Chapter 7 Exponential Functions

7.1 Characteristics of Exponential Functions

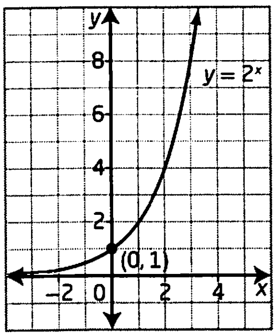
KEY IDEAS

- An exponential function models a type of non-linear change. These types of functions have the form $y = c^x$, where c is a constant ($c > 0$). All exponential functions of this form have a y-intercept of 1.

Why does c have to be positive?

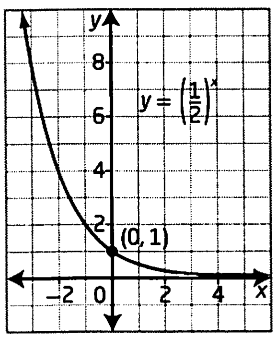
Why is the y-intercept for all exponential functions of this form equal to 1?

- When $c > 1$ in an exponential function of the form $y = c^x$, the exponential function is increasing.



How can you tell from the graph that this is an increasing function? Does this situation represent growth or decay?

- When c is between 0 and 1 (that is, $0 < c < 1$) in an exponential function of the form $y = c^x$, the exponential function is decreasing.



How can you tell from the graph that this is a decreasing function? Does this situation represent growth or decay?

- When $c = 1$ in an exponential function of the form $y = c^x$, the exponential function is neither increasing nor decreasing.

How does the graph of $y = 1^x$ reflect a function that is neither increasing nor decreasing?

- Exponential functions of the form $y = c^x$ have domain $\{x \mid x \in R\}$, range $\{y \mid y > 0, y \in R\}$, no x-intercepts, and horizontal asymptote at $y = 0$.

How do the graphs above reflect the domain, range, and horizontal asymptote?

Working Example 1: Properties of Exponential Functions

Graph each exponential function. Then, identify the following:

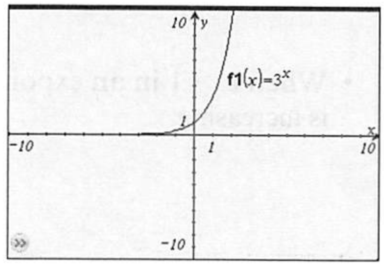
- the domain and range
- the x-intercept and y-intercept, if they exist
- whether the graph represents an increasing or decreasing function
- the equation of the horizontal asymptote

a) $y = 3^x$

b) $y = \left(\frac{1}{3}\right)^x$

Solution

a) Use technology to graph the function.



The function is defined for all real values of x, so the domain is ________________.

All of the y-values of the function are positive, so the range is {________________, $y \in R$}.

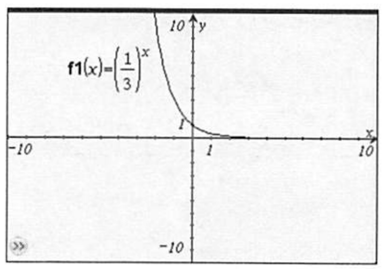
The graph has no x-intercept because the graph never intersects the x-axis.

Since $3^0 = 1$, the y-intercept of the graph is ________________.

The function values get larger as x-values get larger, so the graph is ________________.
(*increasing* or *decreasing*)

The graph of the function gets closer and closer to the ________________ as the x-values decrease, so there is a horizontal asymptote at ________________.

b) Use technology to graph the function.



The domain of the function is ________________.

The range of the function is ________________.

The graph has no x-intercept because the graph never intersects the x-axis.

The y-intercept of the graph is ________________ because $\left(\frac{1}{3}\right)^0 =$ ________.

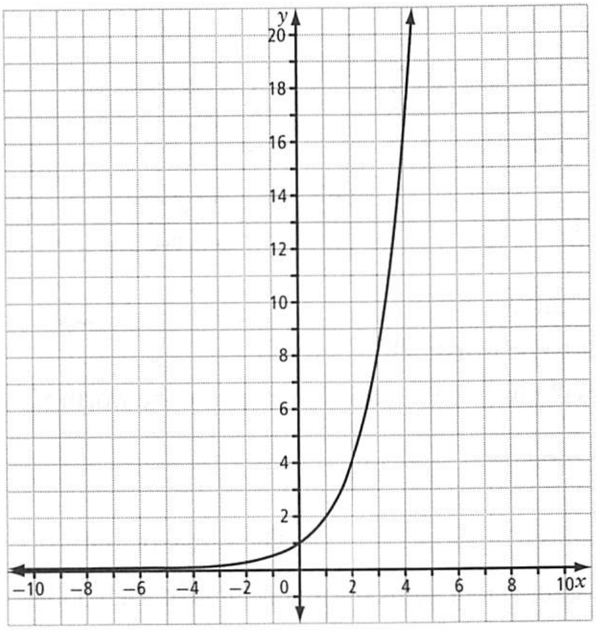
The function values get ________________ as x-values get larger, so the graph is decreasing.

There is a horizontal asymptote at ________________.

See pages 336–338 of *Pre-Calculus 12* for an example using similar concepts.

Working Example 2: Determine an Exponential Function From Its Graph

What exponential function can be used to describe the graph below?



Solution

An exponential function has the form $y = c^x$. When $x = 1$, the value of the function is c.

The given graph includes the ordered pair (1, ________), so you can conclude that $c =$ ________.

Therefore, the exponential function that can be used to describe the graph is ______________.

Choose a point other than (0, 1) to substitute into the function to verify if the function is correct. Try (4, 16).

Check:

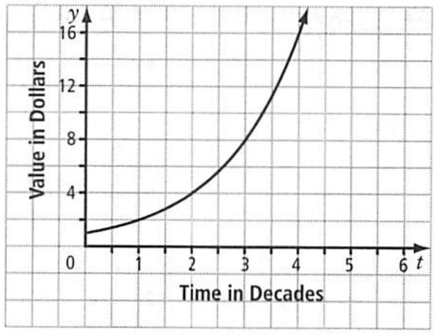
Left Side	Right Side
y	2^x
= ________	$= 2^{\square}$
	= ________

The left side ________ right side, so ______________________.
(= or ≠)

See pages 338–340 of *Pre-Calculus 12* for a similar example.

Working Example 3: An Application of Exponential Functions

Fong has just graduated from university and wants to begin her retirement planning. Her investment advisor suggests an investment that he expects to double in value every decade. The estimated growth rate is modelled by the exponential graph shown.



a) State the domain and range of the function. Explain the significance of the y-intercept.

b) Write an exponential equation that expresses the value of each dollar invested, after t decades.

c) What is the value of a dollar invested for ten years?

d) How long will it take for \$1 in this investment to be worth \$8?

e) Use a table of values to determine how long it will take for \$1 in this investment to be worth \$26.

Solution

a) Since time is graphed on the t-axis, the domain is ______________. The range of the function is ______________. The y-intercept at (0, 1) represents the initial value of each dollar.

b) Since the point (1, ________) is on the graph, the equation is $y = \boxed{}^{t}$.

c) Ten years is one decade. The point (1, 2) is on the graph, so in ten years the value of \$1 is 2^1, or \$________.

d) The point (________, 8) is on the graph, so it will take ________ decades, or ________ years, for each dollar of the investment to be worth \$8.

How might you use the equation to solve part d)?

e) Create a table of values for $V(t) = \boxed{}^{t}$.

t	$V(t)$
1	
2	
3	
4	
5	

The table shows that the number of decades is between ________ and ________. You can determine a better estimate for t by looking at values between these numbers. Use systematic trial to find a value for t where $V \approx 26$:

Therefore, it will take approximately ________ decades, or ________ years, for \$1 in this investment to be worth \$26.

See pages 340–341 of *Pre-Calculus 12* for more examples.

Check Your Understanding

Practise

1. State whether each of the following is an exponential function. Justify your answers.

   a) $y = x^5$

   b) $y = 0.1^x$

   c) $y = 12^x$

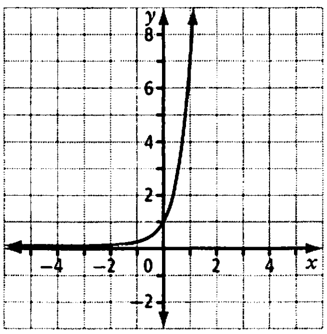
   d) $y = \sqrt[3]{x}$

   e) $y = x^{0.5}$

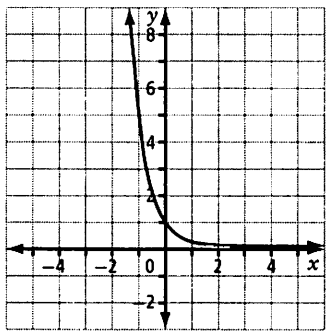
2. Match each exponential function to its graph.

   a) $y = 5^x$ **b)** $y = 7^x$ **c)** $y = \left(\frac{3}{4}\right)^x$ **d)** $y = 0.2^x$

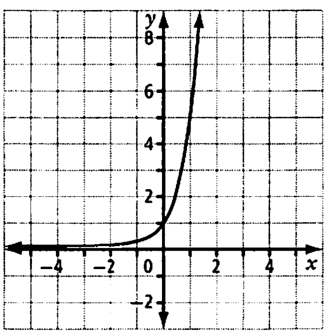
A



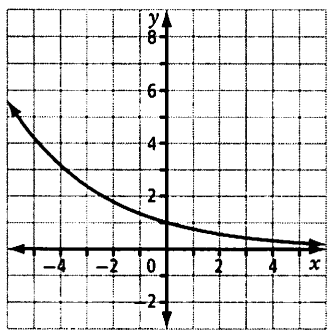
B



C

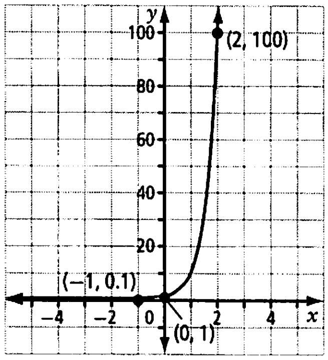


D



3. Write the equation of each exponential function graphed below.

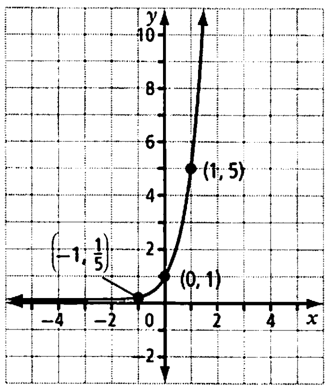
a)



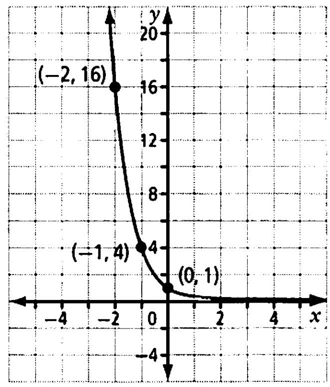
The point (0, 1) does not help determine the equation because ______________________________.

However, since you know that ________^{2} = 100, you can conclude that the base of the exponential function is ________. Thus, the equation is ______________.

b)

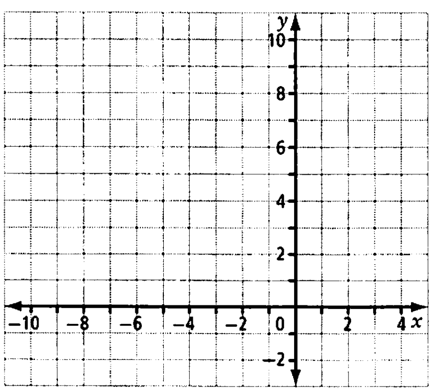


c)

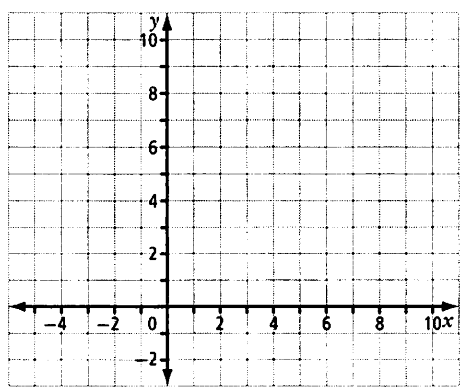


4. Sketch the graph of each exponential function. Identify the domain and range, the y-intercept, whether the graph is increasing or decreasing, and the equation of the horizontal asymptote.

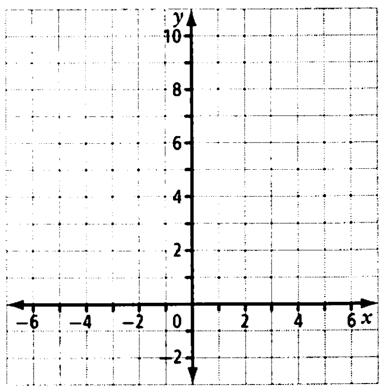
a) $f(x) = 8^x$



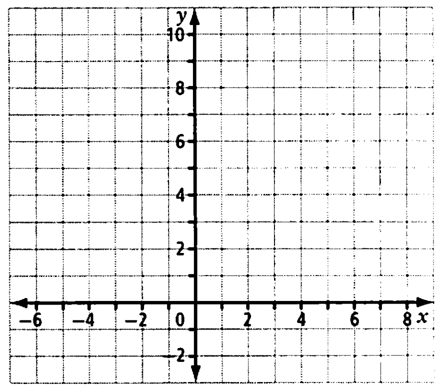
b) $f(x) = 0.5^x$



c) $g(x) = \left(\frac{2}{3}\right)^x$



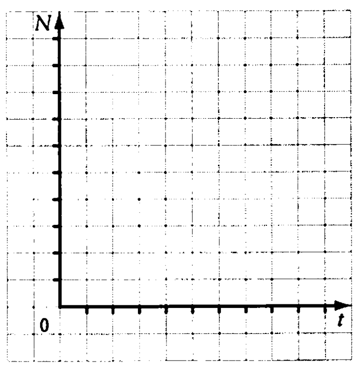
d) $g(x) = \left(\frac{3}{2}\right)^x$



Apply

5. The number of transistors on a computer chip, N, doubles approximately every two years. If originally there is one transistor on a chip, then this can be modelled by the function $N = 2^t$, where t is the number of two-year periods that have passed.

   a) Graph the function. Is the function increasing or decreasing?



   b) What are the domain and range of the function?

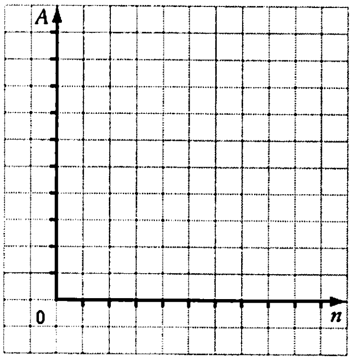
   c) How many transistors are on a chip after 2 years? 10 years? 20 years?

How many 2-year periods are there in 10 years? 20 years? How might you use a table of values to solve this question?

6. A filter removes 75% of the impurities of water that is passed through it. Multiple filters can be used together to increase the purity of the water. This situation is represented by $A = 0.25^n$, where A is the proportion of impurities remaining and n is the number of filters used.

   a) Explain why the base of the exponential function is 0.25.

   b) Graph the exponential function.



> What are the benefits and drawbacks of using a graph? What other methods could you use? What are the benefits and drawbacks of using an equation or table of values?

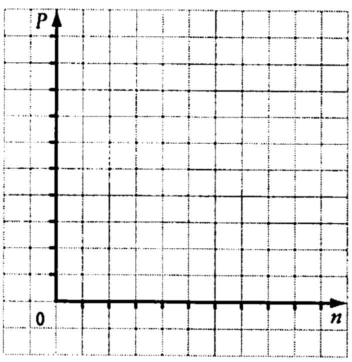
   c) What are the domain and range?

   d) What proportion of impurities remains when four filters are used, to the nearest ten thousandth?

7. The population of a particular insect decreases by half each night that the temperature drops below freezing.

   a) Write an exponential function to model the proportion of the population remaining, P, after n nights of freezing temperatures.

   b) Graph the function.



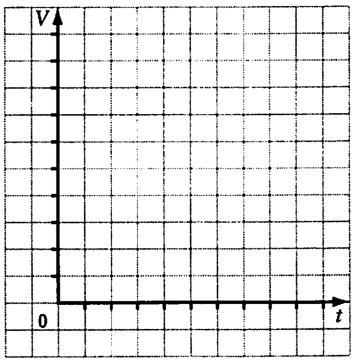
c) What percent of the population is remaining after five freezing nights, to the nearest hundredth of a percent? After eight freezing nights?

d) Will the population ever reach zero? Explain.

8. One dollar is invested at 4.5% interest compounded annually.

a) Write an exponential function to represent the value of the investment, V, after t years.

b) Graph the function from part a).



c) Determine the value of the investment after 15 years.

d) Use the graph to determine the time needed for the value of the investment to reach $3. Round your answer to the nearest year.

Connect

9. Consider the exponential function $y = c^x$, where $c > 0$.

a) Explain why the domain and range of all such exponential functions are the same.

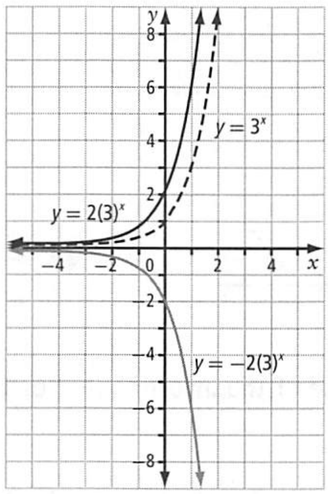
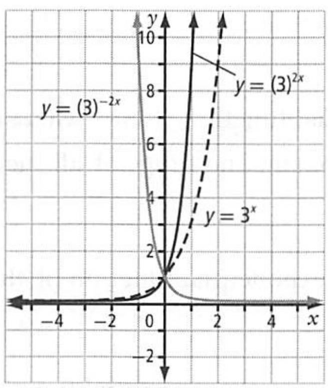
b) Explain why every exponential function of this form has the same y-intercept, 1.

c) State one characteristic of exponential functions that depends on the value of c.

7.2 Transformations of Exponential Functions

KEY IDEAS

- You can use transformed exponential functions to model real-world applications of exponential growth or decay.
- To graph an exponential function of the form $y = a(c)^{b(x - h)} + k$, apply transformations to the base function, $y = c^x$, where $c > 0$. Each of the parameters, a, b, h, and k, is associated with a particular transformation.

Parameter	Transformation	Example
a	• vertical stretch about the x-axis by a factor of $\lvert a \rvert$ • $a < 0$ results in a reflection in the x-axis • $(x, y) \rightarrow (x, ay)$	For $a = 2$, the equation of the transformed base function is $y = 2(3)^x$. 
b	• horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ • $b < 0$ results in a reflection in the y-axis • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	For $b = 2$, the equation of the transformed base function is $y = (3)^{2x}$. 

<table>
<tr><th>Parameter</th><th>Transformation</th><th>Example</th></tr>
<tr><td>h</td><td>

- horizontal translation left or right, depending on the sign: $+h$ shifts the graph left, and $-h$ shifts the graph right
- $(x, y) \rightarrow (x + h, y)$

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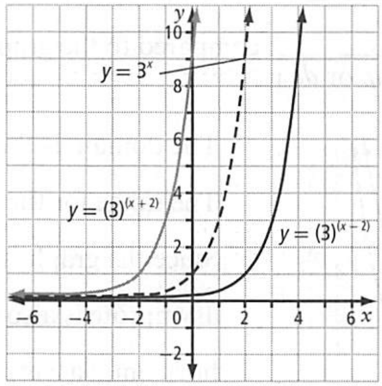
For $h = \pm 2$, the equation of the transformed base function is $y = (3)^{(x \pm 2)}$.



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<tr><td>k</td><td>

- vertical translation up or down, depending on the sign: $+k$ shifts the graph up, and $-k$ shifts the graph down
- $(x, y) \rightarrow (x, y + k)$

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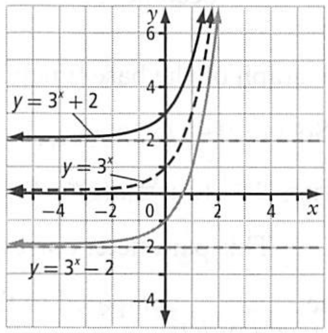
For $k = \pm 2$, the equation of the transformed base function is $y = (3)^x \pm 2$.



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- When applying transformations, you must apply parameters a and b before parameters h and k.

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Working Example 1: Translations of Exponential Functions

Consider the exponential function $y = 2^x$. For each of the following transformed functions,

- state the parameter and describe the transformation
- graph the base function and the transformed function on the same grid
- describe any changes to the domain, range, intercepts, and equation of the horizontal asymptote
- explain the effect of the transformation on an arbitrary point, (x, y), on the graph of the base function

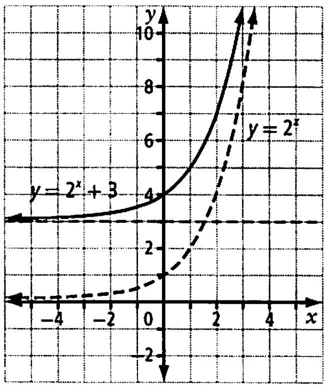
a) $y = 2^x + 3$

b) $y = 2^{x-5}$

c) $y - 4 = 2^{x+1}$

Solution

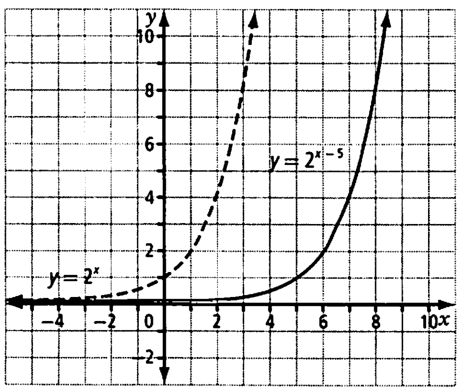
a) Compare the function $y = 2^x + 3$ to $y = a(c)^{b(x-h)} + k$ to determine the value of the parameter: $k =$ ______. This transformation indicates a ____________ translation of ______ units ______ compared to the graph of $y = 2^x$.
(*up* or *down*)



The domain is the same for both graphs: ____________.
The range for the transformed graph is $\{y \mid y > 3, y \in \mathrm{R}\}$. Since the graph is translated up, the ____________ asymptote is also translated up. Thus, the equation of the horizontal asymptote for $y = 2^x + 3$ is ______. Similarly, the y-intercept of the transformed graph is translated up to become ______.

Since each point on the graph of the base function moves up three units, each y-coordinate increases by ______. So, $(x, y) \rightarrow (x, y +$ ___$)$.

b) Compare the function $y = 2^{x-5}$ to $y = a(c)^{b(x-h)} + k$ to determine the value of the parameter: $h =$ ______. This parameter corresponds to a ____________ translation of ______ units to the ______ compared to the graph of $y = 2^x$.
(*right* or *left*)

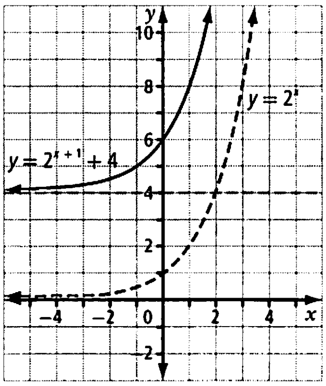


The domain is $\{x \mid x \in \mathrm{R}\}$ for both graphs. The range, which is unchanged for the transformed graph, is ____________. A horizontal translation has no effect on the horizontal asymptote. The y-intercept of $y = 2^{x-5}$ corresponds to the point $\left(5, \frac{1}{32}\right)$ on $y = 2^x$, so the y-intercept of the transformed function is $\frac{1}{32}$.

Since each point on the graph of the base function moves right five units, each x-coordinate moves ______ by ______ units. So, $(x, y) \rightarrow ($______, ______$)$.
(*right* or *left*)

c) Begin by writing $y - 4 = 2^{x+1}$ in the form $y =$ ____________. The parameters in the transformed function are

- $h =$ ______, which corresponds to a ____________ translation by ______ units to the ______.
  (*right* or *left*)
- $k =$ ______, which corresponds to a ____________ translation by ______ units ______.
  (*up or down*)



The domain is ____________ for both graphs. The range of the transformed graph is ____________. The equation of the horizontal asymptote for the transformed function is ______. You can determine the y-intercept of the transformed function, $y = 2^{x+1} + 4$, by substituting $x =$ ______ into the equation to calculate the new y-intercept: ____________.

For this transformation, each point $(x, y) \rightarrow$ (______, ______).

Working Example 2: Stretches of Exponential Functions

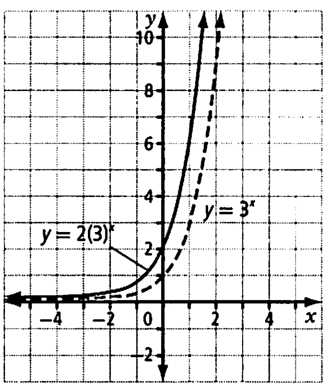
Consider the exponential function $y = 3^x$. For each of the following transformations,

- state the parameter and describe the corresponding transformation
- graph the base function and the transformed function on the same grid
- describe any changes to the domain, range, intercepts, and equation of the horizontal asymptote
- show what happens to an arbitrary point, (x, y), on the graph of the base function

a) $y = 2(3)^x$ **b)** $y = 3^{2x}$

Solution

a) Compare the function $y = 2(3)^x$ with $y = a(c)^{b(x-h)} + k$ to determine the value of the parameter:

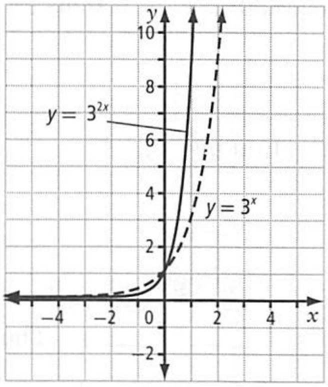


$a =$ ______, which corresponds to a ____________ stretch by a factor of ______.

The domain, range, and equation of the horizontal asymptote are unchanged under this transformation. However, the y-intercept of the transformed graph is ____________ because the y-coordinates of every point on the base function are multiplied by 2.

For this transformation, $(x, y) \rightarrow$ (______, ______).

b) Compare the function $y = 3^{2x}$ with $y = a(c)^{b(x - h)} + k$ to determine the value of the parameter:



$b =$ ________, which corresponds to a ________________ stretch by a factor of ________.

The domain and range are unchanged. The y-intercept is the invariant point of the transformation, so it remains the same. A horizontal stretch has no effect on the horizontal asymptote.

For this transformation, $(x, y) \rightarrow$ (________, ________).

Working Example 3: Combining Transformations of Exponential Functions

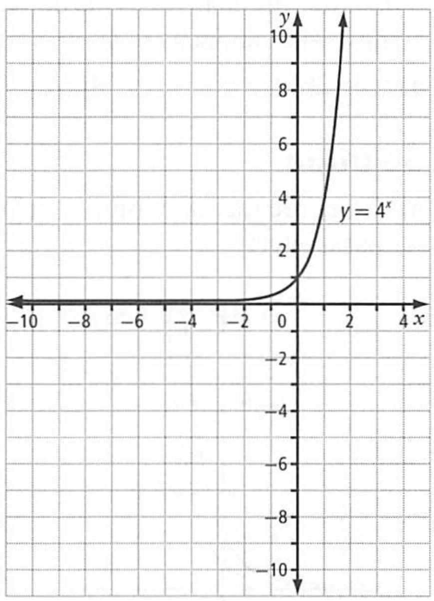
Describe the transformations of $y = 4^x$ caused by $y = -4^{\frac{1}{3}(x + 2)} - 1$. Then, graph the base function and transformed function on the same axes.

Solution

Compare the function $y = -4^{\frac{1}{3}(x + 2)} - 1$ to $y = a(c)^{b(x - h)} + k$ to determine the values of the parameters.

- $a =$ ______, which corresponds to a ________________ in the ________-axis.
- $b =$ ______, which corresponds to a ________________ stretch by a factor of ________.
- $h =$ ______, which corresponds to a ________________ translation of ________ units ________.
- $k =$ ______, which corresponds to a ________________ translation of ________ units ________.

The figure on the right shows the graph of the base function. Sketch the transformed function and the horizontal asymptote on the same grid as the base function.When combining transformations, recall that you must first perform stretches and reflections, in any order, and then translations.



See pages 349–351 of *Pre-Calculus 12* for an example that explores transformations.

Working Example 4: Use Transformations of an Exponential Function to Model a Situation

The real estate board in a city announces that the current average price of a house in the city is \$400 000. It predicts that average prices will double every 15 years.

a) Write a transformed exponential function in the form $y = a(c)^{b(x-h)} + k$ to model this situation. Justify your answer.

b) Describe how each of the parameters in the transformed function relates to the information provided.

c) Use technology to graph the function. Use the graph to predict the value of a house after 10 years.

Solution

a) Since the average price of a house doubles over a certain time interval, the base function is $P(t) = 2^t$, where P is the price of the house and t is the time. The time is in intervals of ________ years, so t can be replaced by the rational exponent $\frac{r}{\square}$, where r represents the number of years. Therefore the function becomes $P(r) =$ ______________.

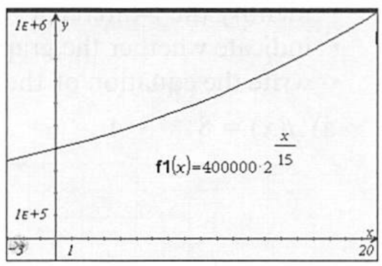
The current price of a home is ________, so the P-intercept is (________, ________).

This means that there must be a vertical stretch by a factor of ________. Therefore, the transformed function that models the price of the house is $P(r) =$ ______________.

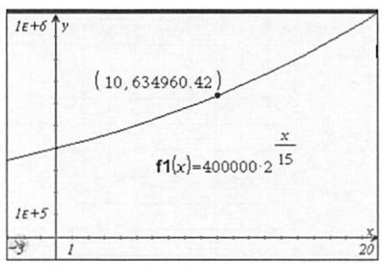
b) Based on the function $y = a(c)^{b(x-h)} + k$, the parameters of the function are

- $b = \frac{\square}{\square}$, representing ______________________
- $a =$ ________, representing ______________________

c) The graph represents the change in value of a house in this city.



Determine the point on the curve corresponding to $x = 10$.



When $r = 10$, $y =$ ______________. So, in 10 years, the average value of a house in this city will be ______________.

See pages 352–353 of *Pre-Calculus 12* for more examples.

Check Your Understanding

Practise

1. State whether each function shows a vertical translation of $y = 5^x$.
   a) $y = 5^{x-2}$
   b) $y = 5^x - 2$
   c) $y = 2(5)^x$
   d) $y = 5^{3x}$

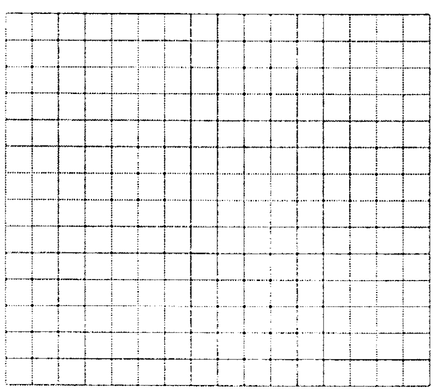
2. State whether each function shows a horizontal stretch of $y = 5^x$.
   a) $y = 5^{x-2}$
   b) $y = 5^x - 2$
   c) $y = 2(5)^x$
   d) $y = 5^{3x}$

3. State whether each function shows a reflection in the y-axis of $y = 5^x$.
   a) $y = 5^{x-2}$
   b) $y = -5^x - 2$
   c) $y = 2(5)^{-x}$
   d) $y = 5^{\frac{x}{3}}$

4. Identify all transformations for each function.
   a) $y = 4^{2(x-5)} - 6$
   b) $y = \frac{2}{3}\left(\frac{1}{2}\right)^{-x} + 9$
   c) $y = -2(1.06)^{\frac{1}{4}x}$
   d) $y = 500\left(\frac{5}{2}\right)^{2x+6} - 8$

5. Sketch the graph of each exponential function without using technology. For each function,
   - state the domain and range
   - identify the y-intercept
   - indicate whether the graph is increasing or decreasing
   - write the equation of the horizontal asymptote

   a) $f(x) = 8^{x-2} + 4$



b) $g(x) = -2^x + 3$

c) $f(x) = 0.5(3)^{x+2} - 5$

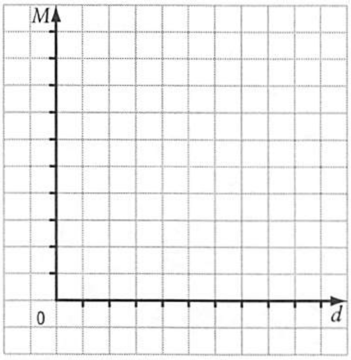
Apply

6. Iodine-131 has a half-life of 8 days. This means that after 8 days, half of the original mass of the isotope will have decayed. Suppose a sample of iodine-131 has a mass of 250 grams.

a) Write an exponential equation that models the amount, M, of iodine-131 remaining after d days. State the transformations that are represented by your function.

The equation is $M(d) =$ ________ $\left(\frac{1}{2}\right)^{\frac{d}{\square}}$. This represents a ____________________ (*horizontal* or *vertical*) stretch about the ________-axis by a factor of ________, and a horizontal stretch about the ________-axis by a factor of ________.

b) Graph the exponential equation you wrote. State the domain, range, equation of the horizontal asymptote, and M-intercept of the graph.



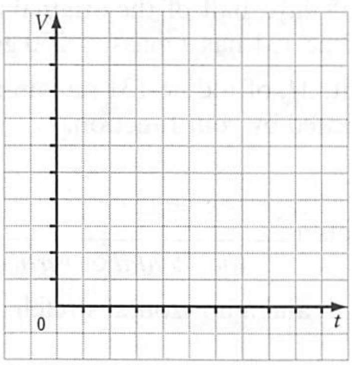
c) What mass of iodine-131 remains after 3 days? Explain how you arrived at your answer.

Question 9 on page 356 of *Pre-Calculus 12* is related to this question.

7. Lyndsay bought a classic car 15 years ago for $12 500. The car has tripled in value in that time.

a) Write an exponential function that models the value, V, of Lyndsay's car after t years. State the transformations represented by the function.

b) Graph the exponential function. What are the domain and range?

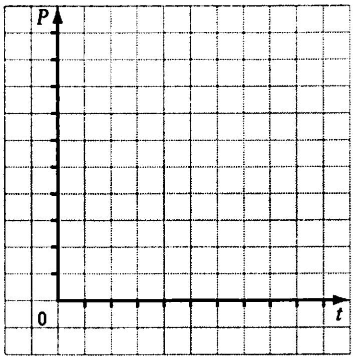


c) What might Lyndsay expect her car to be worth in 40 years? What assumption are you making?

d) Use your graph to approximate when the car will be worth $50 000.

8. The population of a town is decreasing by 2% per year. The current population of the town is 11 568.

a) Write an exponential function to model the population remaining, P, after t years. Graph the function.



b) State the transformations in your exponential function.

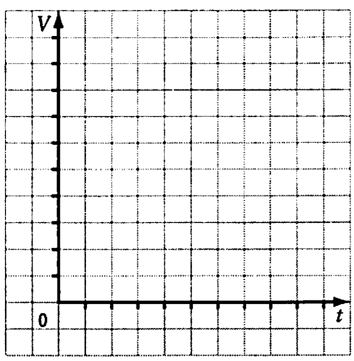
c) Use your equation to determine the population remaining after 5 years.

d) Use the graph to approximate the population of the town after 20 years.

e) According to your equation, will the population ever reach zero? Is this reasonable? Explain.

9. The exponential function $V = 500(1.0225)^{2t}$ models the value of \$500, after t years, that is invested at 4.5% compounded semi-annually.

a) State the transformations represented by the function. Graph the function.



b) State the V-intercept of the function. Explain what this point represents in this situation.

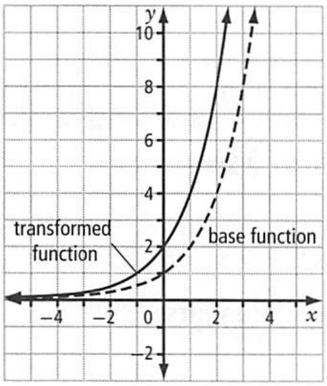
c) Determine the value of the investment after 15 years.

d) Use the graph to determine the time needed for the value of the investment to reach $1000.

Completing #7 to #9 should help you complete #10 to #12 on page 356 of *Pre-Calculus 12*.

Connect

10. Two students are discussing the exponential functions graphed below.



a) David says that he believes that the functions graphed are $y = 2^x$ and $y = 2(2)^x$. Jodi believes that the functions graphed are $y = 2^x$ and $y = 2^{x+1}$. Who is correct? Explain.

b) Show algebraically why $y = 2^{x+1}$ and $y = 2(2)^x$ have the same graph.

c) Write two other transformations that look different but have the same graph. Show algebraically that the two transformations are equivalent.

See pages 354–357 of *Pre-Calculus 12* for more questions.

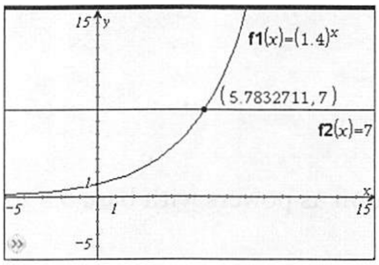
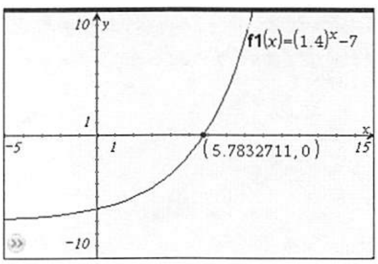
7.3 Solving Exponential Equations

KEY IDEAS

Strategies for Solving Exponential Equations With a Common Base

Description	Example	
For equations that begin with terms on both sides of the equal sign that have the same base ...	$11^{(2x + 6)} = 11^2$ $2x + 6 = 2$ $2x = -4$ $x = -2$	Bases are the same. Equate the exponents. Solve for x.
For equations that begin with terms on each side of the equal sign that have different bases, but that can be rewritten as the same base ...	$9^{(x + 3)} = 81^{(2x + 9)}$ $3^{2(x + 3)} = 3^{4(2x + 9)}$ $2(x + 3) = 4(2x + 9)$ $2x + 6 = 8x + 36$ $-6x = 30$ $x = -5$	 Rewrite terms so they have the same base. Equate the exponents. Solve for x.

Strategies for Solving Exponential Equations That Do Not Have a Common Base

Description	Example
Systematic trial	Consider the equation $7 = 1.4^x$. Guess 1: $x = 5$: $1.4^5 = 5.37824$ (less than 7) Guess 2: $x = 7$: $1.4^7 = 10.5413504$ (greater than 7) Guess 3: $x = 6$: $1.4^6 = 7.529536$ (approximately 7) So, x is approximately 6.
Graphing • Method 1: Point of Intersection Graph $y = 7$ and $y = 1.4^x$ on the same axes, and find the point of intersection. • Method 2: x-Intercept Graph $y = 1.4^x - 7$, and determine the x-intercept.	Consider the equation $7 = 1.4^x$.   So, $x = 5.7832711$.

Working Example 1: Rewriting Powers With a Specified Base

Rewrite each of the following with a base of 2.

a) 32 **b)** 16^3 **c)** $\sqrt[3]{16}$ **d)** $\left(\frac{1}{64}\right)^{\frac{1}{3}}$

Solution

a) $32 = 2^5$

b) Since $16 = 2^4$, $16^3 = (\_\_\_\_\_\_\_)^3$. Thus, $16^3 = 2^{\square}$.

c) $\sqrt[3]{16} = (2^4)^{\square} = \_\_\_\_\_\_\_$

d) Since $64 = \_\_\_\_\_\_\_$, $\frac{1}{64} = 2^{\square}$. So $\left(\frac{1}{64}\right)^{\frac{1}{3}} = (\_\_\_\_\_\_\_)^{\frac{1}{3}} = 2^{-2}$.

Working Example 2: Solving Exponential Equations by Changing Bases

Solve each equation.

a) $10^{x+4} = 1000^{x-4}$ **b)** $25^{2x} = 125^{x-1}$

Solution

a) Express both sides of the equation as powers with base 10.

$10^{x+4} = 1000^{x-4}$

$10^{x+4} = (10^3)^{x-4}$

$10^{x+4} = 10^{3x-12}$

Equate the powers:

$x + 4 = 3x - 12$

$\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_$

$x = \_\_\_\_\_\_\_$

b) Express both sides of the equation as powers with base 5.

$25^{2x} = 125^{x-1}$

$\left(5^{\square}\right)^{2x} = \left(5^{\square}\right)^{x-1}$

$5^{\square} = 5^{\square}$

$\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_$

Equate the powers:

$4x = \_\_\_\_\_\_\_$

$x = \_\_\_\_\_\_\_$

See pages 360–361 of *Pre-Calculus 12* for similar examples.

Working Example 3: Solving Exponential Equations With Different Bases

Solve for x. Express your answer to the nearest tenth.

$7^x = 3579$

Solution

3579 is not an integer power of 7, so you cannot use the same techniques as in Working Examples 1 and 2. Both sides cannot be expressed with the same base. Instead, you can either use systematic trial or technology. When using technology, you can determine x by graphing one or two functions.

Method 1: Systematic Trial

Use systematic trial to find the approximate value of x that satisfies the equation.

Try $x = 4$.	Try $x = 5$.
$7^4 = 2401$ (less than 3570)	$7^5 = 16\,807$ (greater than 3570)

The solution is between 4 and 5, but is obviously closer to 4 than 5.

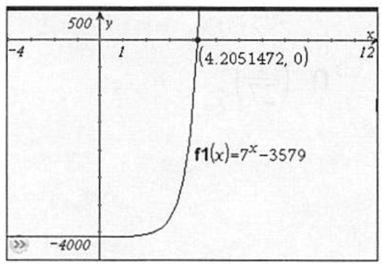
Try $x = 4.2$.	Try $x = 4.3$.
$7^{4.2} \approx 3543$ (less than 3579)	$7^{4.3} \approx 4304$ (greater than 3579)

The solution is between $x = 4.2$ and 4.3, and probably much closer to $x = 4.2$. You could continue this process to get an even closer approximation, but this is probably close enough.

Method 2: Using Technology, Find the x-Intercept of a Single Function

Use your graphing calculator to graph the function $f(x) = 7^x - 3579$. Find the x-intercept of the graph:

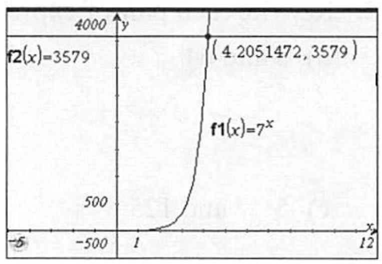
$x =$ ________.



Method 3: Using Technology, Find the Point of Intersection of the Graphs of Two Functions

Graph the functions $f(x) = 7^x$ and $g(x) = 3579$ in the same calculator window. Use the trace or intersection feature of the graphing calculator to determine the coordinates of the point of intersection:

$x =$ ________.



Which method do you prefer?

See pages 362–363 of *Pre-Calculus* 12 for more examples.

Check Your Understanding

Practise

1. Express each of the following with base 3.

a) 81

b) 27^5

c) $3\sqrt{3}$

d) $\sqrt[3]{243}$

e) $9\sqrt[3]{81^2}$

Work with 9 and $\sqrt[3]{81^2}$ separately.

$9 = 3^{\square}$

$\sqrt[3]{81^2} = \left(\left(3^{\square}\right)^{\square}\right)^{\square}$

$= \_\_\_\_\_\_$

Thus, $9\sqrt[3]{81^2} = 3^{\square}3^{\square}$.

$= \_\_\_\_\_\_$

f) $\left(\frac{1}{27}\right)^2$

g) $\left(\frac{\sqrt{3}}{81}\right)^{-3}$

2. Rewrite each pair of expressions to have the same base.

a) 8 and 64

b) 3^2 and 9^3

c) 5^{x+6} and 125

d) 2^{3x} and 8^{2x+4}

e) 27^{5x+4} and $\left(\frac{1}{9}\right)^{x+3}$

f) $\left(\frac{1}{4}\right)^{x+7}$ and 8^{-3x}

3. Solve using systematic trial. Round answers to one decimal place. Check your answers using technology.

a) $175 = 5^x$

b) $12 = 2.1^x$

4. Solve the following.

a) $4^{2x} = 4096$

b) $2^{3x-5} = 128$

c) $6^{x+3} = \frac{1}{216}$

d) $10^{5x+6} = 0.0001$

5. Solve the following.

a) $64^{4x} = 16^{x+5}$

b) $9^{x-7} = 27^{2x-9}$

c) $125^{6x+2} = 25^{8x+1}$

d) $8^{x+2} = \left(\frac{1}{4}\right)^{x+3}$

e) $5(3)^x = 135$

_______ = _______ Divide each side by 5.

3^x = _______ Express as base 3.

_______ = _______ Equate powers and solve.

6. Solve each of the following graphically, using technology. Sketch a diagram of the graph. Where necessary, round answers to the nearest hundredth. Use both methods described in Working Example 3 at least once.

a) $5^x = 32$

b) $10^{2x} = 439$

c) $25\left(\frac{1}{2}\right)^{4x} = 5$

d) $200(1.05)^{12x} = 1250$

Apply

7. A type of bacterium doubles each hour.

a) If there are 4 bacteria in a sample, write a exponential function that models the sample's growth over time.

b) Use your equation to determine the time it takes for the sample to become 4096 bacteria.

8. A painting doubles in value every 8 years. It is currently worth $1000.

a) Write an exponential function that models the value of the painting.

b) Use your equation to determine the time needed for the painting to be worth $3200.

9. The student council of a school notices that their membership is growing by 3% per year.

   a) The membership is currently 350 students. Write an exponential function to model the size of the student council.

   b) Use your equation to determine the time needed until the student council has 560 members. Round your answer to the nearest whole number.

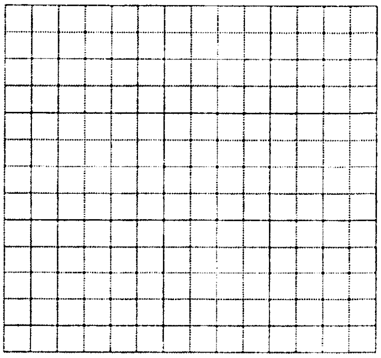
Connect

10. Keegan invests $1000 at 3.75% compounded annually.

   a) Write an exponential function to model the growth of Keegan's investment.

   b) Describe the two methods that you have learned that you can use to find an approximate solution to this equation.

   c) Use your preferred method to determine the time needed, to the nearest year, that it would take for Keegan's investment to be worth $2500. If you choose a graphical method, sketch your graph.



   d) Why are you unable to solve your equation algebraically more precisely? Describe the difficulty that arises. (You will learn to overcome this difficulty in the Chapter 8.)

Chapter 7 Review

7.1 Characteristics of Exponential Functions, pages 229–237

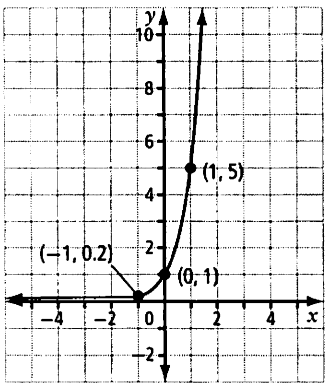
1. For each exponential function, state the domain, range, y-intercept, horizontal asymptote, and whether a graph of the function would be increasing or decreasing. Verify your answers by using technology to graph the functions.

   a) $y = 4.5^x$

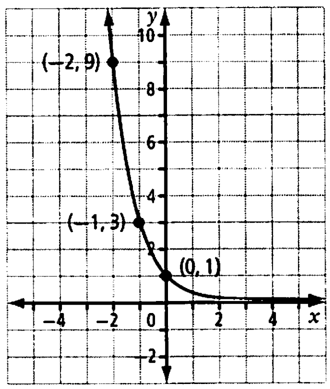
   b) $y = \left(\frac{2}{3}\right)^x$

2. State the exponential function represented by each graph.

   a)



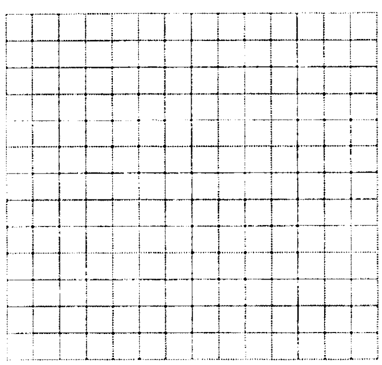
   b)



3. A photographer uses filters to change the look of the photographs she takes. A particular filter reduces the light transmitted through it by 15%. The photographer may use more than one of this type of filter at a time.

   a) Write an exponential function that shows the proportion of light, P, passing through n filters.

   b) Graph your function and use the graph to estimate the number of filters needed to reduce the light transmitted to half the original intensity.



7.2 Transformations of Exponential Functions, pages 238–248

4. Identify all the transformations in each exponential function below.

   a) $y = 2(3)^x - 3$

   b) $y = 5^{x+3}$

   c) $y = 10^{2x-8} + 1$

   d) $y = 5(8)^{6x+12}$

5. Write the equation for each of the following transformations to the function $y = 4^x$. Then, state the domain and range of the transformed function.

   a) vertically stretched by a factor of $\frac{1}{2}$, translated 2 units left and 6 units down

   b) horizontally stretched by a factor of $\frac{1}{3}$, vertically stretched by a factor of 5

   c) horizontally stretched by a factor of 2, translated 3 units right and 1 unit down

7.3 Solving Exponential Equations, pages 249–255

6. Solve each of the following equations algebraically. Use graphing technology to check your answer.

   a) $5^{x+2} = 3125$

   b) $2^{3x-2} = 16^x$

   c) $\left(\frac{1}{9}\right)^{x-6} = 27^{2x-1}$

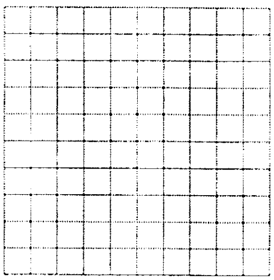
   d) $(\sqrt{3})^x = 9^{2x+5}$

7. The half-life of a radioactive substance is 4 days.

   a) Write an exponential function that models the proportion, P, of the substance remaining after t days.

   b) Use your function to determine the time that must pass until there is 25% of the substance remaining.

   c) Use graphing technology to determine the time that must pass until there is 7% of the substance remaining. State your answer to the nearest tenth. Sketch the graph.



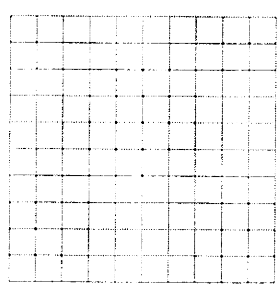
8. The number of bacteria in a sample doubles every 10 h. Initially, there are 64 colonies present.

   a) Write an exponential function that models the number of bacteria colonies, N, present after t hours.

   b) Use your function to determine the number of colonies present after 24 h.

   c) Determine the time that must pass until there are 1024 colonies present.

   d) Use graphing technology to determine the time, correct to the nearest hour, that passes before 1500 colonies are present. Sketch the graph.



Chapter 7 Skills Organizer

Complete the organizer to review the concepts you have learned in this chapter.

Graphs and Transformations

$y = c^x$
domain: $x \in R$
range: $y > 0$
y-intercept: ______
horizontal asymptote: ______

$y = a(c)^x$
transformation: ______
domain: ______
range: ______

$y = c^{x-h}$
transformation: ______
domain: ______
range: ______

$y = c^{bx}$
transformation: ______
domain: ______
range: ______

$y = c^x + k$
transformation: ______
domain: ______
range: ______

Solving Exponential Equations

If the bases are the same,

If the powers can be rewritten to have the same base,

If the powers cannot be written with the same base,

Chapter 8 Logarithmic Functions

8.1 Understanding Logarithms

KEY IDEAS

- A logarithm is the exponent to which a fixed base must be raised to obtain a specific value.

  Example: $5^3 = 125$. The logarithm of 125 is the exponent that must be applied to base 5 to obtain 125. In this example, the logarithm is 3: $\log_5 125 = 3$.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	**Logarithmic Form**
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$.
- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in \mathrm{R}\}$
  - the range is $\{y \mid y \in \mathrm{R}\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms: $\log_{10} x = \log x$

Working Example 1: Graph the Inverse of an Exponential Function

The graph of $y = 2^x$ is shown at right. State the inverse of the function. Then, sketch the graph of the inverse function and identify the following characteristics of the graph:

- domain and range
- x-intercept, if it exists
- y-intercept, if it exists
- the equation of any asymptotes