paper or graphing technology

1. Each water fountain jet creates a parabolic stream of water. You can represent this curve by the quadratic function $h(x) = -6(x - 1)^2 + 6$, where h is the height of the jet of water and x is the horizontal distance of the jet of water from the nozzle, both in metres.



a) Graph the quadratic function $h(x) = -6(x - 1)^2 + 6$.

b) How far from the nozzle should the underwater lights be placed? Explain your reasoning.

2. You can control the height and horizontal distance of the jet of water by changing the water pressure. Suppose that the quadratic function $h(x) = -x^2 + 12x$ models the path of a jet of water at maximum pressure. The quadratic function $h(x) = -3x^2 + 12x$ models the path of the same jet of water at a lower pressure.

a) Graph these two functions on the same set of axes as in step 1.

b) Describe what you notice about the x-intercepts and height of the two graphs compared to the graph in step 1.

c) Why do you think the x-intercepts of the graph are called the zeros of the function?

Reflect and Respond

3. a) If the water pressure in the fountain must remain constant, how else could you control the path of the jets of water?

b) Could two jets of water at constant water pressure with different parabolic paths land on the same spot? Explain your reasoning.

Did You Know?

The Dubai Fountain at the Burj Khalifa in Dubai is the largest in the world. It can shoot about 22 000 gal of water about 500 ft into the air and features over 6600 lights and 25 colour projectors.

Link the Ideas

quadratic equation

- a second-degree equation with standard form $ax^2 + bx + c = 0$, where $a \neq 0$
- for example, $2x^2 + 12x + 16 = 0$

root(s) of an equation

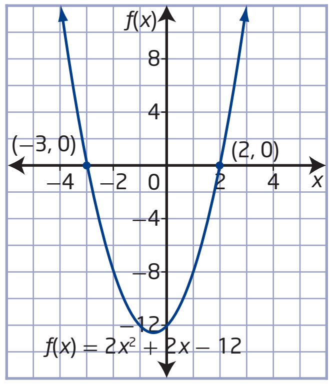
- the solution(s) to an equation

zero(s) of a function

- the value(s) of x for which $f(x) = 0$
- related to the x-intercept(s) of the graph of a function, $f(x)$

You can solve a **quadratic equation** of the form $ax^2 + bx + c = 0$ by graphing the corresponding quadratic function, $f(x) = ax^2 + bx + c$. The solutions to a quadratic equation are called the **roots** of the equation. You can find the roots of a quadratic equation by determining the x-intercepts of the graph, or the **zeros** of the corresponding quadratic function.

For example, you can solve the quadratic equation $2x^2 + 2x - 12 = 0$ by graphing the corresponding quadratic function, $f(x) = 2x^2 + 2x - 12$. The graph shows that the x-intercepts occur at $(-3, 0)$ and $(2, 0)$ and have values of -3 and 2. The zeros of the function occur when $f(x) = 0$. So, the zeros of the function are -3 and 2. Therefore, the roots of the equation are -3 and 2.



Example 1

Quadratic Equations With One Real Root

What are the roots of the equation $-x^2 + 8x - 16 = 0$?

Solution

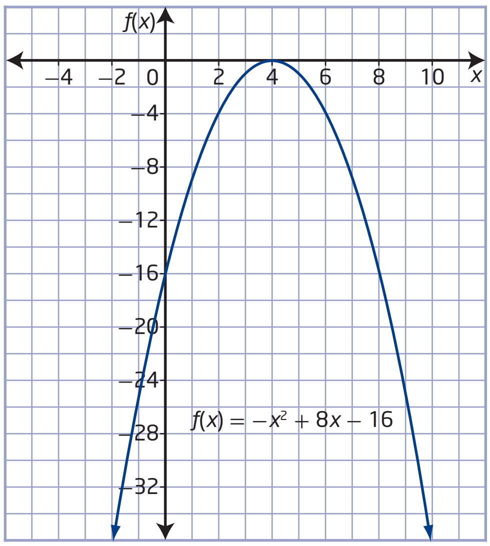
To solve the equation, graph the corresponding quadratic function, $f(x) = -x^2 + 8x - 16$, and determine the x-intercepts.

Method 1: Use Paper and Pencil

Create a table of values. Plot the coordinate pairs and use them to sketch the graph of the function.

Why were these values of x chosen?

x	$f(x)$
−2	−36
−1	−25
0	−16
1	−9
2	−4
3	−1
4	0
5	−1
6	−4
7	−9
8	−16
9	−25
10	−36



How do you know that there is only one root for this quadratic equation?

The graph meets the x-axis at the point (4, 0), the vertex of the corresponding quadratic function.
The x-intercept of the graph occurs at (4, 0) and has a value of 4.
The zero of the function is 4.
Therefore, the root of the equation is 4.

Method 2: Use a Spreadsheet

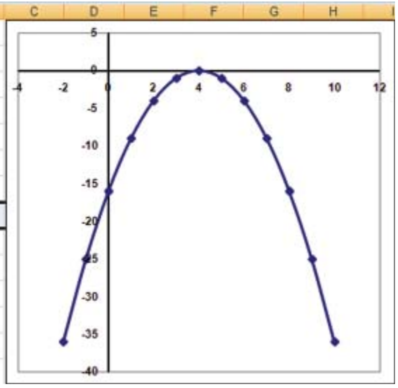
In a spreadsheet, enter the table of values shown. Then, use the spreadsheet's graphing features.

The x-intercept of the graph occurs at (4, 0) and has a value of 4.

The zero of the function is 4.

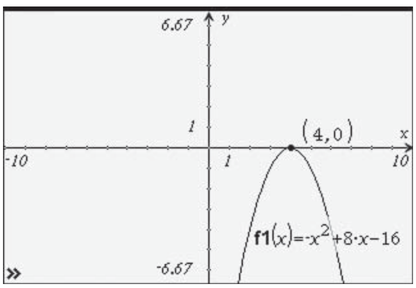
Therefore, the root of the equation is 4.

	A	B
1	x	y
2	-2	-36
3	-1	-25
4	0	-16
5	1	-9
6	2	-4
7	3	-1
8	4	0
9	5	-1
10	6	-4
11	7	-9
12	8	-16
13	9	-25
14	10	-36



Method 3: Use a Graphing Calculator

Graph the function using a graphing calculator. Then, use the trace or zero function to identify the x-intercept.



Compare the three methods. Which do you prefer? Why?

The x-intercept of the graph occurs at (4, 0) and has a value of 4.
The zero of the function is 4.
Therefore, the root of the equation is 4.

Check for Methods 1, 2, and 3:
Substitute $x = 4$ into the equation $-x^2 + 8x - 16 = 0$.

Left Side	Right Side
$-x^2 + 8x - 16$	0
$= -(4)^2 + 8(4) - 16$	
$= -16 + 32 - 16$	
$= 0$	

Left Side = Right Side

The solution is correct.

Your Turn

Determine the roots of the quadratic equation $x^2 - 6x + 9 = 0$.

Example 2

Quadratic Equations With Two Distinct Real Roots

The manager of Jasmine's Fine Fashions is investigating the effect that raising or lowering dress prices has on the daily revenue from dress sales. The function $R(x) = 100 + 15x - x^2$ gives the store's revenue R, in dollars, from dress sales, where x is the price change, in dollars. What price changes will result in no revenue?

Solution

When there is no revenue, $R(x) = 0$. To determine the price changes that result in no revenue, solve the quadratic equation $0 = 100 + 15x - x^2$.

Graph the corresponding revenue function. On the graph, the x-intercepts will correspond to the price changes that result in no revenue.

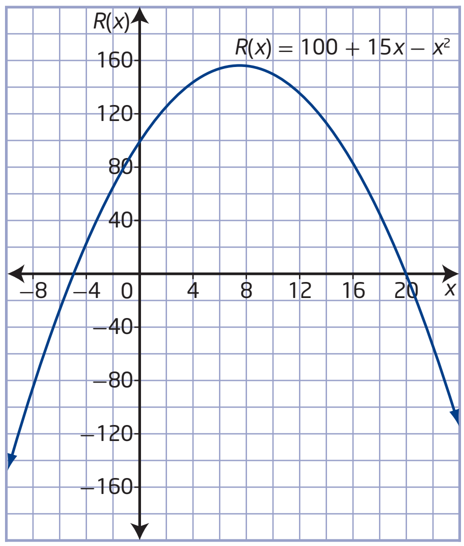
What do the values of x that are not the x-intercepts represent?

Method 1: Use Paper and Pencil

Create a table of values. Plot the coordinate pairs and use them to sketch the graph of the function.

Why do the values of x in the table begin with negative values?

Price Change, x	Revenue, $R(x)$
−10	−150
−8	−84
−6	−26
−4	24
−2	66
0	100
2	126
4	144
6	154
8	156
10	150
12	136
14	114
16	84
18	46
20	0
22	−54



How effective is graphing by hand in this situation?

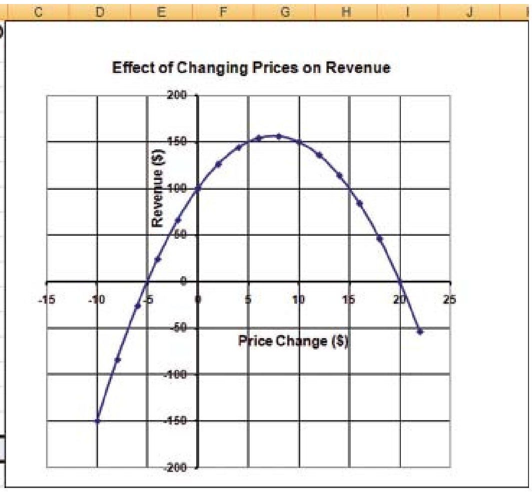
How do you know there are two roots for this quadratic equation?

The graph appears to cross the x-axis at the points $(-5, 0)$ and $(20, 0)$. The x-intercepts of the graph, or zeros of the function, are -5 and 20. Therefore, the roots of the equation are -5 and 20.

Why do the roots of the equation result in no revenue?

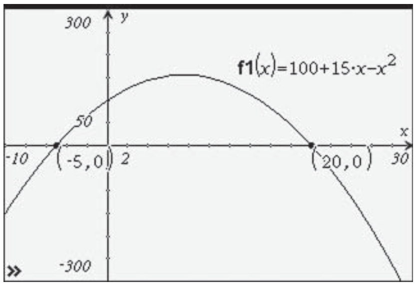
Method 2: Use a Spreadsheet
In a spreadsheet, enter the table of values shown.
Then, use the spreadsheet's graphing features.

	A	B
1	Price Change ($)	Revenue ($)
2	-10	-150
3	-8	-84
4	-6	-26
5	-4	24
6	-2	66
7	0	100
8	2	126
9	4	144
10	6	154
11	8	156
12	10	150
13	12	136
14	14	114
15	16	84
16	18	46
17	20	0
18	22	-54



The graph crosses the x-axis at the points $(-5, 0)$ and $(20, 0)$. The x-intercepts of the graph, or zeros of the function, are -5 and 20. Therefore, the roots of the equation are -5 and 20.

Method 3: Use a Graphing Calculator
Graph the revenue function using a graphing calculator. Adjust the window settings of the graph until you see the vertex of the parabola and the x-intercepts. Use the trace or zero function to identify the x-intercepts of the graph.



The graph crosses the x-axis at the points $(-5, 0)$ and $(20, 0)$.
The x-intercepts of the graph, or zeros of the function, are -5 and 20.
Therefore, the roots of the equation are -5 and 20.

Check for Methods 1, 2, and 3:
Substitute the values $x = -5$ and $x = 20$ into the equation $0 = 100 + 15x - x^2$.

Left Side	Right Side
0	$100 + 15x - x^2$
	$= 100 + 15(-5) - (-5)^2$
	$= 100 - 75 - 25$
	$= 0$

Left Side = Right Side

Left Side	Right Side
0	$100 + 15x - x^2$
	$= 100 + 15(20) - (20)^2$
	$= 100 + 300 - 400$
	$= 0$

Left Side = Right Side

Both solutions are correct. A dress price increase of \$20 or a decrease of \$5 will result in no revenue from dress sales.

Why is one price change an increase and the other a decrease? Do both price changes make sense? Why or why not?

Your Turn

The manager at Suzie's Fashion Store has determined that the function $R(x) = 600 - 6x^2$ models the expected weekly revenue, R, in dollars, from sweatshirts as the price changes, where x is the change in price, in dollars. What price increase or decrease will result in no revenue?

Example 3

Quadratic Equations With No Real Roots

Solve $2x^2 + x = -2$ by graphing.

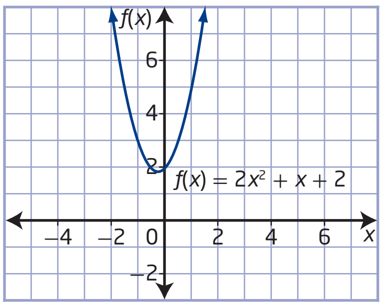
Solution

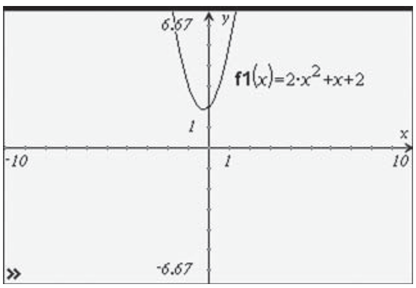
Rewrite the equation in the form $ax^2 + bx + c = 0$.

$2x^2 + x + 2 = 0$

Why do you rewrite the equation in the form $ax^2 + bx + c = 0$?

Graph the corresponding quadratic function $f(x) = 2x^2 + x + 2$.





The graph does not intersect the x-axis.
There are no zeros for this function.

Therefore, the quadratic equation has no real roots.

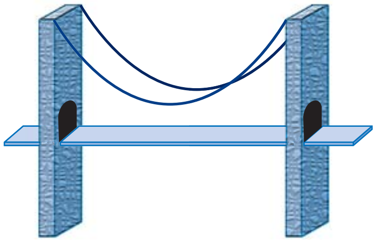
Your Turn

Solve $3m^2 - m = -2$ by graphing.

Example 4

Solve a Problem Involving Quadratic Equations

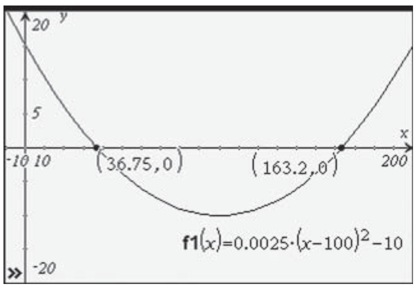
The curve of a suspension bridge cable attached between the tops of two towers can be modelled by the function $h(d) = 0.0025(d - 100)^2 - 10$, where h is the vertical distance from the top of a tower to the cable and d is the horizontal distance from the left end of the bridge, both in metres. What is the horizontal distance between the two towers? Express your answer to the nearest tenth of a metre.



Solution

At the tops of the towers, $h(d) = 0$. To determine the locations of the two towers, solve the quadratic equation $0 = 0.0025(d - 100)^2 - 10$. Graph the cable function using graphing technology. Adjust the dimensions of the graph until you see the vertex of the parabola and the x-intercepts. Use the trace or zero function to identify the x-intercepts of the graph.

The x-intercepts of the graph occur at approximately (36.8, 0) and (163.2, 0). The zeros of the function are approximately 36.8 and 163.2. Therefore, the roots of the equation are approximately 36.8 and 163.2.



What does the x-axis represent?

The first tower is located approximately 36.8 m from the left end of the bridge.

The second tower is located approximately 163.2 m from the left end of the bridge.

Subtract to determine the distance between the two towers.
$163.2 - 36.8 = 126.4$

The horizontal distance between the two towers is approximately 126.4 m.

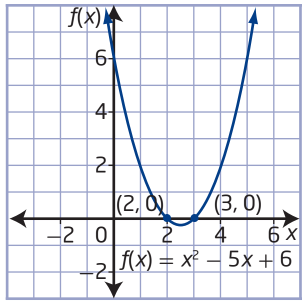
Your Turn

Suppose the cable of the suspension bridge in Example 4 is modelled by the function $h(d) = 0.0025(d - 100)^2 - 12$. What is the horizontal distance between the two towers? Express your answer to the nearest tenth of a metre.

Key Ideas

- One approach to solving a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, is to graph the corresponding quadratic function, $f(x) = ax^2 + bx + c$. Then, determine the x-intercepts of the graph.
- The x-intercepts of the graph, or the zeros of the quadratic function, correspond to the solutions, or roots, of the quadratic equation.

For example, you can solve $x^2 - 5x + 6 = 0$ by graphing the corresponding function, $f(x) = x^2 - 5x + 6$, and determining the x-intercepts.



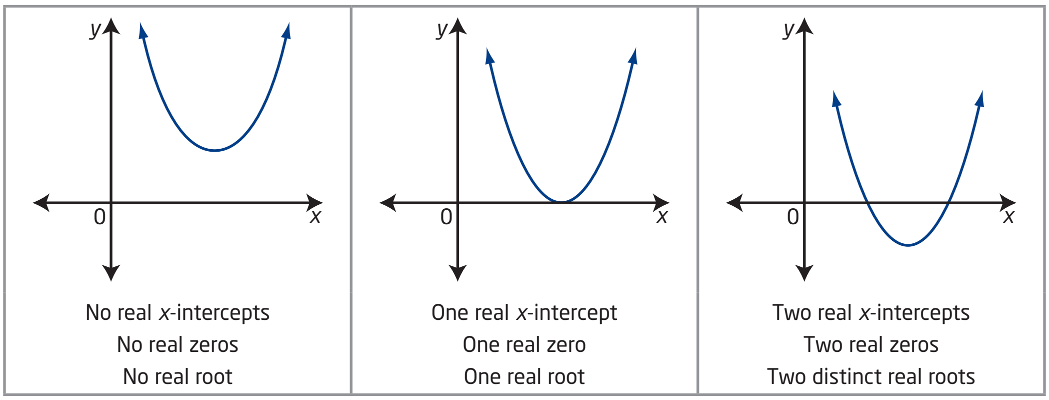
The x-intercepts of the graph and the zeros of the function are 2 and 3. So, the roots of the equation are 2 and 3.

Check:
Substitute the values $x = 2$ and $x = 3$ into the equation $x^2 - 5x + 6 = 0$.

Left Side	Right Side	Left Side	Right Side
$x^2 - 5x + 6$	0	$x^2 - 5x + 6$	0
$= (2)^2 - 5(2) + 6$		$= (3)^2 - 5(3) + 6$	
$= 4 - 10 + 6$		$= 9 - 15 + 6$	
$= 0$		$= 0$	
Left Side = Right Side		Left Side = Right Side	

Both solutions are correct.

- The graph of a quadratic function can have zero, one, or two real x-intercepts. Therefore, the quadratic function has zero, one, or two real zeros, and correspondingly the quadratic equation has zero, one, or two real roots.

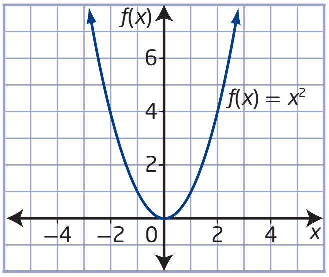


Check Your Understanding

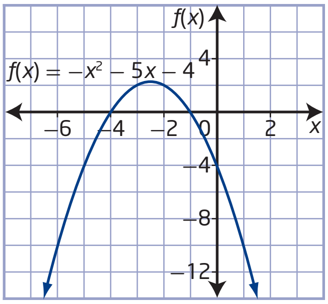
Practise

1. How many x-intercepts does each quadratic function graph have?

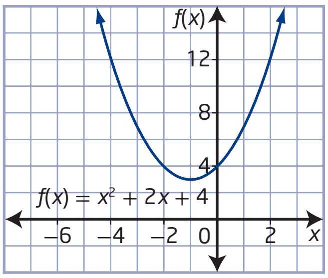
a)



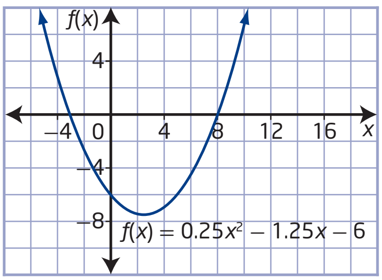
b)



c)



d)



2. What are the roots of the corresponding quadratic equations represented by the graphs of the functions shown in #1? Verify your answers.

3. Solve each equation by graphing the corresponding function.

a) $0 = x^2 - 5x - 24$

b) $0 = -2r^2 - 6r$

c) $h^2 + 2h + 5 = 0$

d) $5x^2 - 5x = 30$

e) $-z^2 + 4z = 4$

f) $0 = t^2 + 4t + 10$

4. What are the roots of each quadratic equation? Where integral roots cannot be found, estimate the roots to the nearest tenth.

a) $n^2 - 10 = 0$

b) $0 = 3x^2 + 9x - 12$

c) $0 = -w^2 + 4w - 3$

d) $0 = 2d^2 + 20d + 32$

e) $0 = v^2 + 6v + 6$

f) $m^2 - 10m = -21$

Apply

5. In a Canadian Football League game, the path of the football at one particular kick-off can be modelled using the function $h(d) = -0.02d^2 + 2.6d - 66.5$, where h is the height of the ball and d is the horizontal distance from the kicking team's goal line, both in yards. A value of $h(d) = 0$ represents the height of the ball at ground level. What horizontal distance does the ball travel before it hits the ground?

6. Two numbers have a sum of 9 and a product of 20.

a) What single-variable quadratic equation in the form $ax^2 + bx + c = 0$ can be used to represent the product of the two numbers?

b) Determine the two numbers by graphing the corresponding quadratic function.

7. Two consecutive even integers have a product of 168.

   a) What single-variable quadratic equation in the form $ax^2 + bx + c = 0$ can be used to represent the product of the two numbers?

   b) Determine the two numbers by graphing the corresponding quadratic function.

8. The path of the stream of water coming out of a fire hose can be approximated using the function $h(x) = -0.09x^2 + x + 1.2$, where h is the height of the water stream and x is the horizontal distance from the firefighter holding the nozzle, both in metres.

   a) What does the equation $-0.09x^2 + x + 1.2 = 0$ represent in this situation?

   b) At what maximum distance from the building could a firefighter stand and still reach the base of the fire with the water? Express your answer to the nearest tenth of a metre.

   c) What assumptions did you make when solving this problem?

9. The HSBC Celebration of Light is an annual pyro-musical fireworks competition that takes place over English Bay in Vancouver. The fireworks are set off from a barge so they land on the water. The path of a particular fireworks rocket is modelled by the function $h(t) = -4.9(t - 3)^2 + 47$, where h is the rocket's height above the water, in metres, at time, t, in seconds.

   a) What does the equation $0 = -4.9(t - 3)^2 + 47$ represent in this situation?

   b) The fireworks rocket stays lit until it hits the water. For how long is it lit, to the nearest tenth of a second?

10. A skateboarder jumps off a ledge at a skateboard park. His path is modelled by the function $h(d) = -0.75d^2 + 0.9d + 1.5$, where h is the height above ground and d is the horizontal distance the skateboarder travels from the ledge, both in metres.

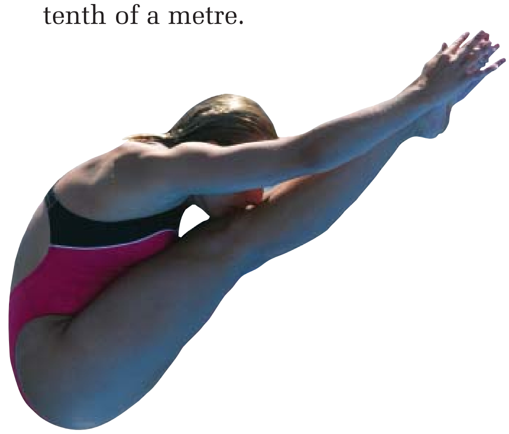
    a) Write a quadratic equation to represent the situation when the skateboarder lands.

    b) At what distance from the base of the ledge will the skateboarder land? Express your answer to the nearest tenth of a metre.

11. Émilie Heymans is a three-time Canadian Olympic diving medallist. Suppose that for a dive off the 10-m tower, her height, h, in metres, above the surface of the water is given by the function $h(d) = -2d^2 + 3d + 10$, where d is the horizontal distance from the end of the tower platform, in metres.

    a) Write a quadratic equation to represent the situation when Émilie enters the water.

    b) What is Émilie's horizontal distance from the end of the tower platform when she enters the water? Express your answer to the nearest tenth of a metre.



Did You Know?

Émilie Heymans, from Montréal, Québec, is only the fifth Canadian to win medals at three consecutive Olympic Games.

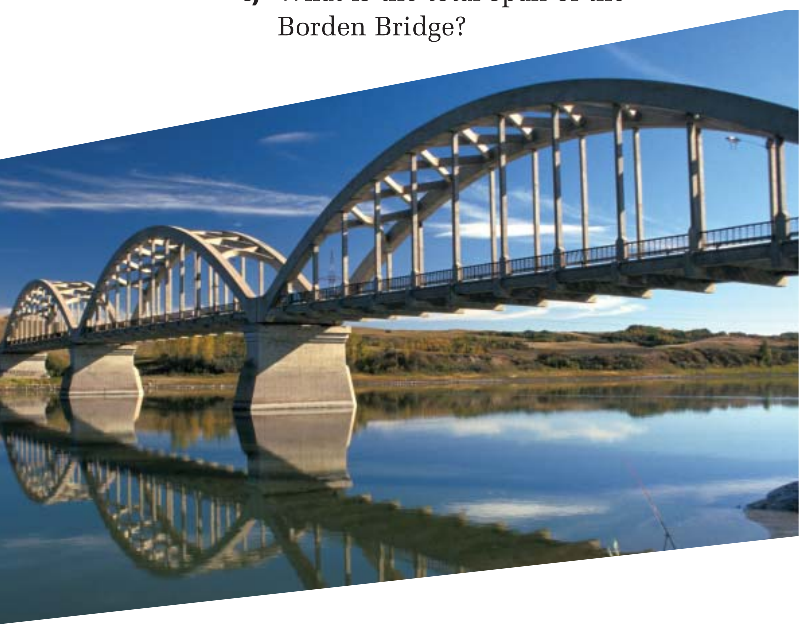
12. Matthew is investigating the old Borden Bridge, which spans the North Saskatchewan River about 50 km west of Saskatoon. The three parabolic arches of the bridge can be modelled using quadratic functions, where h is the height of the arch above the bridge deck and x is the horizontal distance of the bridge deck from the beginning of the first arch, both in metres.

First arch:
$h(x) = -0.01x^2 + 0.84x$
Second arch:
$h(x) = -0.01x^2 + 2.52x - 141.12$
Third arch:
$h(x) = -0.01x^2 + 4.2x - 423.36$

a) What are the zeros of each quadratic function?

b) What is the significance of the zeros in this situation?

c) What is the total span of the Borden Bridge?



Extend

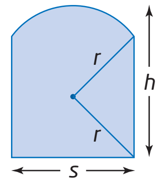
13. For what values of k does the equation $x^2 + 6x + k = 0$ have

a) one real root?

b) two distinct real roots?

c) no real roots?

14. The height of a circular arch is represented by $4h^2 - 8hr + s^2 = 0$, where h is the height, r is the radius, and s is the span of the arch, all in feet.



a) How high must an arch be to have a span of 64 ft and a radius of 40 ft?

b) How would this equation change if all the measurements were in metres? Explain.

15. Two new hybrid vehicles accelerate at different rates. The Ultra Range's acceleration can be modelled by the function $d(t) = 1.5t^2$, while the Edison's can be modelled by the function $d(t) = 5.4t^2$, where d is the distance, in metres, and t is the time, in seconds. The Ultra Range starts the race at 0 s. At what time should the Edison start so that both cars are at the same point 5 s after the race starts? Express your answer to the nearest tenth of a second.

Did You Know?

A hybrid vehicle uses two or more distinct power sources. The most common hybrid uses a combination of an internal combustion engine and an electric motor. These are called hybrid electric vehicles or HEVs.

Create Connections

16. Suppose the value of a quadratic function is negative when $x = 1$ and positive when $x = 2$. Explain why it is reasonable to assume that the related equation has a root between 1 and 2.

17. The equation of the axis of symmetry of a quadratic function is $x = 0$ and one of the x-intercepts is -4. What is the other x-intercept? Explain using a diagram.

18. The roots of the quadratic equation $0 = x^2 - 4x - 12$ are 6 and -2. How can you use the roots to determine the vertex of the graph of the corresponding function?

4.2

Factoring Quadratic Equations

Focus on...

- factoring a variety of quadratic expressions
- factoring to solve quadratic equations
- solving problems involving quadratic equations

Football, soccer, basketball, and volleyball are just a few examples of sports that involve throwing, kicking, or striking a ball. Each time a ball or projectile sails through the air, it follows a trajectory that can be modelled with a quadratic function.

Each of these sports is played on a rectangular playing area. The playing area for each sport can be modelled by a quadratic equation.

Investigate Solving Quadratic Equations by Factoring

Materials

- grid paper or graphing technology

1. For women's indoor competition, the length of the volleyball court is twice its width. If x represents the width, then $2x$ represents the length. The area of the court is 162 m^2.

a) Write a quadratic equation in standard form, $A(x) = 0$, to represent the area of the court.

b) Graph the corresponding quadratic function. How many x-intercepts are there? What are they?

c) From your graph, what are the roots of the quadratic equation you wrote in part a)? How do you know these are the roots of the equation?

d) In this context, are all the roots acceptable? Explain.

2. a) Factor the left side of the quadratic equation you wrote in step 1a).

b) Graph the corresponding quadratic function in factored form. Compare your graph to the graph you created in step 1b).

c) How is the factored form of the equation related to the x-intercepts of the graph?

d) How can you use the x-intercepts of a graph, $x = r$ and $x = s$, to write a quadratic equation in standard form?

3. For men's sitting volleyball, a Paralympic sport, the length of the court is 4 m more than the width. The area of the court is 60 m^2.

a) If x represents the width, write a quadratic equation in standard form to represent the area of the court.

b) Graph the corresponding quadratic function. How many x-intercepts are there? What are they?

4. a) Use the x-intercepts, $x = r$ and $x = s$, of your graph in step 3 to write the quadratic equation $(x - r)(x - s) = 0$.

b) Graph the corresponding quadratic function. Compare your graph to the graph you created in step 3.

Reflect and Respond

5. How does the factored form of a quadratic equation relate to the x-intercepts of the graph, the zeros of the quadratic function, and the roots of the equation?

6. Describe how you can factor the quadratic equation $0 = x^2 - 5x - 6$ to find the roots.

7. The roots of a quadratic equation are 3 and -5. What is a possible equation?

Did You Know?

Volleyball is the world's number two participation sport. Which sport do you think is number one?

Link the Ideas

Factoring Quadratic Expressions

To factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 0$, first factor out common factors, if possible.

For example,

$$\begin{aligned} 4x^2 - 2x - 12 &= 2(2x^2 - x - 6) \\ &= 2(2x^2 - 4x + 3x - 6) \\ &= 2[2x(x - 2) + 3(x - 2)] \\ &= 2(x - 2)(2x + 3) \end{aligned}$$

You can factor perfect square trinomials of the forms $(ax)^2 + 2abx + b^2$ and $(ax)^2 - 2abx + b^2$ into $(ax + b)^2$ and $(ax - b)^2$, respectively.

For example,

$$\begin{aligned} 4x^2 + 12x + 9 &= (2x + 3)(2x + 3) \\ &= (2x + 3)^2 \end{aligned} \qquad \begin{aligned} 9x^2 - 24x + 16 &= (3x - 4)(3x - 4) \\ &= (3x - 4)^2 \end{aligned}$$

You can factor a difference of squares, $(ax)^2 - (by)^2$, into $(ax - by)(ax + by)$.

For example,

$$\frac{4}{9}x^2 - 16y^2 = \left(\frac{2}{3}x - 4y\right)\left(\frac{2}{3}x + 4y\right)$$

Factoring Polynomials Having a Quadratic Pattern

You can extend the patterns established for factoring trinomials and a difference of squares to factor polynomials in quadratic form. You can factor a polynomial of the form $a(P)^2 + b(P) + c$, where P is any expression, as follows:

- Treat the expression P as a single variable, say r, by letting $r = P$.
- Factor as you have done before.
- Replace the substituted variable r with the expression P.
- Simplify the expression.

For example, in $3(x + 2)^2 - 13(x + 2) + 12$, substitute r for $x + 2$ and factor the resulting expression, $3r^2 - 13r + 12$.

$3r^2 - 13r + 12 = (3r - 4)(r - 3)$

Once the expression in r is factored, you can substitute $x + 2$ back in for r.

The resulting expression is

$$\begin{aligned}[3(x + 2) - 4][(x + 2 - 3) &= (3x + 6 - 4)(x - 1)\\ &= (3x + 2)(x - 1)\end{aligned}$$

You can factor a polynomial in the form of a difference of squares, as $P^2 - Q^2 = (P - Q)(P + Q)$ where P and Q are any expressions.

For example,

$$\begin{aligned}(3x + 1)^2 - (2x - 3)^2 &= [(3x + 1) - (2x - 3)][(3x + 1) + (2x - 3)]\\ &= (3x + 1 - 2x + 3)(3x + 1 + 2x - 3)\\ &= (x + 4)(5x - 2)\end{aligned}$$

Example 1

Factor Quadratic Expressions

Factor.

a) $2x^2 - 2x - 12$

b) $\frac{1}{4}x^2 - x - 3$

c) $9x^2 - 0.64y^2$

Solution

a) Method 1: Remove the Common Factor First

Factor out the common factor of 2.

$2x^2 - 2x - 12 = 2(x^2 - x - 6)$

Find two integers with a product of -6 and a sum of -1.

Factors of -6	Product	Sum
1, -6	-6	-5
2, -3	-6	-1
3, -2	-6	1
6, -1	-6	5

The factors are $x + 2$ and $x - 3$.

$$2x^2 - 2x - 12 = 2(x^2 - x - 6)$$
$$= 2(x + 2)(x - 3)$$

Method 2: Factor the Trinomial First by Grouping

To factor, $2x^2 - 2x - 12$, find two integers with

- a product of $(2)(-12) = -24$
- a sum of -2

The two integers are -6 and 4.
Write $-2x$ as the sum $-6x + 4x$.
Then, factor by grouping.

$$2x^2 - 2x - 12 = 2x^2 - 6x + 4x - 12$$
$$= 2x(x - 3) + 4(x - 3)$$
$$= (2x + 4)(x - 3)$$
$$= 2(x + 2)(x - 3)$$

Factor out the common factor of 2.

b) Factor out the common factor of $\frac{1}{4}$ first.

$$\frac{1}{4}x^2 - x - 3 = \frac{1}{4}(x^2 - 4x - 12)$$
$$= \frac{1}{4}(x + 2)(x - 6)$$

How does factoring out the common factor of $\frac{1}{4}$ help you?

How can you determine the factors for the trinomial $x^2 - 4x - 12$?

c) The binomial $9x^2 - 0.64y^2$ is a difference of squares.

The first term is a perfect square: $(3x)^2$
The second term is a perfect square: $(0.8y)^2$

$$9x^2 - 0.64y^2 = (3x)^2 - (0.8y)^2$$
$$= (3x - 0.8y)(3x + 0.8y)$$

Your Turn

Factor.

a) $3x^2 + 3x - 6$

b) $\frac{1}{2}x^2 - x - 4$

c) $0.49j^2 - 36k^2$

Did You Know?

When the leading coefficient of a quadratic polynomial is not an integer, you can factor out the rational number as a common factor.

For example,
$\frac{1}{2}x^2 - 5x + 1$
$= \frac{1}{2}(x^2 - 10x + 2)$

What do you need to multiply $\frac{1}{2}$ by to get 5?

What do you need to multiply $\frac{1}{2}$ by to get 1?

Example 2

Factor Polynomials of Quadratic Form

Factor each polynomial.

a) $12(x + 2)^2 + 24(x + 2) + 9$
b) $9(2t + 1)^2 - 4(s - 2)^2$

Solution

a) $12(x + 2)^2 + 24(x + 2) + 9$

Treat the term $x + 2$ as a single variable.
Substitute $r = x + 2$ into the quadratic expression and factor as usual.

$12(x + 2)^2 + 24(x + 2) + 9$	Substitute r for $x + 2$.
$= 12r^2 + 24r + 9$	
$= 3(4r^2 + 8r + 3)$	Factor out the common factor of 3.
$= 3(4r^2 + 2r + 6r + 3)$	Find two integers with a product of (4)(3) = 12 and a sum of 8. The integers 2 and 6 work.
$= 3[(4r^2 + 2r) + (6r + 3)]$	Factor by grouping.
$= 3[2r(2r + 1) + 3(2r + 1)]$	
$= 3(2r + 1)(2r + 3)$	
$= 3[2(x + 2) + 1][2(x + 2) + 3]$	Replace r with $x + 2$.
$= 3(2x + 4 + 1)(2x + 4 + 3)$	Simplify.
$= 3(2x + 5)(2x + 7)$	

The expression $12(x + 2)^2 + 24(x + 2) + 9$ in factored form is $3(2x + 5)(2x + 7)$.

b) $9(2t + 1)^2 - 4(s - 2)^2$
Each term of the polynomial is a perfect square.
Therefore, this is a difference of squares of the form $P^2 - Q^2 = (P - Q)(P + Q)$ where P represents $3(2t + 1)$ and Q represents $2(s - 2)$.

Use the pattern for factoring a difference of squares.

$$\begin{aligned}
&9(2t + 1)^2 - 4(s - 2)^2 \\
&= [3(2t + 1) - 2(s - 2)][3(2t + 1) + 2(s - 2)] \\
&= (6t + 3 - 2s + 4)(6t + 3 + 2s - 4) \\
&= (6t - 2s + 7)(6t + 2s - 1)
\end{aligned}$$

The expression $9(2t + 1)^2 - 4(s - 2)^2$ in factored form is $(6t - 2s + 7)(6t + 2s - 1)$.

Your Turn

Factor each polynomial.

a) $-2(n + 3)^2 + 12(n + 3) + 14$
b) $4(x - 2)^2 - 0.25(y - 4)^2$

Solving Quadratic Equations by Factoring

Some quadratic equations that have real-number solutions can be factored easily.

The *zero product property* states that if the product of two real numbers is zero, then one or both of the numbers must be zero. This means that if $de = 0$, then at least one of d and e is 0.

The roots of a quadratic equation occur when the product of the factors is equal to zero. To solve a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, factor the expression and then set either factor equal to zero. The solutions are the roots of the equation.

For example, rewrite the quadratic equation $3x^2 - 2x - 5 = 0$ in factored form.

$$3x^2 - 2x - 5 = 0$$
$$(3x - 5)(x + 1) = 0$$

$$3x - 5 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = \frac{5}{3} \qquad\qquad x = -1$$

The roots are $\frac{5}{3}$ and -1.

Example 3

Solve Quadratic Equations by Factoring

Determine the roots of each quadratic equation. Verify your solutions.

a) $x^2 + 6x + 9 = 0$ **b)** $x^2 + 4x - 21 = 0$ **c)** $2x^2 - 9x - 5 = 0$

Solution

a) To solve $x^2 + 6x + 9 = 0$, determine the factors and then solve for x.

$$x^2 + 6x + 9 = 0$$
$$(x + 3)(x + 3) = 0$$

This is a perfect square trinomial.

$$(x + 3) = 0 \quad \text{or} \quad (x + 3) = 0$$
$$x = -3 \qquad\qquad x = -3$$

For the quadratic equation to equal 0, one of the factors must equal 0.

This equation has two equal real roots. Since both roots are equal, the roots may be viewed as one distinct real root. Check by substituting the solution into the original quadratic equation.

For $x = -3$:

Left Side	Right Side
$x^2 + 6x + 9$	0
$= (-3)^2 + 6(-3) + 9$	
$= 9 - 18 + 9$	
$= 0$	

Left Side = Right Side

The solution is correct. The roots of the equation are -3 and -3, or just -3.

b) To solve $x^2 + 4x - 21 = 0$, first determine the factors, and then solve for x.

$$x^2 + 4x - 21 = 0$$
$$(x - 3)(x + 7) = 0$$

Two integers with a product of -21 and a sum of 4 are -3 and 7.

Set each factor equal to zero and solve for x.

$x - 3 = 0$ or $x + 7 = 0$

$x = 3$ $\qquad x = -7$

The equation has two distinct real roots. Check by substituting each solution into the original quadratic equation.

For $x = 3$:

Left Side	Right Side
$x^2 + 4x - 21$	0
$= 3^2 + 4(3) - 21$	
$= 9 + 12 - 21$	
$= 0$	

Left Side = Right Side

For $x = -7$:

Left Side	Right Side
$x^2 + 4x - 21$	0
$= (-7)^2 + 4(-7) - 21$	
$= 49 - 28 - 21$	
$= 0$	

Left Side = Right Side

Both solutions are correct. The roots of the quadratic equation are 3 and -7.

c) To solve $2x^2 - 9x - 5 = 0$, first determine the factors, and then solve for x.

Method 1: Factor by Inspection

$2x^2$ is the product of the first terms, and -5 is the product of the second terms.

$2x^2 - 9x - 5 = (2x + ■)(x + ■)$

The last term, -5, is negative. So, one factor of -5 must be negative. Try factor pairs of -5 until the sum of the products of the outer and inner terms is $-9x$.

Factors of -5	Product	Middle Term
-5, 1	$(2x - 5)(x + 1) = 2x^2 + 2x - 5x - 5$ $= 2x^2 - 3x - 5$	$-3x$ is not the correct middle term.
1, -5	$(2x + 1)(x - 5) = 2x^2 - 10x + 1x - 5$ $= 2x^2 - 9x - 5$	Correct.

Therefore, $2x^2 - 9x - 5 = (2x + 1)(x - 5)$.

$$2x^2 - 9x - 5 = 0$$
$$(2x + 1)(x - 5) = 0$$

Set each factor equal to zero and solve for x.

$2x + 1 = 0$ or $x - 5 = 0$

$2x = -1$ $\qquad x = 5$

$x = -\frac{1}{2}$

The roots are $-\frac{1}{2}$ and 5.

Method 2: Factor by Grouping

Find two integers with a product of $(2)(-5) = -10$ and a sum of -9.

Factors of -10	Product	Sum
1, -10	-10	-9
2, -5	-10	-3
5, -2	-10	3
10, -1	-10	9

Write $-9x$ as $x - 10x$. Then, factor by grouping.

$$\begin{aligned} 2x^2 - 9x - 5 &= 0 \\ 2x^2 + x - 10x - 5 &= 0 \\ (2x^2 + x) + (-10x - 5) &= 0 \\ x(2x + 1) - 5(2x + 1) &= 0 \\ (2x + 1)(x - 5) &= 0 \end{aligned}$$

Set each factor equal to zero and solve for x.

$$\begin{aligned} 2x + 1 &= 0 \quad \text{or} \quad & x - 5 &= 0 \\ 2x &= -1 & x &= 5 \\ x &= -\frac{1}{2} \end{aligned}$$

The roots are $-\frac{1}{2}$ and 5.

Check for both Methods 1 and 2:

The equation has two distinct real roots. Check by substituting each root into the original quadratic equation.

For $x = -\frac{1}{2}$:

Left Side: $2x^2 - 9x - 5$

$= 2\left(-\frac{1}{2}\right)^2 - 9\left(-\frac{1}{2}\right) - 5$

$= 2\left(\frac{1}{4}\right) + \frac{9}{2} - 5$

$= \frac{1}{2} + \frac{9}{2} - \frac{10}{2}$

$= 0$

Right Side: 0

Left Side = Right Side

For $x = 5$:

Left Side: $2x^2 - 9x - 5$

$= 2(5)^2 - 9(5) - 5$

$= 50 - 45 - 5$

$= 0$

Right Side: 0

Left Side = Right Side

Both solutions are correct.

The roots of the quadratic equation are $-\frac{1}{2}$ and 5.

Your Turn

Determine the roots of each quadratic equation.

a) $x^2 - 10x + 25 = 0$

b) $x^2 - 16 = 0$

c) $3x^2 - 2x - 8 = 0$

Example 4

Apply Quadratic Equations

Did You Know?

Dock jumping competitions started in 2000 and have spread throughout the world, with events in Canada, United States, Great Britain, Japan, Australia, and Germany. The current world record holder jumped 29 ft 1 in. (8.86 m).

Dock jumping is an exciting dog event in which dogs compete for the longest jumping distance from a dock into a body of water. The path of a Jack Russell terrier on a particular jump can be approximated by the quadratic function $h(d) = -\frac{3}{10}d^2 + \frac{11}{10}d + 2$, where h is the height above the surface of the water and d is the horizontal distance the dog travels from the base of the dock, both in feet. All measurements are taken from the base of the dog's tail. Determine the horizontal distance of the jump.

Solution

When the dog lands in the water, the dog's height above the surface is 0 m. To solve this problem, determine the roots of the quadratic equation $-\frac{3}{10}d^2 + \frac{11}{10}d + 2 = 0$.

$$-\frac{3}{10}d^2 + \frac{11}{10}d + 2 = 0$$

$$-\frac{1}{10}(3d^2 - 11d - 20) = 0$$

Factor out the common factor of $-\frac{1}{10}$.

$$-\frac{1}{10}(3d + 4)(d - 5) = 0$$

$$3d + 4 = 0 \quad \text{or} \quad d - 5 = 0$$
$$3d = -4 \qquad\qquad d = 5$$
$$d = -\frac{4}{3}$$

Solve for d to determine the roots of the equation. Why does the factor $-\frac{1}{10}$ neither result in a root nor affect the other roots of the equation?

Since d represents the horizontal distance of the dog from the base of the dock, it cannot be negative.

So, reject the root $-\frac{4}{3}$.

Check the solution by substituting $d = 5$ into the original quadratic equation.

For $d = 5$:

Left Side	Right Side
$-\frac{3}{10}d^2 + \frac{11}{10}d + 2$	0
$= -\frac{3}{10}(5)^2 + \frac{11}{10}(5) + 2$	
$= -\frac{15}{2} + \frac{11}{2} + \frac{4}{2}$	
$= 0$	

Left Side = Right Side

The solution is correct.
The dog travels a horizontal distance of 5 ft.

Your Turn

A waterslide ends with the slider dropping into a deep pool of water. The path of the slider after leaving the lower end of the slide can be approximated by the quadratic function $h(d) = -\frac{1}{6}d^2 - \frac{1}{6}d + 2$, where h is the height above the surface of the pool and d is the horizontal distance the slider travels from the lower end of the slide, both in feet. What is the horizontal distance the slider travels before dropping into the pool after leaving the lower end of the slide?

Example 5

Write and Solve a Quadratic Equation

The length of an outdoor lacrosse field is 10 m less than twice the width. The area of the field is 6600 m². Determine the dimensions of an outdoor lacrosse field.

Did You Know?

Lacrosse is one of the oldest team sports in North America. The game of lacrosse was developed more than 500 years ago and is referred to as *The Creator's Game*. It is based on the First Nations game *baggataway*. Traditional games could go on for days. Hundreds of players from different tribes took turns playing. Today, amateur and professional teams throughout North America play lacrosse.

Solution

Let w represent the width of the field.
Then, the length of the field is $2w - 10$.

Use the area formula.

$$\begin{aligned} A &= lw \\ 6600 &= (2w - 10)(w) \\ 6600 &= 2w^2 - 10w \\ 0 &= 2w^2 - 10w - 6600 \\ 0 &= 2(w^2 - 5w - 3300) \\ 0 &= w^2 - 5w - 3300 \\ 0 &= (w - 60)(w + 55) \end{aligned}$$

$$w - 60 = 0 \quad \text{or} \quad w + 55 = 0$$
$$w = 60 \qquad\qquad w = -55$$

Since the width of the field cannot be negative, $w = -55$ is rejected. The width of the field is 60 m. The length of the field is $2(60) - 10$ or 110 m.

Check:
The area of the field is (60)(110) or 6600 m².

Your Turn

The area of a rectangular Ping-Pong table is 45 ft². The length is 4 ft more than the width. What are the dimensions of the table?

Key Ideas

- You can solve some quadratic equations by factoring.
- If two factors of a quadratic equation have a product of zero, then by the zero product property one of the factors must be equal to zero.
- To solve a quadratic equation by factoring, first write the equation in the form $ax^2 + bx + c = 0$, and then factor the left side. Next, set each factor equal to zero, and solve for the unknown.

  For example,

$$\begin{aligned} x^2 + 8x &= -12 \\ x^2 + 8x + 12 &= 0 \\ (x + 2)(x + 6) &= 0 \end{aligned}$$

$$\begin{aligned} x + 2 &= 0 \quad \text{or} \quad x + 6 = 0 \\ x &= -2 \qquad\qquad x = -6 \end{aligned}$$

- The solutions to a quadratic equation are called the roots of the equation.
- You can factor polynomials in quadratic form.
  - Factor trinomials of the form $a(P)^2 + b(P) + c$, where $a \neq 0$ and P is any expression, by replacing the expression for P with a single variable. Then substitute the expression for P back into the factored expression. Simplify the final factors, if possible.

    For example, factor $2(x + 3)^2 - 11(x + 3) + 15$ by letting $r = x + 3$.

$$\begin{aligned} 2(x + 3)^2 - 11(x + 3) + 15 &= 2r^2 - 11r + 15 \\ &= 2r^2 - 5r - 6r + 15 \\ &= (2r^2 - 5r) + (-6r + 15) \\ &= r(2r - 5) - 3(2r - 5) \\ &= (2r - 5)(r - 3) \\ &= [2(x + 3) - 5][(x + 3) - 3] \\ &= (2x + 1)(x) \\ &= x(2x + 1) \end{aligned}$$

  - Factor a difference of squares, $P^2 - Q^2$, where P and Q are any expressions, as $[P - Q][P + Q]$.

Check Your Understanding

Practise

1. Factor completely.

a) $x^2 + 7x + 10$

b) $5z^2 + 40z + 60$

c) $0.2d^2 - 2.2d + 5.6$

2. Factor completely.

a) $3y^2 + 4y - 7$

b) $8k^2 - 6k - 5$

c) $0.4m^2 + 0.6m - 1.8$

3. Factor completely.

a) $x^2 + x - 20$

b) $x^2 - 12x + 36$

c) $\frac{1}{4}x^2 + 2x + 3$

d) $2x^2 + 12x + 18$

4. Factor each expression.

a) $4y^2 - 9x^2$

b) $0.36p^2 - 0.49q^2$

c) $\frac{1}{4}s^2 - \frac{9}{25}t^2$

d) $0.16t^2 - 16s^2$

5. Factor each expression.

a) $(x + 2)^2 - (x + 2) - 42$

b) $6(x^2 - 4x + 4)^2 + (x^2 - 4x + 4) - 1$

c) $(4j - 2)^2 - (2 + 4j)^2$

6. What are the factors of each expression?

a) $4(5b - 3)^2 + 10(5b - 3) - 6$

b) $16(x^2 + 1)^2 - 4(2x)^2$

c) $-\frac{1}{4}(2x)^2 + 25(2y^3)^2$

7. Solve each factored equation.

a) $(x + 3)(x + 4) = 0$

b) $(x - 2)\left(x + \frac{1}{2}\right) = 0$

c) $(x + 7)(x - 8) = 0$

d) $x(x + 5) = 0$

e) $(3x + 1)(5x - 4) = 0$

f) $2(x - 4)(7 - 2x) = 0$

8. Solve each quadratic equation by factoring. Check your answers.

a) $10n^2 - 40 = 0$

b) $\frac{1}{4}x^2 + \frac{5}{4}x + 1 = 0$

c) $3w^2 + 28w + 9 = 0$

d) $8y^2 - 22y + 15 = 0$

e) $d^2 + \frac{5}{2}d + \frac{3}{2} = 0$

f) $4x^2 - 12x + 9 = 0$

9. Determine the roots of each quadratic equation. Verify your answers.

a) $k^2 - 5k = 0$

b) $9x^2 = x + 8$

c) $\frac{8}{3}t + 5 = -\frac{1}{3}t^2$

d) $\frac{25}{49}y^2 - 9 = 0$

e) $2s^2 - 4s = 70$

f) $4q^2 - 28q = -49$

10. Solve each equation.

a) $42 = x^2 - x$

b) $g^2 = 30 - 7g$

c) $y^2 + 4y = 21$

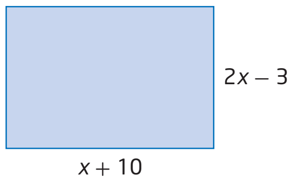
d) $3 = 6p^2 - 7p$

e) $3x^2 + 9x = 30$

f) $2z^2 = 3 - 5z$

Apply

11. A rectangle has dimensions $x + 10$ and $2x - 3$, where x is in centimetres. The area of the rectangle is 54 cm^2.



a) What equation could you use to determine the value of x?

b) What is the value of x?

12. An osprey, a fish-eating bird of prey, dives toward the water to catch a salmon. The height, h, in metres, of the osprey above the water t seconds after it begins its dive can be approximated by the function $h(t) = 5t^2 - 30t + 45$.

a) Determine the time it takes for the osprey to reach a height of 20 m.

b) What assumptions did you make? Are your assumptions reasonable? Explain.

13. A flare is launched from a boat. The height, h, in metres, of the flare above the water is approximately modelled by the function $h(t) = 150t - 5t^2$, where t is the number of seconds after the flare is launched.

a) What equation could you use to determine the time it takes for the flare to return to the water?

b) How many seconds will it take for the flare to return to the water?

14. The product of two consecutive even integers is 16 more than 8 times the smaller integer. Determine the integers.

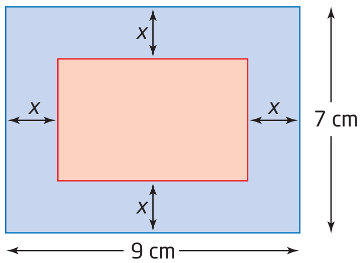
15. The area of a square is tripled by adding 10 cm to one dimension and 12 cm to the other. Determine the side length of the square.

16. Ted popped a baseball straight up with an initial upward velocity of 48 ft/s. The height, h, in feet, of the ball above the ground is modelled by the function $h(t) = 3 + 48t - 16t^2$. How long was the ball in the air if the catcher catches the ball 3 ft above the ground? Is your answer reasonable in this situation? Explain.

Did You Know?

Many Canadians have made a positive impact on Major League Baseball. Players such as Larry Walker of Maple Ridge, British Columbia, Jason Bay of Trail, British Columbia, and Justin Morneau of Westminster, British Columbia have had very successful careers in baseball's highest league.

17. A rectangle with area of 35 cm^2 is formed by cutting off strips of equal width from a rectangular piece of paper.



a) What is the width of each strip?

b) What are the dimensions of the new rectangle?

18. Without factoring, state if the binomial is a factor of the trinomial. Explain why or why not.

a) $x^2 - 5x - 36$, $x - 5$

b) $x^2 - 2x - 15$, $x + 3$

c) $6x^2 + 11x + 4$, $4x + 1$

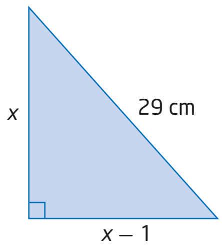
d) $4x^2 + 4x - 3$, $2x - 1$

19. Solve each equation.

a) $x(2x - 3) - 2(3 + 2x) = -4(x + 1)$

b) $3(x - 2)(x + 1) - 4 = 2(x - 1)^2$

20. The hypotenuse of a right triangle measures 29 cm. One leg is 1 cm shorter than the other. What are the lengths of the legs?



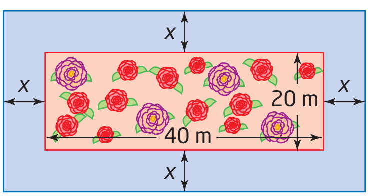
21. A field is in the shape of a right triangle. The fence around the perimeter of the field measures 40 m. If the length of the hypotenuse is 17 m, find the length of the other two sides.

22. The width of the top of a notebook computer is 7 cm less than the length. The surface area of the top of the notebook is 690 cm^2.

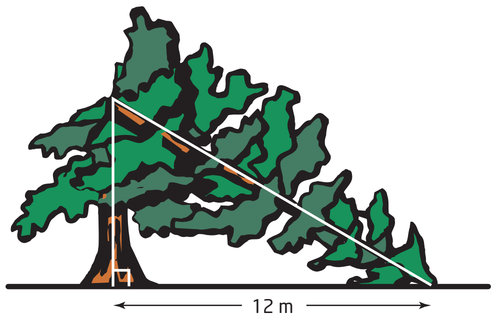
a) Write an equation to represent the surface area of the top of the notebook computer.

b) What are the dimensions of the top of the computer?

23. Stephan plans to build a uniform walkway around a rectangular flower bed that is 20 m by 40 m. There is enough material to make a walkway that has a total area of 700 m^2. What is the width of the walkway?



24. An 18-m-tall tree is broken during a severe storm, as shown. The distance from the base of the trunk to the point where the tip touches the ground is 12 m. At what height did the tree break?



25. The pressure difference, P, in newtons per square metre, above and below an airplane wing is described by the formula $P = \left(\frac{1}{2}d\right)(v_1)^2 - \left(\frac{1}{2}d\right)(v_2)^2$, where d is the density of the air, in kilograms per cubic metre; v_1 is the velocity, in metres per second, of the air passing above; and v_2 is the velocity, in metres per second, of the air passing below. Write this formula in factored form.

26. Carlos was asked to factor the trinomial $6x^2 - 16x + 8$ completely. His work is shown below.

Carlos's solution:

$$\begin{aligned} &6x^2 - 16x + 8 \\ &= 6x^2 - 12x - 4x + 8 \\ &= 6x(x - 2) - 4(x - 2) \\ &= (x - 2)(6x - 4) \end{aligned}$$

Is Carlos correct? Explain.

27. Factor each expression.

a) $3(2z + 3)^2 - 9(2z + 3) - 30$

b) $16(m^2 - 4)^2 - 4(3n)^2$

c) $\frac{1}{9}y^2 - \frac{1}{3}yx + \frac{1}{4}x^2$

d) $-28\left(w + \frac{2}{3}\right)^2 + 7\left(3w - \frac{1}{3}\right)^2$

Extend

28. A square has an area of $(9x^2 + 30xy + 25y^2)$ square centimetres. What is an expression for the perimeter of the square?

29. Angela opened a surf shop in Tofino, British Columbia. Her accountant models her profit, P, in dollars, with the function $P(t) = 1125(t - 1)^2 - 10\,125$, where t is the number of years of operation. Use graphing or factoring to determine how long it will take for the shop to start making a profit.

Pete Devries

Did You Know?

Pete Devries was the first Canadian to win an international surfing competition. In 2009, he outperformed over 110 world-class surfers to win the O'Neill Cold Water Classic Canada held in Tofino, British Columbia.

Create Connections

30. Write a quadratic equation in standard form with the given root(s).

a) -3 and 3

b) 2

c) $\frac{2}{3}$ and 4

d) $\frac{3}{5}$ and $-\frac{1}{2}$

31. Create an example of a quadratic equation that cannot be solved by factoring. Explain why it cannot be factored. Show the graph of the corresponding quadratic function and show where the roots are located.

32. You can use the difference of squares pattern to perform certain mental math shortcuts. For example,

$$\begin{aligned} 81 - 36 &= (9 - 6)(9 + 6) \\ &= (3)(15) \\ &= 45 \end{aligned}$$

a) Explain how this strategy works. When can you use it?

b) Create two examples to illustrate the strategy.

Project Corner — Avalanche Safety

- Experts use avalanche control all over the world above highways, ski resorts, railroads, mining operations, and utility companies, and anywhere else that may be threatened by avalanches.
- Avalanche control is the intentional triggering of avalanches. People are cleared away to a safe distance, then experts produce more frequent, but smaller, avalanches at controlled times.
- Because avalanches tend to occur in the same zones and under certain conditions, avalanche experts can predict when avalanches are likely to occur.
- Charges are delivered by launchers, thrown out of helicopters, or delivered above the avalanche starting zones by an avalanche control expert on skis.
- What precautions would avalanche control experts need to take to ensure public safety?

4.3

Solving Quadratic Equations by Completing the Square

Focus on...

- solving quadratic equations by completing the square

Rogers Pass gets up to 15 m of snow per year. Because of the steep mountains, over 130 avalanche paths must be monitored during the winter. To keep the Trans-Canada Highway open, the Royal Canadian Artillery uses 105-mm howitzers to create controlled avalanches. The Artillery must aim the howitzer accurately to operate it safely. Suppose that the quadratic function that approximates the trajectory of a shell fired by a howitzer at an angle of 45° is $h(x) = -\frac{1}{5}x^2 + 2x + \frac{1}{20}$, where h is the height of the shell and x is the horizontal distance from the howitzer to where the shell lands, both in kilometres. How can this function be used to determine where to place the howitzer to fire at a specific spot on the mountainside?

Investigate Solving Quadratic Equations by Completing the Square

Materials

- grid paper, graphing calculator, or computer with graphing software

Sometimes factoring quadratic equations is not practical. In Chapter 3, you learned how to complete the square to analyse and graph quadratic functions. You can complete the square to help solve quadratic equations such as $-\frac{1}{5}x^2 + 2x + \frac{1}{20} = 0$.

1. Graph the function $f(x) = -\frac{1}{5}x^2 + 2x + \frac{1}{20}$.
2. What are the x-intercepts of the graph? How accurate are your answers? Why might it be important to determine more accurate zeros for the function?
3. a) Rewrite the function in the form $h(x) = a(x - p)^2 + q$ by completing the square.
   b) Set $h(x)$ equal to zero. Solve for x. Express your answers as exact values.

Reflect and Respond

4. What are the two roots of the quadratic equation for projectile motion, $0 = -\frac{1}{5}x^2 + 2x + \frac{1}{20}$? What do the roots represent in this situation?

5. To initiate an avalanche, the howitzer crew must aim the shell up the slope of the mountain. The shot from the howitzer lands 750 m above where the howitzer is located. How could the crew determine the horizontal distance from the point of impact at which the howitzer must be located? Explain your reasoning. Calculate the horizontal distances involved in this scenario. Include a sketch of the path of the projectile.

6. At which horizontal distance from the point of impact would you locate the howitzer if you were in charge of setting off a controlled avalanche? Explain your reasoning.

Did You Know?

Parks Canada operates the world's largest mobile avalanche control program to keep the Trans-Canada Highway and the Canadian Pacific Railway operating through Rogers Pass.

Taking slope angle measurement.

Link the Ideas

You can solve quadratic equations of the form $ax^2 + bx + c = 0$, where $b = 0$, or of the form $a(x - p)^2 + q = 0$, where $a \neq 0$, that have real-number solutions by isolating the squared term and taking the square root of both sides. The square root of a positive real number can be positive or negative, so there are two possible solutions to these equations.

To solve $x^2 = 9$, take the square root of both sides.

$$x^2 = 9$$
$$\pm\sqrt{x^2} = \pm\sqrt{9}$$
$$x = \pm 3$$

Read $\pm$ as "plus or minus."

3 is a solution to the equation because (3)(3) = 9.
−3 is a solution to the equation because (−3)(−3) = 9.

To solve $(x - 1)^2 - 49 = 0$, isolate the squared term and take the square root of both sides.

$$(x - 1)^2 - 49 = 0$$
$$(x - 1)^2 = 49$$
$$x - 1 = \pm 7$$
$$x = 1 \pm 7$$

$x = 1 + 7$ or $x = 1 - 7$
$x = 8$ $\qquad$ $x = -6$

Check:
Substitute $x = 8$ and $x = -6$ into the original equation.

Left Side	Right Side	Left Side	Right Side
$(x - 1)^2 - 49$	0	$(x - 1)^2 - 49$	0
$= (8 - 1)^2 - 49$		$= (-6 - 1)^2 - 49$	
$= 7^2 - 49$		$= (-7)^2 - 49$	
$= 49 - 49$		$= 49 - 49$	
$= 0$		$= 0$	
Left Side = Right Side		Left Side = Right Side	

Both solutions are correct. The roots are 8 and −6.

Did You Know?

Around 830 C.E., Abu Ja'far Muhammad ibn Musa al-Khwarizmi wrote *Hisab al-jabr w'al-muqabala*. The word *al-jabr* from this title is the basis of the word we use today, *algebra*. In his book, al-Khwarizmi describes how to solve a quadratic equation by completing the square.

Web Link

To learn more about al-Khwarizmi, go to www.mhrprecalc11.ca and follow the links.

Many quadratic equations cannot be solved by factoring. In addition, graphing the corresponding functions may not result in exact solutions. You can write a quadratic function expressed in standard form, $y = ax^2 + bx + c$, in vertex form, $y = a(x - p)^2 + q$, by completing the square. You can also use the process of completing the square to determine exact solutions to quadratic equations.

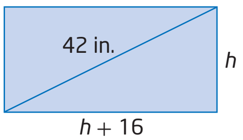
Example 1

Write and Solve a Quadratic Equation by Taking the Square Root

A wide-screen television has a diagonal measure of 42 in. The width of the screen is 16 in. more than the height. Determine the dimensions of the screen, to the nearest tenth of an inch.

Solution

Draw a diagram. Let h represent the height of the screen. Then, $h + 16$ represents the width of the screen.



Use the Pythagorean Theorem.

$$\begin{aligned}
h^2 + (h + 16)^2 &= 42^2 \\
h^2 + (h^2 + 32h + 256) &= 1764 \\
2h^2 + 32h + 256 &= 1764 \\
2h^2 + 32h &= 1508 \\
h^2 + 16h &= 754 && \text{Isolate the variable terms on the left side.} \\
h^2 + 16h + 64 &= 754 + 64 && \text{Add the square of half the coefficient of } h \text{ to both sides.} \\
(h + 8)^2 &= 818 && \text{Factor the perfect square trinomial on the left side.} \\
h + 8 &= \pm\sqrt{818} && \text{Take the square root of both sides.} \\
h &= -8 \pm \sqrt{818}
\end{aligned}$$

$$\begin{aligned}
h &= -8 + \sqrt{818} \quad \text{or} \quad & h &= -8 - \sqrt{818} \\
h &\approx 20.6 & h &\approx -36.6
\end{aligned}$$

Since the height of the screen cannot be negative, $h = -36.6$ is an **extraneous root**.

extraneous root

- a number obtained in solving an equation, which does not satisfy the initial restrictions on the variable

Thus, the height of the screen is approximately 20.6 in., and the width of the screen is approximately 20.6 + 16 or 36.6 in..

Hence, the dimensions of a 42-in. television are approximately 20.6 in. by 36.6 in..

Check:
$20.6^2 + 36.6^2$ is 1763.92, and $\sqrt{1763.92}$ is approximately 42, the diagonal of the television, in inches.

Your Turn

The circular Canadian two-dollar coin consists of an aluminum and bronze core and a nickel outer ring. If the radius of the inner core is 0.84 cm and the area of the circular face of the coin is 1.96π cm^2, what is the width of the outer ring?

Example 2

Solve a Quadratic Equation by Completing the Square When $a = 1$

Solve $x^2 - 21 = -10x$ by completing the square. Express your answers to the nearest tenth.

Can you solve this equation by factoring? Explain.

Solution

$$\begin{aligned} x^2 - 21 &= -10x \\ x^2 + 10x &= 21 \\ x^2 + 10x + 25 &= 21 + 25 \\ (x + 5)^2 &= 46 \\ x + 5 &= \pm\sqrt{46} \end{aligned}$$

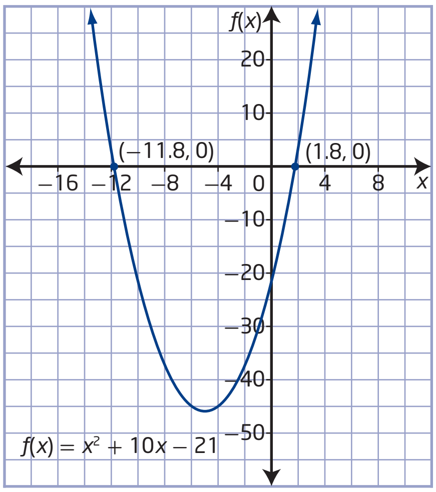
Solve for x.

$$\begin{aligned} x + 5 &= \sqrt{46} & \quad \text{or} \quad x + 5 &= -\sqrt{46} \\ x &= -5 + \sqrt{46} & x &= -5 - \sqrt{46} \\ x &= 1.7823\ldots & x &= -11.7823\ldots \end{aligned}$$

The exact roots are $-5 + \sqrt{46}$ and $-5 - \sqrt{46}$.
The roots are 1.8 and -11.8, to the nearest tenth.

You can also see the solutions to this equation graphically as the x-intercepts of the graph of the function $f(x) = x^2 + 10x - 21$.

These occur at approximately $(-11.8, 0)$ and $(1.8, 0)$ and have values of -11.8 and 1.8, respectively.



Your Turn

Solve $p^2 - 4p = 11$ by completing the square. Express your answers to the nearest tenth.

Example 3

Solve a Quadratic Equation by Completing the Square When $a \neq 1$

Determine the roots of $-2x^2 - 3x + 7 = 0$, to the nearest hundredth. Then, use technology to verify your answers.

Solution

$$-2x^2 - 3x + 7 = 0$$

$$x^2 + \frac{3}{2}x - \frac{7}{2} = 0 \qquad \text{Divide both sides by a factor of } -2.$$

$$x^2 + \frac{3}{2}x = \frac{7}{2} \qquad \text{Isolate the variable terms on the left side.}$$

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{7}{2} + \frac{9}{16} \qquad \text{Why is } \frac{9}{16} \text{ added to both sides?}$$

$$\left(x + \frac{3}{4}\right)^2 = \frac{65}{16}$$

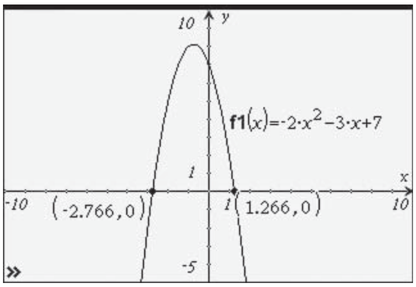
$$x + \frac{3}{4} = \pm\sqrt{\frac{65}{16}} \qquad \text{Solve for } x.$$

$$x = -\frac{3}{4} \pm \frac{\sqrt{65}}{4}$$

$$x = \frac{-3 \pm \sqrt{65}}{4}$$

The exact roots are $\frac{-3 + \sqrt{65}}{4}$ and $\frac{-3 - \sqrt{65}}{4}$.

The roots are 1.27 and −2.77, to the nearest hundredth.



Your Turn

Determine the roots of the equation $-2x^2 - 5x + 2 = 0$, to the nearest hundredth. Verify your solutions using technology.

Example 4

Apply Completing the Square

A defender kicks a soccer ball away from her own goal. The path of the kicked soccer ball can be approximated by the quadratic function $h(x) = -0.06x^2 + 3.168x - 35.34$, where x is the horizontal distance travelled, in metres, from the goal line and h is the height, in metres.

a) You can determine the distance the soccer ball is from the goal line by solving the corresponding equation, $-0.06x^2 + 3.168x - 35.34 = 0$. How far is the soccer ball from the goal line when it is kicked? Express your answer to the nearest tenth of a metre.

b) How far does the soccer ball travel before it hits the ground?

Solution

a) Solve the equation $-0.06x^2 + 3.168x - 35.34 = 0$ by completing the square.

$$-0.06x^2 + 3.168x - 35.34 = 0$$

$$x^2 - 52.8x + 589 = 0$$ Divide both sides by a common factor of -0.06.

$$x^2 - 52.8x = -589$$ Isolate the variable terms on the left side.

$$x^2 - 52.8x + \left(\frac{52.8}{2}\right)^2 = -589 + \left(\frac{52.8}{2}\right)^2$$ Complete the square on the left side.

$$x^2 - 52.8x + 696.96 = -589 + 696.96$$

$$(x - 26.4)^2 = 107.96$$

$$x - 26.4 = \pm\sqrt{107.96}$$ Take the square root of both sides.

Solve for x.

$$x - 26.4 = \sqrt{107.96} \quad \text{or} \quad x - 26.4 = -\sqrt{107.96}$$

$$x = 26.4 + \sqrt{107.96} \qquad x = 26.4 - \sqrt{107.96}$$

$$x = 36.7903\ldots \qquad x = 16.0096\ldots$$

The roots of the equation are approximately 36.8 and 16.0.
The ball is kicked approximately 16.0 m from the goal line.

b) From part a), the soccer ball is kicked approximately 16.0 m from the goal line. The ball lands approximately 36.8 m from the goal line. Therefore, the soccer ball travels 36.8 − 16.0, or 20.8 m, before it hits the ground.

Your Turn

How far does the soccer ball in Example 4 travel if the function that models its trajectory is $h(x) = -0.016x^2 + 1.152x - 15.2$?

Key Ideas

- Completing the square is the process of rewriting a quadratic polynomial from the standard form, $ax^2 + bx + c$, to the vertex form, $a(x - p)^2 + q$.
- You can use completing the square to determine the roots of a quadratic equation in standard form.

For example,

$$
\begin{aligned}
2x^2 - 4x - 2 &= 0 \\
x^2 - 2x - 1 &= 0 && \text{Divide both sides by a common factor of 2.} \\
x^2 - 2x &= 1 && \text{Isolate the variable terms on the left side.} \\
x^2 - 2x + 1 &= 1 + 1 && \text{Complete the square on the left side.} \\
(x - 1)^2 &= 2 \\
x - 1 &= \pm\sqrt{2} && \text{Take the square root of both sides.}
\end{aligned}
$$

$$
\begin{aligned}
x - 1 &= \sqrt{2} \quad \text{or} \quad & x - 1 &= -\sqrt{2} && \text{Solve for } x. \\
x &= 1 + \sqrt{2} & x &= 1 - \sqrt{2} \\
x &\approx 2.41 & x &\approx -0.41
\end{aligned}
$$

- Express roots of quadratic equations as exact roots or as decimal approximations.

Check Your Understanding

Practise

1. What value of c makes each expression a perfect square?

a) $x^2 + x + c$

b) $x^2 - 5x + c$

c) $x^2 - 0.5x + c$

d) $x^2 + 0.2x + c$

e) $x^2 + 15x + c$

f) $x^2 - 9x + c$

2. Complete the square to write each quadratic equation in the form $(x + p)^2 = q$.

a) $2x^2 + 8x + 4 = 0$

b) $-3x^2 - 12x + 5 = 0$

c) $\frac{1}{2}x^2 - 3x + 5 = 0$

3. Write each equation in the form $a(x - p)^2 + q = 0$.

a) $x^2 - 12x + 9 = 0$

b) $5x^2 - 20x - 1 = 0$

c) $-2x^2 + x - 1 = 0$

d) $0.5x^2 + 2.1x + 3.6 = 0$

e) $-1.2x^2 - 5.1x - 7.4 = 0$

f) $\frac{1}{2}x^2 + 3x - 6 = 0$

4. Solve each quadratic equation. Express your answers as exact roots.

a) $x^2 = 64$

b) $2s^2 - 8 = 0$

c) $\frac{1}{3}t^2 - 1 = 11$

d) $-y^2 + 5 = -6$

5. Solve. Express your answers as exact roots.

a) $(x - 3)^2 = 4$

b) $(x + 2)^2 = 9$

c) $\left(d + \frac{1}{2}\right)^2 = 1$

d) $\left(h - \frac{3}{4}\right)^2 = \frac{7}{16}$

e) $(s + 6)^2 = \frac{3}{4}$

f) $(x + 4)^2 = 18$

6. Solve each quadratic equation by completing the square. Express your answers as exact roots.

a) $x^2 + 10x + 4 = 0$

b) $x^2 - 8x + 13 = 0$

c) $3x^2 + 6x + 1 = 0$

d) $-2x^2 + 4x + 3 = 0$

e) $-0.1x^2 - 0.6x + 0.4 = 0$

f) $0.5x^2 - 4x - 6 = 0$

7. Solve each quadratic equation by completing the square. Express your answers to the nearest tenth.

a) $x^2 - 8x - 4 = 0$

b) $-3x^2 + 4x + 5 = 0$

c) $\frac{1}{2}x^2 - 6x - 5 = 0$

d) $0.2x^2 + 0.12x - 11 = 0$

e) $-\frac{2}{3}x^2 - x + 2 = 0$

f) $\frac{3}{4}x^2 + 6x + 1 = 0$

Apply

8. Dinahi's rectangular dog kennel measures 4 ft by 10 ft. She plans to double the area of the kennel by extending each side by an equal amount.

a) Sketch and label a diagram to represent this situation.

b) Write the equation to model the new area.

c) What are the dimensions of the new dog kennel, to the nearest tenth of a foot?

9. Evan passes a flying disc to a teammate during a competition at the Flatland Ultimate and Cups Tournament in Winnipeg. The flying disc follows the path $h(d) = -0.02d^2 + 0.4d + 1$, where h is the height, in metres, and d is the horizontal distance, in metres, that the flying disc has travelled from the thrower. If no one catches the flying disc, the height of the disc above the ground when it lands can be modelled by $h(d) = 0$.

a) What quadratic equation can you use to determine how far the disc will travel if no one catches it?

b) How far will the disc travel if no one catches it? Express your answer to the nearest tenth of a metre.

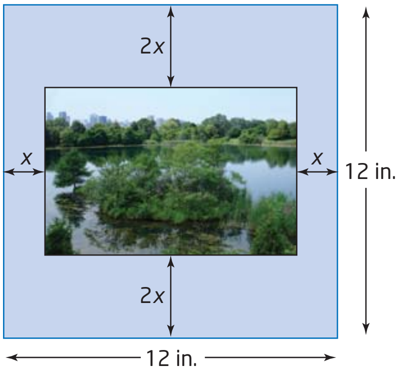
Did You Know?

Each August, teams compete in the Canadian Ultimate Championships for the national title in five different divisions: juniors, masters, mixed, open, and women's. This tournament also determines who will represent Canada at the next world championships.

10. A model rocket is launched from a platform. Its trajectory can be approximated by the function $h(d) = -0.01d^2 + 2d + 1$, where h is the height, in metres, of the rocket and d is the horizontal distance, in metres, the rocket travels. How far does the rocket land from its launch position? Express your answer to the nearest tenth of a metre.

11. Brian is placing a photograph behind a 12-in. by 12-in. piece of matting. He positions the photograph so the matting is twice as wide at the top and bottom as it is at the sides.

The visible area of the photograph is 54 sq. in. What are the dimensions of the photograph?



12. The path of debris from fireworks when the wind is about 25 km/h can be modelled by the quadratic function $h(x) = -0.04x^2 + 2x + 8$, where h is the height and x is the horizontal distance travelled, both measured in metres. How far away from the launch site will the debris land? Express your answer to the nearest tenth of a metre.

Extend

13. Write a quadratic equation with the given roots.

a) $\sqrt{7}$ and $-\sqrt{7}$

b) $1 + \sqrt{3}$ and $1 - \sqrt{3}$

c) $\dfrac{5 + \sqrt{11}}{2}$ and $\dfrac{5 - \sqrt{11}}{2}$

14. Solve each equation for x by completing the square.

a) $x^2 + 2x = k$

b) $kx^2 - 2x = k$

c) $x^2 = kx + 1$

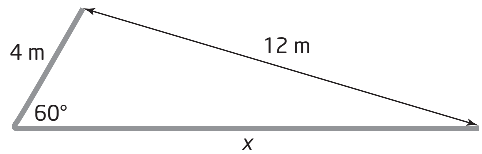
15. Determine the roots of $ax^2 + bx + c = 0$ by completing the square. Can you use this result to solve any quadratic equation? Explain.

16. The sum of the first n terms, S_n, of an arithmetic series can be found using the formula $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$, where t_1 is the first term and d is the common difference.

a) The sum of the first n terms in the arithmetic series $6 + 10 + 14 + \cdots$ is 3870. Determine the value of n.

b) The sum of the first n consecutive natural numbers is 780. Determine the value of n.

17. A machinist in a fabrication shop needs to bend a metal rod at an angle of 60° at a point 4 m from one end of the rod so that the ends of the rod are 12 m apart, as shown.



a) Using the cosine law, write a quadratic equation to represent this situation.

b) Solve the quadratic equation. How long is the rod, to the nearest tenth of a metre?

Create Connections

18. The solution to $x^2 = 9$ is $x = \pm 3$. The solution to the equation $x = \sqrt{9}$ is $x = 3$. Explain why the solutions to the two equations are different.

19. Allison completed the square to determine the vertex form of the quadratic function $y = x^2 - 6x - 27$. Her method is shown.

Allison's method:

$y = x^2 - 6x - 27$
$y = (x^2 - 6x) - 27$
$y = (x^2 - 6x + 9 - 9) - 27$
$y = [(x - 3)^2 - 9] - 27$
$y = (x - 3)^2 - 36$

Riley completed the square to begin to solve the quadratic equation $0 = x^2 - 6x - 27$. His method is shown.

Riley's method:

$$\begin{aligned} 0 &= x^2 - 6x - 27 \\ 27 &= x^2 - 6x \\ 27 + 9 &= x^2 - 6x + 9 \\ 36 &= (x - 3)^2 \\ \pm 6 &= x - 3 \end{aligned}$$

Describe the similarities and differences between the two uses of the method of completing the square.

20. Compare and contrast the following strategies for solving $x^2 - 5x - 6 = 0$.
- completing the square
- graphing the corresponding function
- factoring

21. Write a quadratic function in the form $y = a(x - p)^2 + q$ satisfying each of the following descriptions. Then, write the corresponding quadratic equation in the form $0 = ax^2 + bx + c$. Use graphing technology to verify that your equation also satisfies the description.

a) two distinct real roots

b) one distinct real root, or two equal real roots

c) no real roots

Project Corner **Avalanche Blasting**

- An avalauncher is a two-chambered compressed-gas cannon used in avalanche control work. It fires projectiles with trajectories that can be varied by altering the firing angle and the nitrogen pressure.
- The main disadvantages of avalaunchers, compared to powerful artillery such as the howitzer, are that they have a short range and poor accuracy in strong winds.
- Which would you use if you were an expert initiating a controlled avalanche near a ski resort, a howitzer or an avalauncher? Why?


Howitzer


Avalauncher

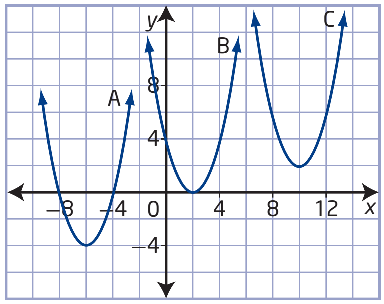
4.4

The Quadratic Formula

Focus on...

- developing the quadratic formula
- solving quadratic equations using the quadratic formula
- using the discriminant to determine the nature of the roots of a quadratic equation
- selecting an appropriate method for solving a quadratic equation
- solving problems involving quadratic equations

You can solve quadratic equations graphically, by factoring, by determining the square root, and by completing the square. Are there other ways? The Greek mathematicians Pythagoras (500 B.C.E.) and Euclid (300 B.C.E.) both derived geometric solutions to a quadratic equation. A general solution for quadratic equations using numbers was derived in about 700 C.E. by the Hindu mathematician Brahmagupta. The general formula used today was derived in about 1100 C.E. by another Hindu mathematician, Bhaskara. He was also the first to recognize that any positive number has two square roots, one positive and one negative.



For each parabola shown, how many roots does the related quadratic equation have?

Investigate the Quadratic Formula

By completing the square, you can develop a formula that allows you to solve any quadratic equation in standard form.

quadratic formula

- a formula for determining the roots of a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$
- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

1. Copy the calculations. Describe the steps in the following example of the **quadratic formula**.

$$2x^2 + 7x + 1 = 0$$
$$x^2 + \frac{7}{2}x + \frac{1}{2} = 0$$
$$x^2 + \frac{7}{2}x = -\frac{1}{2}$$
$$x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = -\frac{1}{2} + \left(\frac{7}{4}\right)^2$$
$$\left(x + \frac{7}{4}\right)^2 = -\frac{8}{16} + \frac{49}{16}$$
$$\left(x + \frac{7}{4}\right)^2 = \frac{41}{16}$$
$$x + \frac{7}{4} = \pm\sqrt{\frac{41}{16}}$$
$$x = -\frac{7}{4} \pm \frac{\sqrt{41}}{4}$$
$$x = \frac{-7 \pm \sqrt{41}}{4}$$

2. Repeat the steps using the general quadratic equation in standard form $ax^2 + bx + c = 0$.

Reflect and Respond

3. a) Will the quadratic formula work for any quadratic equation written in any form?

b) When do you think it is appropriate to use the quadratic formula to solve a quadratic equation?

c) When is it appropriate to use a different method, such as graphing the corresponding function, factoring, determining the square root, or completing the square? Explain.

4. What is the maximum number of roots the quadratic formula will give? How do you know this?

5. Describe the conditions for a, b, and c that are necessary for the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to result in only one possible root.

6. Is there a condition relating a, b, and c that will result in no real solution to a quadratic equation? Explain.

Link the Ideas

You can solve quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$, using the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

For example, in the quadratic equation $3x^2 + 5x - 2 = 0$, $a = 3$, $b = 5$, and $c = -2$.

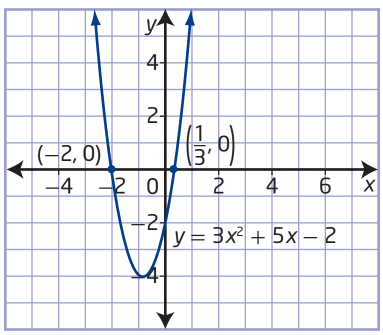
Substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(-2)}}{2(3)}$$
$$x = \frac{-5 \pm \sqrt{25 + 24}}{6}$$
$$x = \frac{-5 \pm \sqrt{49}}{6}$$
$$x = \frac{-5 \pm 7}{6}$$

Determine the two roots.

$$x = \frac{-5 + 7}{6} \quad \text{or} \quad x = \frac{-5 - 7}{6}$$
$$x = \frac{1}{3} \qquad\qquad x = \frac{-12}{6}$$
$$x = -2$$

The roots are $\frac{1}{3}$ and -2.



Check:

Substitute $x = \frac{1}{3}$ and $x = -2$ into the original equation.

Left Side	Right Side	Left Side	Right Side
$3x^2 + 5x - 2$	0	$3x^2 + 5x - 2$	0
$= 3\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) - 2$		$= 3(-2)^2 + 5(-2) - 2$	
$= \frac{1}{3} + \frac{5}{3} - \frac{6}{3}$		$= 12 - 10 - 2$	
$= 0$		$= 0$	
Left Side = Right Side		Left Side = Right Side	

Both solutions are correct. The roots of the equation are $\frac{1}{3}$ and -2.

You can determine the nature of the roots for a quadratic equation by the value of the **discriminant**.

discriminant

- the expression $b^2 - 4ac$ located under the radical sign in the quadratic formula
- use its value to determine the nature of the roots for a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$

- When the value of the discriminant is positive, $b^2 - 4ac > 0$, there are two distinct real roots.
- When the value of the discriminant is zero, $b^2 - 4ac = 0$, there is one distinct real root, or two equal real roots.
- When the value of the discriminant is negative, $b^2 - 4ac < 0$, there are no real roots.

You can see that this is true by testing the three different types of values of the discriminant in the quadratic formula.

Example 1

Use the Discriminant to Determine the Nature of the Roots

Use the discriminant to determine the nature of the roots for each quadratic equation. Check by graphing.

a) $-2x^2 + 3x + 8 = 0$
b) $3x^2 - 5x = -9$
c) $\frac{1}{4}x^2 - 3x + 9 = 0$

Solution

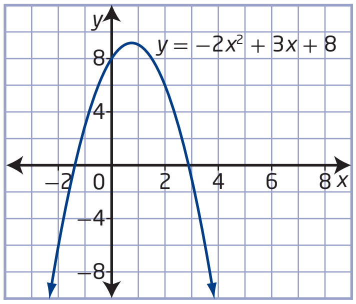
To determine the nature of the roots for each equation, substitute the corresponding values for a, b, and c into the discriminant expression, $b^2 - 4ac$.

a) For $-2x^2 + 3x + 8 = 0$, $a = -2$, $b = 3$, and $c = 8$.

$b^2 - 4ac = 3^2 - 4(-2)(8)$
$b^2 - 4ac = 9 + 64$
$b^2 - 4ac = 73$

Since the value of the discriminant is positive, there are two distinct real roots.

The graph of the corresponding quadratic function, $y = -2x^2 + 3x + 8$, confirms that there are two distinct x-intercepts.



b) First, rewrite $3x^2 - 5x = -9$ in the form $ax^2 + bx + c = 0$.

$3x^2 - 5x + 9 = 0$

For $3x^2 - 5x + 9 = 0$, $a = 3$, $b = -5$, and $c = 9$.

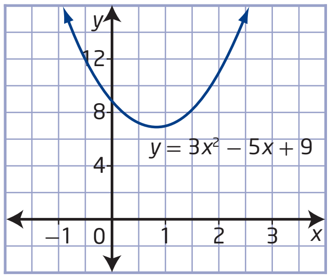
$b^2 - 4ac = (-5)^2 - 4(3)(9)$

$b^2 - 4ac = 25 - 108$

$b^2 - 4ac = -83$

Since the value of the discriminant is negative, there are no real roots. The square root of a negative number does not result in a real number.

The graph of the corresponding quadratic function, $y = 3x^2 - 5x + 9$, shows that there are no x-intercepts.



c) For $\frac{1}{4}x^2 - 3x + 9 = 0$, $a = \frac{1}{4}$, $b = -3$, and $c = 9$.

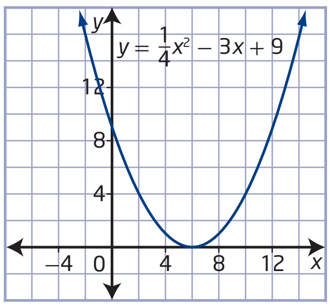
$b^2 - 4ac = (-3)^2 - 4\left(\frac{1}{4}\right)(9)$

$b^2 - 4ac = 9 - 9$

$b^2 - 4ac = 0$

Since the value of the discriminant is zero, there is one distinct real root, or two equal real roots.

The graph of the corresponding quadratic function, $y = \frac{1}{4}x^2 - 3x + 9$, confirms that there is only one x-intercept because it touches the x-axis but does not cross it.



Your Turn

Use the discriminant to determine the nature of the roots for each quadratic equation. Check by graphing.

a) $x^2 - 5x + 4 = 0$ **b)** $3x^2 + 4x + \frac{4}{3} = 0$ **c)** $2x^2 - 8x = -9$

Example 2

Use the Quadratic Formula to Solve Quadratic Equations

Use the quadratic formula to solve each quadratic equation. Express your answers to the nearest hundredth.

a) $9x^2 + 12x = -4$
b) $5x^2 - 7x - 1 = 0$

Solution

a) First, write $9x^2 + 12x = -4$ in the form $ax^2 + bx + c = 0$.
$9x^2 + 12x + 4 = 0$

For $9x^2 + 12x + 4 = 0$, $a = 9$, $b = 12$, and $c = 4$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-12 \pm \sqrt{12^2 - 4(9)(4)}}{2(9)}$$
$$x = \frac{-12 \pm \sqrt{144 - 144}}{18}$$
$$x = \frac{-12 \pm \sqrt{0}}{18}$$
$$x = \frac{-12}{18}$$
$$x = -\frac{2}{3}$$

Since the value of the discriminant is zero, there is only one distinct real root, or two equal real roots.

Check:

Substitute $x = -\frac{2}{3}$ into the original equation.

Left Side	Right Side
$9x^2 + 12x$	-4
$= 9\left(-\frac{2}{3}\right)^2 + 12\left(-\frac{2}{3}\right)$	
$= 9\left(\frac{4}{9}\right) - 8$	
$= 4 - 8$	
$= -4$	

Left Side = Right Side

How could you use technology to check your solution graphically?

The root is $-\frac{2}{3}$, or approximately -0.67.

b) For $5x^2 - 7x - 1 = 0$, $a = 5$, $b = -7$, and $c = -1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(5)(-1)}}{2(5)}$$

$$x = \frac{7 \pm \sqrt{49 + 20}}{10}$$

$$x = \frac{7 \pm \sqrt{69}}{10}$$

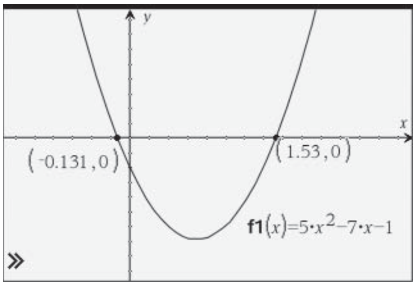
Since the value of the discriminant is positive, there are two distinct real roots.

$$x = \frac{7 + \sqrt{69}}{10} \quad \text{or} \quad x = \frac{7 - \sqrt{69}}{10}$$

$$x = 1.5306\ldots \qquad x = -0.1306\ldots$$

The roots are $\frac{7 + \sqrt{69}}{10}$ and $\frac{7 - \sqrt{69}}{10}$, or approximately 1.53 and -0.13.

Check:
The graph of the corresponding function, $y = 5x^2 - 7x - 1$, shows the zeros at approximately $(-0.13, 0)$ and $(1.53, 0)$.



Therefore, both solutions are correct.

Your Turn

Determine the roots for each quadratic equation. Express your answers to the nearest hundredth.

a) $3x^2 + 5x - 2 = 0$

b) $\frac{t^2}{2} - t - \frac{5}{2} = 0$

Example 3

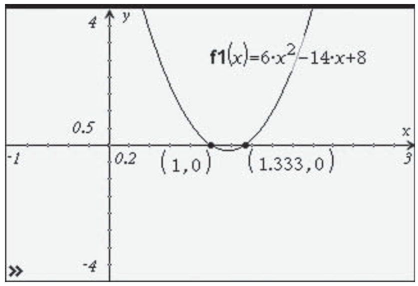
Select a Strategy to Solve a Quadratic Equation

a) Solve $6x^2 - 14x + 8 = 0$ by
  i) graphing the corresponding function
  ii) factoring the equation
  iii) completing the square
  iv) using the quadratic formula

b) Which strategy do you prefer? Justify your reasoning.

Solution

a) **i)** Graph the function $f(x) = 6x^2 - 14x + 8$, and then determine the x-intercepts. The x-intercepts are 1 and approximately 1.33. Therefore, the roots are 1 and approximately 1.33.



ii) Factor the equation.

$$6x^2 - 14x + 8 = 0$$
$$3x^2 - 7x + 4 = 0$$
$$(3x - 4)(x - 1) = 0$$

By inspection, $3x^2 - 7x + 4 = (3x - \blacksquare)(x - \blacksquare)$. What factors of 4 give the correct middle term?

$3x - 4 = 0$ or $x - 1 = 0$

$3x = 4$ $\qquad x = 1$

$x = \frac{4}{3}$

iii) Complete the square.

$$6x^2 - 14x + 8 = 0$$
$$x^2 - \frac{7}{3}x + \frac{4}{3} = 0$$
$$x^2 - \frac{7}{3}x = -\frac{4}{3}$$
$$x^2 - \frac{7}{3}x + \frac{49}{36} = -\frac{4}{3} + \frac{49}{36}$$
$$\left(x - \frac{7}{6}\right)^2 = \frac{1}{36}$$
$$x - \frac{7}{6} = \pm\sqrt{\frac{1}{36}}$$
$$x = \frac{7}{6} \pm \frac{1}{6}$$

$x = \frac{7}{6} + \frac{1}{6}$ or $x = \frac{7}{6} - \frac{1}{6}$

$x = \frac{8}{6}$ $\qquad x = \frac{6}{6}$

$x = \frac{4}{3}$ $\qquad x = 1$

iv) Use the quadratic formula. For $6x^2 - 14x + 8 = 0$, $a = 6$, $b = -14$, and $c = 8$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(6)(8)}}{2(6)}$$
$$x = \frac{14 \pm \sqrt{196 - 192}}{12}$$
$$x = \frac{14 \pm \sqrt{4}}{12}$$
$$x = \frac{14 \pm 2}{12}$$

$$x = \frac{14 + 2}{12} \quad \text{or} \quad x = \frac{14 - 2}{12}$$
$$x = \frac{16}{12} \qquad\qquad x = \frac{12}{12}$$
$$x = \frac{4}{3} \qquad\qquad x = 1$$

Check for methods ii), iii), and iv):

Substitute $x = \frac{4}{3}$ and $x = 1$ into the equation $6x^2 - 14x + 8 = 0$.

For $x = \frac{4}{3}$:

Left Side	Right Side
$6x^2 - 14x + 8$	0
$= 6\left(\frac{4}{3}\right)^2 - 14\left(\frac{4}{3}\right) + 8$	
$= 6\left(\frac{16}{9}\right) - \frac{56}{3} + \frac{24}{3}$	
$= \frac{32}{3} - \frac{56}{3} + \frac{24}{3}$	
$= -\frac{24}{3} + \frac{24}{3}$	
$= 0$	

Left Side = Right Side

For $x = 1$:

Left Side	Right Side
$6x^2 - 14x + 8$	0
$= 6(1)^2 - 14(1) + 8$	
$= 6 - 14 + 8$	
$= -8 + 8$	
$= 0$	

Left Side = Right Side

Both solutions are correct. The roots are $\frac{4}{3}$ and 1.

b) While all four methods produce the same solutions, factoring is probably the most efficient strategy for this question, since the quadratic equation is not difficult to factor. If the quadratic equation could not be factored, either graphing using technology or using the quadratic formula would be preferred. Using the quadratic formula will always produce an exact answer.

Your Turn

Which method would you use to solve $0.57x^2 - 3.7x - 2.5 = 0$? Justify your choice. Then, solve the equation, expressing your answers to the nearest hundredth.

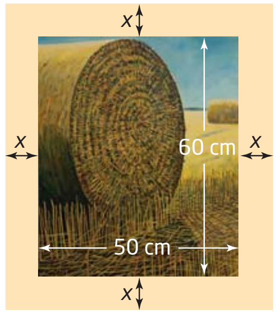
Example 4

Apply the Quadratic Formula

Leah wants to frame an oil original painted on canvas measuring 50 cm by 60 cm. Before framing, she places the painting on a rectangular mat so that a uniform strip of the mat shows on all sides of the painting. The area of the mat is twice the area of the painting. How wide is the strip of exposed mat showing on all sides of the painting, to the nearest tenth of a centimetre?

Solution

Draw a diagram.



Round Bale by Jill Moloy
Lethbridge, Alberta

Let x represent the width of the strip of exposed mat showing on all sides of the painting. Then, the length of the mat is $2x + 60$ and the width of the mat is $2x + 50$.

Use the area formula.
Let A represent the area of the mat.

$$\begin{aligned} A &= lw \\ 2(60)(50) &= (2x + 60)(2x + 50) \\ 6000 &= 4x^2 + 220x + 3000 \\ 0 &= 4x^2 + 220x - 3000 \\ 0 &= 4(x^2 + 55x - 750) \\ 0 &= x^2 + 55x - 750 \end{aligned}$$

Substitute into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(55) \pm \sqrt{(55)^2 - 4(1)(-750)}}{2(1)}$$

$$x = \frac{-55 \pm \sqrt{6025}}{2}$$

$$x = \frac{-55 + \sqrt{6025}}{2} \quad \text{or} \quad x = \frac{-55 - \sqrt{6025}}{2}$$

$x = 11.310...$ $\qquad$ $x = -66.310...$

So, $x \approx 11.3$ or $x \approx -66.3$.

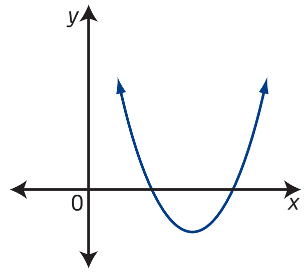
Since $x > 0$, reject $x \approx -66.3$. Therefore, the width of the strip of exposed mat is approximately 11.3 cm. The approximate dimensions of the mat are $2(11.3) + 60$ by $2(11.3) + 50$ or 82.6 cm by 72.6 cm. The approximate area of the mat is 82.6×72.6 or 5996.76 cm^2, which is about 6000 cm^2, twice the area of the painting.

Your Turn

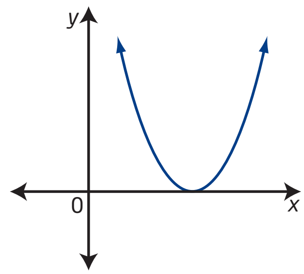
A picture measures 30 cm by 21 cm. You crop the picture by removing strips of the same width from the top and one side of the picture. This reduces the area to 40% of the original area. Determine the width of the removed strips.

Key Ideas

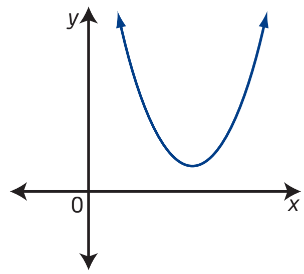
- You can solve a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, for x using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Use the discriminant to determine the nature of the roots of a quadratic equation.
  - When $b^2 - 4ac > 0$, there are two distinct real roots. The graph of the corresponding function has two different x-intercepts.



  - When $b^2 - 4ac = 0$, there is one distinct real root, or two equal real roots. The graph of the corresponding function has one x-intercept.



  - When $b^2 - 4ac < 0$, there are no real roots. The graph of the corresponding function has no x-intercepts.



- You can solve quadratic equations in a variety of ways. You may prefer some methods over others depending on the circumstances.

Check Your Understanding

Practise

1. Use the discriminant to determine the nature of the roots for each equation. Do not solve the equations. Check your answers graphically.
   a) $x^2 - 7x + 4 = 0$
   b) $s^2 + 3s - 2 = 0$
   c) $r^2 + 9r + 6 = 0$
   d) $n^2 - 2n + 1 = 0$
   e) $7y^2 + 3y + 2 = 0$
   f) $4t^2 + 12t + 9 = 0$

2. Without graphing, determine the number of zeros for each function.
   a) $f(x) = x^2 - 2x - 14$
   b) $g(x) = -3x^2 + 0.06x + 4$
   c) $f(x) = \frac{1}{4}x^2 - 3x + 9$
   d) $f(v) = -v^2 + 2v - 1$
   e) $f(x) = \frac{1}{2}x^2 - x + \frac{5}{2}$
   f) $g(y) = -6y^2 + 5y - 1$

3. Use the quadratic formula to solve each quadratic equation. Express your answers as exact roots.
   a) $7x^2 + 24x + 9 = 0$
   b) $4p^2 - 12p - 9 = 0$
   c) $3q^2 + 5q = 1$
   d) $2m^2 + 4m - 7 = 0$
   e) $2j^2 - 7j = -4$
   f) $16g^2 + 24g = -9$

4. Use the quadratic formula to solve each equation. Express your answers to the nearest hundredth.
   a) $3z^2 + 14z + 5 = 0$
   b) $4c^2 - 7c - 1 = 0$
   c) $-5u^2 + 16u - 2 = 0$
   d) $8b^2 + 12b = -1$
   e) $10w^2 - 45w = 7$
   f) $-6k^2 + 17k + 5 = 0$

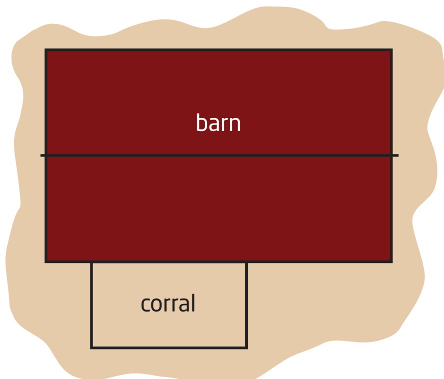
5. Determine the roots of each quadratic equation. Express your answers as exact values and to the nearest hundredth.
   a) $3x^2 + 6x + 1 = 0$
   b) $h^2 + \frac{h}{6} - \frac{1}{2} = 0$
   c) $0.2m^2 = -0.3m + 0.1$
   d) $4y^2 + 7 - 12y = 0$
   e) $\frac{x}{2} + 1 = \frac{7x^2}{2}$
   f) $2z^2 = 6z - 1$

6. Marge claims that the most efficient way to solve all quadratic equations is to use the quadratic formula. Do you agree with her? Explain with examples.

7. Solve using an appropriate method. Justify your choice of method.
   a) $n^2 + 2n - 2 = 0$
   b) $-y^2 + 6y - 9 = 0$
   c) $-2u^2 + 16 = 0$
   d) $\frac{x^2}{2} - \frac{x}{3} = 1$
   e) $x^2 - 4x + 8 = 0$

Apply

8. To save materials, Choma decides to build a horse corral using the barn for one side. He has 30 m of fencing materials and wants the corral to have an area of 100 m². What are the dimensions of the corral?



9. A mural is being painted on an outside wall that is 15 m wide and 12 m tall. A border of uniform width surrounds the mural. The mural covers 75% of the area of the wall. How wide is the border? Express your answer to the nearest hundredth of a metre.

10. Subtracting a number from half its square gives a result of 11. What is the number? Express your answers as exact values and to the nearest hundredth.

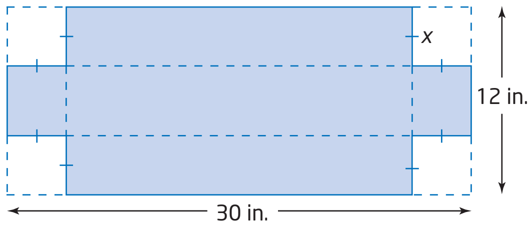
11. The mural *Northern Tradition and Transition*, located in the Saskatchewan Legislature, was painted by Métis artist Roger Jerome to honour the province of Saskatchewan's 100th anniversary in 2005. The mural includes a parabolic arch. The approximate shape of the arch can be modelled by the function $h(d) = -0.4(d - 2.5)^2 + 2.5$, where h is the height of the arch, in metres, and d is the distance, in metres, from one end of the arch. How wide is the arch at its base?

Did You Know?

Roger Jerome included the arch shape to symbolize the unity of northern and southern Saskatchewan.

Northern Tradition and Transition by Roger Jerome

12. An open-topped box is being made from a piece of cardboard measuring 12 in. by 30 in. The sides of the box are formed when four congruent squares are cut from the corners, as shown in the diagram. The base of the box has an area of 208 sq. in..



a) What equation represents the surface area of the base of the box?

b) What is the side length, x, of the square cut from each corner?

c) What are the dimensions of the box?

13. A car travelling at a speed of v kilometres per hour needs a stopping distance of d metres to stop without skidding. This relationship can be modelled by the function $d(v) = 0.0067v^2 + 0.15v$. At what speed can a car be travelling to be able to stop in each distance? Express your answer to the nearest tenth of a kilometre per hour.

a) 42 m

b) 75 m

c) 135 m

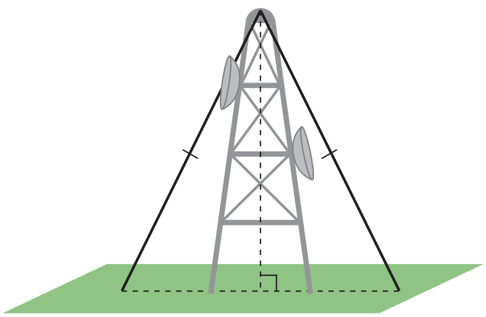
14. A study of the air quality in a particular city suggests that t years from now, the level of carbon monoxide in the air, A, in parts per million, can be modelled by the function $A(t) = 0.3t^2 + 0.1t + 4.2$.

a) What is the level, in parts per million, of carbon monoxide in the air now, at $t = 0$?

b) In how many years from now will the carbon monoxide level be 8 parts per million? Express your answer to the nearest tenth of a year.

15. A sporting goods store sells 90 ski jackets in a season for \$275 each. Each \$15 decrease in the price results in five more jackets being sold. What is the lowest price that would produce revenues of at least \$19 600? How many jackets would be sold at this price?

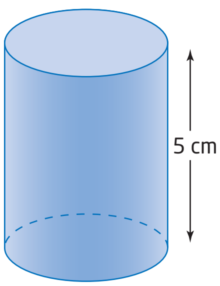
16. Two guy wires are attached to the top of a telecommunications tower and anchored to the ground on opposite sides of the tower, as shown. The length of the guy wire is 20 m more than the height of the tower. The horizontal distance from the base of the tower to where the guy wire is anchored to the ground is one-half the height of the tower. How tall is the tower, to the nearest tenth of a metre?



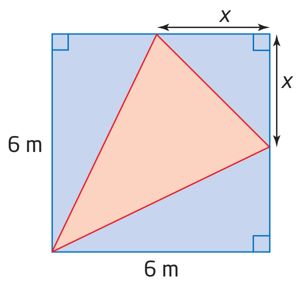
Extend

17. One root of the equation $2x^2 + bx - 24 = 0$ is -8. What are the possible values of b and the other root?

18. A cylinder has a height of 5 cm and a surface area of 100 cm^2. What is the radius of the cylinder, to the nearest tenth of a centimetre?



19. In the diagram, the square has side lengths of 6 m. The square is divided into three right triangles and one acute isosceles triangle. The areas of the three right triangles are equal.



a) Determine the exact value of x.

b) What is the exact area of the acute isosceles triangle?

20. Two small private planes take off from the same airport. One plane flies north at 150 km/h. Two hours later, the second plane flies west at 200 km/h. How long after the first plane takes off will the two planes be 600 km apart? Express your answer to the nearest tenth of an hour.

Create Connections

21. Determine the error(s) in the following solution. Explain how to correct the solution.

Solve $-3x^2 - 7x + 2 = 0$.

Line 1: $x = \dfrac{-7 \pm \sqrt{(-7)^2 - 4(-3)(2)}}{2(-3)}$

Line 2: $x = \dfrac{-7 \pm \sqrt{49 - 24}}{-6}$

Line 3: $x = \dfrac{-7 \pm \sqrt{25}}{-6}$

Line 4: $x = \dfrac{-7 \pm 5}{-6}$

Line 5: So, $x = 2$ or $x = \dfrac{1}{3}$.

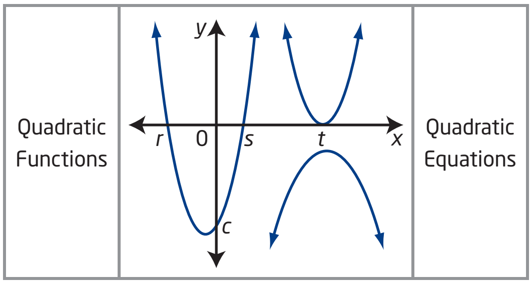
22. Pierre calculated the roots of a quadratic equation as $x = \dfrac{3 \pm \sqrt{25}}{2}$.

a) What are the x-intercepts of the graph of the corresponding quadratic function?

b) Describe how to use the x-intercepts to determine the equation of the axis of symmetry.

23. You have learned to solve quadratic equations by graphing the corresponding function, determining the square roots, factoring, completing the square, and applying the quadratic formula. In what circumstances would one method of solving a quadratic equation be preferred over another?

24. Create a mind map of how the concepts you have learned in Chapters 3 and 4 are connected. One is started below. Make a larger version and add any details that help you.

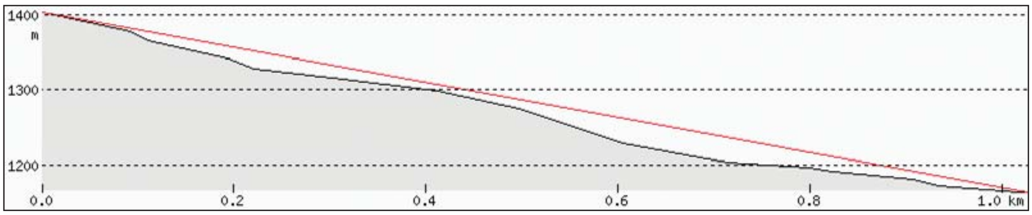


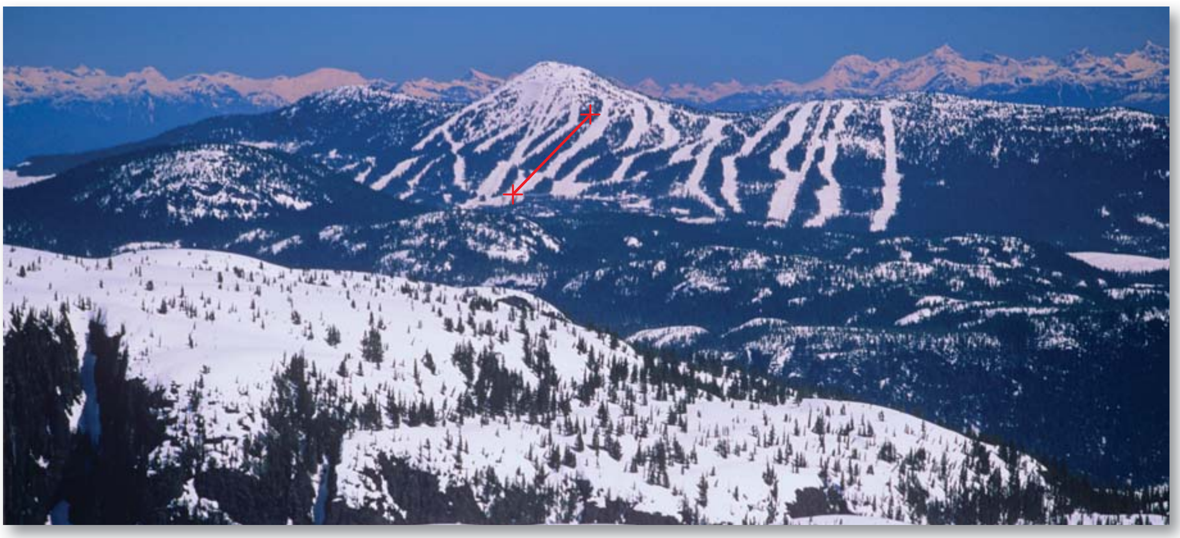
Project Corner — Contour Maps

- Contour lines are lines on a map that connect points of equal elevation.
- Contour maps show the elevations above sea level and the surface features of the land using contour lines.
- A profile view shows how the elevation changes when a line is drawn across part of a contour map.

Web Link

To explore generating a profile view, go to www.mhrprecalc11.ca and follow the links.





Chapter 4 Review

4.1 Graphical Solutions of Quadratic Equations, pages 206–217

1. Solve each quadratic equation by graphing the corresponding quadratic function.

a) $0 = x^2 + 8x + 12$

b) $0 = x^2 - 4x - 5$

c) $0 = 3x^2 + 10x + 8$

d) $0 = -x^2 - 3x$

e) $0 = x^2 - 25$

2. Use graphing technology to determine which of the following quadratic equations has different roots from the other three.

A $0 = 3 - 3x - 3x^2$

B $0 = x^2 + x - 1$

C $0 = 2(x - 1)^2 + 6x - 4$

D $0 = 2x + 2 + 2x^2$

3. Explain what must be true about the graph of the corresponding function for a quadratic equation to have no real roots.

4. A manufacturing company produces key rings. Last year, the company collected data about the number of key rings produced per day and the corresponding profit. The data can be modelled by the function $P(k) = -2k^2 + 12k - 10$, where P is the profit, in thousands of dollars, and k is the number of key rings, in thousands.

a) Sketch a graph of the function.

b) Using the equation $-2k^2 + 12k - 10 = 0$, determine the number of key rings that must be produced so that there is no profit or loss. Justify your answer.

5. The path of a soccer ball can be modelled by the function $h(d) = -0.1d^2 + 0.5d + 0.6$, where h is the height of the ball and d is the horizontal distance from the kicker, both in metres.

a) What are the zeros of the function?

b) You can use the quadratic equation $0 = -0.1d^2 + 0.5d + 0.6$ to determine the horizontal distance that a ball travels after being kicked. How far did the ball travel downfield before it hit the ground?

4.2 Factoring Quadratic Equations, pages 218–233

6. Factor.

a) $4x^2 - 13x + 9$

b) $\frac{1}{2}x^2 - \frac{3}{2}x - 2$

c) $3(v + 1)^2 + 10(v + 1) + 7$

d) $9(a^2 - 4)^2 - 25(7b)^2$

7. Solve by factoring. Check your solutions.

a) $0 = x^2 + 10x + 21$

b) $\frac{1}{4}m^2 + 2m - 5 = 0$

c) $5p^2 + 13p - 6 = 0$

d) $0 = 6z^2 - 21z + 9$

8. Solve.

a) $-4g^2 + 6 = -10g$

b) $8y^2 = -5 + 14y$

c) $30k - 25k^2 = 9$

d) $0 = 2x^2 - 9x - 18$

9. Write a quadratic equation in standard form with the given roots.

a) 2 and 3

b) -1 and -5

c) $\frac{3}{2}$ and -4

10. The path of a paper airplane can be modelled approximately by the function $h(t) = -\frac{1}{4}t^2 + t + 3$, where h is the height above the ground, in metres, and t is the time of flight, in seconds. Determine how long it takes for the paper airplane to hit the ground, $h(t) = 0$.

11. The length of the base of a rectangular prism is 2 m more than its width, and the height of the prism is 15 m.

a) Write an algebraic expression for the volume of the rectangular prism.

b) The volume of the prism is 2145 m^3. Write an equation to model the situation.

c) Solve the equation in part b) by factoring. What are the dimensions of the base of the rectangular prism?

12. Solve the quadratic equation $x^2 - 2x - 24 = 0$ by factoring and by graphing. Which method do you prefer to use? Explain.

4.3 Solving Quadratic Equations by Completing the Square, pages 234–243

13. Determine the value of k that makes each expression a perfect square trinomial.

a) $x^2 + 4x + k$

b) $x^2 + 3x + k$

14. Solve. Express your answers as exact values.

a) $2x^2 - 98 = 0$

b) $(x + 3)^2 = 25$

c) $(x - 5)^2 = 24$

d) $(x - 1)^2 = \frac{5}{9}$

15. Complete the square to determine the roots of each quadratic equation. Express your answers as exact values.

a) $-2x^2 + 16x - 3 = 0$

b) $5y^2 + 20y + 1 = 0$

c) $4p^2 + 2p = -5$

16. In a simulation, the path of a new aircraft after it has achieved weightlessness can be modelled approximately by $h(t) = -5t^2 + 200t + 9750$, where h is the altitude of the aircraft, in metres, and t is the time, in seconds, after weightlessness is achieved. How long does the aircraft take to return to the ground, $h(t) = 0$? Express your answer to the nearest tenth of a second.

17. The path of a snowboarder after jumping from a ramp can be modelled by the function $h(d) = -\frac{1}{2}d^2 + 2d + 1$, where h is the height above the ground and d is the horizontal distance the snowboarder travels, both in metres.

a) Write a quadratic equation you would solve to determine the horizontal distance the snowboarder has travelled when she lands.

b) What horizontal distance does the snowboarder travel? Express your answer to the nearest tenth of a metre.

4.4 The Quadratic Formula, pages 244–257

18. Use the discriminant to determine the nature of the roots for each quadratic equation. Do not solve the equation.

a) $2x^2 + 11x + 5 = 0$

b) $4x^2 - 4x + 1 = 0$

c) $3p^2 + 6p + 24 = 0$

d) $4x^2 + 4x - 7 = 0$

19. Use the quadratic formula to determine the roots for each quadratic equation. Express your answers as exact values.

a) $-3x^2 - 2x + 5 = 0$

b) $5x^2 + 7x + 1 = 0$

c) $3x^2 - 4x - 1 = 0$

d) $25x^2 + 90x + 81 = 0$

20. A large fountain in a park has 35 water jets. One of the streams of water shoots out of a metal rod and follows a parabolic path. The path of the stream of water can be modelled by the function $h(x) = -2x^2 + 6x + 1$, where h is the height, in metres, at any horizontal distance x metres from its jet.

a) What quadratic equation would you solve to determine the maximum horizontal distance the water jet can reach?

b) What is the maximum horizontal distance the water jet can reach? Express your answer to the nearest tenth of a metre.

21. A ferry carries people to an island airport. It carries 2480 people per day at a cost of \$3.70 per person. Surveys have indicated that for every \$0.05 decrease in the fare, 40 more people will use the ferry. Use x to represent the number of decreases in the fare.

a) Write an expression to model the fare per person.

b) Write an expression to model the number of people that would use the ferry per day.

c) Determine the expression that models the revenue, R, for the ferry, which is the product of the number of people using the ferry per day and the fare per person.

d) Determine the number of fare decreases that result in a revenue of \$9246.

22. Given the quadratic equation in standard form, $ax^2 + bx + c = 0$, arrange the following algebraic steps and explanations in the order necessary to derive the quadratic formula.

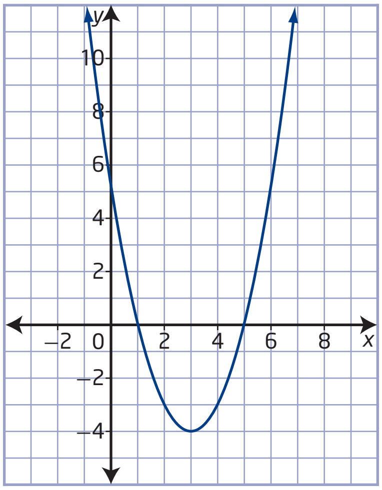
Algebraic Steps	Explanations
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Complete the square.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Solve for x.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Subtract c from both sides.
$ax^2 + bx = -c$	Take the square root of both side.
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$	Divide both sides by a.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Factor the perfect square trinomial.

Chapter 4 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. What points on the graph of this quadratic function represent the locations of the zeros of the function?



A (0, 5) and (1, 0)

B (0, 1) and (0, 5)

C (1, 0) and (5, 0)

D (5, 0) and (0, 1)

2. What is one of the factors of $x^2 - 3x - 10$?

A $x + 5$ **B** $x - 5$

C $x - 10$ **D** $x + 10$

3. What integral values of k will make $2x^2 + kx - 1$ factorable?

A -1 and 2 **B** -2 and 2

C -2 and 1 **D** -1 and 1

4. The roots, to the nearest hundredth, of $0 = -\frac{1}{2}x^2 + x + \frac{7}{2}$ are

A 1.83 and 3.83

B -1.83 and 3.83

C 1.83 and -3.83

D -1.83 and -3.83

5. The number of baseball games, G, that must be scheduled in a league with n teams can be modelled by the function $G(n) = \frac{n^2 - n}{2}$, where each team plays every other team exactly once. Suppose a league schedules 15 games. How many teams are in the league?

A 5

B 6

C 7

D 8

Short Answer

6. Determine the roots of each quadratic equation. If the quadratic equation does not have real roots, use a graph of the corresponding function to explain.

a) $0 = x^2 - 4x + 3$

b) $0 = 2x^2 - 7x - 15$

c) $0 = -x^2 - 2x + 3$

7. Solve the quadratic equation $0 = 3x^2 + 5x - 1$ by completing the square. Express your answers as exact roots.

8. Use the quadratic formula to determine the roots of the equation $x^2 + 4x - 7 = 0$. Express your answers as exact roots in simplest radical form.

9. Without solving, determine the nature of the roots for each quadratic equation.

a) $x^2 + 10x + 25 = 0$

b) $2x^2 + x = 5$

c) $2x^2 + 6 = 4x$

d) $\frac{2}{3}x^2 + \frac{1}{2}x - 3 = 0$

10. The length of the hypotenuse of a right triangle is 1 cm more than triple that of the shorter leg. The length of the longer leg is 1 cm less than triple that of the shorter leg.

a) Sketch and label a diagram with expressions for the side lengths.

b) Write an equation to model the situation.

c) Determine the lengths of the sides of the triangle.

Extended Response

11. A pebble is tossed upward from a scenic lookout and falls to the river below. The approximate height, h, in metres, of the pebble above the river t seconds after being tossed is modelled by the function $h(x) = -5t^2 + 10t + 35$.

a) After how many seconds does the pebble hit the river? Express your answer to the nearest tenth of a second.

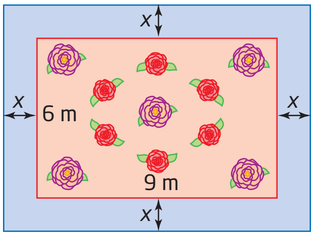
b) How high is the scenic lookout above the river?

c) Which method did you choose to solve the quadratic equation? Justify your choice.

12. Three rods measure 20 cm, 41 cm, and 44 cm. If the same length is cut off each piece, the remaining lengths can be formed into a right triangle. What length is cut off?

13. A rectangular piece of paper has a perimeter of 100 cm and an area of 616 cm². Determine the dimensions of the paper.

14. The parks department is planning a new flower bed. It will be rectangular with dimensions 9 m by 6 m. The flower bed will be surrounded by a grass strip of constant width with the same area as the flower bed.



a) Write a quadratic equation to model the situation.

b) Solve the quadratic equation. Justify your choice of method.

c) Calculate the perimeter of the outside of the path.

12. Examples:

a) The vertex form of the function $C(v) = 0.004v^2 - 0.62v + 30$ is $C(v) = 0.004(v - 77.5)^2 + 5.975$. The most efficient speed would be 77.5 km/h and will produce a fuel consumption of 5.975 L/100 km.

b) By completing the square and determining the vertex of the function, you can determine the most efficient fuel consumption and at what speed it occurs.

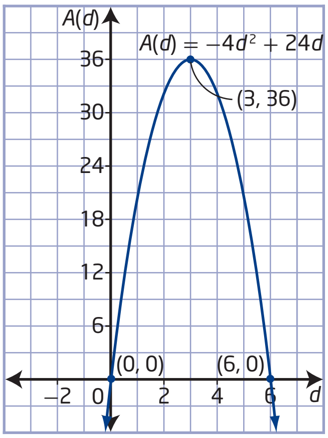
13. a) The maximum height of the flare is 191.406 25 m, 6.25 s after being shot.

b) Example: Complete the square to produce the vertex form and use the value of q to determine the maximum height and the value of p to determine when it occurs, or use the fact that the x-coordinate of the vertex of a quadratic function in standard form is $x = -\frac{b}{2a}$ and substitute this value into the function to find the corresponding y-coordinate, or graph the function to find the vertex.

14. a) $A(d) = -4d^2 + 24d$

b) Since the function is a polynomial of degree two, it satisfies the definition of a quadratic function.

c)



Example: By completing the square, determine the vertex, find the y-intercept and its corresponding point, plot the three points, and join them with a smooth curve.

d) (3, 36); the maximum area of 36 m^2 happens when the fence is extended to 3 m from the building.

e) domain: $\{d \mid 0 \le d \le 6, d \in \mathrm{R}\}$, range: $\{A \mid 0 \le A \le 36, A \in \mathrm{R}\}$; negative distance and area do not have meaning in this situation.

f) Yes; the maximum value is 36 when d is 3, and the minimum value is 0 when d is 0 or 6.

g) Example: Assume that any real-number distance can be used to build the fence.

15. a) $f(x) = -0.03x^2$

b) $f(x) = -0.03x^2 + 12$

c) $f(x) = -0.03(x + 20)^2 + 12$

d) $f(x) = -0.03(x - 28)^2 - 3$

16. a) $R = (2.25 - 0.05x)(120 + 8x)$

b) Expand and complete the square to get the vertex form of the function. A price of $1.50 gives the maximum revenue of $360.

c) Example: Assume that any whole number of price decreases can occur.

Chapter 4 Quadratic Equations

4.1 Graphical Solutions of Quadratic Equations, pages 215 to 217

1. a) 1 **b)** 2 **c)** 0 **d)** 2

2. a) 0 **b)** −1 and −4

c) none **d)** −3 and 8

3. a) $x = -3, x = 8$ **b)** $r = -3, r = 0$

c) no real solutions **d)** $x = 3, x = -2$

e) $z = 2$ **f)** no real solutions

4. a) $n \approx -3.2, n \approx 3.2$ **b)** $x = -4, x = 1$

c) $w = 1, w = 3$ **d)** $d = -8, d = -2$

e) $v \approx -4.7, v \approx -1.3$ **f)** $m = 3, m = 7$

5. 60 yd

6. a) $-x^2 + 9x - 20 = 0$ or $x^2 - 9x + 20 = 0$

b) 4 and 5

7. a) $x^2 + 2x - 168 = 0$

b) $x = 12$ and $x = 14$ or $x = -12$ and $x = -14$

8. a) Example: Solving the equation leads to the distance from the firefighter that the water hits the ground. The negative solution is not part of this situation.

b) 12.2 m

c) Example: Assume that aiming the hose higher would not reach farther. Assume that wind does not affect the path of the water.

9. a) Example: Solving the equation leads to the time that the fireworks hit the ground. The negative solution is not part of the situation.

b) 6.1 s

10. a) $-0.75d^2 + 0.9d + 1.5 = 0$ **b)** 2.1 m

11. a) $-2d^2 + 3d + 10 = 0$ **b)** 3.1 m

12. a) first arch: $x = 0$ and $x = 84$, second arch: $x = 84$ and $x = 168$, third arch: $x = 168$ and $x = 252$

b) The zeros represent where the arches reach down to the bridge deck.

c) 252 m

13. a) $k = 9$ **b)** $k < 9$ **c)** $k > 9$

14. a) 64 ft

b) The relationship between the height, radius, and span of the arch stays the same. Input the measures in metres and solve.

15. about 2.4 s

16. For the value of the function to change from negative to positive, it must cross the x-axis and therefore there must be an x-intercept between the two values of x.

17. The other x-intercept would have to be 4.

18. The x-coordinate of the vertex is halfway between the two roots. So, it is at 2. You can then substitute $x = 2$ into the equation to find the minimum value of -16.

4.2 Factoring Quadratic Equations, pages 229 to 233

1. a) $(x + 2)(x + 5)$ **b)** $5(z + 2)(z + 6)$

c) $0.2(d - 4)(d - 7)$

2. a) $(y - 1)(3y + 7)$ **b)** $(4k - 5)(2k + 1)$

c) $0.2(2m - 3)(m + 3)$

3. a) $(x + 5)(x - 4)$ **b)** $(x - 6)^2$

c) $\frac{1}{4}(x + 2)(x + 6)$ **d)** $2(x + 3)^2$

4. a) $(2y + 3x)(2y - 3x)$

b) $(0.6p + 0.7q)(0.6p - 0.7q)$

c) $\left(\frac{1}{2}s + \frac{3}{5}t\right)\left(\frac{1}{2}s - \frac{3}{5}t\right)$

d) $(0.4t + 4s)(0.4t - 4s)$

5. a) $(x + 8)(x - 5)$

b) $(2x^2 - 8x + 9)(3x^2 - 12x + 11)$

c) $(-4)(8j)$

6. a) $(10b)(10b - 7)$

b) $16(x^2 - x + 1)(x^2 + x + 1)$

c) $(10y^3 - x)(10y^3 + x)$

7. a) $x = -3, x = -4$ **b)** $x = 2, x = -\frac{1}{2}$

c) $x = -7, x = 8$ **d)** $x = 0, x = -5$

e) $x = -\frac{1}{3}, x = \frac{4}{5}$ **f)** $x = 4, x = \frac{7}{2}$

8. a) $n = -2, n = 2$ **b)** $x = -4, x = -1$

c) $w = -9, x = -\frac{1}{3}$ **d)** $y = \frac{5}{4}, y = \frac{3}{2}$

e) $d = -\frac{3}{2}, d = -1$ **f)** $x = \frac{3}{2}$

9. a) 0 and 5 **b)** $-\frac{8}{9}$ and 1

c) -5 and -3 **d)** $-\frac{21}{5}$ and $\frac{21}{5}$

e) -5 and 7 **f)** $\frac{7}{2}$

10. a) -6 and 7 **b)** -10 and 3

c) -7 and 3 **d)** $-\frac{1}{3}$ and $\frac{3}{2}$

e) -5 and 2 **f)** -3 and $\frac{1}{2}$

11. a) $(x + 10)(2x - 3) = 54$ **b)** 3.5 cm

12. a) 1 s and 5 s

b) Assume that the mass of the fish does not affect the speed at which the osprey flies after catching the fish. This may not be a reasonable assumption for a large fish.

13. a) $150t - 5t^2 = 0$ **b)** 30 s

14. 8 and 10 or 0 and -2

15. 15 cm

16. 3 s; this seems a very long time considering the ball went up only 39 ft.

17. a) 1 cm

b) 7 cm by 5 cm

18. a) No; $(x - 5)$ is not a factor of the expression $x^2 - 5x - 36$, since $x = 5$ does not satisfy the equation $x^2 - 5x - 36 = 0$.

b) Yes; $(x + 3)$ is a factor of the expression $x^2 - 2x - 15$, since $x = -3$ satisfies the equation $x^2 - 2x - 15 = 0$.

c) No; $(4x + 1)$ is not a factor of the expression $6x^2 + 11x + 4$, since $x = -\frac{1}{4}$ does not satisfy the equation $6x^2 + 11x + 4 = 0$.

d) Yes; $(2x - 1)$ is a factor of the expression $4x^2 + 4x - 3$, since $x = \frac{1}{2}$ satisfies the equation $4x^2 + 4x - 3 = 0$.

19. a) $-\frac{1}{2}$ and 2 **b)** -4 and 3

20. 20 cm and 21 cm

21. 8 m and 15 m

22. a) $x(x - 7) = 690$ **b)** 30 cm by 23 cm

23. 5 m

24. 5 m

25. $P = \frac{1}{2}d(v_1 + v_2)(v_1 - v_2)$

26. No; the factor $6x - 4$ still has a common factor of 2.

27. a) $6(z - 1)(2z + 5)$

b) $4(2m^2 - 8 - 3n)(2m^2 - 8 + 3n)$

c) $\frac{1}{36}(2y - 3x)^2$

d) $7\left(w - \frac{5}{3}\right)(5w + 1)$

28. $4(3x + 5y)$ centimetres

29. The shop will make a profit after 4 years.

30. a) $x^2 - 9 = 0$ **b)** $x^2 - 4x + 4 = 0$

c) $3x^2 - 14x + 8 = 0$

d) $10x^2 - x - 3 = 0$

31. Example: $x^2 - x + 1 = 0$

32. a) Instead of evaluating $81 - 36$, use the difference of squares pattern to rewrite the expression as $(9 - 6)(9 + 6)$ and then simplify. You can use this method when a question asks you to subtract a square number from a square number.

b) Examples:

$$144 - 25 = (12 - 5)(12 + 5) = (7)(17) = 119$$

$$256 - 49 = (16 - 7)(16 + 7) = (9)(23) = 207$$

4.3 Solving Quadratic Equations by Completing the Square, pages 240 to 243

1. a) $c = \frac{1}{4}$ **b)** $c = \frac{25}{4}$
c) $c = 0.0625$ **d)** $c = 0.01$
e) $c = \frac{225}{4}$ **f)** $c = \frac{81}{4}$

2. a) $(x + 2)^2 = 2$ **b)** $(x + 2)^2 = \frac{17}{3}$
c) $(x - 3)^2 = -1$

3. a) $(x - 6)^2 - 27 = 0$ **b)** $5(x - 2)^2 - 21 = 0$
c) $-2\left(x - \frac{1}{4}\right)^2 - \frac{7}{8} = 0$
d) $0.5(x + 2.1)^2 + 1.395 = 0$
e) $-1.2(x + 2.125)^2 - 1.981\ 25 = 0$
f) $\frac{1}{2}(x + 3)^2 - \frac{21}{2} = 0$

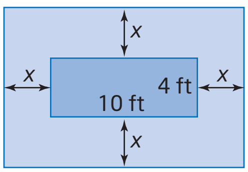
4. a) $x = \pm 8$ **b)** $s = \pm 2$
c) $t = \pm 6$ **d)** $y = \pm\sqrt{11}$

5. a) $x = 1, x = 5$ **b)** $x = -5, x = 1$
c) $d = -\frac{3}{2}, d = \frac{1}{2}$ **d)** $h = \frac{3 \pm \sqrt{7}}{4}$
e) $s = \frac{-12 \pm \sqrt{3}}{2}$ **f)** $x = -4 \pm 3\sqrt{2}$

6. a) $x = -5 \pm \sqrt{21}$ **b)** $x = 4 \pm \sqrt{3}$
c) $x = -1 \pm \sqrt{\frac{2}{3}}$ or $\frac{-3 \pm \sqrt{6}}{3}$
d) $x = 1 \pm \sqrt{\frac{5}{2}}$ or $\frac{2 \pm \sqrt{10}}{2}$
e) $x = -3 \pm \sqrt{13}$ **f)** $x = 4 \pm 2\sqrt{7}$

7. a) $x = 8.5, x = -0.5$ **b)** $x = -0.8, x = 2.1$
c) $x = 12.8, x = -0.8$ **d)** $x = -7.7, x = 7.1$
e) $x = -2.6, x = 1.1$ **f)** $x = -7.8, x = -0.2$

8. a)



b) $4x^2 + 28x - 40 = 0$
c) 12.4 ft by 6.4 ft

9. a) $-0.02d^2 + 0.4d + 1 = 0$
b) 22.2 m

10. 200.5 m

11. 6 in. by 9 in.

12. 53.7 m

13. a) $x^2 - 7 = 0$ **b)** $x^2 - 2x - 2 = 0$
c) $4x^2 - 20x + 14 = 0$ or $2x^2 - 10x + 7 = 0$

14. a) $x = -1 \pm \sqrt{k + 1}$ **b)** $x = \frac{1 \pm \sqrt{k^2 + 1}}{k}$
c) $x = \frac{k \pm \sqrt{k^2 + 4}}{2}$

15. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ No. Some will result in a negative in the radical, which means the solution(s) are not real.

16. a) $n = 43$ **b)** $n = 39$

17. a) $12^2 = 4^2 + x^2 - 2(4)(x) \cos (60°)$
b) 13.5 m

18. Example: In the first equation, you must take the square root to isolate or solve for x. This creates the $\pm$ situation. In the second equation, $\sqrt{9}$ is already present, which means the principle or positive square root only.

19. Example: Allison did all of her work on one side of the equation; Riley worked on both sides. Both end up at the same solution but by different paths.

20. Example:
- Completing the square requires operations with rational numbers, which could lead to arithmetic errors.
- Graphing the corresponding function using technology is very easy. Without technology, the manual graph could take a longer amount of time.
- Factoring should be the quickest of the methods.

All of the methods lead to the same answers.

21. a) Example: $y = 2(x - 1)^2 - 3$, $0 = 2x^2 - 4x - 1$
b) Example: $y = 2(x + 2)^2$, $0 = 2x^2 + 8x + 8$
c) Example: $y = 3(x - 2)^2 + 1$, $0 = 3x^2 - 12x + 13$

4.4 The Quadratic Formula, pages 254 to 257

1. a) two distinct real roots
b) two distinct real roots
c) two distinct real roots
d) one distinct real root
e) no real roots
f) one distinct real root

2. a) 2 **b)** 2 **c)** 1
d) 1 **e)** 0 **f)** 2

3. a) $x = -3, x = -\frac{3}{7}$ **b)** $p = \frac{3 \pm 3\sqrt{2}}{2}$
c) $q = \frac{-5 \pm \sqrt{37}}{6}$ **d)** $m = \frac{-2 \pm 3\sqrt{2}}{2}$
e) $j = \frac{7 \pm \sqrt{17}}{4}$ **f)** $g = -\frac{3}{4}$

4. a) $z = -4.28, z = -0.39$
b) $c = -0.13, c = 1.88$
c) $u = 0.13, u = 3.07$
d) $b = -1.41, b = -0.09$
e) $w = -0.15, w = 4.65$
f) $k = -0.27, k = 3.10$

5. a) $x = \frac{-3 \pm \sqrt{6}}{3}$, -0.18 and -1.82

b) $h = \dfrac{-1 \pm \sqrt{73}}{12}$, -0.80 and 0.63

c) $m = \dfrac{-0.3 \pm \sqrt{0.17}}{0.4}$, -1.78 and 0.28

d) $y = \dfrac{3 \pm \sqrt{2}}{2}$, 0.79 and 2.21

e) $x = \dfrac{1 \pm \sqrt{57}}{14}$, -0.47 and 0.61

f) $z = \dfrac{3 \pm \sqrt{7}}{2}$, 0.18 and 2.82

6. Example: Some are easily solved so they do not require the use of the quadratic formula. $x^2 - 9 = 0$

7. a) $n = -1 \pm \sqrt{3}$; complete the square
b) $y = 3$; factor
c) $u = \pm 2\sqrt{2}$; square root
d) $x = \dfrac{1 \pm \sqrt{19}}{3}$; quadratic formula
e) no real roots; graphing

8. 5 m by 20 m or 10 m by 10 m

9. 0.89 m

10. $1 \pm \sqrt{23}$, -3.80 and 5.80

11. 5 m

12. a) $(30 - 2x)(12 - 2x) = 208$
b) 2 in.
c) 8 in. by 26 in. by 2 in.

13. a) 68.8 km/h **b)** 95.2 km/h
c) 131.2 km/h

14. a) 4.2 ppm **b)** 3.4 years

15. \$155, 130 jackets

16. 169.4 m

17. $b = 13$, $x = \dfrac{3}{2}$

18. 2.2 cm

19. a) $(-3 + 3\sqrt{5})$ m **b)** $(-45 + 27\sqrt{5})$ m^2

20. 3.5 h

21. Error in Line 1: The $-b$ would make the first number $-(-7) = 7$.
Error in Line 2: $-4(-3)(2) = +24$ not -24.
The correct solution is $x = \dfrac{-7 \pm \sqrt{73}}{6}$.

22. a) $x = -1$ and $x = 4$
b) Example: The axis of symmetry is halfway between the roots. $\dfrac{-1 + 4}{2} = \dfrac{3}{2}$. Therefore, the equation of the axis of symmetry is $x = \dfrac{3}{2}$.

23. Example: If the quadratic is easily factored, then factoring is faster. If it is not easily factored, then using the quadratic formula will yield exact answers. Graphing with technology is a quick way of finding out if there are real solutions.

24. Answers may vary.

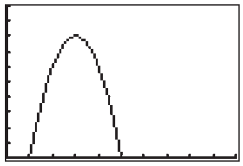
Chapter 4 Review, pages 258 to 260

1. a) $x = -6$, $x = -2$ **b)** $x = -1$, $x = 5$
c) $x = -2$, $x = -\dfrac{4}{3}$ **d)** $x = -3$, $x = 0$
e) $x = -5$, $x = 5$

2. D

3. Example: The graph cannot cross over or touch the x-axis.

4. a) Example:



b) 1000 key rings or 5000 key rings produce no profit or loss because the value of P is 0 then.

5. a) -1 and 6 **b)** 6 m

6. a) $(x - 1)(4x - 9)$ **b)** $\dfrac{1}{2}(x + 1)(x - 4)$
c) $(3v + 10)(v + 2)$
d) $(3a^2 - 12 + 35b)(3a^2 - 12 - 35b)$

7. a) $x = -7$, $x = -3$ **b)** $m = -10$, $m = 2$
c) $p = -3$, $p = \dfrac{2}{5}$ **d)** $z = \dfrac{1}{2}$, $z = 3$

8. a) $g = 3$, $g = -\dfrac{1}{2}$ **b)** $y = \dfrac{1}{2}$, $y = \dfrac{5}{4}$
c) $k = \dfrac{3}{5}$ **d)** $x = -\dfrac{3}{2}$, $x = 6$

9. a) Example: $0 = x^2 - 5x + 6$
b) Example: $0 = x^2 + 6x + 5$
c) Example: $0 = 2x^2 + 5x - 12$

10. 6 s

11. a) $V = 15(x)(x + 2)$ **b)** $2145 = 15x(x + 2)$
c) 11 m by 13 m

12. $x = -4$ and $x = 6$. Example: Factoring is fairly easy and exact.

13. a) $k = 4$ **b)** $k = \dfrac{9}{4}$

14. a) $x = \pm 7$ **b)** $x = 2$, $x = -8$
c) $x = 5 \pm 2\sqrt{6}$ **d)** $x = \dfrac{3 \pm \sqrt{5}}{3}$

15. a) $x = 4 \pm \sqrt{\dfrac{29}{2}}$ or $\dfrac{8 \pm \sqrt{58}}{2}$
b) $y = -2 \pm \sqrt{\dfrac{19}{5}}$ or $\dfrac{-10 \pm \sqrt{95}}{5}$
c) no real solutions

16. 68.5 s

17. a) $0 = -\dfrac{1}{2}d^2 + 2d + 1$ **b)** 4.4 m

18. a) two distinct real roots
b) one distinct real root
c) no real roots
d) two distinct real roots

19. a) $x = -\dfrac{5}{3}$, $x = 1$ **b)** $x = \dfrac{-7 \pm \sqrt{29}}{10}$
c) $x = \dfrac{2 \pm \sqrt{7}}{3}$ **d)** $x = -\dfrac{9}{5}$

20. a) $0 = -2x^2 + 6x + 1$ **b)** 3.2 m

21. a) $3.7 - 0.05x$ **b)** $2480 + 40x$

c) $R = -2x^2 + 24x + 9176$

d) 5 or 7

22.

Algebraic Steps	Explanations
$ax^2 + bx = -c$	Subtract c from both sides.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Divide both sides by a.
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$	Complete the square.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Factor the perfect square trinomial.
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Take the square root of both sides.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Solve for x.

Chapter 4 Practice Test, pages 261 to 262

1. C

2. B

3. D

4. B

5. B

6. a) $x = 3, x = 1$ **b)** $x = -\frac{3}{2}, x = 5$

c) $x = -3, x = 1$

7. $x = \frac{-5 \pm \sqrt{37}}{6}$

8. $x = -2 \pm \sqrt{11}$

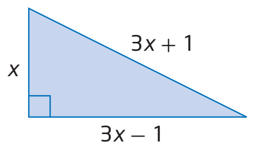
9. a) one distinct real root

b) two distinct real roots

c) no real roots

d) two distinct real roots

10. a)



b) $x^2 + (3x - 1)^2 = (3x + 1)^2$

c) 12 cm, 35 cm, and 37 cm

11. a) 3.8 s

b) 35 m

c) Example: Choose graphing with technology so you can see the path and know which points correspond to the situation.

12. 5 cm

13. 22 cm by 28 cm

14. a) $(9 + 2x)(6 + 2x) = 108$ or $4x^2 + 30x - 54 = 0$

b) $x = 1.5$
Example: Factoring is the most efficient strategy.

c) 42 m

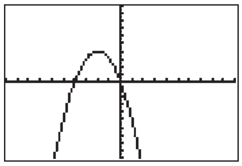
Cumulative Review, Chapters 3–4, pages 264 to 265

1. a) C **b)** A **c)** D **d)** B

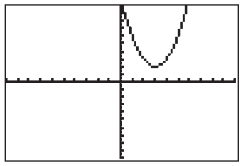
2. a) not quadratic **b)** quadratic

c) not quadratic **d)** quadratic

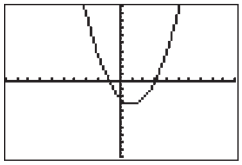
3. a) Example:



b) Example:



c) Example:



4. a) vertex: $(-4, -3)$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq -3, y \in \mathrm{R}\}$, axis of symmetry: $x = -4$, x-intercepts occur at approximately $(-5.7, 0)$ and $(-2.3, 0)$, y-intercept occurs at $(0, 13)$

b) vertex: $(2, 1)$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq 1, y \in \mathrm{R}\}$, axis of symmetry: $x = 2$, x-intercepts occur at $(1, 0)$ and $(3, 0)$, y-intercept occurs at $(0, -3)$

c) vertex: $(0, -6)$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq -6, y \in \mathrm{R}\}$, axis of symmetry: $x = 0$, no x-intercepts, y-intercept occurs at $(0, -6)$

d) vertex: $(-8, 6)$, domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \geq 6, y \in \mathrm{R}\}$, axis of symmetry: $x = -8$, no x-intercepts, y-intercept occurs at $(0, 38)$

5. a) $y = (x - 5)^2 - 7$; the shapes of the graphs are the same with the parabola of $y = (x - 5)^2 - 7$ being translated 5 units to the right and 7 units down.

b) $y = -(x - 2)^2 - 3$; the shapes of the graphs are the same with the parabola of $y = -(x - 2)^2 - 3$ being reflected in the x-axis and translated 2 units to the right and 3 units down.

c) $y = 3(x - 1)^2 + 2$; the shape of the graph of $y = 3(x - 1)^2 + 2$ is narrower by a multiplication of the y-values by a factor of 3 and translated 1 unit to the right and 2 units up.

d) $y = \frac{1}{4}(x + 8)^2 + 4$; the shape of the graph of $y = \frac{1}{4}(x + 8)^2 + 4$ is wider by a multiplication of the y-values by a factor of $\frac{1}{4}$ and translated 8 units to the left and 4 units up.

6. a) 22 m b) 2 m c) 4 s

7. In order: roots, zeros, x-intercepts

8. a) $(3x + 4)(3x - 2)$ b) $(4r - 9s)(4r + 9s)$
c) $(x + 3)(2x + 9)$ d) $(xy + 4)(xy - 9)$
e) $5(a + b)(13a + b)$ f) $(11r + 20)(11r - 20)$

9. 7, 8, 9 or $-9, -8, -7$

10. 15 seats per row, 19 rows

11. 3.5 m

12. Example: Dallas did not divide the 2 out of the -12 in the first line or multiply the 36 by 2 and thus add 72 to the right side instead of 36 in line two. Doug made a sign error on the -12 in the first line. He should have calculated 200 as the value in the radical, not 80. When he simplified, he took $\sqrt{80}$ divided by 4 to get $\sqrt{20}$, which is not correct.

The correct answer is $3 \pm \frac{5}{\sqrt{2}}$ or $\frac{6 \pm 5\sqrt{2}}{2}$.

13. a) Example: square root, $x = \pm\sqrt{2}$
b) Example: factor, $m = 2$ and $m = 13$
c) Example: factor, $s = -5$ and $s = 7$
d) Example: use quadratic formula, $x = -\frac{1}{16}$ and $x = 3$

14. a) two distinct real roots
b) one distinct real root
c) no real roots

15. a) $85 = x^2 + (x + 1)^2$
b) Example: factoring, $x = -7$ and $x = 6$
c) The top is 7-in. by 7-in. and the bottom is 6-in. by 6-in.
d) Example: Negative lengths are not possible.

Unit 2 Test, pages 266 to 267

1. A
2. D
3. D
4. B
5. B
6. 76
7. $900
8. 0.18
9. a) 53.5 cm b) 75.7 cm c) No
10. a) 47.5 m b) 6.1 s
11. 12 cm by 12 cm
12. a) $3x^2 + 6x - 672 = 0$
b) $x = -16$ and $x = 14$
c) 14 in., 15 in., and 16 in.
d) Negative lengths are not possible.

Chapter 5 Radical Expressions and Equations

5.1 Working With Radicals, pages 278 to 281

1.

Mixed Radical Form	Entire Radical Form
$4\sqrt{7}$	$\sqrt{112}$
$5\sqrt{2}$	$\sqrt{50}$
$-11\sqrt{8}$	$-\sqrt{968}$
$-10\sqrt{2}$	$-\sqrt{200}$

2. a) $2\sqrt{14}$ b) $15\sqrt{3}$
c) $2\sqrt[3]{3}$ d) $cd\sqrt{c}$

3. a) $6m^2\sqrt{2}$, $m \in \mathrm{R}$ b) $2q\sqrt[3]{3q^2}$, $q \in \mathrm{R}$
c) $-4st\sqrt[5]{5t}$, $s, t \in \mathrm{R}$

4.

Mixed Radical Form	Entire Radical Form
$3n\sqrt{5}$	$\sqrt{45n^2}, n \geq 0$ or $-\sqrt{45n^2}, n < 0$
$-6\sqrt[3]{2}$	$\sqrt[3]{-432}$
$\frac{1}{2a}\sqrt[3]{7a}$	$\sqrt[3]{\frac{7}{8a^2}}, a \neq 0$
$4x\sqrt[3]{2x}$	$\sqrt[3]{128x^4}$

5. a) $15\sqrt{5}$ and $40\sqrt{5}$ b) $32z^4\sqrt{7}$ and $48z^2\sqrt{7}$
c) $-35\sqrt[4]{w^2}$ and $9w^2(\sqrt[4]{w^2})$
d) $6\sqrt[3]{2}$ and $18\sqrt[3]{2}$

6. a) $3\sqrt{6}, 7\sqrt{2}, 10$
b) $-3\sqrt{2}, -4, -2\sqrt{\frac{7}{2}}, -2\sqrt{3}$
c) $\sqrt[3]{21}, 2.8, 2\sqrt[3]{5}, 3\sqrt[3]{2}$

7. Example: Technology could be used.

8. a) $4\sqrt{5}$ b) $10.4\sqrt{2} - 7$
c) $-4\sqrt[4]{11} + 14$ d) $-\frac{2}{3}\sqrt{6} + 2\sqrt{10}$

9. a) $12\sqrt{3}$ b) $6\sqrt{2} + 6\sqrt{7}$
c) $-28\sqrt{5} + 22.5$ d) $\frac{13}{4}\sqrt[3]{3} - 7\sqrt{11}$

10. a) $8a\sqrt{a}$, $a \geq 0$ b) $9\sqrt{2x} - \sqrt{x}$, $x \geq 0$
c) $2(r - 10)\sqrt[3]{5r}$, $r \in \mathrm{R}$
d) $\frac{4w}{5} - 6\sqrt{2w}$, $w \geq 0$

11. $25.2\sqrt{3}$ m/s

12. $12\sqrt{2}$ cm

13. $12\sqrt[3]{3025}$ million kilometres

14. $2\sqrt{30}$ m/s $\approx$ 11 m/s

15. a) $2\sqrt{38}$ m b) $8\sqrt{19}$ m

16. $\sqrt{1575}$ mm^2, $15\sqrt{7}$ mm^2

17. $7\sqrt{5}$ units

18. $14\sqrt{2}$ m

19. Brady is correct. The answer can be further simplified to $10y^2\sqrt{y}$.

20. $4\sqrt{58}$
Example: Simplify each radical to see which is not a like radical to $12\sqrt{6}$.

21. $\sqrt{2} - \sqrt{3}$ m