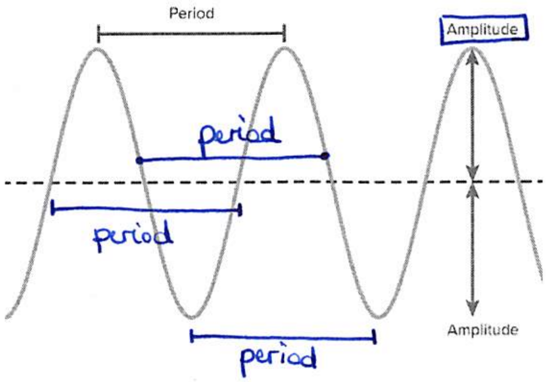
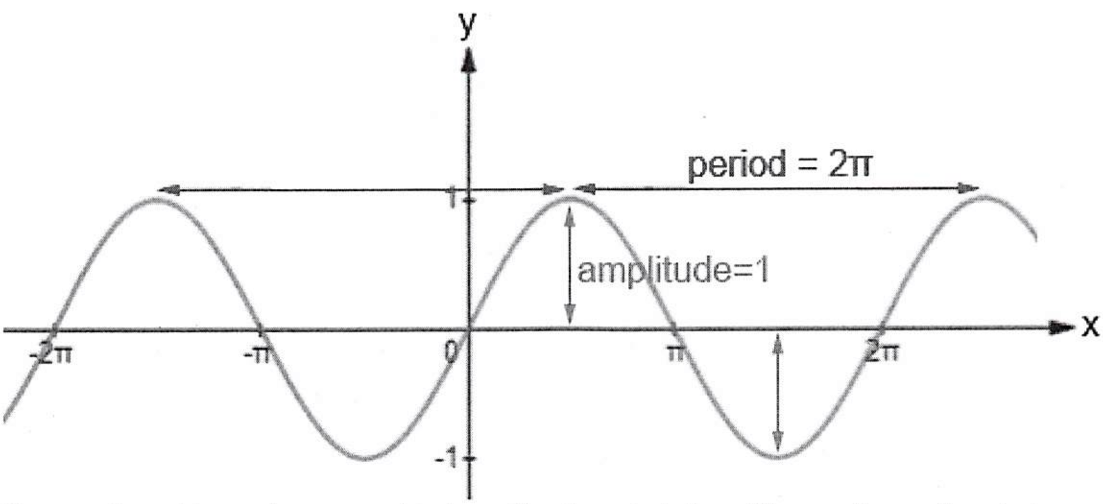
5.1 – GRAPHING SINE AND COSINE FUNCTIONS

The graphs of $y = cos\, x$ and $y = sin\, x$ look a lot alike.
They are both sinusoidal curves with a period 2π.

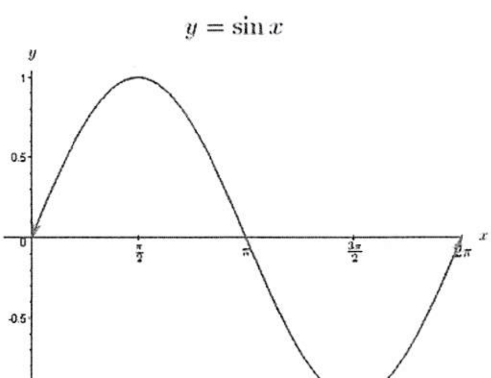
Sinusoidal means that the curve oscillates repeatedly up and down from a centre line.
A period is the number of units every which the graph repeats itself exactly.



I – Graph of $y = \sin x$.



To reproduce this graph, you need to learn the characteristics of the graph over 1 period :



Max : 1 Min : -1

Amplitude : 1

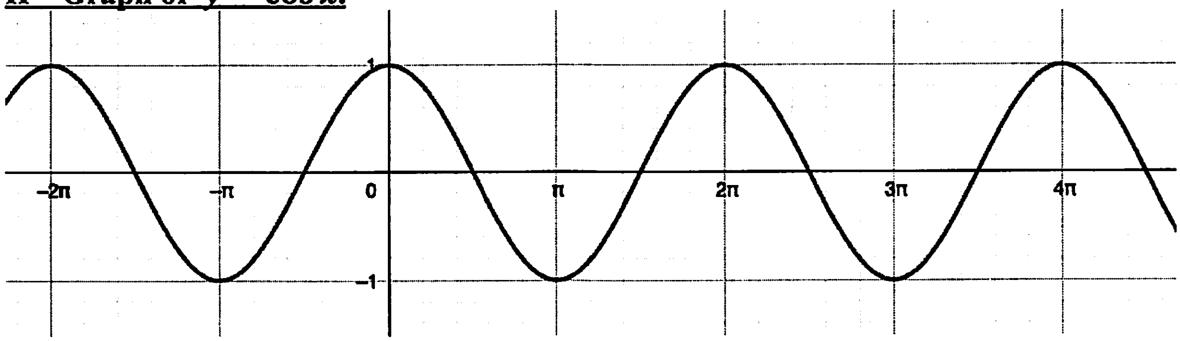
Period 2π or 360^o

y –intercept : 0

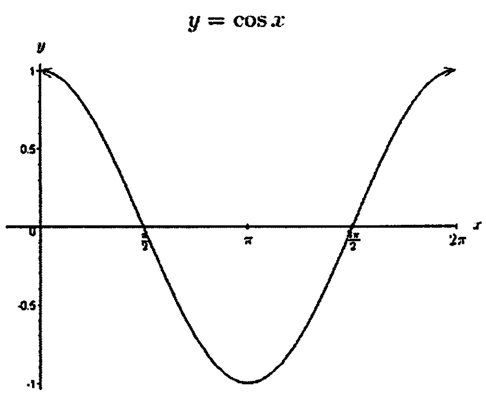
x –intercepts : $0, \pi, 2\pi$ (on the cycle showed)

Domain : $\mathbb{R}$ Range : $[-1,1]$

II – Graph of $y = \cos x$.



To reproduce this graph, you need to learn the characteristics of the graph over 1 period:



Max : 1 Min : -1

Amplitude : 1

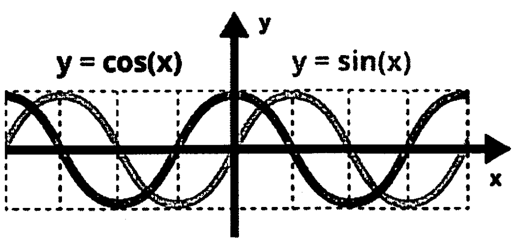
Period 2π or 360^o

y –intercept : 1

x –intercepts : $\frac{\pi}{2}, \frac{3\pi}{2}$ (on the cycle showed)

Domain : $\mathbb{R}$ Range : $[-1,1]$

The two graphs are very similar. One is a horizontal translation of the other one.



Hwk: p 233 # 1 – 11, 14, 15, 17, 20, 22 – 24.

5.2 – TRANSFORMATIONS OF SINUSOIDAL FUNCTIONS

will affect the period

$$\boxed{y = a\sin(b(x-h)) + k}$$

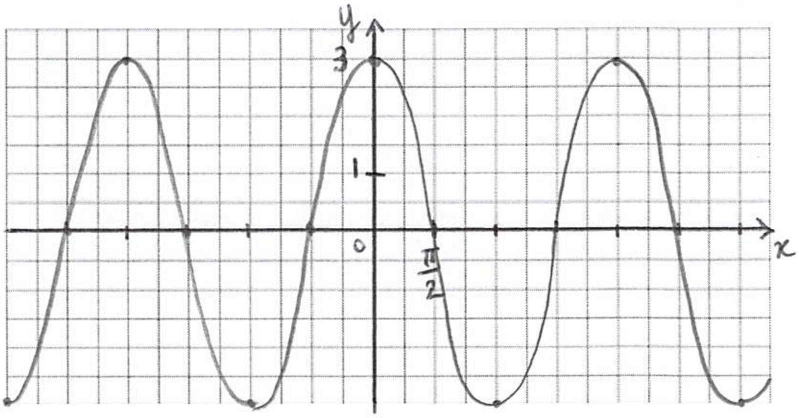
will affect the centre line

will affect the amplitude

will affect the "starting" point of the cycle

Examples:

a) $y = 3\cos x$

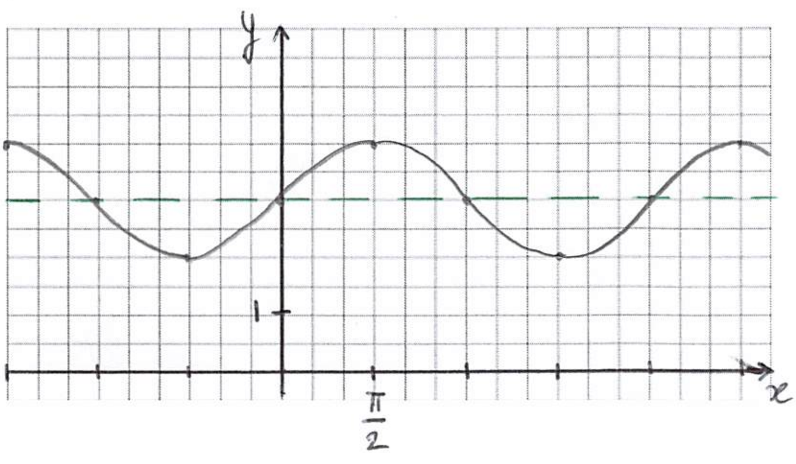


Nothing changed except the amplitude which is now 3.

$$\boxed{\text{Amplitude}: \ |a|}$$

Note: If a was negative, there would also be a reflection around the centre line (the graph would "start" at a minimum).

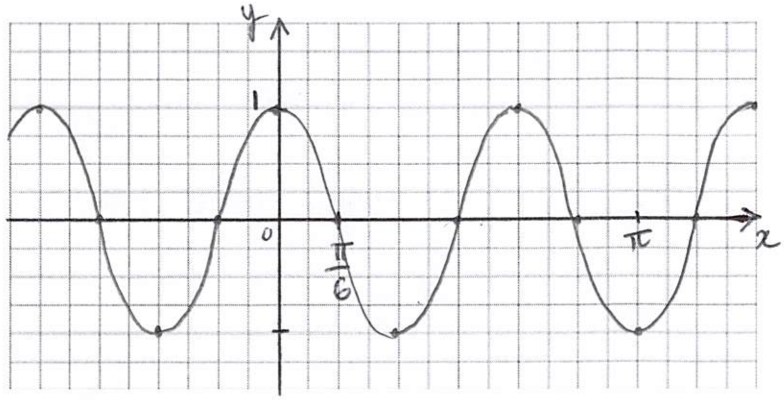
b) $y = \sin x + 3$



Nothing changed except the centre line which is $y = 3$.

$$\boxed{\text{Centre line}: \ y = k}$$

c) $y = \cos(3x)$

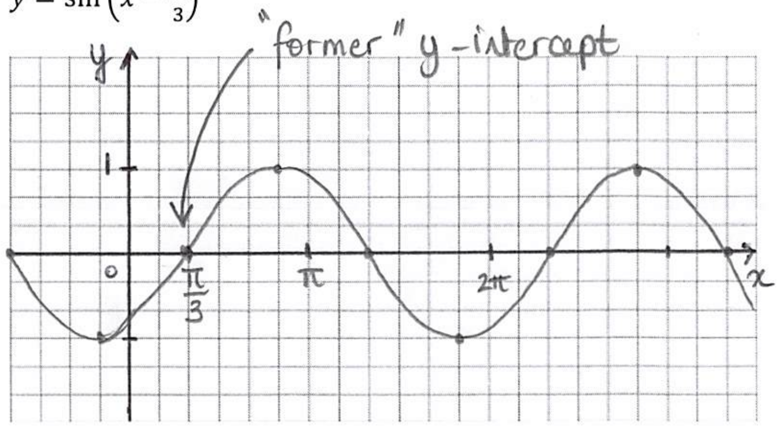


Nothing changed except the period which is now $\frac{2\pi}{3}$.

$$\boxed{\text{Period : } \frac{2\pi}{|b|} \text{ or } \frac{360^o}{|b|}}$$

Note: in each cycle, there are 4 easy points to plot (equidistant from one another): 1 max, 1 min and 2 x-intercepts. So, by dividing the period by 4, we know how "often" we are going to plot a point. That helps choosing our scale on the x-axis. In this example, dividing the period by 4 gives us $\frac{\pi}{6}$. That's how often we plotted a point.

d) $y = \sin\left(x - \frac{\pi}{3}\right)$



The graph is shifted $\frac{\pi}{3}$ units to the right. We call it the **Phase Shift**.

$$\boxed{\text{Phase Shift : } h}$$

Note: We need to choose our scale on the x-axis so that the phase shift (starting point from our cycle) is easy to plot. So, we need something in common with the 4 easy points previously considered. In radians a common denominator will be the easiest.

Example: $y = \cos 3\left(x - \frac{\pi}{4}\right)$

→ phase shift : $\frac{\pi}{4}$

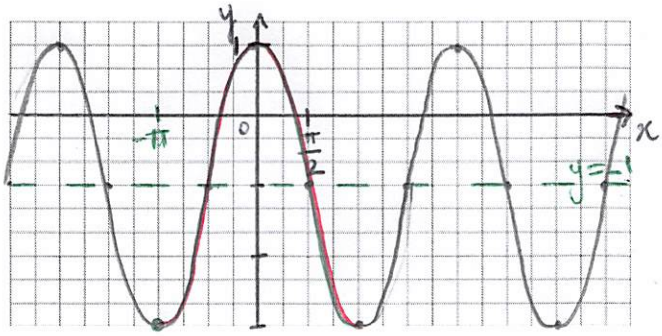
→ period : $\frac{2\pi}{3}$ ----> plot a point every $\frac{\pi}{6}$

} common denom : $\frac{\pi}{12}$

⇒ phase shift : $\frac{3\pi}{12}$ & plot a point every $\frac{2\pi}{12}$

Combining all the transformations:

a) $y = -2\cos(x + \pi) - 1$



a : amplitude 2 & reflected cos

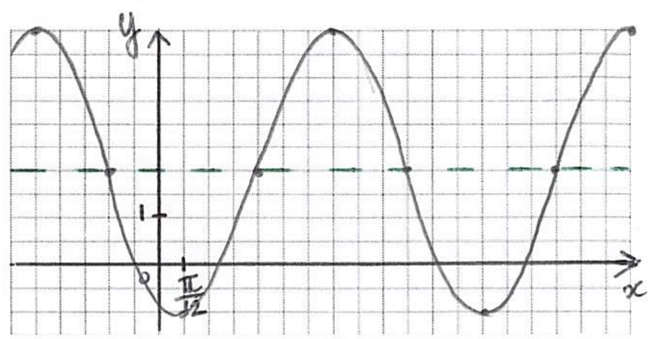
b : period 2π ...> a pt every $\frac{\pi}{2}$

h : phase shift : $-\pi$

$-\pi$

k : centre line : $y = -1$ $\Rightarrow$ min = -3
max = 1

b) $y = 3\sin\left(2x - \frac{2\pi}{3}\right) + 2$



a: amplitude 3

b: period π ....> a pt every $\frac{\pi}{4}$

h : phase shift : $\frac{\pi}{3}$

$\frac{\pi}{3}$

k : centre line : $y = 2$ => min : -1
max : 5

scale $\frac{\pi}{12}$

⚠ $y = 3\sin\left(2\left(x - \frac{\pi}{3}\right)\right) + 2$

(phase shift : $\frac{4\pi}{12}$
a point every $\frac{3\pi}{12}$)

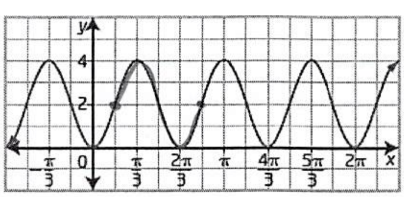
Determining an equation from a graph:

Example:

The graph shows the function $y = f(x)$.

a) Write the equation of the function in the form $y = a \sin b(x - c) + d$, $a > 0$.

b) Write the equation of the function in the form $y = a \cos b(x - c) + d$, $a > 0$.



a) $y = 2\sin 3\left(x - \frac{\pi}{6}\right) + 2$

b) $y = 2\cos 3\left(x - \frac{\pi}{3}\right) + 2$

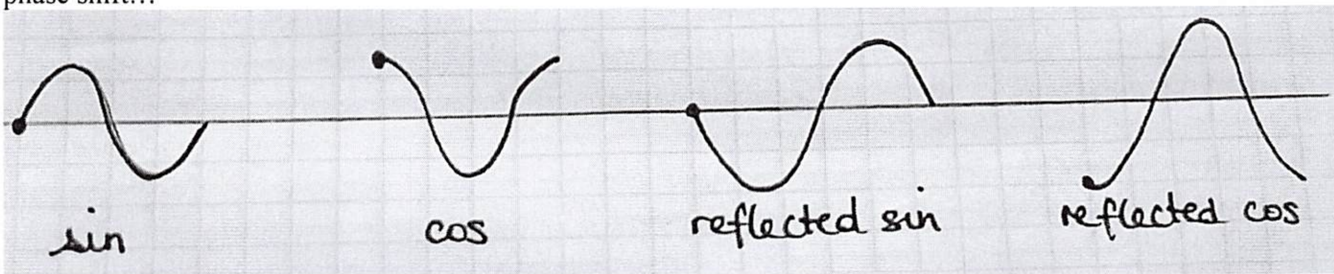
$a = 2$

$b = 3$

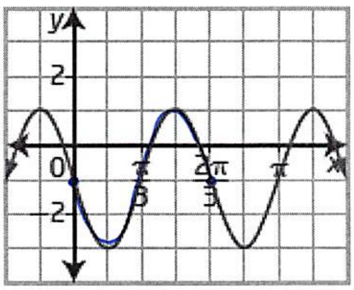
$h = \frac{\pi}{6}$ for sin and $\frac{\pi}{3}$ for cos.

$k = 2$

You can usually choose between a cos or a sin curve. The easiest is usually to pick the one that doesn't involve a phase shift…



Your turn



$y = -2\sin 3x - 1$

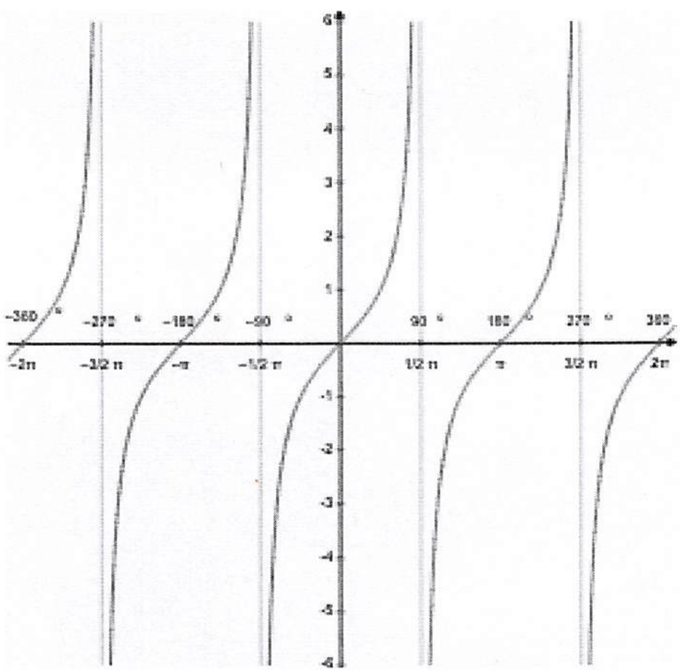
$a = -2$
$b = 3$
$h = 0$
$k = -1$

<u>Hwk</u>: p 250 # 1acef, 2acf, 3, 5 – 7, 9, 13 – 17, 22, 24.

5.3 – THE TANGENT FUNCTION

Reminder : $\tan x = \dfrac{\sin x}{\cos x}$

As a consequence, $\tan x$ is undefined when $\cos x = 0$, i.e. when $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.



Characteristics :

Period : π

Domain : $\left\{x \in \mathbb{R}, x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

Range : $\mathbb{R}$ (No min, no max)

Vertical asymptotes : $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

x-intercepts : $x = k\pi, k \in \mathbb{Z}$

y-intercept : 0

Application : (example 2 p 260)

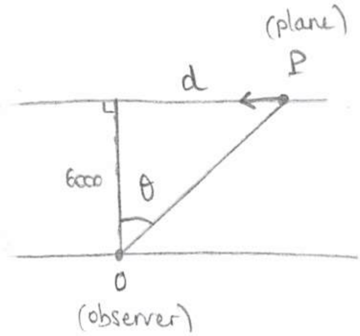
A small plane is flying at a constant altitude of 6000 m directly toward an observer. Assume that the ground is flat in the region close to the observer.

a) Determine the relation between the horizontal distance, in metres, from the observer to the plane and the angle, in degrees, formed from the vertical to the plane.

b) Sketch the graph of the function.

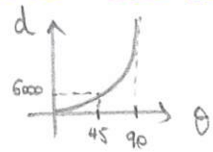
c) Where are the asymptotes located in this graph? What do they represent?

d) Explain what happens when the angle is equal to 0°.



↳ a) $\tan\theta = \dfrac{d}{6000}$ i.e $d = 6000 \tan\theta$

b) domain : $[0, 90)$



c) $\theta = 90$: impossible: even if the plane is really far, the angle will be close to 90°, but never 90°

Hwk : p 262 # 1 – 4, 9 – 12.

d) when $\theta = 0°$, the plane is right above the observer.

5.4 – Equations and Graphs of Trigonometric Functions

I – Solving Trigonometric Equations

1. Graphically

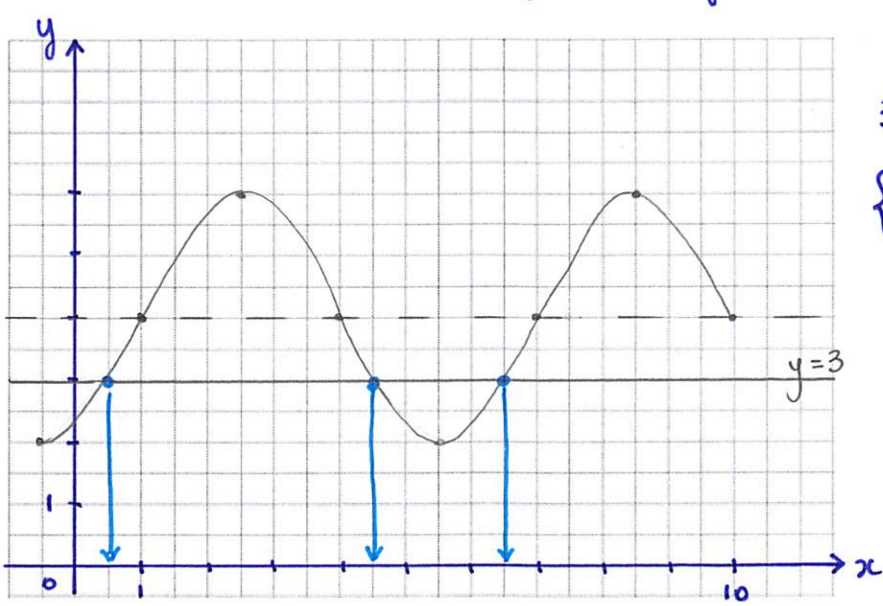
Example 1 : Solve $2\sin\left(\frac{\pi}{3}(x-1)\right)+4=3$ for $x \in [0; 10]$

You need to graph (by hand or with your graphing calculator) both sides of the equation : $y = 2\sin\left(\frac{\pi}{3}(x-1)\right)+4$ *and* $y = 3$ *and look for the x coordinates of the points of intersection on the domain.*

$y = 2\sin\left(\frac{\pi}{3}(x-1)\right)+4 \Rightarrow$ centre line: $y = 4$ → max : 6, min : 2

period: $\frac{2\pi}{\frac{\pi}{3}} = 2\pi \times \frac{3}{\pi} = 6$ (a pt every 1.5)

phase shift: 1



3 solutions :

$\left\{\frac{1}{2}, \frac{9}{2}, \frac{13}{2}\right\}$

Note 1 : If you use our graphing calculator, don't forget to choose the window to match the domain and range, and use the proper mode (Degrees or Radians).

Note 2 : This method is time consuming and will not give exact values if the solutions are not easy to read…

2. Algebraically

Example 2 : Solve $2\sin\left(\frac{\pi}{3}(x-1)\right)+4=3$ for $x \in [0;10]$

Treat the argument of the trig function as an angle (you can name it θ)

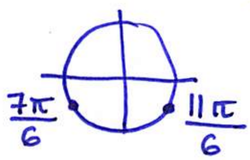
$$2\sin\theta+4=3$$

Solve the more basic equation and find all the solutions for 1 period.

$$2\sin\theta=-1$$

$$\sin\theta=-\frac{1}{2}$$

$$\Rightarrow \theta=\frac{7\pi}{6} \text{ or } \theta=\frac{11\pi}{6}$$



Then solve for x for each solution and get all the solutions using the transformed period.

$\div\frac{\pi}{3}$ $\quad \times\frac{3}{\pi}$

$$\frac{\pi}{3}(x-1)=\frac{7\pi}{6}$$

$$x-1=\frac{7}{2}$$

$$x=\frac{9}{2}$$

$$\frac{\pi}{3}(x-1)=\frac{11\pi}{6}$$

$$x-1=\frac{11}{2}$$

$$x=\frac{13}{2}$$

period : $\frac{2\pi}{\frac{\pi}{3}}=6 = \frac{12}{2}$

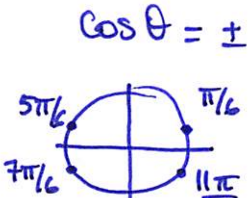
$\Rightarrow$ solutions (on the given interval) $\left\{\frac{1}{2},\frac{9}{2},\frac{13}{2}\right\}$

Example 3 : $4\cos^2\left(2\left(x-\frac{\pi}{4}\right)\right)=3$ for $x \in [0,2\pi]$

- $4\cos^2\theta=3$

$$\cos^2\theta=\frac{3}{4}$$

$$\cos\theta=\pm\frac{\sqrt{3}}{2}$$



* $2\left(x-\frac{\pi}{4}\right)=\frac{\pi}{6}$

$$x-\frac{\pi}{4}=\frac{\pi}{12}$$

$$x=\frac{4\pi}{12}$$

$$x=\frac{\pi}{3}$$

* $2\left(x-\frac{\pi}{4}\right)=\frac{5\pi}{6}$

$$x-\frac{\pi}{4}=\frac{5\pi}{12}$$

$$x=\frac{8\pi}{12}$$

$$x=\frac{2\pi}{3}$$

$p=\pi$

$$\boxed{\text{Sol: } \left\{\frac{\pi}{3},\frac{2\pi}{3},\frac{5\pi}{6},\frac{7\pi}{6},\frac{4\pi}{3},\frac{\pi}{6},\frac{5\pi}{3},\frac{11\pi}{6}\right\}}$$

* $2\left(x-\frac{\pi}{4}\right)=\frac{7\pi}{6}$

$$x-\frac{\pi}{4}=\frac{7\pi}{12}$$

$$x=\frac{5\pi}{6}$$

* $2\left(x-\frac{\pi}{4}\right)=\frac{11\pi}{6}$

$$x-\frac{\pi}{4}=\frac{11\pi}{12}$$

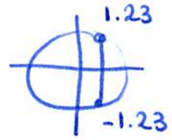
$$x=\frac{7\pi}{6}$$

<u>Example 4 :</u> Solve $6\cos\left(\frac{\pi}{6}x\right) + 14 = 16$ for $x \in [0; 20]$

• $6\cos\theta + 14 = 16$

$6\cos\theta = 2$

$\cos\theta = \frac{1}{3}$



• $p = \frac{2\pi}{\frac{\pi}{6}} = 12$

* $\frac{\pi}{6}x \simeq 1.23$

$x \simeq 2.35$

$x = 2.35 + 12n, n \in \mathbb{Z}$

* $\frac{\pi}{6}x \simeq -1.23$

$x \simeq -2.35$

$x = -2.35 + 12n, n \in \mathbb{Z}$

$\Rightarrow$ solutions : $\{2.35 ; 9.65 , 14.35\}$

<u>Your turn :</u>

a) $\tan(\pi(x-2)) = -1$ for $x \in [0; 5]$

b) $12\cos(2x - 90^o) + 8 = 10$ for $x \in [0; 500]$

solutions :

a) {2.35, 9.65, 14.35}

b) {4.8°, 85.2°, 184.8°, 265.2°, 364.8°, 445.2°}

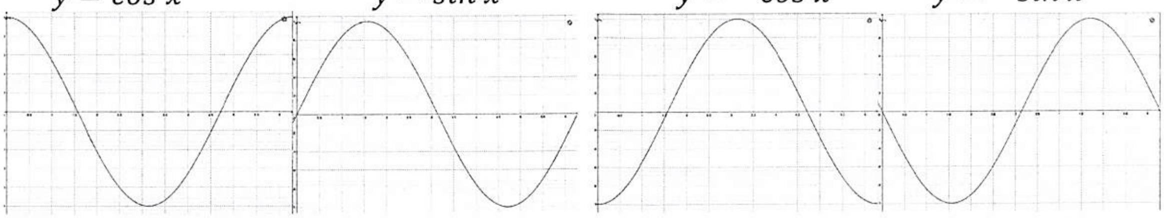
<u>II – Modeling a Real-Life situation with a Trigonometric Equation</u>

Many periodic phenomena can be modeled by trigonometric functions.
In 5.3, we've seen that shadows, rotating lights, … can often be modeled by the tangent function. Most problems involving waves, heartbeat, circular motion, … can be modeled with the sine or cosine function.

*From the information in the word problem, you need to determine the period (the time it takes for a full cycle to happen), the min and the max values (which will give you the centre line and the amplitude) as well as when the first cycle starts (which will give you the phase shift).
From there you can determine the coefficients a, b, h and k of your function.*

TIP : It's usually your choice to use a sin or cos function. It's usually a good idea to choose, when possible, the one that doesn't give a phase shift…
If it starts at a max or a min, choose a cos function.
If it starts on the centre line, choose a sin function.

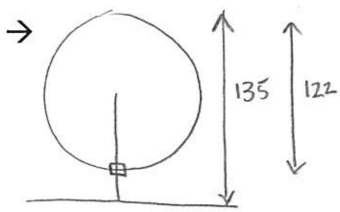
$y = cos\,x$ $\quad$ $y = sin\,x$ $\quad$ $y = -cos\,x$ $\quad$ $y = -sin\,x$

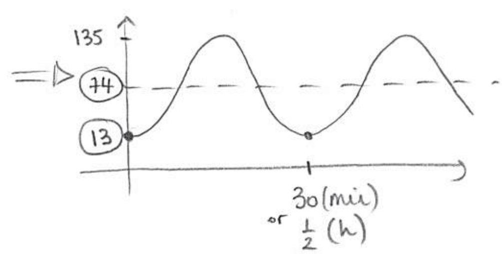


Example: You're boarding a Ferris wheel at the bottom. The height of the Ferris Wheel is 135m, but the diameter of the wheel is 122m. The Wheel makes 2 rotations per hour (It's a sight-seeing one).

a) Model the height of a passenger as a function of time.
b) For how long is the passenger going to stay above 130m in order to see his house during the first rotation?

a)





$a = 61$

$b = \frac{2\pi}{30} = \frac{\pi}{15}$ if we choose min

$h = 0$ if we choose $y = -\cos x$

$K = 74$

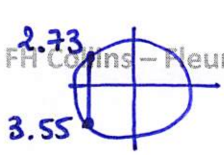
$$\boxed{h(t) = -61\cos\left(\frac{\pi}{15}t\right) + 74}$$

m $\quad$ min

b) • $-61\cos\left(\frac{\pi}{15}t\right) + 74 = 130$

$-61\cos\left(\frac{\pi}{15}t\right) = 56$

$\cos\theta = -\frac{56}{61}$



$\theta_R \simeq 0.41$

• $\frac{\pi}{15}t = 2.73$ $\quad$ $\frac{\pi}{15}t = 3.55$

$t \simeq 13.05$ $\quad$ $t \simeq 16.95$

• $p = 30$ min

$\Rightarrow T = 16.95 - 13.05 = 3.9$

He will be above 130 m for 3.9 min