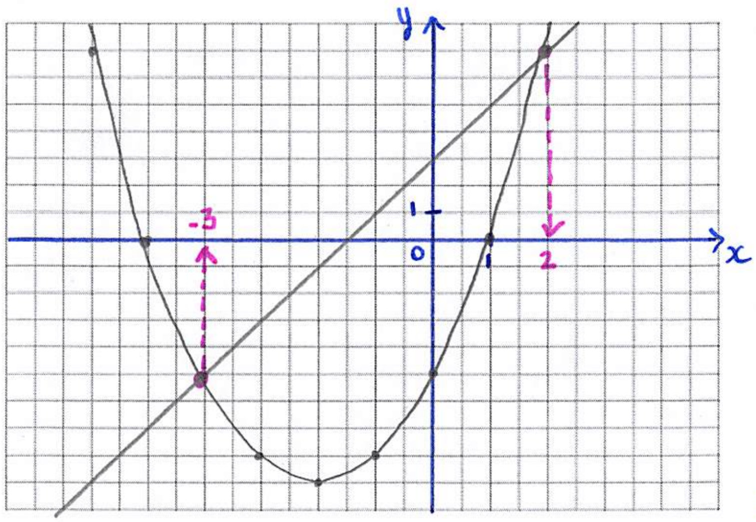
# 4.1 – Graphical Solutions of Quadratic Equations

REMINDER: Solving an equation graphically means graphing the expressions on each side of the equality and look for the values of the variable for which the 2 graphs intersect…
If the equation is quadratic, at least one of the graphs will be a parabola and the other one will be another parabola or a straight line.

Example: Solve $x^2 + 4x - 5 = 3x + 1$ graphically.




• $y = x^2 + 4x - 5$
vertex: $\frac{-b}{2a} = -2$ $(-2; -9)$

| $x$ | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| $y$ | -8 | -5 | 0 | 7 |

• $y = 3x + 1$
y-intercept: 1
slope: $\frac{3}{1}$

$\Rightarrow \boxed{\text{Sol}: \{-3; 2\}}$

Note: You can verify your solutions by replacing the variable by the value and check if the equality is true:

→ if $x = -3$: $x^2 + 4x - 5 = 3x + 1$

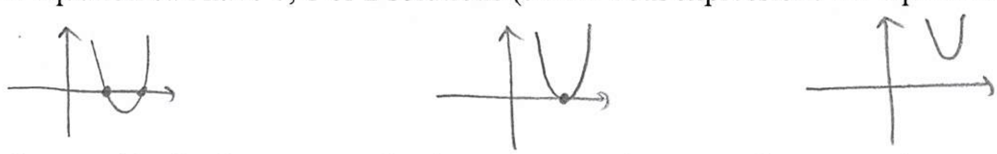
| $x^2+4x-5$ | $3x+1$ |
|---|---|
| $(-3)^2 + 4(-3) - 5$ | $3(-3) + 1$ |
| $9 - 12 - 5$ | $-9 + 1$ |
| $-8$ ✓ | $-8$ ✓ |

if $x = 2$: $x^2 + 4x - 5 = 3x + 1$

| $x^2+4x-5$ | $3x+1$ |
|---|---|
| $2^2 + 4(2) - 5$ | $3(2) + 1$ |
| $4 + 8 - 5$ | $6 + 1$ |
| $7$ ✓ | $7$ ✓ |

Vocabulary: It is always possible to "move" all the terms on the same side of the equation. In the previous example, we would get: $x^2 + x - 6 = 0$.
The **solutions** of such an equation are also called **zeros** (or $x$-intercepts) of the function $y = x^2 + x - 6$ or **roots** of the polynomial $P = x^2 + x - 6$.

A quadratic equation can have 0, 1 or 2 solutions (unless both expressions are equivalent…).



Note: Solving graphically doesn't usually give us exact values. In order to get a better approximation, we can use our graphing calculator … (CALC – zeros or intersect).

Note: In order to see the zeros on a graph, it is important to choose an appropriate window (scale and position of the axis). That might require some calculations (vertex…) even when using a calculator.

Hwk : p 215 # 1, 2, 3abe, 4ab, 5, 7, 8, 11, 13, 17.

# 4.2 – FACTORING QUADRATIC EQUATIONS

To solve quadratic equations algebraically, the methods we used to use for linear equations can't work…
Indeed, you can't isolate "$x$" … ex: $2x^2 = 4x - 1$
We need to find another way…

An important use for factoring is to determine zeros of an expression. Indeed, a product can only be zero if one of its factors is zero…

Example 1: Solve $(2x-3)(x+1) = 0$

$\rightarrow$ $2x - 3 = 0$ $\quad$ $x + 1 = 0$
$2x = 3$ $\quad$ $x = -1$
$x = \frac{3}{2}$

sol : $\{-1 ; \frac{3}{2}\}$

Example 2: Solve $4(x-5)^2 = 0$

$\rightarrow$ $x - 5 = 0$
$x = 5$ $\quad$ sol: $\{5\}$

METHOD: To solve an equation by factoring, you need to write all the terms on the same side of the "=", factor the expression in order to find all the possible zeros.

Example: Solve $x^2 + x - 2 = 2x + 4$ algebraically by factoring

$\rightarrow$ $x^2 - x - 6 = 0$
$(x-3)(x+2) = 0$

$x - 3 = 0$ $\quad$ $x + 2 = 0$
$x = 3$ $\quad$ $x = -2$

sol : $\{-2 ; 3\}$

Hwk : p 229 # 1, 3, 4, 7, 8abc, 9abcd, 11, 12, 16, 17, 19, 20, 23, 24.

Unfortunately, some expressions will be harder to factor than the ones seen in grade 10…

ADDITIONAL METHODS TO FACTOR:

Example 1: $2(x+4)^2 - 11(x+4) + 15 = 0$

Let $t = x + 4$

$\otimes\ 30,\ \oplus\ -11 \ \}\ -5\ \&\ -6$

$2t^2 - 11t + 15 = 0$

$2t^2 - 5t - 6t + 15 = 0$

$t(2t-5) - 3(2t-5) = 0$

$(t-3)(2t-5) = 0$

$(x+4-3)(2(x+4)-5) = 0$

$(x+1)(2x+3) = 0$

$x+1=0 \qquad 2x+3=0$

$\boxed{x=-1} \qquad \boxed{x=-3/2}$

Example 2: $(x^2-6)^2 + 7(x^2-6) - 30 = 0$

Let $t = x^2 - 6$

$\otimes\ -30,\ \oplus\ 7 \ \}\ 10\ \&\ -3$

$t^2 + 7t - 30 = 0$

$(t+10)(t-3) = 0$

$(x^2-6+10)(x^2-6-3) = 0$

$(x^2+4)(x^2-9) = 0$

$(x^2+4)(x+3)(x-3) = 0$

no zero $\qquad \boxed{x=-3} \qquad \boxed{x=3}$

Example 3: $(2x^2+4)^2 - 16(x^2-1)^2 = 0$

$\big((2x^2+4) + 4(x^2-1)\big)\big((2x^2+4) - 4(x^2-1)\big) = 0$

$(6x^2)(-2x^2+8) = 0$

$\boxed{x=0}$

$-2(x^2-4) = 0$

$-2(x+2)(x-2) = 0$

$\boxed{x=-2} \qquad \boxed{x=2}$

Example 4: $3x^4(x+4)^3 - 3x^2(x+4)^4 - 24x^2(x+4)^3 = 0$

$3x^2(x+4)^3\left[x^2 - (x+4) - 8\right] = 0$

$3x^2(x+4)^3(x^2 - x - 12) = 0$

$3x^2(x+4)^3(x-4)(x+3) = 0$

$\boxed{x=0} \qquad \boxed{x=-4} \qquad \boxed{x=4} \qquad \boxed{x=-3}$

Hwk : p 229 # 5, 6 + extra practice worksheet

ADDITIONAL METHODS TO FACTOR: **use a zero…**

Property: If $\propto$ is a zero of an expression, then the expression can be factored by $(x - \propto)$

Application : Show that $2x^2 - 9x - 5$ can be factored by $(x - 5)$.

$\rightarrow 2(5)^2 - 9(5) - 5 = 50 - 45 - 5 = 0 \Rightarrow$ yes!

Examples : a) Is $(x - 3)$ a factor of $3x^2 - 4x - 15$ ?

$3(3)^2 - 4(3) - 15 = 27 - 12 - 15 = 0 \Rightarrow$ yes!

b) Is $(x - 4)$ a factor of $2x^2 - 5x + 1$ ?

$2(4)^2 - 5(4) + 1 = 32 - 20 + 1 \neq 0 \Rightarrow$ no!

c) Is $(x + 1)$ a factor of $4x^2 + x - 3$ ?

$4(-1)^2 + (-1) - 3 = 4 - 1 - 3 = 0 \Rightarrow$ yes!

Note: It is possible to find the missing factor by looking at how we expand products…

$\underline{\underline{2x^2}} - 9x - 5 = (x - 5)(\underline{\underline{2x}} + 1)$ $\quad$ $3x^2 - 4x - 15 = (x - 3)(3x + 5)$

$4x^2 + x - 3 = (x + 1)(4x - 3)$

Your turn: Same questions with:

a) $(x - 7)$ and $2x^2 - 15x + 7$ yes $\quad 2x^2 - 15x + 7 = (x - 7)(2x - 1)$
b) $(x - 2)$ and $4x^2 - 3x + 1$ no
c) $(x + 3)$ and $x^2 - 2x - 15$ yes $\quad x^2 - 2x - 15 = (x + 3)(x - 5)$

Note: To find zeros of an expression, we can look at the table of values of our graphing calculator…
Therefore, by looking at the table of values on our calculator, we can determine some factors…

Example: factor $5x^2 + 3x - 14$ and $2x^2 - 13x + 21$ using graphing technology.

$\rightarrow 5x^2 + 3x - 14 = (x + 2)(5x - 7)$ $\quad$ $2x^2 - 13x + 21 = \underline{(x - 3)}(2x - 7)$

found -2 on table

found 3 on table

Hwk : p 229 # 18

# 4.3 – SOLVING A QUADRATIC EQUATION IN VERTEX FORM

Solve: $x^2 = 9$ $\qquad$ $x^2 = 10$ $\qquad$ $x^2 = 0$ $\qquad$ $x^2 = -4$

$x = \pm 3$ $\qquad$ $x = \pm\sqrt{10}$ $\qquad$ $x = 0$ $\qquad$ no solut°

The equation $x^2 = a$ has 2 solutions if $a > 0$
1 solution (double) if $a = 0$
No real solution if $a < 0$

If a quadratic equation is in vertex form, we use the previous property to determine the solutions:

Applications: Solve

1) $(x-3)^2 - 16 = 0$

1. isolate the perfect square
2. 16 is positive ⟹ 2 sol

$(x-3)^2 = 16$

$x - 3 = \pm 4$ $\qquad$ $x = 3 \pm 4$ i.e $\boxed{x = -1 \text{ or } x = 7}$

2) $3(x+5)^2 - 40 = 0$

$3(x+5)^2 = 40$

$(x+5)^2 = \frac{40}{3}$

$x + 5 = \pm\sqrt{\frac{40}{3}}$

$\boxed{x = -5 \pm \sqrt{\frac{40}{3}}}$ exact values

$x \simeq -8.7$ or $x \simeq 1.3$ (approx)

Note: Some quadratic equations can't be factored. It doesn't mean that they don't have solutions, but that the solutions aren't rational…
It is always possible to write a quadratic expression in vertex form. The equation will have 2 solutions if the perfect square equals a positive number, 1 solutions if it equals 0 and no solution if it equals a negative number.

Examples : Solve

a) $-2x^2 + 4x - 1 = 0$

$\frac{-b}{2a} = \frac{-4}{-4} = 1$ $\qquad$ $(1, 1)$

$-2(x-1)^2 + 1 = 0$

$-2(x-1)^2 = -1$

$(x-1)^2 = \frac{1}{2}$

$x - 1 = \pm\sqrt{\frac{1}{2}}$

$\boxed{x = 1 \pm \sqrt{\frac{1}{2}}}$

b) $2x^2 - 4x + 3 = 0$

$\frac{-b}{2a} = \frac{4}{4} = 1$ $\qquad$ $(1, 1)$

$2(x-1)^2 + 1 = 0$

$(x-1)^2 = -\frac{1}{2}$

no sol.

Hwk : p 240 # 3, 4, 6, 8, 9, 11, 13, 14

# 4.4 – THE QUADRATIC FORMULA

Any quadratic equation can be written in general form : $ax^2 + bx + c = 0$.

The sign of the **discriminant** $\Delta$ of a quadratic expression tells us the number of zeros that it has.

$$\boxed{\Delta = b^2 - 4ac}$$

Property: If $\Delta > 0$, the expression has 2 distinct real roots.
If $\Delta = 0$, the expression has 1 double real root.
If $\Delta < 0$, the expression has no real root.

Examples: Determine the number of solutions of the following equations:

a) $-2x^2 + 3x + 8 = 0$

$\Delta = 9 - 4(-2)(8)$
$= 73 \Rightarrow 2$ sol

b) $3x^2 - 5x = -9$

$3x^2 - 5x + 9 = 0$
$\Delta = 25 - 4(3)(9) = -83$
$\Rightarrow$ no sol.

c) $\frac{1}{4}x^2 - 3x + 9 = 0$

$\Delta = 9 - 4\left(\frac{1}{4}\right)(9) = 0$
$\Rightarrow 1$ sol.

The values of the roots are given by the **quadratic formula**:

$$\boxed{x = \frac{-b \pm \sqrt{\Delta}}{2a}}$$

Examples : a) $x^2 + 5x + 6 = 0$

$\Delta = 25 - 4(1)(6) = 1$

$x = \frac{-5 \pm \sqrt{1}}{2}$ $\boxed{x = -3}$ $\boxed{x = -2}$

b) $x^2 - 3x - 2 = 0$

$\Delta = 9 - 4(1)(-2)$
$= 17$

$\boxed{x = \frac{3 \pm \sqrt{17}}{2}}$ exact values

$x \approx -0.56$
$x \approx 3.56$

c) $-5x^2 + x - 3 = x - 4$

$-5x^2 + 1 = 0$
$\Delta = 0 - 4(-5)(1) = 20$

$x = \frac{0 \pm \sqrt{20}}{-10}$ $\boxed{x = \pm \frac{\sqrt{20}}{10}}$

Note: If the discriminant is a perfect square, the roots will be rational. It also means that you could have solved by factoring.

Hwk : p 254 # 2, 3, 5, 7, 9, 10, 12, 14, 15, 17 – 20.