6.1 – RECIPROCAL, QUOTIENT AND PYTHAGOREAN IDENTITIES

I – Identities vs. Equations

An **identity** is an equality that is true for all values of the variable(s) on their domain. Ex: $2x + 3x = 5x$
An **Equation** is an equality that is true only for some values of the variable on the domain. Ex: $2x + 3 = 5$

If you want to prove that an equality is true for a certain value of the variable, you need to evaluate each side of the equality **SEPARATELY** and compare them.

Example: Prove that $2\sin x = 7 - 3\csc x$ for $x = \frac{\pi}{6}$

L.S: $2\sin x = 2\sin\frac{\pi}{6} = 2 \times \frac{1}{2} = 1$ ✓

R.S: $7 - 3\csc x = 7 - 3\csc\frac{\pi}{6} = 7 - 3 \times 2 = 1$ ✓

Your turn: Prove that $\frac{\csc x}{\tan x + \cot x} = \cos x$ for $x = \frac{\pi}{3}$

L.S: $\frac{\csc x}{\tan x + \cot x} = \frac{\frac{2}{\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}} = \frac{\frac{2}{\sqrt{3}}}{\frac{3+1}{\sqrt{3}}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{4} = \frac{1}{2}$ ✓

R.S: $\cos x = \frac{1}{2}$ ✓

values:
$\cos\frac{\pi}{3} = \frac{1}{2}$
$\csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$
$\tan\frac{\pi}{3} = \sqrt{3}$
$\cot \pi/3 = \frac{1}{\sqrt{3}}$

BE CAREFUL, proving that an equality is true for some (or even many) values doesn't prove that it's an identity! The previous example would have been false for $x = \frac{\pi}{2}$ for example. To prove an identity, we will have to transform expressions using other identities that have been proven or definitions.

II – Non-permissible values of a trigonometric expression:

Most of the times, you will just need to make sure that denominators (visible of hidden)don't equal zero.
The hidden denominators exist when the expression includes $tan\ x, \cot x, \sec x$ & $\csc x$.

Example 1: Determine the restrictions of $\frac{\tan x}{1-\cos x}$.

- $1 - \cos x \neq 0$
  $\cos x \neq 1$
  $x \neq 0 + 2\pi n, n \in \mathbb{Z}$
- $\cos x \neq 0$ ← "hidden" denom.
  $x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$

Your turn:

a) $\frac{\cos x}{\sin x}$ b) $\frac{\sin x}{\tan x}$

c) $\frac{\cot x}{1 - \sin x}$ d) $\frac{\tan x}{\cos x + 1}$

a) $x \neq 0 + \pi n, n \in \mathbb{Z}$

b) $x \neq 0 + \frac{\pi}{2}n, n \in \mathbb{Z}$

c) $x \neq 0 + \pi n, n \in \mathbb{Z}$
$x \neq \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z}$

d) $x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$
$x \neq \pi + 2\pi n, n \in \mathbb{Z}$

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Reminder on fractions: $\begin{cases} \frac{A}{B} \text{ exists when } B \neq 0 \\ \frac{A}{B} = 0 \text{ when } A = 0 \text{ (and } B \neq 0 \text{ otherwise it wouldn't even exist...)} \end{cases}$

Example 2: Determine the restrictions of $\frac{1+\cos^2 x}{\tan x}$.

• $\tan x \neq 0$ • $\cos x \neq 0$

$\sin x \neq 0$

$\Rightarrow x \neq 0 + \frac{\pi}{2}n, \ n \in \mathbb{Z}$

III – Using Identities to simplify expressions:

In this section, we will use the first 8 identities on the formula sheet.
Most of the time, we rewrite everything in terms of $\sin x$ and $\cos x$.

Example 1: Simplify $\csc x \,.\, \tan x \,.\, \sec x \,.\, \cos x$ $= \frac{1}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{\cos x} \cdot \frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1} = \frac{1}{\cos x} = \sec x$

Example 2: Simplify $\frac{\sec x}{\tan x + \cot x}$ $= \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{\frac{1}{\cos x}}{\frac{\sin^2 x + \cos^2 x}{\sin x \cdot \cos x}} = \frac{\frac{1}{\cos x}}{\frac{1}{\sin x \cos x}}$

$= \frac{1}{\cos x} \cdot \frac{\sin x \cdot \cos x}{1}$

$= \sin x$

NOTE: The Pythagorean identity is used in 3 different ways:

$$\boxed{\begin{aligned} &\cos^2 x + \sin^2 x = 1 \\ &\cos^2 x = 1 - \sin^2 x \\ &\sin^2 x = 1 - \cos^2 x \end{aligned}}$$

Example 3: Simplify $\frac{1}{1+\sin x} + \frac{1}{1-\sin x}$

$= \frac{1 - \sin x}{(1+\sin x)(1-\sin x)} + \frac{1+\sin x}{(1+\sin x)(1-\sin x)} = \frac{2}{1-\sin^2 x} = \frac{2}{\cos^2 x} = 2\sec^2 x$

Hwk: p 296 # 3 – 17.

6.2 – SUM, DIFFERENCE and DOUBLE-ANGLE IDENTITIES

They are the rest of the identities on our formula sheet (1st column).

In particular, it is important to notice that the trig functions are not linear, which means for example that:

$$\sin(A+B) \neq \sin A + \sin B$$

$$\sin(A-B) \neq \sin A - \sin B$$

$$\sin(2A) \neq 2\sin A$$

Applications:

1. Write these expressions as a single trigonometric ratio:

   a) $\sin 48 . \cos 17 - \cos 48 . \sin 17 = \sin(48-17) = \sin 31°$

   b) $\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3} = \cos\left(2\times\frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

   Your turn:

   $\cos 88° \cos 35° + \sin 88° \sin 35° = \cos(88-35) = \cos 53°$

   $2\sin\frac{\pi}{12}\cos\frac{\pi}{12} = \sin\left(2\times\frac{\pi}{12}\right) = \sin\frac{\pi}{6}$

2. Determine the restrictions and prove that $\frac{1-\cos 2x}{\sin 2x} = tanx$

- Restrictions: $\sin 2x \neq 0$ & $\cos x \neq 0$

  $2x \neq 0 + \pi n, n \in \mathbb{Z}$ $\quad$ $x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$

  $x \neq 0 + \frac{\pi}{2}n, n \in \mathbb{Z}$

  $x \neq 0 + \frac{\pi}{2}n, n \in \mathbb{Z}$

- $\frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x . \cos x}$

  $= \frac{2\sin^2 x}{2\sin x \cos x}$

  $= \frac{\sin x}{\cos x}$

  $= \tan x$ ✓

3. Determine the exact value of

a) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

see the angle as a sum or difference of "special" angles.

$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$= \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$\boxed{\sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}}$

b) $\tan 105^o = \tan(60 + 45)$

$= \frac{\tan 60 + \tan 45}{1 - \tan 60 \cdot \tan 45}$

$= \frac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1}$

$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$ — we can simplify by rationalizing the denominator.

$= \frac{(1 + \sqrt{3})^2}{1 - 3}$

$= \frac{1 + 2\sqrt{3} + 3}{-2}$

$= -2 - \sqrt{3}$

Your turn:

a) $\cos 165°$

$= \cos(120 + 45)$

$= \cos 120 \cdot \cos 45 - \sin 120 \cdot \sin 45$

$= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$= -\frac{\sqrt{2} + \sqrt{6}}{4}$

b) $\tan\frac{11\pi}{12} = \tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$\frac{\pi}{3} = \frac{4\pi}{12}$

$\frac{\pi}{4} = \frac{3\pi}{12}$

$= \frac{\tan\frac{2\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{2\pi}{3} \cdot \tan\frac{\pi}{4}}$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{2\pi}{3} = -\frac{1}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$= \frac{-\sqrt{3} + 1}{1 + \sqrt{3} \cdot 1}$

$= \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$

$= \frac{(1 - \sqrt{3})^2}{1 - 3}$

$= \frac{4 - 2\sqrt{3}}{-2}$

$= -2 + \sqrt{3}$

<u>Hwk</u>: p 306 # 1 - 23

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6.3 – PROVING IDENTITIES

Proving an identity means showing that the equality is true for all permissible values of the variable.

Testing the equality for some values of the variable is not a proof.

To prove an identity, you need to work with each side SEPARATELY! You can either work with one side and transform it into the other side, OR simplify both sides separately until you get the same expression on both sides.

Examples:

1. Prove that $\frac{1-\cos 2x}{\sin 2x} = \tan x$ is an identity.

$$\text{L.S}: \frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x}$$
$$= \frac{2\sin^2 x}{2\sin x\cos x}$$
$$= \frac{\sin x}{\cos x}$$
$$= \tan x$$

2. Prove that $\frac{1-\sin^2 x}{\cos x} = \frac{\sin 2x}{2\sin x}$ is an identity.

$= \frac{\cos^2 x}{\cos x}$	$= \frac{2\sin x\cos x}{2\sin x}$
$= \cos x$ ✓	$= \cos x$ ✓

Here are some strategies to help when you feel stuck:

- Simplify each expression starting with the most complicated one.
- If there are quadratic expressions, think about the Pythagorean identities or try to factor. (Remember you can only simplify a common factor top and bottom, not terms…)
- Rewrite the expressions using sin and cos only, and using single angles unless all the angles are the same.
- When you feel stuck and there is nothing left to simplify, think about multiplying top and bottom by a conjugate expression.
- Keep an eye on the other side to remember where you are trying to go…

Examples: Prove the following identities

1. $\frac{2}{cosx} = \frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}$

$$\frac{\cos x(1+\sin x) + \cos x(1-\sin x)}{1-\sin^2 x}$$

$$\frac{2\cos x}{\cos^2 x}$$

$$\frac{2}{\cos x} \checkmark$$

2. $\frac{\sin x+\sin^2 x}{\cos x+\sin x.\cos x} = \tan x$

$$\frac{\sin x(1+\sin x)}{\cos x(1+\sin x)}$$

$$\tan x \checkmark$$

can be factored like $2a^2 - a - 1$

3. $\cot x - \csc x = \frac{\cos 2x - \cos x}{\sin 2x + \sin x}$

$$\frac{\cos x}{\sin x} - \frac{1}{\sin x}$$

$$\frac{\cos x - 1}{\sin x} \checkmark$$

$$\frac{2\cos^2 x - 1 - \cos x}{2\sin x\cos x + \sin x} = \frac{2\cos^2 x - \cos x - 1}{2\sin x\cos x + \sin x}$$

$$\frac{(2\cos x+1)(\cos x - 1)}{\sin x(2\cos x + 1)}$$

$$\frac{\cos x - 1}{\sin x} \checkmark$$

4. $\frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$

$$\frac{\sin x(1-\cos x)}{1-\cos^2 x}$$

$$\frac{\sin x(1-\cos x)}{\sin^2 x}$$

$$\frac{1-\cos x}{\sin x} \checkmark$$

Remember to look and anticipate where you want to go in order to avoid working in circles…

<u>Hwk</u>: p 314 # 1 – 18 + extra practice package

6.4 – SOLVING TRIGONOMETRIC EQUATIONS USING IDENTITIES

To solve a trigonometric equation, you need to simplify it into solving basic equations using identities if needed, but you need to determine the restrictions before transforming them.

Examples: Solve the following equations on the given domain.

a) $1 - \cos^2 x = 3\sin x - 2$ over $D = [0, 2\pi)$ (no restriction)

$\sin^2 x - 3\sin x + 2 = 0$

$(\sin x - 1)(\sin x - 2) = 0$

$\sin x = 1$ $\sin x = 2$

impossible

$x = \frac{\pi}{2}$

$\Rightarrow$ Solution: $\left\{\frac{\pi}{2}\right\}$

b) $\cos 2x + 1 - \cos x = 0.$ over $D = [0, 2\pi)$ (no restriction)

$2\cos^2 x - 1 + 1 - \cos x = 0$

$2\cos^2 x - \cos x = 0$

$\cos x\,(2\cos x - 1) = 0$

$\cos x = 0$ $\cos x = \frac{1}{2}$

$\Rightarrow$ sol: $\left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}\right\}$

c) $2\sin x = 7 - 3\csc x$ Restrict°: $x \neq 0 + \pi n, n \in \mathbb{Z}$

$2\sin x = 7 - \frac{3}{\sin x}$

$2\sin^2 x - 7\sin x + 3 = 0$

$\Delta = 49 - 4(2)(3) = 25$

$\sin x = \frac{7 \pm 5}{4}$

$\sin x = 3$ no sol.

$\sin x = \frac{1}{2}$

sol: $\left\{\frac{\pi}{6} + 2\pi n, n \in \mathbb{Z}, \frac{5\pi}{6} + 2\pi n, n \in \mathbb{Z}\right\}$

Hwk: p 320 # 1 – 19