## Chapter 7 Skills Organizer

Complete the organizer to review the concepts you have learned in this chapter.

### Graphs and Transformations

$y = c^x$
domain: $x \in R$
range: $y > 0$
$y$-intercept: ______
horizontal asymptote: ______

$y = a(c)^x$
transformation: ______
domain: ______
range: ______

$y = c^{x - h}$
transformation: ______
domain: ______
range: ______

$y = c^{bx}$
transformation: ______
domain: ______
range: ______

$y = c^x + k$
transformation: ______
domain: ______
range: ______

### Solving Exponential Equations

If the bases are the same,

If the powers can be rewritten to have the same base,

If the powers cannot be written with the same base,

# Chapter 8 Logarithmic Functions

## 8.1 Understanding Logarithms

### KEY IDEAS

- A logarithm is the exponent to which a fixed base must be raised to obtain a specific value.

  Example: $5^3 = 125$. The logarithm of 125 is the exponent that must be applied to base 5 to obtain 125. In this example, the logarithm is 3: $\log_5 125 = 3$.

- Equations in exponential form can be written in logarithmic form and vice versa.

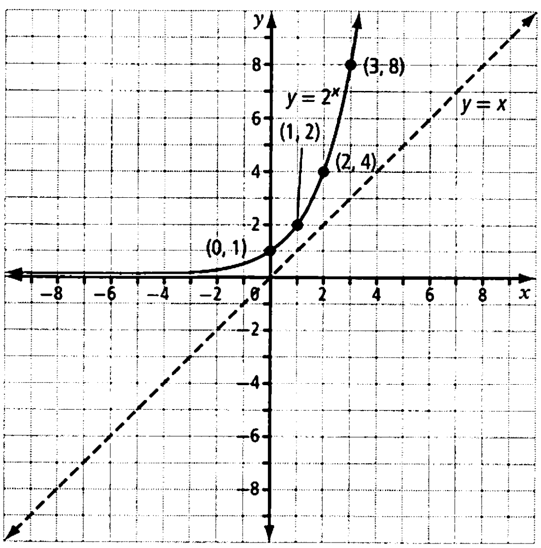
| **Exponential Form** | **Logarithmic Form** |
|---|---|
| $x = c^y$ | $y = \log_c x$ |

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$.
- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in R\}$
  - the range is $\{y \mid y \in R\}$
  - the $x$-intercept is 1
  - the vertical asymptote is $x = 0$, or the $y$-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms: $\log_{10} x = \log x$

## Working Example 1: Graph the Inverse of an Exponential Function

The graph of $y = 2^x$ is shown at right. State the inverse of the function. Then, sketch the graph of the inverse function and identify the following characteristics of the graph:

- domain and range
- $x$-intercept, if it exists
- $y$-intercept, if it exists
- the equation of any asymptotes



## Solution

The inverse of $y = 2^x$ is $x = 2^y$.

In logarithmic form, the inverse function is $y = \log_{\square} x$.

The inverse of a function is its reflection in the line $y = x$. To graph the inverse, sketch this reflection. Alternatively, interchange each pair of coordinates on the graph of the function to obtain coordinates on the inverse graph.

For example, the point (1, 0) is on the graph of $y = 2^x$. Therefore, the point ______ is on the graph of the inverse.

Add the graph of the inverse of the function to the grid on the previous page.

- The domain is ____________.
- The range is ____________.
- The graph has no $y$-intercept.
- The $x$-intercept is ______.
- The vertical asymptote is ______.

## Working Example 2: Change the Form of an Expression

For each expression in exponential form, rewrite it in logarithmic form. For each expression in logarithmic form, rewrite it in exponential form.

**a)** $3^4 = 81$

**b)** $36^{\frac{1}{2}} = 6$

**c)** $\log_5 125 = 3$

**d)** $\log 10\,000 = 4$

## Solution

**a)** The base is 3 and the exponent is 4. The logarithmic form is $\log_{\square} 81 =$ ______.

**b)** 36 is the base, so the logarithmic form is $\log_{36}$ ______ = ______.

**c)** The base is 5 and the exponent is 3. The exponential form is $5^3 =$ ______.

**d)** When a base is not stated, the base of a logarithm is 10. The exponential form of this expression is ______ = ______.

## Working Example 3: Evaluate and Determine a Value in Logarithmic Expressions

Evaluate each expression.

**a)** $\log_2 32$

**b)** $\log_{100} 10$

**c)** $\log_3 \sqrt{27}$

**d)** $\log_4 x = 3$

**e)** $\log_x 125 = 3$

**f)** $\log 0.01 = x$

## Solution

a) This logarithmic expression asks which exponent is applied to base 2 to produce a result of 32. Since $2^5 = 32$, the value of the logarithmic expression is ________.

b) Since $100^{\frac{1}{2}} = 10$, the value of $\log_{100} 10$ is ________.

c) Written as a power of 3, 27 is equal to ________ and $\sqrt{\ }$ is written as an exponent of ________. This means that $\log_3 \sqrt{27} = \frac{3}{2}$.

d) In exponential form, $\log_4 x = 3$ is equivalent to ________ = ________. So, $x =$ ________.

e) In exponential form, $\log_x 125 = 3$ is equivalent to ________ = ________. So, $x =$ ________.

f) 0.01 can be written as $\frac{1}{100}$, which is $10^{\square}$. So, $\log 0.01 =$ ________.

## Working Example 4: An Application of Logarithmic Functions

The intensity of sound is measured in decibels (dB). The level of a sound, $L$, in decibels, is given by $L = 10\log\left(\frac{I}{I_0}\right)$, where $I$ is the intensity of the sound and $I_0$ is the faintest sound detectable to humans. The sound level inside a particular car is 39 dB when it is idling, and 80 dB at full throttle. How many times more intense is the sound at full throttle?

## Solution

Let $I_i$ be the intensity of the sound at idle and $I_f$ be the intensity at full throttle.

$39 = 10\log\left(\frac{I_i}{I_0}\right)$ or $3.9 = \log\left(\frac{I_i}{I_0}\right)$ $\qquad$ $80 = 10\log\left(\frac{I_f}{I_0}\right)$ or $8 =$ ____________

Rewrite each expression in exponential form.

$\frac{I_i}{I_0} =$ ________ $\qquad$ $\frac{I_f}{I_0} =$ ________

Then, multiply by $I_0$ to obtain $I_i = I_0\,10^{3.9}$ and $I_f =$ ________.

To compare the intensities, divide $I_f$ by $I_i$.

$$\frac{I_f}{I_i} = \frac{I_0 10^{\square}}{I_0 10^{\square}}$$

$\frac{I_f}{I_i} =$ ________

The $I_0$ terms divide out, leaving $10^{\square} =$ ________.

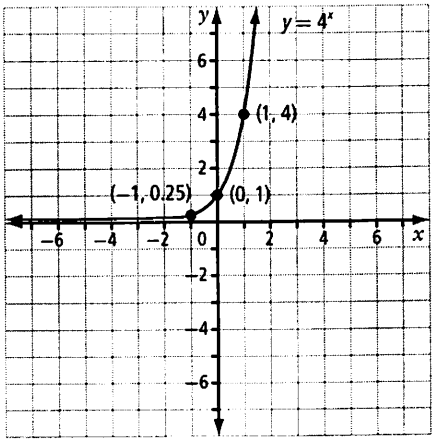
Therefore, the sound at full throttle is about 12 589 times as intense as the sound at idle.

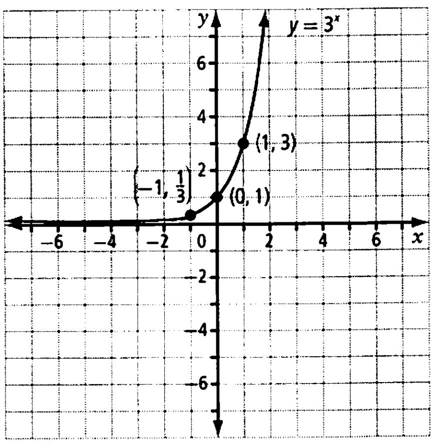
See pages 375–379 of *Pre-Calculus 12* for more examples.

## Check Your Understanding

### Practise

1. For each exponential graph below, sketch the inverse function and state the domain, range, $x$-intercept, and equation of the vertical asymptote. Then, write the equation of the inverse function.

a)



b)



2. Express in logarithmic form.

a) $3^5 = 243$

b) $10^4 = 10\ 000$

c) $16^{\frac{1}{2}} = 4$

d) $8^{-2} = \frac{1}{64}$

e) $10^{-2} = 0.01$

f) $27^{\frac{2}{3}} = 9$

g) $12^x = 2y$

h) $2^{2x-5} = y - 1$

**3.** Express in exponential form.

**a)** $\log_2 32 = 5$

**b)** $\log_8 512 = 3$

**c)** $\log_5 625 = 4$

**d)** $\log 1000 = 3$

**e)** $\log 0.0001 = -4$

**f)** $\log_{\frac{1}{2}} 8 = -3$

**g)** $\log_3 (x + 1) = y$

**h)** $\log_4 2x = y + 1$

**4.** Evaluate.

**a)** $\log_6 36$

**b)** $\log_2 64$

**c)** $\log 100$

**d)** $\log_3 \frac{1}{9}$

This expression asks for the exponent of 3 that gives a value of $\frac{1}{9}$.

**e)** $\log_5 \frac{1}{125}$

**f)** $\log_7 7$

**g)** $\log_5 5^4$

**h)** $\log_2 (8\sqrt{32})$

Writing as powers of 2,

$8 =$ ________ and $\sqrt{32} =$ ________.

By the exponent laws, $(8\sqrt{32}) = 2^{\square}$.

$\log_2 (8\sqrt{32}) =$ ________

**i)** $\log_{12} 1$

**j)** $\log_{25} 5$

Completing these will help you to complete #1–#4 on page 380 of *Pre-Calculus 12*.

5. Put the following in ascending order: $\log_6 400$, $\log_2 100$, $\log_{10} 300$.

## Apply

6. Determine the value of $x$ in each of the following.

a) $4^x = 64$

b) $10^{2x} = 1\,000\,000$

c) $\log_2 x = 4$

d) $\log_5 x = -2$

e) $\log_4 256 = x$

f) $\log_{16} 4 = x$

g) $\log_x 81 = 4$

h) $\log_x 6 = \frac{1}{2}$

i) $\log_x \frac{1}{25} = -2$

j) $\log_x \frac{1}{64} = -3$

7. Evaluate each expression.

a) $10^y$, where $y = \log_{10} 216$

b) $8^y$, where $y = \log_8 4$

c) $6^y$, where $y = \log_6 12$

d) $\log_2 2^7$

e) $\log_5 5^{-8}$

f) $\log_7 7^{10}$

8. The intensity of sound is measured in decibels (dB). The level of a sound, $L$, in decibels, is given by $L = 10\log\left(\frac{I}{I_0}\right)$, where $I$ is the intensity of the sound and $I_0$ is the faintest sound detectable to humans.

a) Determine the level of a sound that is 20 times more intense than $I_0$, correct to the nearest decibel.

b) The level of sound in a quiet bedroom at night might be 30 dB, while normal conversation has a sound level of about 60 dB. How many times more intense is normal conversation than the quiet room?

## Connect

9. Refer to the graph of the logarithmic function $y = \log_2 x$ in Working Example 1, page 260.

a) Predict how the graph of $y = \log_5 x$ will compare to the original graph.

b) Using technology, compare the graph of $y = 2^x$ to the graph of $y = 5^x$.

c) Use your work in part b) to justify or change your prediction in part a).

d) Generalize to explain how logarithmic graphs with different bases will compare.

10. Consider the logarithmic function $y = \log_c x$.

a) Explain why $c > 0$ for all such functions.

b) Explain why $c \neq 1$ for all such functions.

## 8.2 Transformations of Logarithmic Functions

### KEY IDEAS

- To represent real-life situations, you may need to transform the basic logarithmic function, $y = \log_b x$, by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters $a$, $b$, $h$, and $k$ in $y = a \log_c (b(x - h)) + k$ on the graph of the logarithmic function $y = \log_c x$ are described in the table.

| Parameter | Effect |
|---|---|
| $a$ | Vertically stretch by a factor of $\lvert a\rvert$ about the $x$-axis. Reflect in the $x$-axis if $a < 0$. |
| $b$ | Horizontally stretch by a factor of $\left\lvert\frac{1}{b}\right\rvert$ about the $y$-axis. Reflect in the $y$-axis if $b < 0$. |
| $h$ | Horizontally translate $h$ units. |
| $k$ | Vertically translate $k$ units. |

- Only parameter $h$ changes the vertical asymptote and the domain. None of the parameters changes the range.

### Working Example 1: Translations of a Logarithmic Function

**a)** Sketch the graph of $y = \log_4 (x + 4) - 5$.

**b)** State the
- domain and range
- $x$-intercept
- $y$-intercept
- equation of the asymptote

### Solution

**a)** Begin with the graph of $y = \log_4 x$. Identify key points, such as (1, 0), (4, 1), and (16, 2).

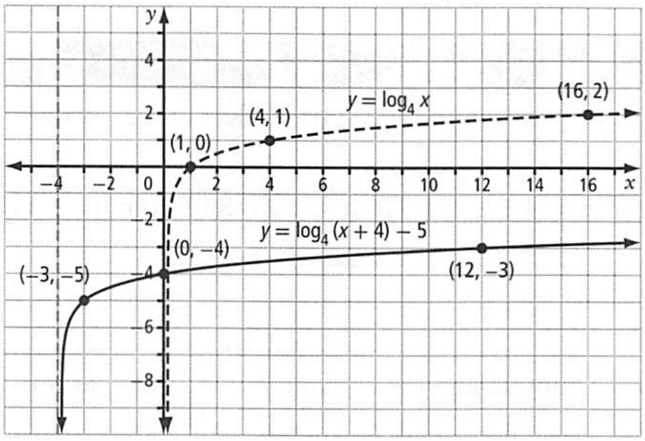
Identify the transformations.

The graph moves ________ units to the left and 4 units ________.

In mapping notation, the key points are transformed as follows:

Key points: $(x, y)$ maps to $(x - 4, y - 5)$

| $(x, y)$ | → | $(x - 4, y - 5)$ |
|---|---|---|
| (1, 0) | → | (−3, −5) |
| (4, 1) | → | (0, −4) |
| (16, 2) | → | (12, −3) |




**b)** The domain of $y = \log_4 x$ is ______________. Since the graph is translated 4 units left, this changes the domain to $\{x \mid x > -4, x \in R\}$. The range of the transformed function is not changed and is ______________.

To determine the $x$-intercept, set $\log_4 (x + 4) - 5 = 0$. Then, solve for $x$.

$\log_4 (x + 4) = 5$

$x + 4 =$ ________ (write in exponential form)

$x =$ ________

To confirm that the $y$-intercept is −4, substitute $x = 0$.

$y = \log_4 (\mathbf{0} + 4) - 5$

$y =$ ________ $- 5$

$y =$ ________

The graph of $y = \log_4 x$ has a vertical asymptote at $x = 0$. Since the graph is translated 4 units left, the asymptote is translated as well.

Thus, $y = \log_4 (x + 4) - 5$ has a vertical asymptote at ________.

## Working Example 2: Reflections and Stretches of Logarithmic Functions

Sketch the graph of each of the following. State any invariant points.

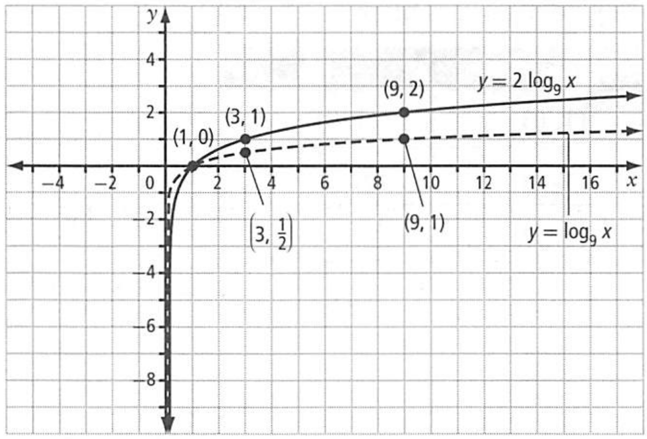
**a)** $y = 2 \log_9 x$ **b)** $y = -\log_2 4x$

## Solution

**a)** The coefficient of 2 indicates a vertical stretch by a factor of 2. Choose key points on the graph of $y = \log_9 x$. Then, use mapping notation to show the transformation of those points.

Key points: $(x, y)$ maps to $(x, 2y)$

| $(x, y)$ | → | $(x, 2y)$ |
|---|---|---|
| $(1, 0)$ | → | $(1, 0)$ |
| $\left(3, \frac{1}{2}\right)$ | → | $(3, 1)$ |
| $(9, 1)$ | → | $(9, 2)$ |

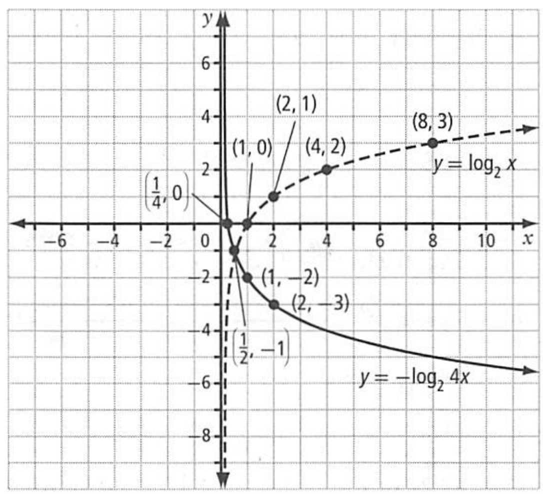



The invariant point is ________.

**b)** The transformations are a reflection in ______________ and a ______________ stretch by a factor of $\frac{1}{4}$. Using key points on $y = \log_2 x$, use mapping notation to express the transformations.

Key points: $(x, y)$ maps to $\left(\frac{1}{4}x, -y\right)$

| $(x, y)$ | → | $\left(\frac{1}{4}x, -y\right)$ |
|---|---|---|
| $(1, 0)$ | → | $\left(\frac{1}{4}, 0\right)$ |
| $(2, 1)$ | → | $\left(\frac{1}{2}, -1\right)$ |
| $(4, 2)$ | → | |
| $(8, 3)$ | → | |




There are no invariant points on the graph.

Why does this set of transformations have no invariant points?

## Working Example 3: Combine Transformations

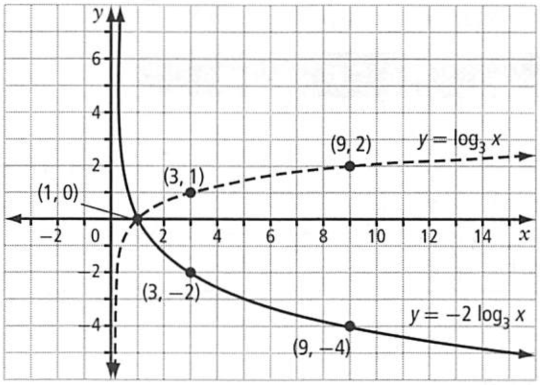
Sketch the graph of $y = -2 \log_3 (x - 3) + 5$.

## Solution

First, apply all stretches and reflections to $y = \log_3 x$, in any order. Then, apply translations. Consider the effect of a reflection in the $x$-axis and a vertical stretch by a factor of 2 on the key points (1, 0), (3, 1), and (9, 2).

Key points: $(x, y)$ maps to $(x, -2y)$

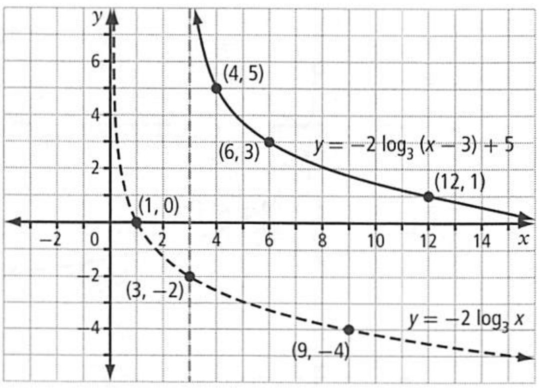
| $y = \log_3 x$ | $y = -2 \log_3 x$ |
|---|---|
| (1, 0) | (1, 0) |
| (3, 1) | (3, −2) |
| (9, 2) | |




Since the graph is translated 3 units right and 5 units up, apply these translations to the points.

Key points: $(x, y)$ maps to $(x + 3, y + 5)$

| $y = -2 \log_3 x$ | $y = -2 \log_3 (x - 3) + 5$ |
|---|---|
| (1, 0) | (4, 5) |
| (3, −2) | |
| (9, −4) | |




See pages 384–389 of *Pre-Calculus 12* for more examples.

## Check Your Understanding

### Practise

1. State the transformations, in order of application, to transform $y = \log_c x$ to each of the following.

   a) $y = \log_4 (x + 1) - 8$

   b) $y = 2 \log (4x)$

   c) $y = -\log_2 (3x)$

   d) $y = 5 \log_6 (-2(x + 4))$

2. Write the equations that correspond to the following transformations of $y = \log_5 x$.

   a) vertically stretched by a factor of 3 and translated 2 units to the right

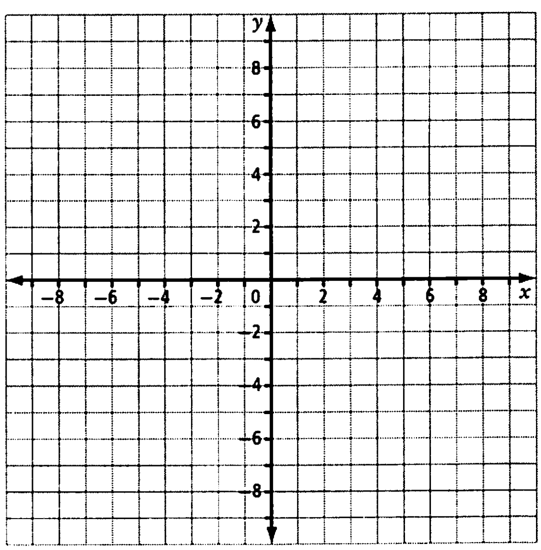
   b) reflected in the $x$-axis and translated 1 unit down and 4 units left

   c) vertically stretched by a factor of $\frac{1}{2}$ and horizontally stretched by a factor of $\frac{1}{2}$

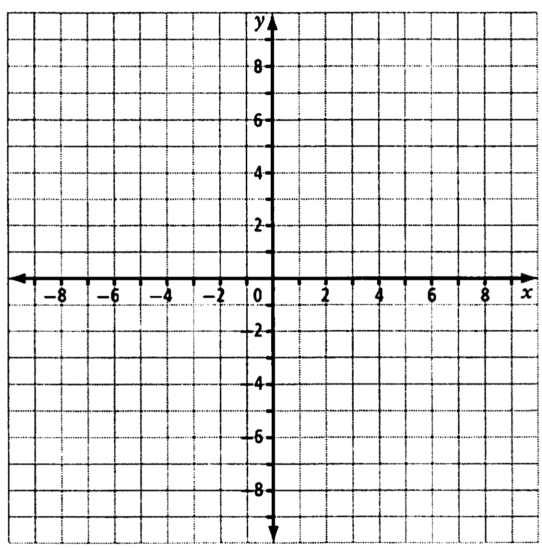
   d) vertically stretched by a factor of 4, reflected in the $y$-axis, and translated 2.5 units down

3. Sketch each of the following transformations of $y = \log_c x$.

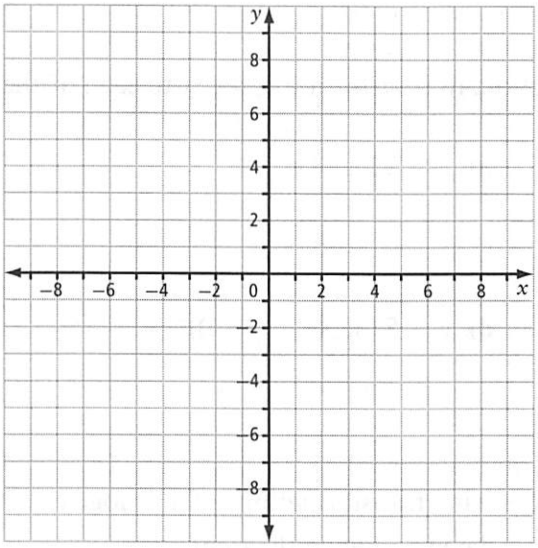
   a) $y = \log_2 (x + 2) - 3$



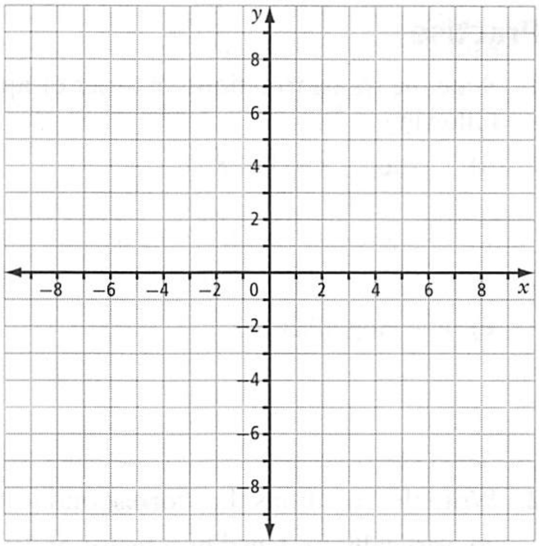

   b) $y = 2 \log_8 x + 4$



c) $y = -\log_5 (x - 1) - 2$




d) $y = \log (2x) + 5$




Completing these will help you complete #1–#4 on pages 389 and 390 of *Pre-Calculus 12*.

**4.** Describe how the graph of each logarithmic function could be obtained from the graph of its base function, $y = \log_c x$.

**a)** $y = \log_6 (2x + 6)$

The graph has been horizontally stretched by a factor of ________.

However, before determining the horizontal translation, factor 2 out of the argument of the logarithm.

> The *argument* of a function is the *input* value. For $y = f(x)$, the argument for the function, $f$, is $x$.

$y = \log_6 (2(\_\_\_\_\_\_\_\_))$

This shows that the graph is translated ________ units to the ________.

**b)** $y = \log_2 (3x - 12)$

**c)** $y = \log \left(\frac{1}{2}x - 3\right)$

**d)** $y = \log_3 \left(\frac{1}{3}x + 6\right)$

## Apply

5. The Shannon-Hartley theorem is used to determine the highest possible rate for transmitting information. The formula is $C = B\log_2(r + 1)$, where $r$ is the signal-to-noise ratio, $B$ is the bandwidth in hertz, and $C$ is the rate in bits per second.

   **a)** Describe the transformations in the theorem, compared to the graph of $C = \log_2 r$.

   **b)** If the bandwidth is 10 000 Hz and the signal-to-noise ratio is 31, determine the transmission rate.

6. For each of the following, state the domain, range, intercepts to the nearest tenth (if they exist), and equation of the vertical asymptote.

   **a)** $y = \log_5(x - 8) - 12$

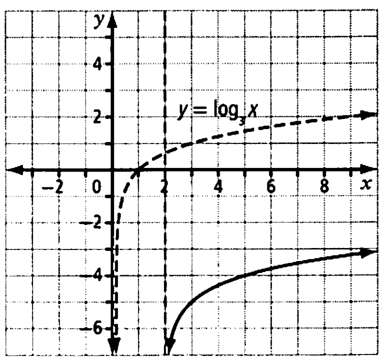
   **b)** $y = -3\log_9(4(x - 1)) + 2$

   **c)** $y = \frac{1}{2}\log_{12}(6x) - 4$

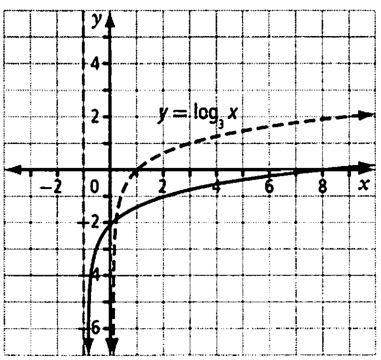
   **d)** $y = \log_2\left(\frac{1}{4}(x + 3)\right) - 5$

7. Each graph has been translated from $y = \log_3 x$. State the translation(s) in each case.

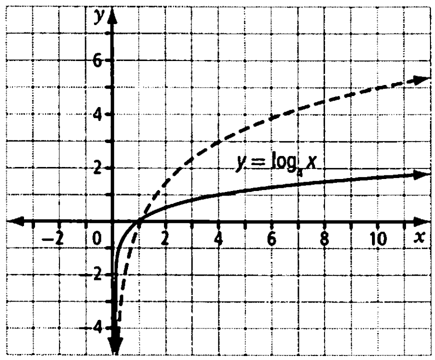
   **a)**

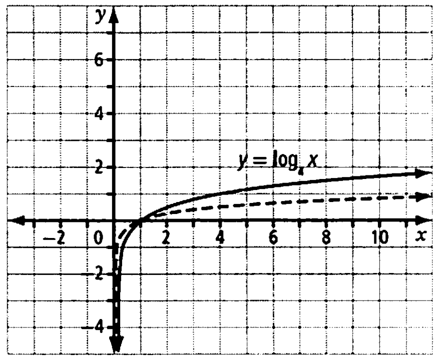



   **b)**



8. Each graph below has been vertically stretched from $y = \log_4 x$. State the stretch in each case.

a) 

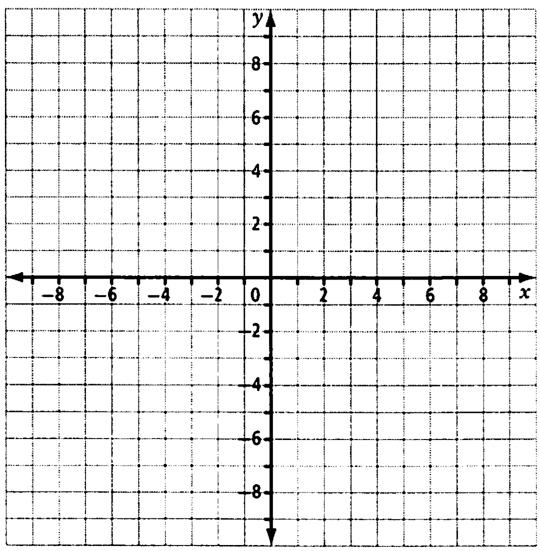

b) 

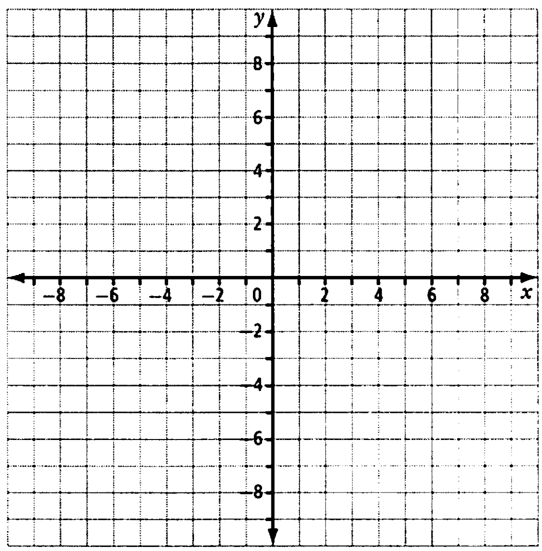

## Connect

9. Consider the following transformations of $y = \log_2 x$.

a) Sketch the result when the graph is vertically stretched by a factor of 2 and then translated 4 units left.

b) Sketch the result when the graph of $y = \log_2 x$ is translated 4 units left and then then vertically stretched by a factor of 2.




c) Compare the transformed graphs in parts a) and b). Does the order of performing a vertical stretch and a horizontal translation matter?

d) Give two other transformations of the graph of $y = \log_2 x$ for which the order of the transformations does not matter.

e) Make a general statement about when the order of performing transformations matters.

## 8.3 Laws of Logarithms

### KEY IDEAS

- Let $P$ be any real number, and $M$, $N$, and $c$ be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

| Name | Law | Description |
|---|---|---|
| Product | $\log_c MN = \log_c M + \log_c N$ | The logarithm of a product of numbers is the sum of the logarithms of the numbers. |
| Quotient | $\log_c \frac{M}{N} = \log_c M - \log_c N$ | The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor. |
| Power | $\log_c M^P = P \log_c M$ | The logarithm of a power of a number is the exponent times the logarithm of the number. |

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

### Working Example 1: Use the Laws of Logarithms to Expand Expressions

Expand each expression using the laws of logarithms.

**a)** $\log_4 \frac{x^3y}{4z}$ **b)** $\log_5 \sqrt{xy^3}$ **c)** $\log \frac{100\sqrt[3]{x^4}}{y^2}$

### Solution

**a)** $\log_4 \frac{x^3y}{4z} = \log_4 \_\_\_\_\_\_ - \log_4 \_\_\_\_\_\_$

$= \log_4 x^3 + \log_4 y - (\log_4 4 + \log_4 z)$

$= 3 \log_4 x + \log_4 y - 1 - \log_4 z$

Why does $\log_4 4 = 1$?

**b)** $\log_5 \sqrt{xy^3} = \log_5 (xy^3)^{\square}$

$= \frac{1}{2} \log_5 (xy^3)$

$= \frac{1}{2}(\log_5 \_\_\_\_\_\_ + \log_5 \_\_\_\_\_\_)$

$= \frac{1}{2}(\log_5 \_\_\_\_\_\_ + \_\_\_\_\_\_ \log_5 \_\_\_\_\_\_)$

$= \frac{1}{2} \log_5 \_\_\_\_\_\_ + \_\_\_\_\_\_ \log_5 \_\_\_\_\_\_$

c) $\log \frac{100\sqrt[3]{x^4}}{y^2} = \log 100\sqrt[3]{x^4} - \log y^2$

$= \log 100 + \log x^{\square} - \_\_\_\_\_\_$

$= 2 + \_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_$

## Working Example 2: Write Expressions With a Single Logarithm

Rewrite each expression using a single logarithm. State the restrictions on the variable.

a) $\log_2 x^3 - 4 \log_2 x - \log_2 \sqrt{x}$

b) $4 \log_6 y^2 + \log_6 y - \frac{2}{3} \log_6 y$

c) $\log (x - 3) + \log (x + 4)$

## Solution

a) $\log_2 x^3 - 4 \log_2 x - \log_2 \sqrt{x}$

$= \log_2 x^3 - \log_2 \_\_\_\_\_\_ - \log_2 x^{\frac{1}{2}}$

$= \log \frac{x^3}{\square}$

$= \log \frac{1}{x^{\frac{3}{2}}}, x > 0$

b) $4 \log_6 y^2 + \log_6 y - \frac{2}{3} \log_6 y$

$= \log_6 \_\_\_\_\_\_ + \log_6 y - \log_6 \_\_\_\_\_\_$

$= \log_6 \frac{y^8 y}{y^{\frac{2}{3}}}$

$= \_\_\_\_\_\_, y > 0$

c) $\log (x - 3) + \log (x + 4)$

$= \log [(x - 3)(x + 4)]$

$= \log \_\_\_\_\_\_\_\_\_\_, x > 3$

## Working Example 3: Evaluate Expressions With the Laws of Logarithms

Evaluate each expression.

a) $\log_4 8 + \log_4 32$

b) $\log_6 216\sqrt[4]{36}$

## Solution

a) $\log_4 8 + \log_4 32$

$= \log_4 (8 \times 32)$

$= \log_4 \_\_\_\_\_\_$

$= \_\_\_\_\_\_$

b) $216 = 6^{\square}$ and $\sqrt[4]{36} = 6^{\square}$

Rewrite $\log_6 216\sqrt[4]{36}$.

$\log_6 216\sqrt[4]{36} = \log_6 6^3 6^{\frac{1}{2}}$

$= \_\_\_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_$

See pages 395–399 of *Pre-Calculus 12* for more examples.

## Check Your Understanding

### Practise

1. Use the laws of logarithms to evaluate.

   a) $\log_4 8 + \log_4 2$

   b) $\log_5 250 - \log_5 2$

   c) $\log_2 96 - \log_2 3$

   d) $\log_6 3 + \log_6 12$

   e) $\log_5 10 + \log_5 10 - \log_5 4$

   f) $\log 25 + 2 \log 4 + \log 5 - \log 2$

   g) $\log_4 4^5$

   h) $\log_9 9^{11}$

   i) $2^{\log_2 16}$

   Since $\log_2 16 =$ ________, this means that $2^{\log_2 16} =$ ________

   j) $10^{\log 1000}$

2. Expand each expression using the laws of logarithms.

   a) $\log_7 x^4\sqrt{y^3}$

   Use the product law to rewrite $\log_7 x^4\sqrt{y^3}$.

   $\log_7$ ________ $+ \log_7$ ________

   Rewrite $\sqrt{y^3}$ using exponents: ________

   Use the power law to rewrite $\log_7 x^4$ and $\log_7 \sqrt{y^3}$.

   $\log_7 x^4 =$ ________ and $\log_7 \sqrt{y^3} =$ ________ $\log_7 y$

   Thus, the expanded expression for $\log_7 x^4\sqrt{y^3}$ is ________________________.

**b)** $\log_{12} (xy^2z^5)^3$

**c)** $\log_8 \frac{x^3}{\sqrt{yz^5}}$

**d)** $\log \sqrt{\frac{x}{y^3}}$

**3.** Expand each expression using the laws of logarithms. Then, evaluate and simplify where possible.

**a)** $\log_7 49\sqrt[3]{x^5}$

**b)** $\log \frac{100}{x^2y^2}$

**c)** $\log_5 \frac{\sqrt[3]{y^7}}{125x}$

**d)** $\log_2 \frac{3x^6}{96y^2}$

4. Write each expression as a single logarithm in simplest form.

a) $\log_6 2x^7 + \log_6 3x^2 + \log_6 \frac{9}{x^5}$

b) $\log_2 5x^2y^3 - \log_2 20x^4y + \log_2 2xy^6$

c) $\log_4 (x^2y)^2 + 5 \log_4 x^3y^4 + \log_4 \left(\frac{1}{x^3y^2}\right)$

d) $6 \log_3 xy - \log_3 xy^2 - \log_3 \sqrt[3]{x^4y}$

e) $\frac{1}{2} \log 4x\sqrt{y} - \log 25x^2\sqrt{y}$

f) $\log_7 x^4 + \frac{1}{3}(\log_7 x^2 - \log_7 \sqrt{5x})$

g) $\frac{\log 16x^8}{4} - \frac{\log 27x}{3}$

h) $\frac{\log_9 x^4y^8}{2} + \frac{\log_9 x^{12}y^{15}}{3}$

Completing #1–#4 will help you with #1–#3 on page 400 of *Pre-Calculus 12*.

## Apply

5. The loudness of a sound, $L$, in decibels, is given by $L = 10 \log \frac{I}{I_0}$, where $I$ is the intensity of the sound, in watts per square metre, and $I_0$ is $10^{-12}$ W/m².

a) Use the laws of logarithms to rewrite the right-hand side of the equation as a difference.

b) Rearrange the equation in part a) to isolate $I$.

6. The pH scale is used to measure the acidity or alkalinity of a substance. The formula for pH is pH = $-\log [H_3O^+]$, where $[H_3O^+]$ is the concentration of the hydronium ion.

   a) Rewrite the equation to isolate $[H_3O^+]$.

   b) A textbook defines pH as $\log \frac{1}{[H_3O^+]}$. Use the laws of logarithms to show that this is equivalent to the definition given above.

7. Decide whether each of the following is true or false. Justify your answer.

   a) $\log_5 (x + 10) = \log_5 x + \log_5 10$

   b) $\frac{\log_2 18}{\log_2 9} = \log_2 2$

   c) $4^{\log_4 y} = y$

   d) $\log_c 1 = 0$

   e) $\log_c xy^2 = 2 \log_c xy$

8. Use the laws of logarithms to isolate $x$ in each expression.

   a) $\log_6 36x = 1$

   b) $\log_3 \frac{27}{x} = 2$

   c) $3 \log_x 4 = 2$

9. Let $\log_5 12 = P$. Write each of the following expressions in terms of $P$.

a) $\log_5 12^7$

b) $\log_5 60$

c) $\log_5 144$

d) $\log_5 \frac{12}{5}$

e) $\log_5 \frac{1}{12}$

f) $\log_5 \sqrt{12}$

## Connect

10. Consider the following transformations of $y = \log_2 x$.

a) Explain how $y = \log_2 x^2$ is a vertical stretch by a factor of 2 of the original graph.

b) Explain how $y = \log_2 3x$ is a horizontal stretch of the original graph. By what factor is the graph stretched?

c) Explain how $y = \log_2 3x$ is a vertical translation of the original graph. By what amount is the graph translated?

d) Is $y = \log_2 \frac{1}{x}$ the reciprocal transformation of the original function? Justify your answer.

## 8.4 Logarithmic and Exponential Equations

### KEY IDEAS

- When solving a logarithmic equation algebraically, start by applying the laws of logarithms to express one side or both sides of the equation as a single logarithm.
- Some useful properties are listed below, where $c$, $L$, $R > 0$ and $c \neq 1$.
  - If $\log_c L = \log_c R$, then $L = R$.
  - The equation $\log_c L = R$ can be written with logarithms on both sides of the equation as $\log_c L = \log_c c^R$.
  - The equation $\log_c L = R$ can be written in exponential form as $L = c^R$.
  - The logarithm of zero or a negative number is undefined. To identify whether a root is extraneous, substitute the root into the original equation and check whether all of the logarithms are defined.
- You can solve an exponential equation algebraically by taking logarithms of both sides of the equation. If $L = R$, then $\log_c L = \log_c R$, where $c$, $L$, $R > 0$ and $c \neq 1$. Then, apply the power law for logarithms to solve for an unknown.
- You can solve an exponential equation or a logarithmic equation using graphical methods.
- Many real-world situations can be modelled with an exponential or a logarithmic equation. A general model for many problems involving exponential growth or decay is

$$\text{Final quantity} = \text{initial quantity} \times (\text{change factor})^{\text{number of changes}}$$

### Working Example 1: Solve Logarithmic Equations

Solve.

**a)** $\log_4 (5x + 1) = \log_4 (x + 17)$

**b)** $\log (5x) - \log (x - 1) = 1$

**c)** $\log_6 (x - 3) + \log_6 (x + 6) = 2$

### Solution

**a)** Since $\log_4 (5x + 1) = \log_4 (x + 17)$, $5x + 1 = x + 17$.
So, $4x = 16$ and $x = 4$.
Check $x = 4$ in the original equation.

| Left Side | Right Side |
|---|---|
| $\log_4 (5(\mathbf{4}) + 1)$ | $\log_4 (\mathbf{4} + 17)$ |
| $= \log_4 21$ | $= \log_4 21$ |

Left Side = Right Side

**b) Method 1: Solve Algebraically by Rewriting in Exponential Form**

Using the laws of logarithms, rewrite log (5$x$) – log ($x$ – 1) as log ______________.

Then, rewrite $\log \frac{5x}{x-1} = 1$ in exponential form.

$\frac{5x}{x-1} =$ ________

Multiply both sides by ($x$ – 1).

$$\frac{5x}{x-1}(x-1) = 10(x-1)$$
$$5x = 10x - 10$$
$$-5x = -10$$
$$x = 2$$

Check:

| Left Side | Right Side |
|---|---|
| log (5(2)) – log (2 – 1)<br>= log 10 – log 1<br>= 1 – 0<br>= 1 | 1 |

Left Side = Right Side

**Method 2: Solve Algebraically by Writing Each Side as a Logarithm**

Begin by rewriting 1 as $\log_{10}$ ________.

So, the equation is log (5$x$) – log ($x$ – 1) = log 10.

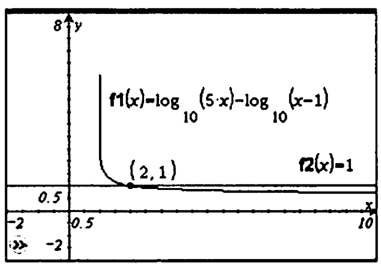
Solve for $x$.

$$\log (5x) - \log (x-1) = 10$$
$$\log \frac{5x}{x-1} = \log 10$$
$$\frac{5x}{x-1} = 10$$
$$5x = 10(x-1)$$
$$5x = 10x - 10$$
$$-5x = -10$$
$$x = 2$$

Compare the algebra in Method 1 to the algebra in Method 2. How are the methods similar? How are they different?

**Method 3: Solve Graphically**

Use technology to graph the equations $y$ = log (5$x$) – log ($x$ – 1) and $y$ = 1 in the same window. Then, find the point of intersection.




The solution is $x$ = 2.

c) Using the laws of logarithms, rewrite the left side of the equation.

$\log_6 (x - 3) + \log_6 (x + 6) = 2$

$\log_6 (x - 3)(x + 6) =$ ____________

**Method 1: Rewrite in Exponential Form**

In exponential form, the equation is equivalent to $(x - 3)(x + 6) = 6^2$.

Expand and simplify the left side of the equation.

____________________ $= 36$

Subtract 36 from each side.

____________________ $= 0$

Factor the left side.

$(x +$ ________$)(x -$ ________$) = 0$

$x =$ ________ or $x =$ ________

The original equation is defined for $x > 3$ and $x > -6$. In other words, both conditions are met when $x > 3$. Thus, $x = -9$ is an extraneous root. Check $x = 6$.

| Left Side | Right Side |
|---|---|
| $\log_6 (6 - 3) + \log_6 (6 + 6)$<br>$= \log_6 3 + \log_6 12$<br>$= \log_6 36$<br>$= 2$ | 2 |

Left Side = Right Side

**Method 2: Rewrite as a Logarithm**

Written as a logarithm with base 6, 2 = ________.

$\log_6 (x - 3) + \log_6 (x + 6) = \log_6 36$

$\log_6 (x - 3)(x + 6) = \log_6 36$

$(x - 3)(x + 6) = 36$

____________________ $= 36$

____________________ $= 0$

Factoring,

____________________ $= 0$

So, $x = -9$ or $x = 6$.
Since $x > 3$, the solution is $x = 6$.

## Working Example 2: Solve Exponential Equations

Solve. Express your answer as an exact value and as a decimal correct to two decimal places.

**a)** $5^x = 200$ **b)** $8^{2x-3} = 15\,109$ **c)** $3^{2x} = 7^{x+1}$

## Solution

**a)** **Method 1: Use the Power Law of Logarithms**

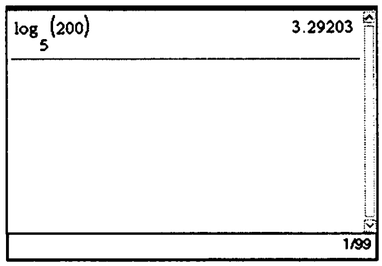
Take the logarithms of both sides of the equation:

$$\log 5^x = \log 200$$
$$x \log 5 = \log 200$$
$$x = \frac{\log 200}{\log 5}$$

As a decimal, $x \approx 3.29$.

**Method 2: Write in Logarithmic Form**

$5^x = 200$ is the same as $\log_5 200 = x$. An exact value for $x$ is $\log_5 200$.
Use technology to calculate the value to two decimal places: $\log_5 200 = 3.29$.




**b)** Take the logarithm of each side: $\log 8^{2x-3} = \log 15\,109$.
Then, use the power law of logarithms.

$$\rule{3cm}{0.4pt} = \log 15\,109$$
$$2x - 3 = \frac{\log 15\,109}{\log 8}$$
$$x = \frac{1}{2}\left(\frac{\log 15\,109}{\log 8} + 3\right)$$
$$x \approx 3.81$$

**c)** Since $3^{2x} = 7^{x+1}$, $\log 3^{2x} = \log 7^{x+1}$.

Use the power law of logarithms.

$$\rule{3cm}{0.4pt} = \rule{3cm}{0.4pt}$$

$$2x \log 3 = x \log 7 + \log 7$$
$$2x \log 3 - x \log 7 = \log 7$$

Factor.

$$x(2 \log 3 - \log 7) = \log 7$$
$$x = \frac{\log 7}{2 \log 3 - \log 7}$$

Written as a decimal, $x \approx 7.74$.

## Working Example 3: Model Exponential Growth

A town has a current population of 12 468. The population is growing by 2% per year.

**a)** Write an exponential equation to model the population growth.

**b)** What will be the town's population in eight years?

**c)** When will the population first reach 20 000?

### Solution

**a)** If $P$ is the population of the town and $t$ is time, in years, then $P = 12\,468(1.02)^t$.

**b)** In eight years, the population will be $P =$ ______________, or 14 608.

**c)** When the population reaches 20 000, $12\,468(1.02)^t = 20\,000$.

Divide each side by 12 468: $1.02^t =$ ______________

Take the logarithm of each side: $\log 1.02^t =$ ______________

Apply the power law of logarithms: ______________ = ______________

$t =$ ______________

Written as a decimal, $t$ is approximately 23.86. It will take about 23.9 years until the town reaches a population of 20 000.

## Working Example 4: Model Exponential Decay

A business invests $450 000 in new equipment. For tax purposes, the equipment is considered to depreciate in value by 20% each year.

**a)** Write an exponential equation to model the value of the equipment.

**b)** What will be the value of the equipment in three years?

**c)** When will the value first drop to $100 000?

### Solution

**a)** If $V$ denotes the value of the equipment, and $t$ is time, in years, then $V = 450\,000(0.8)^t$.

**b)** $V = 450\,000(0.8)^3$

$= 230\,400$

The equipment will be worth $230 400 in three years.

**c)** $450\,000(0.8)^t = 100\,000$

Divide each side by 450 000: ______________ = ______________

Take the logarithm of each side: ______________ = ______________

Use the power law of logarithms to rewrite the equation: ______________ = ______________

Divide each side by log 0.8: ______________ = ______________

After 6.74 years, the equipment will have a value of $100 000 for tax purposes.

See pages 406–411 of *Pre-Calculus 12* for more examples.

## Check Your Understanding

### Practise

1. Solve. Give exact answers.

a) $\log_4 x = 5$

b) $\log_5 x + 6 = 8$

c) $2 \log_2 x = 10$

d) $\log_6 (x + 3) + 2 = 5$

e) $3 \log_5 x = \log_5 125$

f) $2 \log (x - 5) = 6$

2. Solve. Round your answers to two decimal places.

a) $12^{3x} = 1000$

b) $7^{x+2} = 441$

c) $2^{3-x} = 100$

d) $3^{\frac{2x}{3}} = 350$

3. Solve. Express your answers as exact values.

a) $5^x = 205$

Take the logarithm of each side of the equation: ________ = ________

Then, use the power law of logarithms: ________ = ________

Divide each side by log 5: $x =$ ________

b) $4^{x-3} = 311$

c) $10^{2x+1} = 7539$

d) $5(4)^{x+2} = 200$

e) $6^{\frac{x}{2}} = 85$

4. Solve.

a) $3 \log_6 x = \log_6 9 + \log_6 24$

b) $\log_2 x^2 - \log_2 5 = \log_2 20$

c) $\log_4 x + 2 \log_4 x = 6$

d) $5 \log_3 x - \log_3 x = 8$

Completing #1–#4 will help you with #1–#3 on page 412 of *Pre-Calculus 12*.

5. Identify the values of $x$ for which each equation is defined.

a) $\log_9 (x + 4) = \log_9 (2x)$

Since logarithms are only defined for positive values, $x + 4 > 0$, or $x >$ ________.

Similarly, since $2x > 0$, $x >$ ________.

If $x > 0$, both statements are true. So, the equation is defined when $x >$ ________.

b) $\log_7 (3x + 1) - \log_7 (x - 2) = 1$

c) $\log_6 (3 - x) + \log_6 (x - 3) = 2$

## Apply

6. Solve. Express your answers as exact values and as decimal values correct to the nearest hundredth.

a) $5^{x-3} = 10^x$

b) $8^{2x+3} = 12^{2x}$

c) $2^{2x-5} = 6^{x+2}$

d) $2(6)^{x+2} = 3^{2x-3}$

7. Solve.

a) $\log_2 (4x + 10) - \log_2 x = 3$

b) $\log_3 (x + 7) - \log_3 (x - 3) = 2$

c) $\log (2x + 6) = 1 + \log (x - 1)$

d) $\log_5 (4x - 6) - 3 = \log_5 (2x - 3)$

8. Solve.

a) $\log x + \log (x + 3) = 1$

b) $\log_4 (x - 4) + \log_4 (x + 2) = 2$

c) $\log_6 (x + 3) - 2 = -\log_6 (x - 2)$

d) $\log (x + 2) = 2 - \log (7x - 1)$

9. The half-life of plutonium-238 is 88 years. Suppose that a sample of plutonium has a mass of 65 grams.

a) Write an exponential equation to model the mass of plutonium, $m$, present after $t$ years.

b) Determine the mass of plutonium in the sample after 50 years. Round your answer to two decimal places.

c) Determine the time needed for the sample to decay to a mass of 20 grams, to the nearest tenth of a year.

10. The population of a high school is growing by 1.5% per year. Currently there are 974 students in the school.

a) Write an exponential equation to model the population of the school, $p$, after $t$ years.

b) What population should be expected at the high school in five years?

c) When will the population of the school reach 1200 students?

## Connect

**11.** Describe two methods to solve each equation algebraically. Then, solve the equations using each method. Check your solutions numerically or graphically.

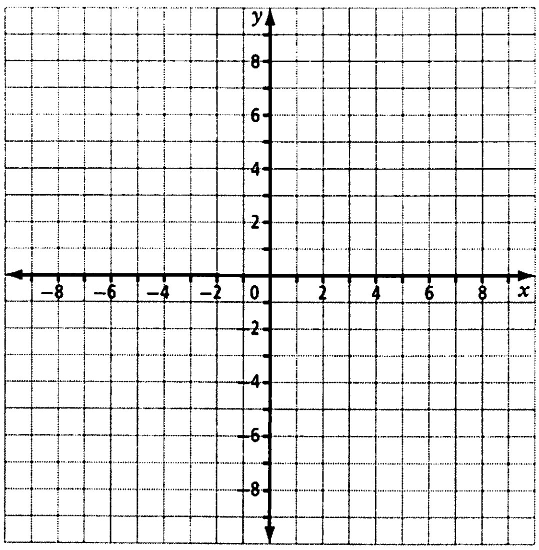
| Equation | Method | Solution |
|---|---|---|
| **a)** $5^{2x(x-1)} = 5^{(3x-3)}$ | | |
| | | |
| **b)** $\log(2x^2 + 3x) = \log 9$ | | |
| | | |

## Chapter 8 Review

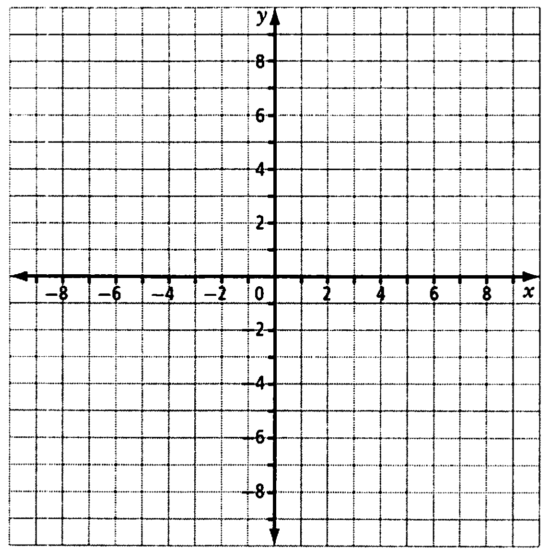
### 8.1 Understanding Logarithms, pages 260–266

**1.** Sketch the graph of each logarithmic function. Then, state the domain, range, $x$-intercept, and vertical asymptote.

**a)** $y = \log_2 x$




**b)** $y = \log_5 x$




**2.** Write each expression in logarithmic form.

**a)** $6^3 = 216$

**b)** $2^{10} = 1024$

**c)** $10^{-3} = 0.001$

**d)** $5^x = 125$

**3.** Write each expression in exponential form.

**a)** $\log_3 81 = 4$

**b)** $\log_{25} 5 = \frac{1}{2}$

**c)** $\log 1 = 0$

**d)** $\log_2 (3x - 4) = 9$

## 8.2 Transformations of Logarithmic Functions, pages 267–274

4. Identify the transformations in each logarithmic function. State the domain, range, and intercepts of the graph of each. Round your answers to one decimal place if necessary.

   a) $y = 2 \log_4 (x + 1)$

   b) $y = \log_7 (x - 3) + 5$

5. Write the equation for each of the following transformations to the function $y = \log x$. Then, state the domain and range of the transformed function.

   a) translation 5 units right and 4 units down

   b) vertical stretch by a factor of 3, translation 2 units left and 6 units down

   c) horizontal stretch by a factor of $\frac{1}{3}$, translation 1 unit up

## 8.3 Laws of Logarithms, pages 275–281

6. Use the laws of logarithms to evaluate each of the following.

   a) $\log_6 9 + \log_6 4$

   b) $\log 2000 - \log 2$

   c) $\log_{12} 9 + \log_{12} 2 + \log_{12} 8$

   d) $\log_7 100 - \log_7 25 - \log_7 4$

7. Expand each of the following.

   a) $\log_5 \left(25x^4 \sqrt[4]{y^3}\right)$

   b) $\log \frac{\sqrt{x}\, y^5}{100x}$

8. Write each of the following as a single logarithm.

a) $\log_4 x^2y^5 + \log_4 xy^{-2}$

b) $\log \frac{x^4}{\sqrt{y}} - \log \frac{y^2}{x}$

## 8.4 Logarithmic and Exponential Equations, pages 282–291

9. Solve. Express each answer as an exact value and as a decimal rounded to two places.

a) $3^x = 100$

b) $7^{x-3} = 517$

c) $10^{2x+1} = 5500$

d) $5^x = 2^{x-4}$

10. Solve.

a) $\log_2 x = 7$

b) $\log_3 (4x + 9) = 5$

c) $\log_2 (6x - 3) - \log_2 x = 4$

d) $\log_8 (6x + 2) + \log_8 (x - 3) = 2$

11. The intensity of sound is measured in decibels (dB). The level of a sound, $L$, is given by $L = 10 \log \frac{I}{I_0}$, where $I$ is the intensity of the sound and $I_0$ is the faintest sound detectable to humans. A sound engineer increases the volume at a concert from 90 decibels (dB) to 93 dB. Show that this increase approximately doubles the intensity of the sound.

12. A strain of bacteria doubles every 4 hours. A sample contains 40 bacteria.

a) Write an exponential equation to determine the number of bacteria present, $N$, after $t$ hours.

b) Determine the time needed until 1000 bacteria are present. Round your answer to two decimal places.

c) Determine the time needed for the number of bacteria in the sample to triple. Does your answer depend on the number of bacteria present at the beginning?

13. A water filter removes 40% of the impurities in a sample of water.

a) Write an exponential equation to determine the percent of impurities remaining, $P$, after the water has passed through $n$ filters.

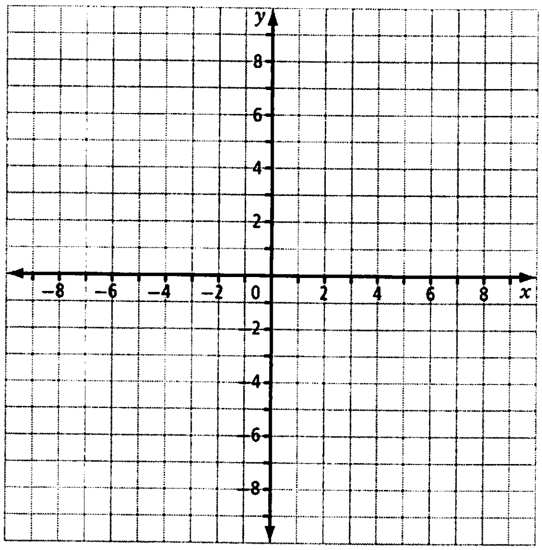
b) What percent of impurities will remain after the water has passed through 3 filters?

c) How many filters are needed to remove at least 99% of impurities in the water?

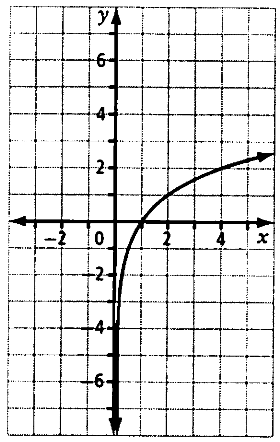
## Chapter 8 Skills Organizer

Complete the following graphic organizer to summarize your knowledge of logarithms.

Sketch the graph of an exponential function and its inverse. State the domain, range, intercepts, and asymptotes for each graph.




Perform at least one of each type of transformation from this chapter on the graph of $y = \log_2 x$ below. Write the equation and sketch the transformation. Give the domain and range of the original function and each transformation.




Logarithms

State the laws of logarithms. Give at least one example of how each law works.

Give examples of situations that exhibit exponential growth, exhibit exponential decay, and use logarithmic scales. Show how logarithms make modelling these situations possible.

Give at least one example to show how each law of logarithms can be used to solve exponential or logarithmic equations.