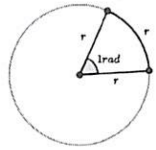
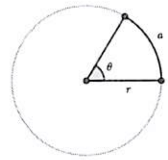
4.1 – ANGLES AND ANGLE MEASURE

Definition: 1 radian is the measure of the central angle subtended by an arc equal in length to the radius of the circle.



Arc length of a circle:
If the angle is expressed in radians, we get: $\boxed{a = r\theta}$

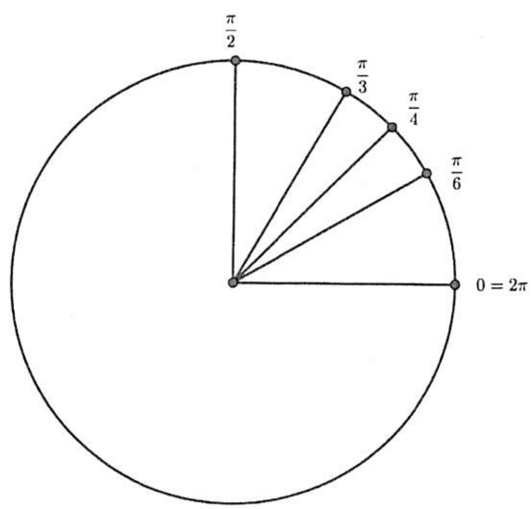


Your turn p 174 (answers: a) $a \approx 11.4$, b) $r \approx 2.6$, c) $\theta = \frac{13}{5}$)

Note, if the circle has radius 1 (unit circle), then the angle in radians and the arc length have the same value.

Conversions to know by heart:

Degrees	Radians
360°	2π
180°	π
90°	$\frac{\pi}{2}$
60°	$\frac{\pi}{3}$
45°	$\frac{\pi}{4}$
30°	$\frac{\pi}{6}$



Conversions:
To convert from radians to degrees or from degrees to radians, we can always use cross multiplication with the reference 180° ←→ π.
But some conversions can be deducted from recognizing the special angles multiples…

Examples:

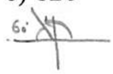
a) 50° $\begin{array}{c|c} \pi & 180^\circ \\ \hline & 50^\circ \end{array}$ $\frac{50\pi}{180} = \frac{5\pi}{18}$

b) $\frac{\pi}{9}$ $\begin{array}{c|c} \pi & 180^\circ \\ \hline \pi/9 & \end{array}$ $\frac{180\pi}{9\pi} = 20^\circ$

c) -213^o $\begin{array}{c|c} \pi & 180^\circ \\ \hline & -213^\circ \end{array}$ $\frac{-213\pi}{180} = -\frac{71\pi}{60}$

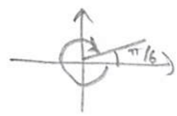
d) 3.2 $\begin{array}{c|c} \pi & 180^\circ \\ \hline 3.2 & \end{array}$ $\frac{3.2 \times 180}{\pi} = \frac{576}{\pi}^\circ$

$\approx 183.3^\circ$

e) 120°  $\frac{\pi}{3} \rightarrow 60^\circ$ $\boxed{\frac{2\pi}{3}}$

g) $\frac{5\pi}{4}$  $180 + 45 = \boxed{225^\circ}$

f) -150°  $30^\circ \rightarrow \frac{\pi}{6}$ $\boxed{-\frac{5\pi}{6}}$

h) $-\frac{11\pi}{6}$  $-360 + 30 = \boxed{-330^\circ}$

Your turn p 169

When 2 angles in standard position have the same terminal arm, we call them **coterminal angles**. They differ by full rotations (360° or 2π or their multiples depending on the unit used).

Examples: 30° and 750° are coterminal angles
$\frac{\pi}{3}$ and $\frac{7\pi}{3}$ are coterminal angles

$750 = 2 \times 360 + 30$

$\frac{7\pi}{3} = \frac{6\pi}{3} + \frac{\pi}{3}$

2π

Example 2 p 170 + your turn p 171

For any angle θ, there is an infinity of coterminal angles. When asked for all of them, we give their **general form**:

$\theta + 360n, n \in \mathbb{Z}$ in degrees or $\theta + 2\pi n, n \in \mathbb{Z}$ in radians

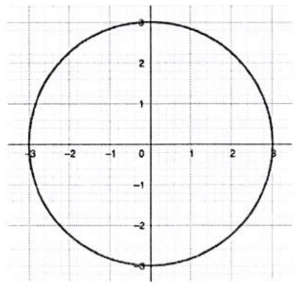
Example 3 p 172 + your turn p 172

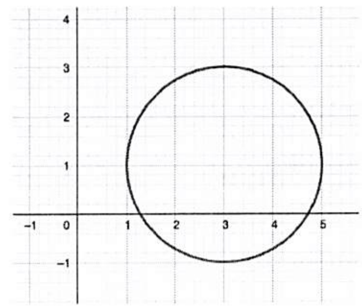
Hwk: p 175 # 2 – 4, 6 – 11, 13 – 16, …

4.2 – THE UNIT CIRCLE

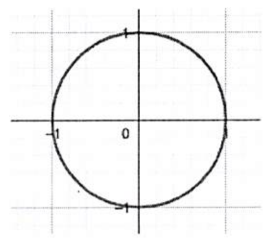
The equation of a circle with centre (a, b) and radius r is: $\boxed{(x-a)^2 + (y-b)^2 = r^2}$

Examples:





The **unit circle** is the circle with centre (0,0) and radius 1.



Its equation is: $\boxed{x^2 + y^2 = 1}$

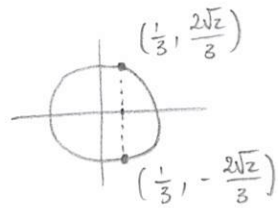
Applications:

a) Prove that $P\left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\right)$ is on the unit circle.

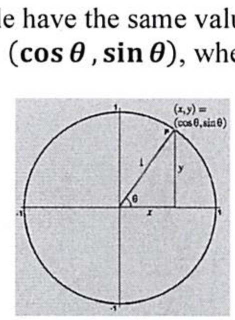
$$\left(\frac{\sqrt{5}}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 = \frac{5}{9} + \frac{4}{9} = 1 \checkmark$$ yes !

b) $A(\frac{1}{3}y)$ is on the unit circle. Determine the possible values of y.

$$\left(\frac{1}{3}\right)^2 + y^2 = 1$$
$$\frac{1}{9} + y^2 = 1$$
$$y^2 = 1 - \frac{1}{9}$$
$$y^2 = \frac{8}{9}$$
$$y = \pm\sqrt{\frac{8}{9}}$$
$$y = \pm\frac{2\sqrt{2}}{3}$$



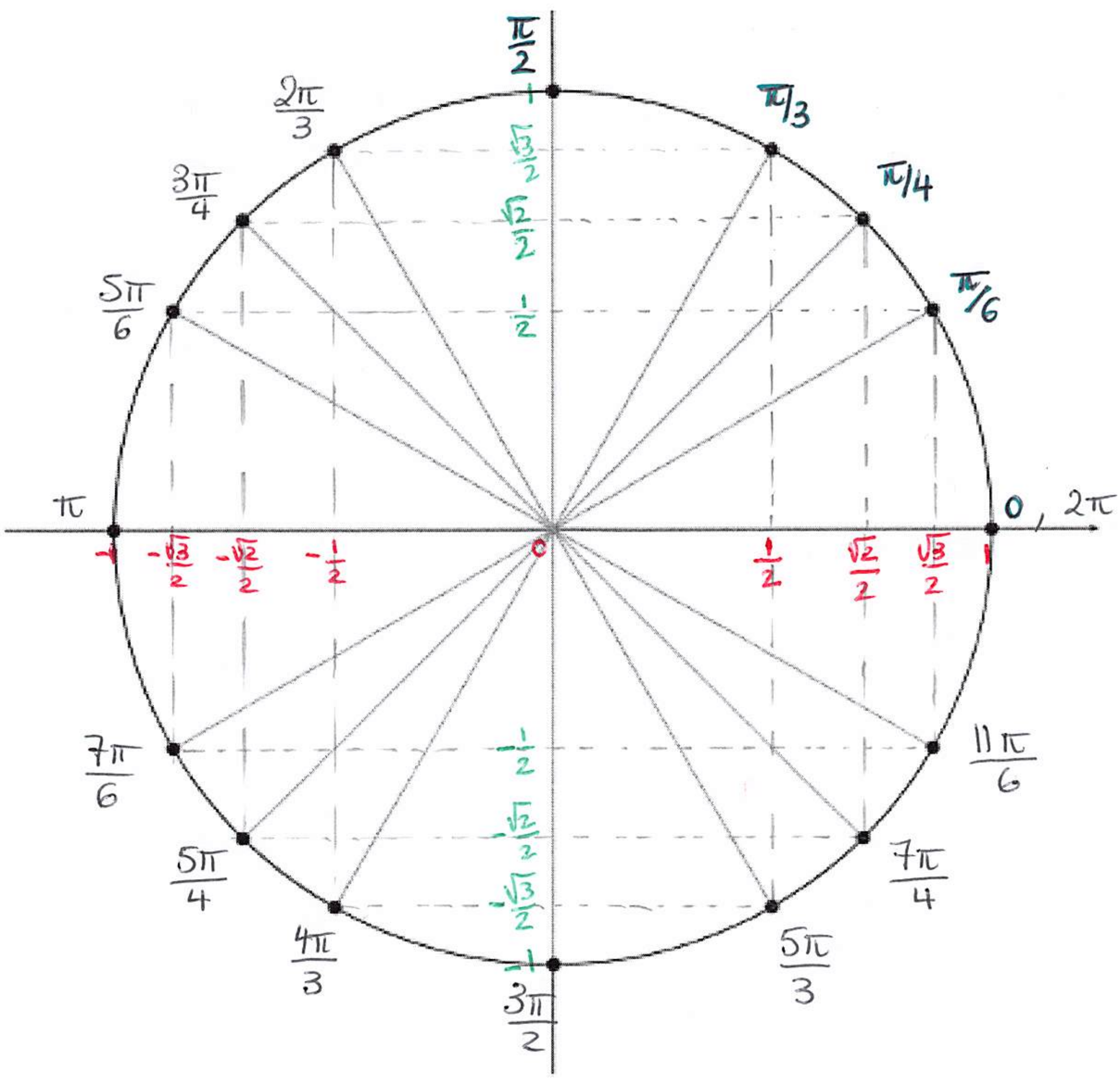
The unit circle is a useful tool, because for any angle in standard position, the angle and the arc length subtended by the angle have the same value. But more importantly, we can notice that **if a point is on the unit circle, its coordinates are $(\cos\theta, \sin\theta)$**, where θ is the angle in standard position that locates the point on the circle.



Remembering the "special" values learned in Precalc 11, we get the coordinates of the special points of the unit circle…

We also need to remember the tangent values that we can get back using the equation: $\boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$

The special "couples" for sin and cos were:	$(0,1)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	and vice versa.
The special values for tan were:	0 or undef	$\frac{1}{\sqrt{3}}$ or $\sqrt{3}$	1	
	quadrantal angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$	$\frac{\pi}{6}$ and $\frac{\pi}{3}$ + multiples	$\frac{\pi}{4}$ + multiples	



<u>Hwk:</u> p 186 # 1 – 7, 9, 10, 12, 13, 17, 19

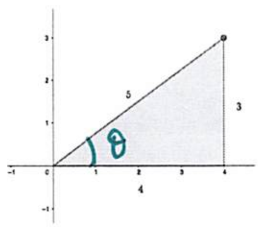
4.3 – TRIGONOMETRIC RATIOS

I – 6 Trigonometric Ratios:

Primary Ratios: $\cos\theta = \dfrac{adj\ \theta_R}{hyp}$ $\quad \sin\ \theta = \dfrac{opp\ \theta_R}{hyp}$ $\quad \tan\ \theta = \dfrac{opp\ \theta_R}{adj\ \theta_R} = \dfrac{\sin\theta}{\cos\theta}$

Reciprocal Ratios: $\boxed{\sec\theta = \dfrac{1}{\cos\theta}}$ $\quad \boxed{\csc\theta = \dfrac{1}{\sin\ \theta}}$ $\quad \boxed{\cot\theta = \dfrac{\cos\theta}{\sin\theta}} = \dfrac{1}{\tan\ \theta}$

Examples:

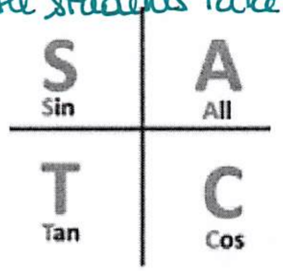


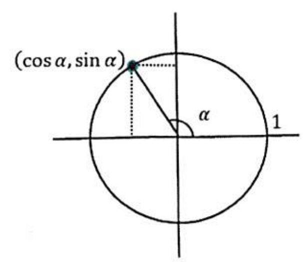
$\cos\theta = \frac{4}{5}$ $\quad \sin\theta = \frac{3}{5}$ $\quad \tan\theta = \frac{3}{4}$

$\sec\theta = \frac{5}{4}$ $\quad \csc\theta = \frac{5}{3}$ $\quad \cot\theta = \frac{4}{3}$

Remember:

All students take Calculus





II – Determining a trigonometric ratio given the angle:

→ **If the angle is a "special" angle** $\left(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\ \textit{and their multiples}\right)$

You are expected to give exact values.

Examples: Determine the following ratios:

a) $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

b) $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

c) $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

d) $\tan 270^o$ undef.

e) $\sec\frac{5\pi}{6}$ $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ so $\sec\frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$ or $-\frac{2\sqrt{3}}{3}$

f) $\csc(-\frac{4\pi}{3})$ $\sin(-\frac{4\pi}{3}) = \frac{\sqrt{3}}{2}$ so $\csc(-\frac{4\pi}{3}) = \frac{2}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$

g) $\cot 315^o$ $\cos 315° = \frac{\sqrt{2}}{2}$ $\sin 315° = -\frac{\sqrt{2}}{2}$ so $\cot 315° = -1$

Your turn p 195

answers : a) undef b) -2 c) $\frac{\sqrt{3}}{2}$ d) 2

→ **<u>If the angle is not "special"</u>**

You will just use your calculator to get the value of the ratios, but the values won't be exact. Be careful to make sure the mode of the calculator matches the unit used…

Examples:

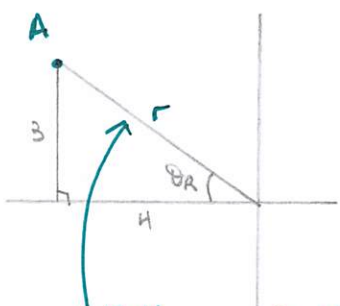
a) $\cos 200^o \approx -0.94$

b) $\tan 3 \approx -0.14$

c) $\sec 130° = \frac{1}{\cos 130°} \approx -2.72$

III – Determining a trigonometric ratio given the coordinates of a point on its terminal arm:

Example: A(-4,3) is on the terminal arm of an angle θ in standard position. Determine the 6 trig ratios of that angle.



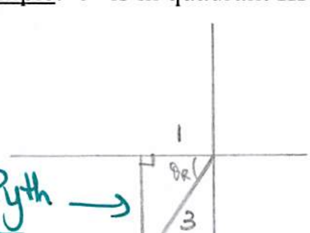
$\cos\theta = -\frac{4}{5}$ $\sin\theta = \frac{3}{5}$ $\tan\theta = -\frac{3}{4}$

$\sec\theta = -\frac{5}{4}$ $\csc\theta = \frac{5}{3}$ $\cot\theta = -\frac{4}{3}$

Pyth : $r^2 = 3^2 + 4^2$
$r^2 = 25$ $r = 5$

IV – Determining a trigonometric ratio given another ratio:

Example: θ is in quadrant III and $\cos\theta = -\frac{1}{3}$. Determine $\sin\theta$ and $\tan\theta$.



Pyth →
$\sqrt{8}$

$\sin\theta = -\frac{\sqrt{8}}{3}$

$\tan\theta = \sqrt{8}$

V – Determining an angle given a trig ratio:

Reminder:

Special values for **sin** and **cos**: $\boxed{0\,;\ \pm\frac{1}{2}\,;\ \pm\frac{1}{\sqrt{2}}\,;\ \pm\frac{\sqrt{2}}{2}\,;\ \pm\frac{\sqrt{3}}{2}\,;\ \pm 1}$

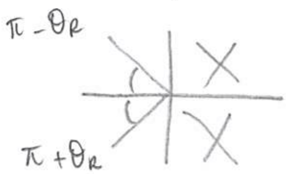
Special values for **tan**: $\boxed{0\,;\ \pm\sqrt{3}\,;\ \pm\frac{1}{\sqrt{3}}\,;\ \pm\frac{\sqrt{3}}{3}\,;\ \pm 1\,;\mathit{undef}}$

For these values, we expect exact angles.

Examples:

① locate quadrants w sign
② ref angle w value
special → exact value
not special → approx calc.
③ answers

a) Solve $\cos\theta = -\frac{1}{2}$ for $0 \le \theta \le 2\pi$.



$\theta_R = \frac{\pi}{3}$

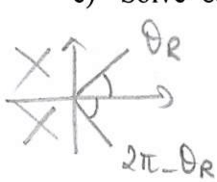
$\Rightarrow \boxed{\frac{2\pi}{3} \text{ and } \frac{4\pi}{3}}$

b) Solve $\sin\theta = 0$ for $0 \le \theta \le 2\pi$.

sol : $\boxed{\{0,\ \pi,\ 2\pi\}}$

c) Solve $\cos\theta = \frac{1}{3}$ for $0 \le \theta \le 2\pi$.

not special

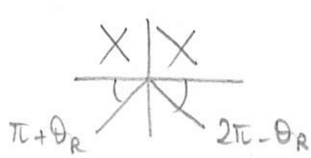


$\theta_R = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.23$

sol : $\boxed{\{1.23\ ,\ 5.05\}}$

d) Solve $\sin\theta = -\frac{3}{4}$ for $0 \le \theta \le 2\pi$.

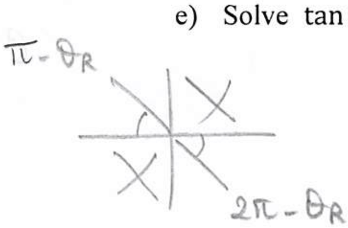
don't use the – if you want the reference angle.



$\theta_R = \sin^{-1}\left(\frac{3}{4}\right) \simeq 0.85$

sol : $\boxed{\{3.99\ ,\ 5.44\}}$

e) Solve $\tan\theta = -\sqrt{3}$ for $0 \le \theta \le 2\pi$.



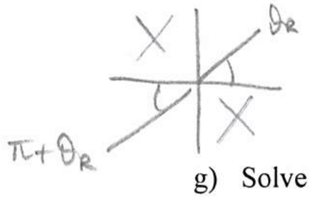
$\theta_R = \frac{\pi}{3}$

$\sqrt{3} = \frac{\sqrt{3}/2}{1/2}$ ← sin, ← cos

special value

sol : $\boxed{\left\{\frac{2\pi}{3},\ \frac{5\pi}{3}\right\}}$

f) Solve $\tan\theta = 0.2$ for $0 \le \theta \le 2\pi$.

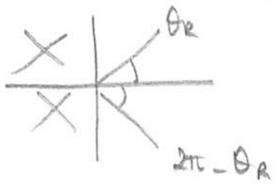


$\theta_R = \tan^{-1}(0.2) \simeq 0.20$

sol : $\boxed{\{0.20,\ 3.34\}}$

g) Solve $\sec\theta = 2$ for $0 \le \theta \le 2\pi$.

$\cos\theta = \frac{1}{2}$



$\theta_R = \frac{\pi}{3}$

sol : $\boxed{\left\{\frac{\pi}{3},\ \frac{5\pi}{3}\right\}}$

NOTE: The values of $\cos\theta$ and $\cos\theta$ can only be between -1 and 1 included.

Therefore an equation like $\cos\theta = 2$ has no solution!

Hwk: p 201 # 1 – 17, 19 – 21.

4.4 – INTRODUCTION TO TRIGONOMETRIC EQUATIONS

To solve trigonometric equations, you always need to simplify them into BASIC trig equations.

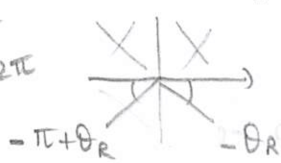
I – BASIC EQUATIONS: They are in the form $prim\ trig\ \theta = a$ $\cos\theta = \frac{\sqrt{2}}{2}$ or $\sin\theta = 0.3$ or $\tan\theta = -1$

As previously seen in 4.3, you need to:
- Locate the possible quadrants.
- Determine the reference angle (preferably its exact value)
- Give all the possible values of θ in the given interval or general form.

Examples:

or find the usual ones and subtract 2π

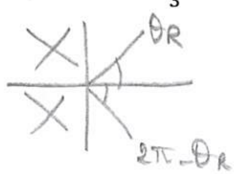
a) $\sin\theta = -\frac{1}{2}$ for $-\pi \le \theta \le \pi$



$\theta_R = \frac{\pi}{6}$

sol: $\boxed{\left\{-\frac{5\pi}{6}, -\frac{\pi}{6}\right\}}$

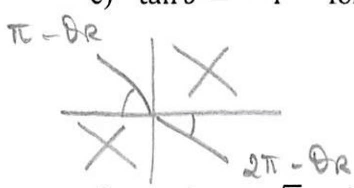
b) $\cos\theta = \frac{1}{3}$ ← no restrictions on the sol ⟹ g^{al} solution



$\theta_R = \cos^{-1}\left(\frac{1}{3}\right) \simeq 1.23$

sol: $\boxed{\{1.23 + 2\pi n, n \in \mathbb{Z};\ 5.05 + 2\pi n, n \in \mathbb{Z}\}}$

c) $\tan\theta = -4$ for $0 \le \theta \le 4\pi$ ← don't take the "−" if you want the reference angle

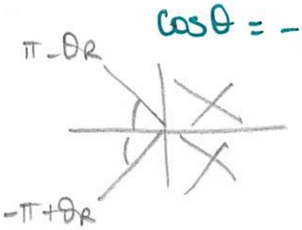


$\theta_R = \tan^{-1}(4) \simeq 1.33$

sol: $\boxed{\{\underbrace{1.82}_{\pi-\theta_R}, \underbrace{4.96}_{2\pi-\theta_R}, \underbrace{8.10}_{1.82+2\pi}, \underbrace{11.24}_{4.96+2\pi}\}}$

d) $\sec\theta = -\sqrt{2}$ for $-\pi \le \theta \le \pi$

$\cos\theta = -\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$



$\theta_R = \frac{\pi}{4}$

sol: $\boxed{\left\{-\frac{3\pi}{4}, \frac{3\pi}{4}\right\}}$

Your turn:

a) $\cos\theta = -\frac{\sqrt{3}}{2}$ for $-\pi \le \theta \le \pi$ $\left\{-\frac{5\pi}{6}, \frac{5\pi}{6}\right\}$

b) $\tan\theta = -\frac{1}{\sqrt{3}}$ G^{al} solution $\left\{\frac{5\pi}{6} + n\pi, n \in \mathbb{Z}\right\}$

c) $\cot\theta = 1$ for $0 \le \theta \le 4\pi$ $\left\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}\right\}$

d) $\csc\theta = -3$ for $-\pi \le \theta \le \pi$ $\{-2.80, -0.34\}$

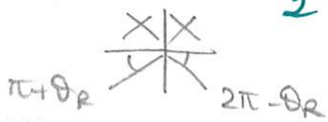
II – OTHER TRIGONOMETRIC EQUATIONS: Your goal is to go back to basic ones!

Example 1: Solve $5\sin\theta + 2 = 1 + 3\sin\theta$ for $0 \le \theta \le 2\pi$

- no restrictions.
- $2\sin\theta = -1$

  $\sin\theta = -\frac{1}{2}$

$\theta_R = \frac{\pi}{6}$



sol: $\left\{ \frac{7\pi}{6}, \frac{11\pi}{6} \right\}$

Your turn p 208

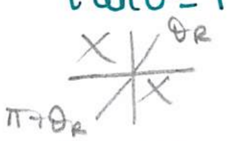
Example 2: Solve $\tan^2\theta - 5\tan\theta + 4 = 0$

- restrictions: $\cos\theta \neq 0$ i.e $\theta \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$
- resolut°: let $t = \tan\theta \Rightarrow t^2 - 5t + 4 = 0$

  $(t-1)(t-4) = 0$

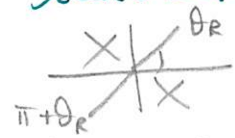
  $t = 1 \qquad t = 4$

$\tan\theta = 1$



$\theta_R = \pi/4$

$\tan\theta = 4$



$\theta_R \simeq 1.33$

$\theta = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}$ or $\theta \simeq 1.33 + n\pi, n \in \mathbb{Z}$

Your turn p 209

Example 3: Solve $\sin^2 x - 1 = 0$

- no restriction
- resolution: $\sin^2 x = 1$

  $\sin x = \pm 1$

sol: $\{ \pi/2 + n\pi, n \in \mathbb{Z} \}$

Your turn p 210

Hwk: p 211 # 1, 3 – 5, 7 – 13, 15, 16, 18 – 23.