6 Geometric Sequences and Series

BUILDING ON

- expressing powers using exponents
- solving equations

KEY LEARNING

- A geometric sequence is created by repeatedly multiplying an initial number by a constant. A geometric series is the sum of the terms of a geometric sequence.
- Any finite series has a sum, but an infinite geometric series may or may not have a sum.

LEADING TO

- applying the properties of geometric sequences and series to exponential functions that illustrate growth and decay

NEW VOCABULARY

term of a sequence or series	divergent and convergent sequences
general term	geometric series
geometric sequence	sigma notation
common ratio	infinite geometric series
finite and infinite geometric sequences	sum to infinity

Discuss the Ideas

RM, CR1

1. How do you determine whether a given sequence is geometric? What assumptions do you make?

2. Which geometric sequences are created when $r = 1$? $r = -1$?

Exercises

A

3. Which sequences could be geometric? If a sequence is geometric, state its common ratio.

a) 1, 2, 4, 8, 16, . . .

b) 4, 9, 16, 25, 36, . . .

c) -3, 2, 7, 12, 17, . . .

d) 6, 0.6, 0.06, 0.006, . . .

e) 10, 100, 1000, 10 000

f) 2, 4, 6, 8, 10, . . .

g) 192, -96, 48, -24, 12, . . .

h) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

4. State the common ratio, then write the next 3 terms of each geometric sequence.

a) $-1, -3, -9, \ldots$ **b)** $48, 24, 12, \ldots$ **c)** $25, -50, 100, \ldots$

d) $4, -2, 1, \ldots$ **e)** $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$ **f)** $-24, 12, -6, \ldots$

5. For each geometric sequence, determine the indicated value.

a) $3, 6, 12, \ldots$; determine t_7 **b)** $18, 9, 4.5, \ldots$; determine t_6

c) $23, -46, 92, \ldots$; determine t_{10} **d)** $2, \frac{1}{2}, \frac{1}{8}, \ldots$; determine t_5

B

CR1 **6.** Write the first 4 terms of each geometric sequence, given the 1st term and the common ratio. Identify the sequence as decreasing, increasing, or neither. Justify your answers.

a) $t_1 = -3; r = 4$ **b)** $t_1 = 5; r = 2$

c) $t_1 = -0.5;\ r = -3$

d) $t_1 = \frac{1}{2};\ r = \frac{2}{3}$

RM, US

7. Write the first 5 terms of a geometric sequence where:

a) the 6th term is 64

b) the 1st term is $\frac{3}{4}$

c) every term is negative

d) every term is an even number

8. Use the given data about each finite geometric sequence to determine the indicated values.

a) Given $t_1 = -1$ and $r = -2$

i) Determine t_9.

ii) The last term is -4096. How many terms are in the sequence?

b) Given $t_1 = 0.002$ and $t_4 = 2$

i) Determine t_7.

ii) Determine which term has the value 20 000.

US, CR2

9. A ball is dropped from a height of 25 m. After each bounce, the ball rises to 80% of the previous height.

a) Write the first 3 terms of a geometric sequence that models the height of the ball in metres.

b) To the nearest centimetre, to what height does the ball rise after the 5th bounce?

c) To the nearest centimetre, to what height does the ball rise after the 10th bounce?

d) After how many bounces does the ball rise to a height less than 1 m?

RM, US, CR1, CR2

10. Suppose a person is given 1¢ on the first day of April; 3¢ on the second day; 9¢ on the third day, and so on. This pattern continues throughout April.

a) About how much money will the person be given on the last day of April?

b) Why might it be difficult to determine the exact amount using a calculator?

11. In a geometric sequence, $t_3 = 9$ and $t_6 = 1.125$; determine t_7 and t_9.

12. A beekeeper starts her business with 200 bees. New bees are hatched at a rate of 104% each week. How many bees were there after week 15?

RM, US, CR1, CR2

13. A ream of paper is about 2 in. thick. Imagine a ream of paper that is repeatedly cut in half and the two halves stacked one on top of the other. How many cuts have to be made before the stack of paper is taller than 318 ft., the height of Le Chateau York in Winnipeg, Manitoba? Justify your answer.

RM, US

14. Between the Canadian censuses in 2001 and 2006, the number of people who could converse in Cree had increased by 7%. In 2006, 87 285 people could converse in Cree. Assume the 5-year increase continues to be 7%. Estimate to the nearest hundred how many people will be able to converse in Cree in 2031.

C

15. A farmer in Saskatchewan wants to estimate the value of a new combine after several years of use. A new combine worth $370 000 depreciates in value by about 10% each year.

a) Estimate the value of the combine at the end of each of the first 5 years. Write the values as a geometric sequence.

b) Predict the value of the combine at the end of 10 years.

RM, US

16. **a)** Show that squaring each term in a geometric sequence produces the same type of sequence. What is the common ratio?

b) Show that raising each term in a geometric sequence to the mth power of each term produces the same type of sequence. What is the common ratio?

Multiple-Choice Questions

1. Which expression below represents the nth term of this geometric sequence: $9, -6, 4, -\frac{8}{3}, \ldots$?

A. $9\left(\frac{2}{3}\right)^{n-1}$ **B.** $9\left(-\frac{2}{3}\right)^{n-1}$ **C.** $\frac{2}{3}(9^{n-1})$ **D.** $-\frac{2}{3}(9^{n-1})$

2. Which geometric sequence does not have a common ratio of -0.5?

A. $-5, 2.5, -1.25, 0.625, \ldots$ **B.** $6, -3, 1.5, -0.75, \ldots$

C. $\frac{1}{200}, -\frac{1}{100}, \frac{1}{50}, -\frac{1}{25}, \ldots$ **D.** $-\frac{1}{3}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{24}, \ldots$

Study Note

CR1, CR2

What is a geometric sequence? Write the first 5 terms of your own geometric sequence and describe it.

ANSWERS

Check Your Understanding

1. a) $-39\,366$ **b)** divergent **3. a)** -21 or 21, -1701 or 1701 **b)** 9 terms
4. a) approximately 31 000 **b)** approximately 41 000

Exercises

3. a) 2 **d)** 0.1 **e)** 10 **g)** $-\frac{1}{2}$ **4. a)** 3; $-27, -81, -243$ **b)** 0.5; 6, 3, 1.5 **c)** -2; $-200, 400, -800$
d) -0.5; $-0.5, 0.25, -0.125$ **e)** $\frac{1}{3}; \frac{1}{54}, \frac{1}{162}, \frac{1}{486}$ **f)** $-\frac{1}{2}$; 3, $-\frac{3}{2}, \frac{3}{4}$ **5. a)** 192 **b)** 0.5625 **c)** $-11\,776$ **d)** $\frac{1}{128}$
6. a) $-3, -12, -48, -192$; decreasing **b)** 5, 10, 20, 40; increasing **c)** $-0.5, 1.5, -4.5, 13.5$; neither
d) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}$; decreasing **8. a) i)** -256 **ii)** 13 **b) i)** 2000 **ii)** t_8 **9. a)** 20, 16, 12.8 **b)** 8.19 m **c)** 2.68 m **d)** 15
10. a) approximately $\$6.863 \times 10^{11}$ **11.** 0.5625; 0.140 625 **12.** 346 **13.** 11 cuts **14.** approximately 122 400 people
15. a) \$333 000, \$299 700, \$269 730, \$242 757, \$218 481 **b)** \$129 011

Multiple Choice

1. B **2.** C

Discuss the Ideas

RM, CR1

1. Why do the terms in some geometric series alternate between positive and negative numbers?

2. How can you identify when a problem may be modelled by a geometric series?

Exercises

A

3. Write a geometric series for each geometric sequence.

a) 1, 4, 16, 64, 256, . . .

b) 20, −10, 5, −2.5, 1.25, . . .

4. Which series appear to be geometric? If the series could be geometric, determine S_5.

a) $2 + 4 + 8 + 16 + 32 + \ldots$

b) $2 - 4 + 8 - 16 + 32 - \ldots$

c) $1 + 4 + 9 + 16 + 25 + \ldots$

d) $-3 + 9 - 27 + 81 - 243 + \ldots$

e) $\sum_{k=1}^{n} 5k$

f) $\sum_{k=1}^{n} 4(3^k)$

5. Use the given data about each geometric series to determine the indicated value.

a) $t_1 = 1$, $r = 0.3$; determine S_8

b) $t_1 = \frac{3}{4}$, $r = \frac{1}{2}$; determine S_4

B

6. Determine S_6 for each geometric series.

a) $2 + 10 + 50 + \ldots$

b) $80 - 40 + 20 - \ldots$

c) $\sum_{k=1}^{n} 3(2)^{k-1}$

d) $\sum_{k=1}^{n} 2(-3)^k$

7. Determine S_{10} for each geometric series. Give the answers to 3 decimal places.

a) $0.1 + 0.01 + 0.001 + 0.0001 + \ldots$

b) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

c) $\sum_{k=1}^{n} 25(0.2)^{k-1}$

d) $\sum_{k=1}^{n} 8\left(-\frac{1}{2}\right)^{k}$

RM, US, CR1

8. a) Explain why this series appears to be geometric:
$1 + 5 + 25 + 125 + \ldots$

b) What information do you need to be certain that this is a geometric series?

c) What assumptions do you make when you identify or extend a geometric series?

RM, US

9. For each geometric series, determine how many terms it has then calculate its sum.

a) $1 - 2 + 4 - 8 + \ldots - 512$

b) $-6561 + 2187 - 729 + 243 - \ldots - 1$

10. Identify the terms in each partial sum of a geometric series.

a) $S_5 = 62, r = 2$

b) $S_8 = 1111.1111; r = 0.1$

RM, US, CR1

11. On Monday, Ian had 3 friends visit his personal profile on a social networking website. On Tuesday, each of these 3 friends had 3 different friends visit Ian's profile. On Wednesday, each of the 9 friends on Tuesday had 3 different friends visit Ian's profile.

a) Write the total number of friends who visited Ian's profile as a geometric series. Represent the series with sigma notation. What is the first term? What is the common ratio?

b) Suppose this pattern continued for 1 week. What is the total number of people who visited Ian's profile? How do you know your answer is correct?

US, CR2

12. Each stroke of a vacuum pump extracts 5% of the air in a 50-m^3 tank. How much air is removed after 50 strokes? Give the answer to the nearest tenth of a cubic metre.

RM, US, CR1

13. Determine S_{11} for the geometric series $\sum_{k=1}^{10} t_1(-2)^{k-1} = 1705$. Explain your reasoning.

C

RM, US, CR1

14. Binary notation is used to represent numbers on a computer. For example, the number 1111 in base two represents $1(2)^3 + 1(2)^2 + 1(2)^1 + 1$, or 15 in base ten.

a) Why is the sum above an example of a geometric series?

b) Represent the series with sigma notation.

c) Which number in base ten is represented by
11 111 111 111 111 111 111 in base two? Explain your reasoning.

RM, US, CR1

15. Show how you can use geometric series to determine this sum:
$1 + 2 + 3 + 4 + 8 + 9 + 16 + 27 + 32 + 64 + 81 + 128 +$
$243 + 256 + 512$

RM, US

16. Determine the common ratio of a geometric series that has these partial sums: $S_3 = -\frac{49}{8}$, $S_4 = -\frac{105}{16}$, $S_5 = -\frac{217}{32}$

Multiple-Choice Questions

1. For which geometric series is -1023 the sum to 10 terms?

A. $1 - 2 + 4 - 8 + \ldots$ **B.** $1 + 2 + 4 + 8 + \ldots$

C. $-1 + 2 - 4 + 8 - \ldots$ **D.** $-1 - 2 - 4 - 8 - \ldots$

2. The sum of the first n terms of a geometric series is: $S_n = 4^n - 1$

For this series:

I. The common ratio is 4.

II. The first 3 terms are 3, 12, and 48.

III. $S_{2n} = 2^{4n} - 1$

A. Statements I, II, and III are correct.

B. Only statements I and II are correct.

C. Only statements II and III are correct.

D. Only statements I and III are correct.

Study Note

CR1, CR2

The rule for the sum of the first n terms of a geometric series has the restriction $r \neq 1$. Identify the geometric series with first term a and $r = 1$, then determine the sum of n terms.

ANSWERS

Check Your Understanding

1. 16 777 215 **2.** -3 **3. a)** $-3, -15, -75, -375$ **b)** $-58\,593$

4. a) 586.25 mg, or just over 586 mg **b)** approximately 588.24 mg, or just over 588 mg

Exercises

3. a) $1 + 4 + 16 + 64 + 256 + \ldots$ **b)** $20 - 10 + 5 - 2.5 + 1.25 - \ldots$

4. a) 62 **b)** 22 **d)** -183 **f)** 1452 **5. a)** approximately 1.428 **b)** approximately 1.406

6. a) 7812 **b)** 52.5 **c)** 189 **d)** 1092 **7. a)** approximately 0.111 **b)** approximately 0.750

c) approximately 31.245 **d)** approximately -2.664 **9. a)** 10 terms; -341

b) 9 terms; -4921 **10. a)** 2, 4, 8, 16, 32 **b)** 1000, 100, 10, 1, 0.1, 0.01, 0.001, 0.0001

11. a) $3 + 9 + 27$; 3; 3 **b)** 3279 people **12.** 46.2 m^3 **13.** -3415 **14. b)** $\sum_{k=1}^{4} 2^3(0.5)^{k-1}$

c) 1 048 575 **15.** 1386 **16.** $\frac{1}{2}$

Multiple Choice

1. D **2.** A

Exercises

A

3. Determine whether each infinite geometric series has a finite sum. How do you know?

a) $2 + 3 + 4.5 + 6.75 + \ldots$

b) $\sum_{k=1}^{\infty} 5(-4)^{k-1}$

c) $-0.5 - 0.05 - 0.005 - 0.0005 - \ldots$

d) $\sum_{k=1}^{\infty} 5\left(-\frac{4}{5}\right)^{k}$

e) $\frac{1}{2} - \frac{3}{8} + \frac{9}{32} - \frac{27}{128} + \ldots$

f) $\sum_{k=1}^{\infty} 4\left(-\frac{5}{4}\right)^{k-1}$

g) $0.1 + 0.2 + 0.4 + 0.8 + \ldots$

h) $\sum_{k=1}^{\infty} 4(-1)^{k}$

4. Write the first 4 terms of each infinite geometric series.

a) $t_1 = -4,\ r = 0.3$

b) $\sum_{k=1}^{\infty} 1(-0.25)^{k-1}$

c) $\sum_{k=1}^{\infty} 4\left(\frac{1}{5}\right)^{k-1}$

d) $t_1 = -\frac{3}{2},\ r = -\frac{3}{8}$

5. Each infinite geometric series converges. Determine each sum.

a) $\sum_{k=1}^{\infty} 8\left(\frac{1}{4}\right)^{k-1}$

b) $-1 - \frac{3}{4} - \frac{9}{16} - \frac{27}{64} - \ldots$

c) $10 - \frac{20}{3} + \frac{40}{9} - \frac{80}{27} + \ldots$

d) $\sum_{k=1}^{\infty} -2\left(-\frac{1}{3}\right)^{k-1}$

B

6. What do you know about the common ratio of an infinite geometric series whose sum is finite?

RM, US

7. Use the given data about each infinite geometric series to determine the indicated value.

a) $t_1 = 21$, $S_\infty = 63$; determine r

b) $r = -\frac{3}{4}$, $S_\infty = \frac{24}{7}$; determine t_1

8. Use an infinite geometric series to express each repeating decimal as a fraction.

a) $0.4\overline{97}$ **b)** $1.\overline{143}$

RM, US, CR1, CR2

9. The hour hand on a clock is pointing to 12. The hand is rotated clockwise 180°, then another 60°, then another 20°, and so on. This pattern continues.

a) Represent this series with sigma notation, then write the first 4 terms.

b) Which number would the hour hand approach if this rotation continued indefinitely? Explain what you did.

c) What assumptions did you make?

RM, US, CR1, CR2

10. Brad has a balance of \$500 in a bank account. Each month he spends 40% of the balance remaining in the account.

a) Express the total amount Brad spends in the first 4 months as a series. Is the series geometric? Explain.

b) Determine the approximate amount Brad spends in 10 months. Explain what you did.

c) Suppose Brad could continue this pattern of spending indefinitely. Would he eventually empty his bank account? Explain.

C

RM, US, CR1

11. Write the product of $0.\overline{a}$ and $0.\overline{b}$ as a fraction, where a and b represent 1-digit natural numbers. Explain your strategy.

RM, US, CR1

12. Create 2 different infinite geometric series with a sum of 4. Explain what you did.

13. Determine the sum of this infinite geometric series:

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{32}} + \frac{1}{\sqrt{128}} + \frac{1}{\sqrt{512}} + \ldots$$

Multiple-Choice Questions

1. What is the sum of this infinite geometric series?

$10 - \frac{20}{3} + \frac{40}{9} - \frac{80}{27} + \ldots$

A. 30 **B.** 4 **C.** 6 **D.** 20

2. Which infinite geometric series has the sum $-8.\overline{3}$?

A. $-5 - 2 - 0.8 - 0.32 - \ldots$

B. $-5 + 2 - 0.8 + 0.32 - \ldots$

C. $5 - 2 + 0.8 - 0.32 + \ldots$

D. $5 + 2 + 0.8 + 0.32 + \ldots$

3. How many of these geometric series have finite sums?

$1 + 0.5 + 0.125 + 0.0625 + \ldots$ $3 - 9 + 27 - 81 + \ldots$

$1 + \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \ldots$ $-12 - 6 - 3 - 1.5 - \ldots$

A. 1 series **B.** 2 series **C.** 3 series **D.** 4 series

Study Note

CR1, CR2

What is a rule for determining the sum of an infinite geometric series? When is it appropriate to apply this rule? When is it not appropriate?

ANSWERS

Check Your Understanding

1. a) $0.\overline{4}$ **b)** does not have a finite sum **c)** $0.\overline{09}$ **2. a)** converges; the sum is $42.\overline{6}$ **b)** converges; the sum is $90.\overline{90}$ **3.** $\frac{1}{6}$

Exercises

3. a) not finite **b)** not finite **c)** finite **d)** finite **e)** finite **f)** not finite **g)** not finite **h)** not finite

4. a) $-4 - 1.2 - 0.36 - 0.108$ **b)** $1 - 0.25 + 0.0625 - 0.015\,625$ **c)** $4 + \frac{4}{5} + \frac{4}{25} + \frac{4}{125}$ **d)** $-\frac{3}{2} + \frac{9}{16} - \frac{27}{128} + \frac{81}{1024}$

5. a) $10.\overline{6}$ **b)** -4 **c)** 6 **d)** -1.5 **7. a)** $\frac{2}{3}$ **b)** 6 **8. a)** $\frac{493}{990}$ **b)** $\frac{1142}{999}$ **9. a)** $\sum_{k=1}^{\infty} 180\left(\frac{1}{3}\right)^{k-1} = 180 + \frac{180}{3} + \frac{180}{3^2} + \frac{180}{3^3}$

b) 9 **10. a)** $\$500(0.4) + \$500(0.6)(0.4) + \$500(0.6)^2(0.4) + \$500(0.6)^3(0.4)$; geometric **b)** \$496.98 **c)** no

11. $\frac{ab}{81}$ **13.** $\frac{2}{\sqrt{2}}$

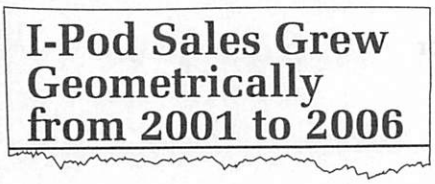
Multiple Choice

1. C **2.** A **3.** B

REVIEW

6.1

CR1 **1.** Explain the meaning of this newspaper headline.



US, CR2 **2.** A soapstone carving was appraised at $2500. The value of the carving is estimated to increase by 12% each year. What will be the approximate value of the carving after 15 years?

6.2

3. Determine the sum of the geometric series below. Give the answer to 3 decimal places.

$-700 + 350 - 175 + \ldots + 5.468\ 75$

6.3

4. Use graphing technology.

Use the series from question 3. Graph the first 5 partial sums. Explain how the graph shows whether the series converges or diverges.

6.4

RM, CR1

5. Explain how you can use the common ratio of a geometric series to identify whether the series is convergent or divergent.

6. Identify each infinite geometric series that converges. Determine the sum of any series that converges.

a) $2 - 3 + 4.5 - 6.75 + \ldots$

b) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \ldots$

c) $\sum_{k=1}^{\infty} 3\left(\frac{1}{4}\right)^{k-1}$

RM, US, CR2

7. A small steel ball bearing is moving vertically between two electromagnets whose relative strength varies each second. The ball bearing moves 10 cm up in the 1st second, then 5 cm down in the 2nd second, then 2.5 cm up in 3rd second, and so on. This pattern continues.

a) Assume the distance the ball bearing moves up is positive; the distance it moves down is negative.

i) Use sigma notation to write a series to represent the distance travelled in 5 s.

ii) Calculate the sum of the series. What does this sum represent?

b) Suppose this process continues indefinitely. What is the sum of the series?

ANSWERS

2. \$13 684 **3.** approximately −464.844 **6. a)** diverges **b)** converges; 1 **c)** converges; 4

7. a) i) $10 - 5 + 2.5 - 1.25 + 0.625$ **ii)** 6.875 cm **b)** $6.\overline{6}$ cm