Chapter –EXPONENTIALS, LOGARITHMS and geometric sequences

I – EXPONENTIAL functions

A function is an exponential function when the variable is in the exponent.

Examples: $y = 2^x$, $y = 3 \times 5^{2x-1} - 4$

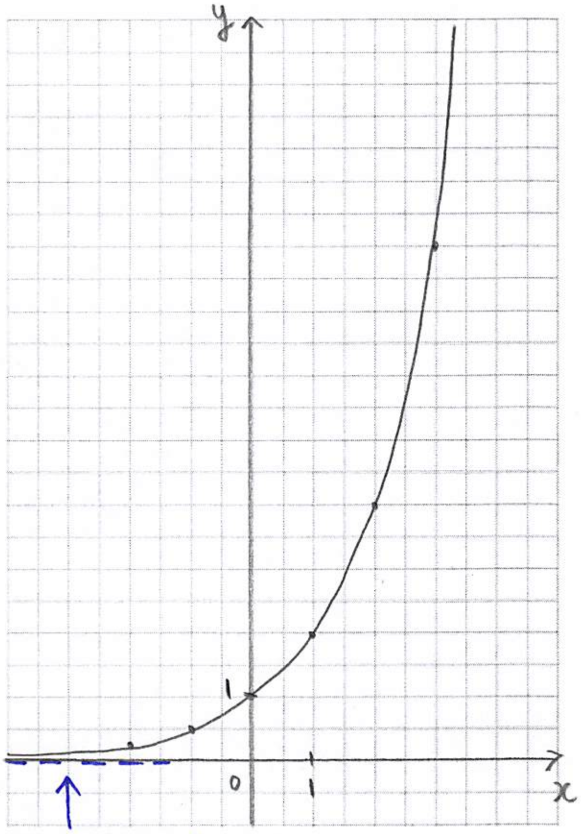
Basic exponential functions (with no transformation) are of the form:

$$y = c^x$$

base of the exponential function. $\boxed{⚠ \; c > 0}$

Examples : a) $y = 2^x$

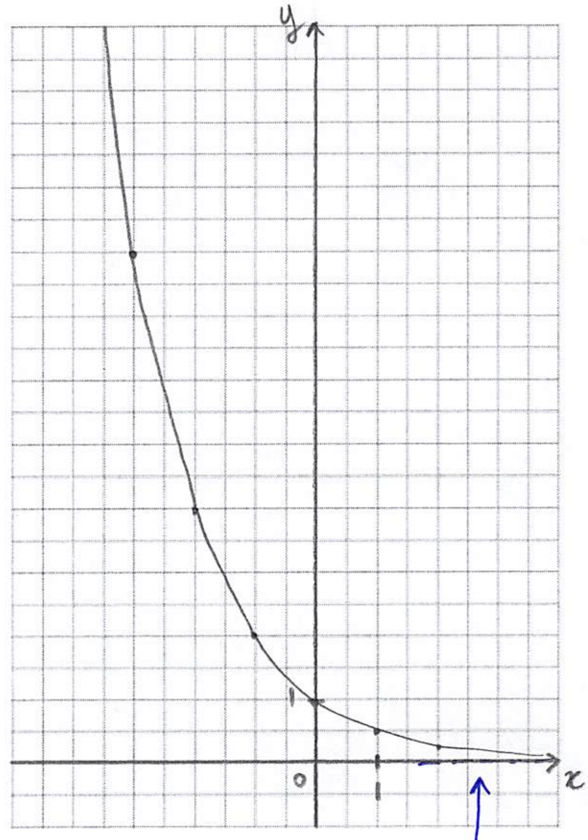
x	-2	-1	0	1	2	3
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



horizontal asymptote $y=0$

b) $y = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



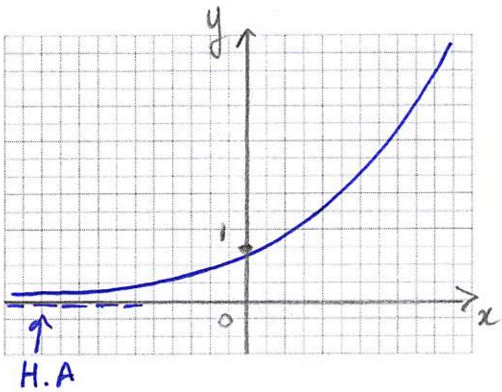
horizontal asymptote $y=0$

We can notice for both graphs that:

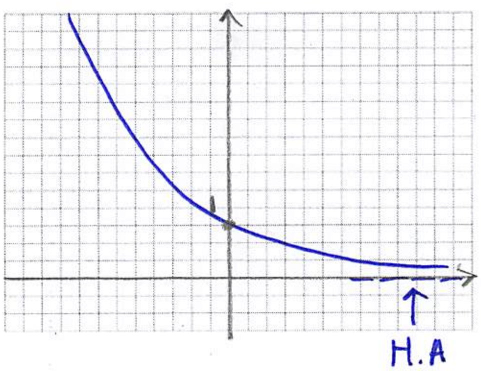
- Domain = $\mathbb{R}$
- Range = $(0, \infty)$
- horizontal asymptote: $y = 0$.
- y-int: 1
- no x-int
- we get the value of the base when $x = 1$.
  we get the reciprocal of the base when $x = -1$.

These are true for all basic exponential functions (not transformed).
Also :

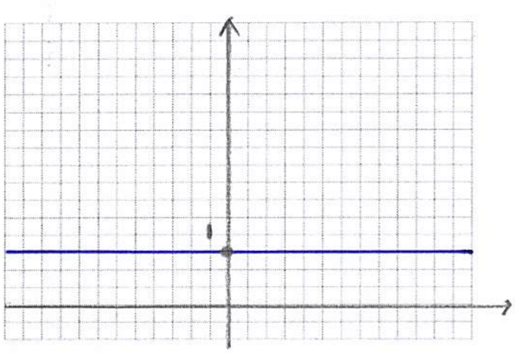
If the base $c > 1$, the function is increasing.



If $0 < c < 1$, the function is decreasing.



If $c = 1$, the function is constant : $y = 1$.



Hwk : p 342 # 1 – 6, 8 – 10, 12

II – LOGARITHMIC functions

a) Definition of logarithms

The inverse functions of exponential functions are called **logarithmic functions**.

$y = \log_3 x$ is the inverse function of $y = 3^x$

Inverse functions are used to express a relation "backwards".
For example, functions $y = \sqrt{x}$ and $y = x^2$ are inverse of each other when $x > 0$.
and $\sqrt{9} = 3$ can be expressed "backwards" as $3^2 = 9$.

The same way, $8 = 2^3$ can be written backwards as: $\log_2 8 = 3$

We notice that the outcome of the logarithm is the exponent on the exponential function…

Examples :

$3^4 = 81$ can be written : $\log_3 81 = 4$

$2^5 = 32$ can be written : $\log_2 32 = 5$

$3^x = 20$ can be written : $\log_3 20 = x$

$3^x = y$ can be written : $\log_3 y = x$

$3^{x-2} = y + 5$ can be written : $\log_3 (y+5) = x - 2$

$\log_3 9 = 2$ can be written : $3^2 = 9$

$\log_5 x = 4$ can be written : $5^4 = x$

Note : you can only change form (exp → log or log → exp) if the exp or log function is ISOLATED!!

Also, since inverse functions cancel the action of each other, we get:

$\boxed{\log_c c^a = a}$ et. $\boxed{c^{\log_c a} = a}$

a log on base c of an exponential on base c

an exponential on base c of a log on base c

Examples :

$\log_2 2^4 = 4$

$\log_2 8 = \log_2 2^3 = 3$

$5^{\log_5 10} = 10$

$7^{\log_7 310} = 310$

Applications: Determine the following exact values:

$\log_7 49 = 2 \quad (7^? = 49)$

$\log_2 \sqrt{8} = \log_2 \sqrt{2^3} = \log_2 2^{3/2} = \frac{3}{2}$

$\log_6 1 = 0 \quad (6^0 = 1)$

$\log_9 27 = x \quad 9^x = 27 \quad 3^{2x} = 3^3 \quad 2x = 3 \quad \boxed{x = 3/2}$

Note 1: When we don't write a base, we assume that the base is 10: $\log\ 0.001 = \log_{10} 10^{-3} = -3$

Note 2: If the argument and the base are powers of the same number, it is possible to evaluate the exact value without a calculator. (cf $\log_9 27$)

Your turn p 374

b) Basic Logarithmic functions (not transformed)

To graph a logarithmic function, we can make a table of values of the corresponding exponential function and then exchange the coordinates <u>or</u> use the symmetry with line $y = x$.

ATTENTION: Exponential functions don't satisfy the horizontal line test if their base is 1. It means that logarithmic functions with base 1 are undefined!!

Also, by properties of inverse functions, domain and Range are swapped.
Therefore,

- $D = (0, \infty)$
- $I = \mathbb{R}$

Examples: a) $y = \log_3 x$

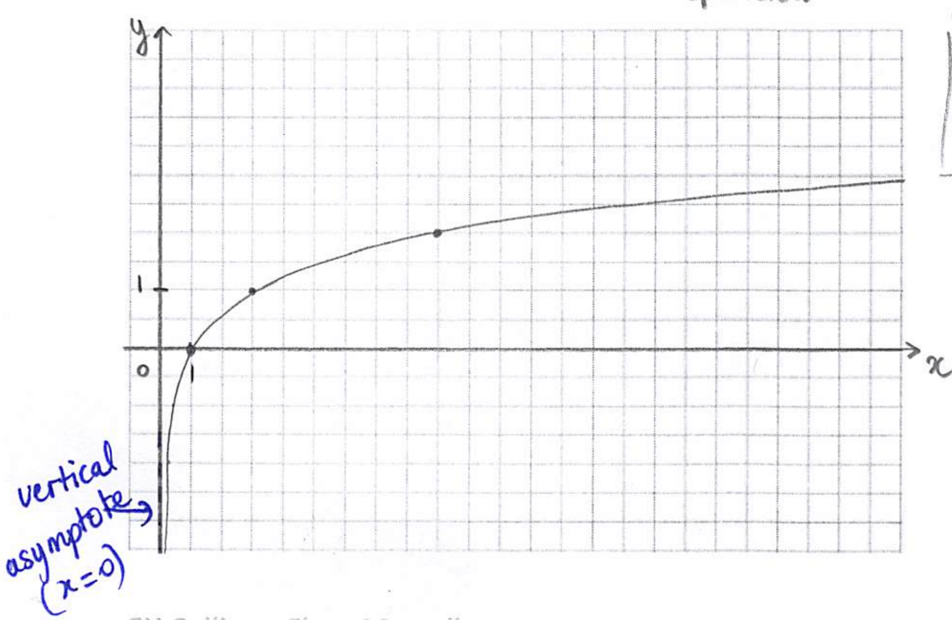
can be written on base 3

x	1/9	1/3	1	3	9	27
y	-2	-1	0	1	2	3

exponents

$y = 3^x$

x	y
-2	1/9
-1	1/3
0	1
1	3
2	9
3	27

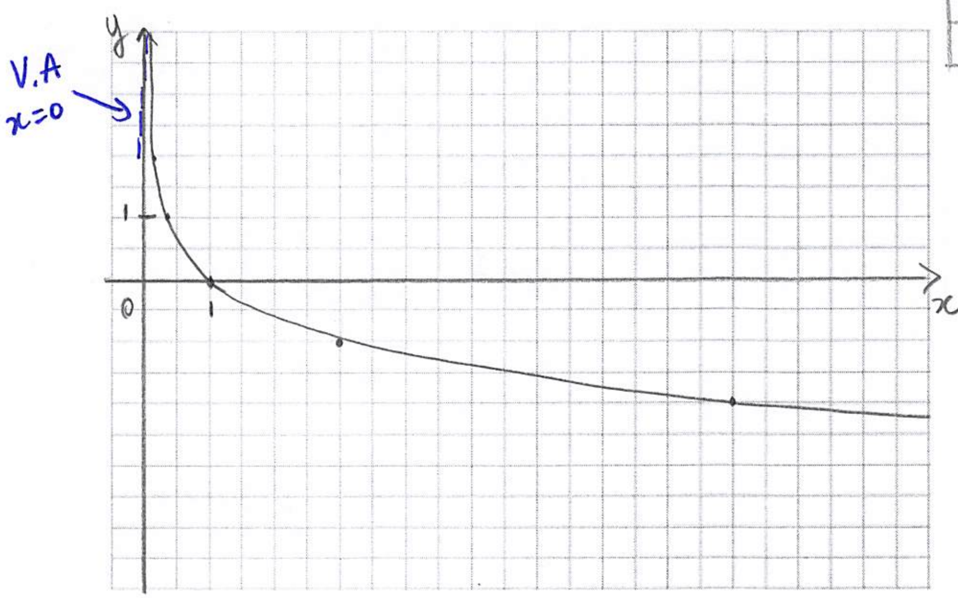


b) $y = \log_{1/3} x$

$y = \left(\frac{1}{3}\right)^x$

x	y
-3	27
-2	9
-1	3
0	1
1	1/3
2	1/9

x	1/9	1/3	1	3	9	27
y	2	1	0	-1	-2	-3



We notice for both graphs that:

- Vertical asymptote: $x = 0$.
- No y-int (it's not on the domain)
- x-int $x = 1$.

- we get the value of the base when y = 1.

$\boxed{\log_c 1 = 0}$

$\boxed{\log_c c = 1}$

These are true for all logarithmic functions (not transformed).

Also :

If the base $\boldsymbol{c > 1}$, the function is increasing.

If $\boldsymbol{0 < c < 1}$, the function is decreasing.

If $\boldsymbol{c = 1}$, the function is undefined!

$c \neq 1$!

Note : With your calculator :

Hwk : p 380 # 1 – 6, 8, 9, 12 – 16, 21

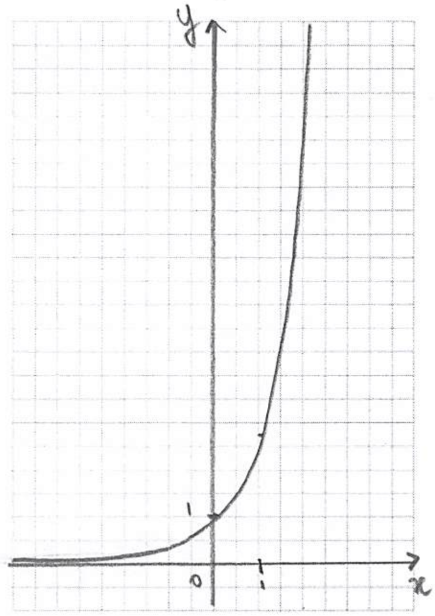
c) Exponentials and logarithms base e

Among the famous irrational numbers (like π), one of them is very important for calculus:

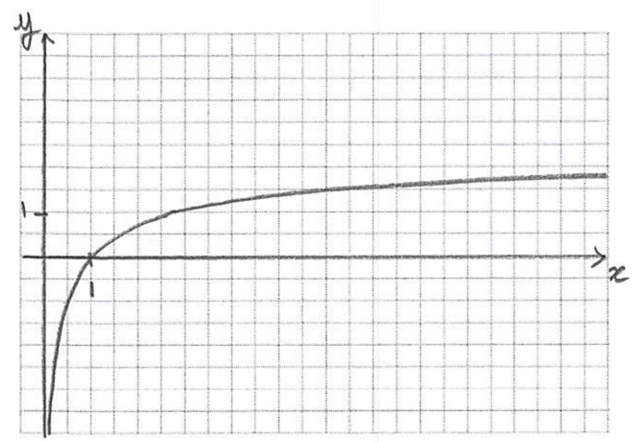
$$e \approx 2.71828$$

The logarithm on base e is called natural logarithm. We write : $\underline{\underline{ln}}(x)$

$y = e^x$



$y = lnx$



$e^{-2} = \frac{1}{e^2}$
$ln1 = 0$
$lne = 1$
$ln(e^x) = x$

III – Transformations of exponential and logarithm functions

$\boxed{y = a\log_c(b(x-h)) + k}$ ← vertical translat°

vertical stretch factor $|a|$
reflection x-axis

horizontal translation

horizontal stretch factor $\frac{1}{|b|}$
reflection y-axis

$$\boxed{y = a \cdot c^{b(x-h)} + k}$$

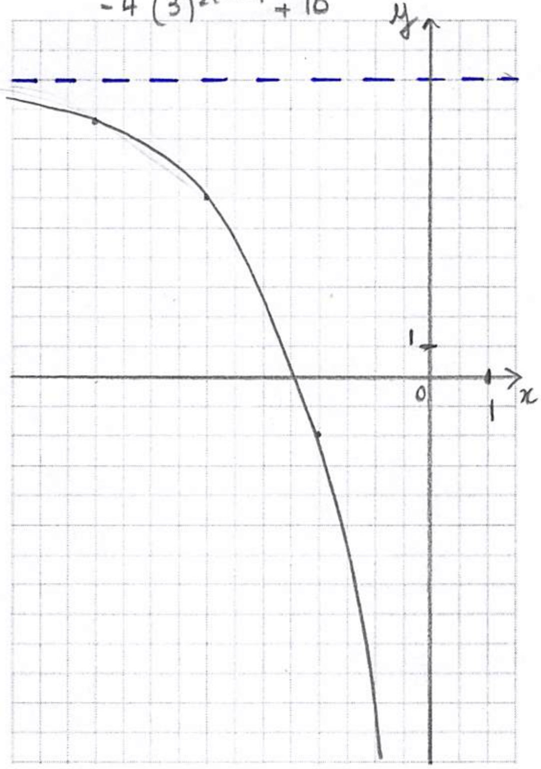
It is important to remember that exponential functions have a horizontal asymptote $y = 0$ that can only be "moved" by applying a vertical translation.

Logarithmic functions have a vertical asymptote $x = 0$ that can only be "moved" by applying a horizontal translation.

Examples :

i) $y = -4 \times 3^{\frac{1}{2}x+2} + 10$

$-4\,(3)^{\frac{1}{2}(x+4)} + 10$



$y = 3^x$

x	y
-1	1/3
0	1
1	3
2	9
~~3~~	~~27~~

too large

H.A $y=0$

$\xrightarrow{\text{reflections and stretches}}$

$x \times 2$

$y \times (-4)$

x	y
-2	-4/3
0	-4
2	-12
4	-36
6	

$\xrightarrow{\text{translations}}$

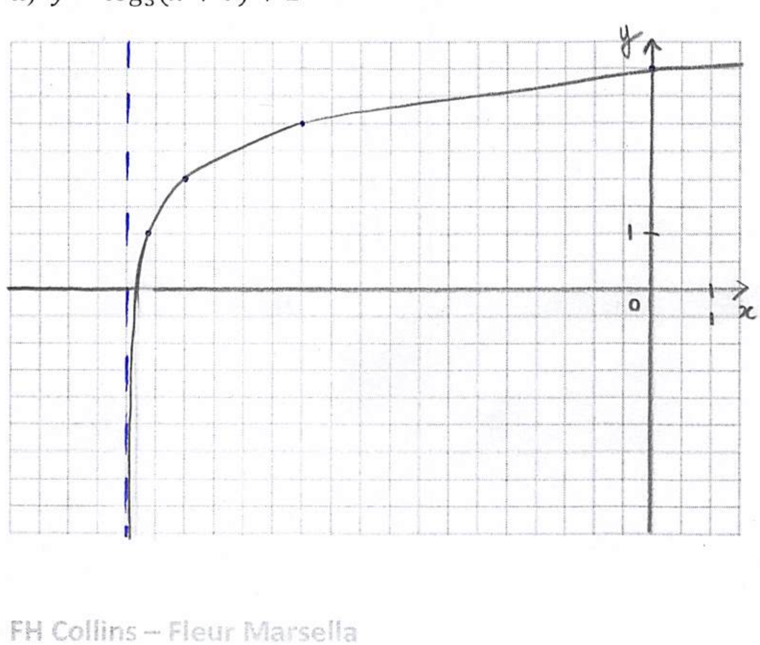
$x - 4$

$y + 10$

x	y
-6	$8.\overline{6}$
-4	6
-2	-2
0	-26

H.A $y = 10$

ii) $y = \log_3(x + 9) + 2$



$y = 3^x$

x	y
-1	1/3
0	1
1	3
2	9

$\longrightarrow$ $y = \log_3 x$

x	y
1/3	-1
1	0
3	1
9	2

VA $x=0$

$\xrightarrow{\text{translations}}$

$x-9$

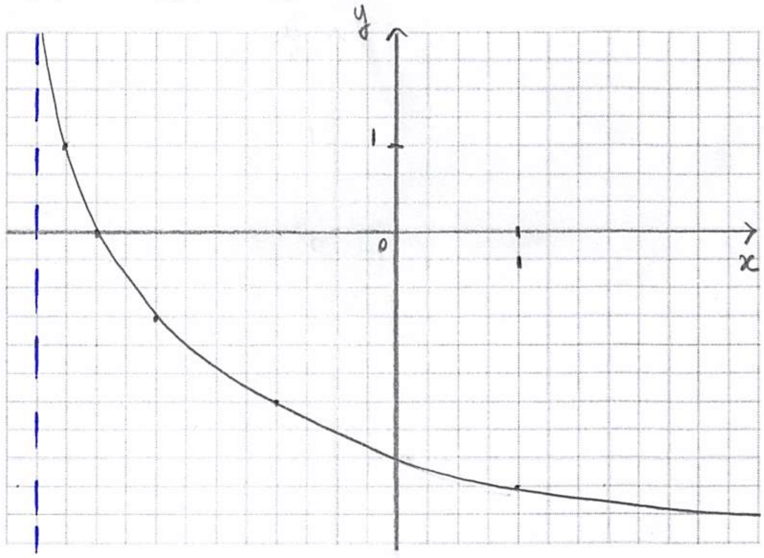
$y+2$

x	y
$-8.\overline{6}$	1
-8	2
-6	3
0	4

V.A $x= -9$

$y = -\log_2(2(x+3))$

iii) $y = -\log_2(2x+6)$



$y = 2^x$ $\longrightarrow$ $y = \log_2 x$

x	y
-1	1/2
0	1
1	2
2	4
3	8

x	y
1/2	-1
1	0
2	1
4	2
8	3

V.A $x = 0$

$\xrightarrow{\text{reflections}}$ & stretches

$x \times \frac{1}{2}$

$y \times (-1)$

x	y
1/4	1
1/2	0
1	-1
2	-2
4	-3

$\xrightarrow{\text{translations}}$

$x - 3$

x	y
-2.75	1
-2.5	0
-2	-1
-1	-2
1	-3

V.A $x = -3$

Hwk : p 354 # 4, 6, 7, 9, 11, 12 & p 389 # 2, 4 – 6, 9, 10, 12 – 14, 17.

IV – The laws of exponents and logarithms

a) Reminder on the exponent laws

$c^x \times c^y = c^{x+y}$

$\frac{c^x}{c^y} = c^{x-y}$

$(c^x)^y = c^{x \cdot y}$

Examples : Simplify

i) $\frac{3^{2x-1} \times 3^5}{3^{x-3}} = \frac{3^{2x-1+5}}{3^{x-3}} = \frac{3^{2x+4}}{3^{x-3}} = 3^{2x+4-(x-3)} = 3^{x+7}$

ii) $(2 \times 5^{2x+1})^3 = 2^3 \times 5^{3(2x+1)} = 8 \times 5^{6x+3}$

iii) $4^{2x+1} - 4^{2x} = 4^{2x}(4-1) = 3 \times 4^{2x}$

iv) $\frac{e^{2x} \times (e^{x+1})^3}{e^{2x-5}} = \frac{e^{2x} \cdot e^{3x+3}}{e^{2x-5}} = \frac{e^{5x+3}}{e^{2x-5}} = e^{3x+8}$

b) The laws of logarithms

- $\log_c(M \times N) = \log_c M + \log_c N$
- $\log_c \frac{M}{N} = \log_c M - \log_c N$
- $\log_c(M^n) = n \times \log_c M$

$(n \in \mathbb{R},\ M, N, c \text{ positive},\ c \neq 1)$

Examples :

i) $\log_5 \frac{xy}{z} = \log_5 x + \log_5 y - \log_5 z$

ii) $\log_7 \sqrt[3]{x} = \log_7 x^{1/3} = \frac{1}{3} \log_7 x$

iii) $\log_6 \frac{1}{x^2} = \log_6 1 - \log_6 x^2$ or $= \log_6 x^{-2}$

$= 0 - 2\log_6 x$ $\qquad = -2\log_6 x$

$= -2\log_6 x$

iv) $\log \frac{x^3}{y\sqrt{z}} = \log x^3 - \log(y\sqrt{z})$

$= 3\log x - (\log y + \log \sqrt{z})$

$= 3\log x - \log y - \frac{1}{2}\log z$

Your turn p 395

v) $\log_6 8 + \log_6 9 - \log_6 2$ $= \log_6 \frac{8\times 9}{2} = \log_6 \frac{72}{2} = \log_6 36 = 2$

vi) $\log_7 7\sqrt{7}$ $= \log_7 7 + \log_7 \sqrt{7}$ or $= \log_7 7^{3/2}$

$= 1 + \frac{1}{2}$ $= \frac{3}{2}$

$= \frac{3}{2}$

vii) $2\log_2 12 - \left(\log_2 6 + \frac{1}{3}\log_2 27\right)$ $= \log_2 12^2 - \log_2 6 - \log_2 \sqrt[3]{27}$

$= \log_2 \frac{12^2}{6\times 3}$

$= \log_2 8$

$= 3$

Your turn p 396

viii) $\log_7 x^2 + \log_7 x - \frac{5\log_7 x}{2}$ $= 2\log_7 x + \log_7 x - \frac{5}{2}\log_7 x$

$= \frac{1}{2}\log_7 x$

or $\log_7 \frac{x^2 \cdot x}{x^{5/2}} = \log_7 x^{3-5/2} = \log_7 x^{1/2} = \frac{1}{2}\log_7 x$

ix) $\log_5(2x-2) - \log_5(x^2+2x-3)$

$= \log_5 \frac{2x-2}{x^2+2x-3}$

$= \log_5 \frac{2(x-1)}{(x+3)(x-1)}$

$= \log_5 \frac{2}{x+3}$

Hwk : p 400 # 1 – 3, 5 – 13, 16, 18, 20.

c) Restrictions on the domain for logarithms

⚠ not covered in your textbook → Restrictions on the domain

Reminder : $y = \log_c(x)$ is defined when $\begin{cases} x > 0 \\ c > 0 \\ c \neq 1 \end{cases}$

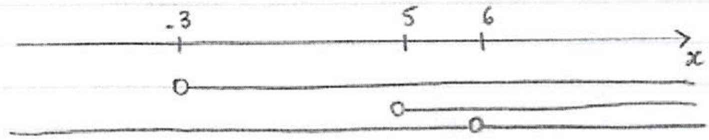
Examples : 1) $y = \log_3(10 - 3x)$

↳ $10 - 3x > 0$ so $\boxed{x < \frac{10}{3}}$ no problem with the base here! $3 \neq 1$ & $3 > 0$

2) $y = \log_{x-5}(x+3)$

↳ • $x + 3 > 0$ and • $x - 5 > 0$ and • $x - 5 \neq 1$

$x > -3$ $x > 5$ $x \neq 6$



$D = \{x \in \mathbb{R} \mid x > 5 \text{ and } x \neq 6\}$

Your turn : • $y = \log_5(x - 3)$ ⟶ $x > 3$

• $y = \log_{2-x}(x - 1)$ ⟶ $1 < x < 2$

• $y = \log_x(x^2 - 1)$ ← ⚠ quadratic inequalities ... ⟶ $x > 1$

Restrictions are MANDATORY when solving an equation!!

ATTENTION, It is important to determine the restrictions before simplifying !!

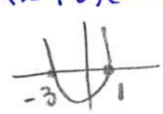
Examples :

a) $\log_7 x^2$ Restriction : $x^2 > 0$ i.e $\boxed{x \neq 0}$

b) $\log_5(2x - 2) - \log_5(x^2 + 2x - 3)$

Restrictions : $2x - 2 > 0$ and $x^2 + 2x - 3 > 0$

$x > 1$ $(x+3)(x-1) > 0$



$x < -3$ or $x > 1$

Conclusion : $\boxed{x > 1}$

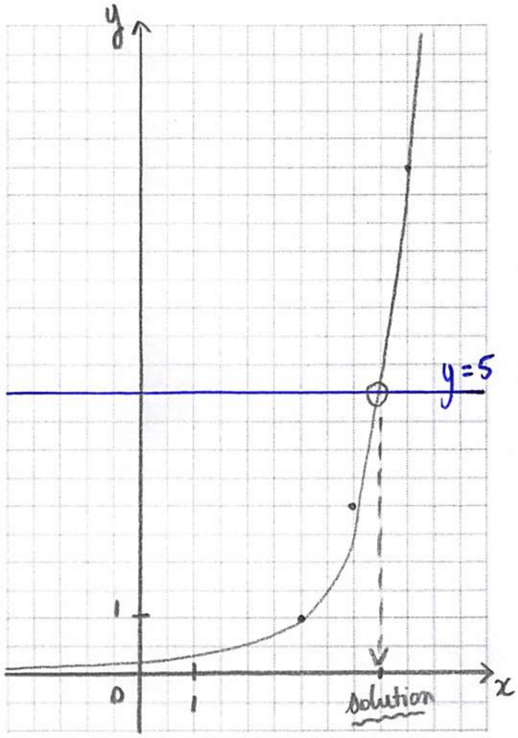
Hwk : Extra Practice

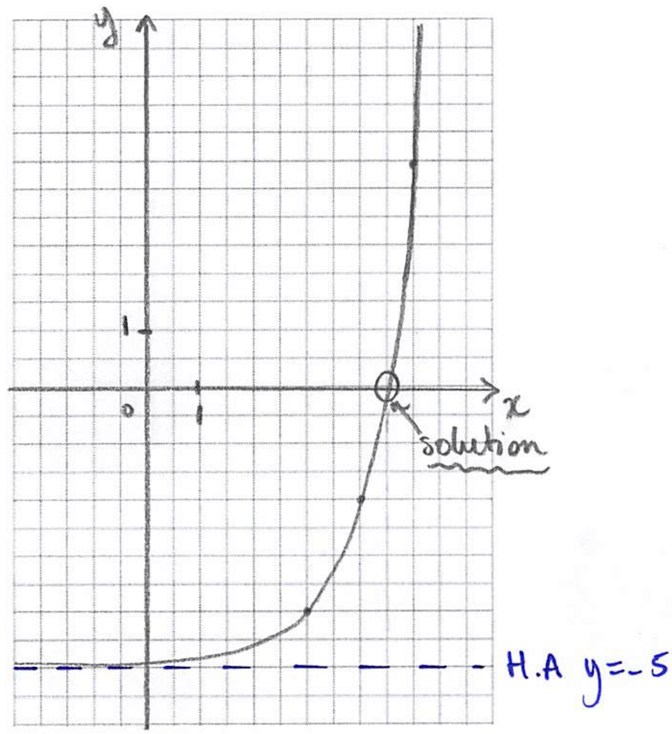
V – Exponential and Logarithmic EQUATIONS

a) Graphically

To solve any type of equation graphically, you need to graph each side of the equation and look for the x-coordinates of the points of intersection OR move all the terms on the same side, graph it and look for the zeros.

Example : Solve $3^{x-3} = 5$ or $3^{x-3} - 5 = 0$





If you don't need to show yoru work, it would be faster to use your graphing calculator…

b) Solving EXPONENTIAL equations algebraically

It's good to notice that exponential equations **don't have any restrictions**.

- **First thing to try: WRITE ALL THE FACTORS ON THE SAME BASE**

Examples : Write the following numbers on base 3

i) $27 = 3^3$

ii) $9^4 = (3^2)^4 = 3^8$

iii) $27^{1/3}\left(\sqrt[3]{81}\right)^2 = (3^3)^{1/3} \cdot \left[(3^4)^{1/3}\right]^2 = 3^1 \cdot 3^{4/3 \times 2}$
$= 3^1 \cdot 3^{8/3} = 3^{11/3}$

Your turn p 360

Applications : Solve :

i) $4^{x+2} = 64^x$

$$4^{x+2} = (4^3)^x$$
$$4^{x+2} = 4^{3x}$$
$$x+2 = 3x$$
$$2 = 2x$$
$$\boxed{x=1}$$

ii) $4^{2x} = 8^{2x-3}$

$$(2^2)^{2x} = (2^3)^{2x-3}$$
$$2^{4x} = 2^{6x-9}$$
$$4x = 6x-9$$
$$9 = 2x$$
$$\boxed{x = 9/2}$$

Attention, it's not always possible!

Your turn p 361

Hwk : p 364 # 1 – 5, 7, 16, 17

- **If you can't write everything on the same base: use LOGARITHMS**

Examples :

i) $4^x = 605$

$$\ln(4^x) = \ln(605)$$
$$x \ln 4 = \ln 605$$
$$\boxed{x = \frac{\ln 605}{\ln 4}}$$
← exact values

[you could choose any other base...]

ex $\log_4(4^x) = \log_4(605)$ ⟶ $x = \log_4(605)$

$(x \simeq 4.62)$

ii) $5(3^{2x}) = 568$

$$\ln(5(3^{2x})) = \ln(568)$$
$$\ln 5 + \ln(3^{2x}) = \ln 568$$
$$\ln 5 + 2x \ln 3 = \ln 568$$
$$2x \ln 3 = \ln 568 - \ln 5$$
$$\boxed{x = \frac{\ln 568 - \ln 5}{2 \ln 3}}$$

$(x \simeq 2.15)$

iii) $2^{4x-3} = 3^{2x-1}$ $\quad \ln(2^{4x-3}) = \ln(3^{2x-1})$

$$(4x-3)\ln 2 = (2x-1)\ln 3$$
$$4x\ln 2 - 3\ln 2 = 2x\ln 3 - \ln 3$$
$$4x\ln 2 - 2x\ln 3 = 3\ln 2 - \ln 3$$
$$x(4\ln 2 - 2\ln 3) = 3\ln 2 - \ln 3$$
$$\boxed{x = \frac{3\ln 2 - \ln 3}{4\ln 2 - 2\ln 3}} \quad (x \simeq 1.70)$$

Hwk : p 412 # 2, 7

c) Solving LOGARITHMIC equations algebraically

FIRST, you need to determine the restrictions on the variable!

The, you need to ISOLATE your logarithms in order to switch to exponential form (in order to "free" your variable from the logarithms…)

Make sure there is no more than 1 log by itself per side…

$\log_c A = \log_c B$	ou	$\log_c A = B$
$\Downarrow$		$\Downarrow$
$A = B$		$c^B = A$

Examples :

i) $\log_2(x+3)^2 = 4$

- Restrictions : $(x+3)^2 > 0$ i.e $x+3 \neq 0$ $\quad x \neq -3$
- Resolution : $2^4 = (x+3)^2$

$$16 = x^2 + 6x + 9$$
$$x^2 + 6x - 7 = 0$$
$$(x+7)(x-1) = 0$$
$$\boxed{x = -7} \text{ or } \boxed{x = 1}$$

Solutions : $\{-7, 1\}$

ii) $\log(8x + 4) = 1 + \log(x + 1)$

- Restrictions: • $8x+4>0$ and $x+1>0$

$\boxed{x > -\frac{1}{2}}$ $\quad x>-1$

- Resolution: $\log(8x+4) = \log 10 + \log(x+1)$

$\log(8x+4) = \log 10(x+1)$

$8x+4 = 10x+10$

$-6 = 2x$

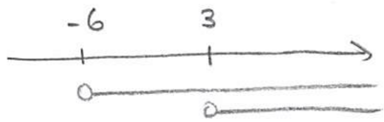
$\cancel{x=-3}$ Restrictions

$\Rightarrow$ no solution

iii) $\log_6(x - 3) + \log_6(x + 6) = 2$

- Restrictions: • $x-3>0$ and $x+6>0$

$\boxed{x>3}$ $\quad x>-6$



- Resolution: $\log_6 (x-3)(x+6) = 2$

$6^2 = (x-3)(x+6)$

$36 = x^2 + 6x - 3x - 18$

$x^2 + 3x - 54 = 0$

$(x+9)(x-6) = 0$

$\cancel{x=-9}$ or $\boxed{x=6}$ $\quad$ solution: $\{6\}$

Restr x

Note : Any number can be written as a logarithm (on any base):

ex : $1 = \log_5 5$ $\qquad 2 = \log_3 9$ $\qquad 3 = \log_2 8$ $\qquad 4 = \log 10000$

Hwk : p 412 # 3, 5, 6, 8

d) Usual word problems

Exponential functions are often used to model real life situations such as:
- evolution of radioactive elements
- evolution of populations
- interests in banking

initial amount (when $t=0$)

$$A = P \times r^{t/n}$$

A: amount at time t

r: rate

t/n: how often the rate is applied (in the same unit as t)

Examples:
- Population of bacterias triples every 10 days. You start with 5 bacterias. How many do you have after t days?

  ↳ $N = 5 \times 3^{t/10}$ t: # days; N: # bacterias after t days

- Radioactive element Americium has a half life of 432 years. If you have 10mg of it, how much will be left after t years?

  ↳ $A = 10 \times \left(\frac{1}{2}\right)^{t/432}$ t: # years; A: amount left in mg.

- The population of a town increases by 5% every year. There are 20000 people today. How many people will live there in t years?

  ↳ $P = 20000 \times 1.05^{t}$ t: # years; P: population

NOTE: increases by 12% $\Rightarrow r = 1.12$
decreases by 7% $\Rightarrow r = 0.93$
is 20% of $\Rightarrow r = 0.2$

- An amount of \$1500 is deposited in a bank paying an annual interest rate of 4.3% compounded quarterly. What will be the balance after 6 years?

↳ $A = 1500 \times 1.01075^{4t}$

$A = 1500 \times 1.01075^{4\times6}$
$\simeq \$1938.84$

means you get a quarter of it 4 times a year! (1.075% each time...)

Note : compounded semi annually would mean that you get half of the interest twice a year...

Other common word problem : Earthquakes (or Sound level):

The magnitude, M (on the Richter scale) and the amplitude, A (or strength) of an earthquake are related by: $\boxed{M = \log\left(\frac{A}{A_0}\right)}$ where A_0 is a constant. ← will be given on the test!

If a first earthquake has a magnitude of 6.4, and a second earthquake is 15 times stronger, what is the magnitude of the second earthquake?

$M_1 = 6.4 \longrightarrow 6.4 = \log\left(\frac{A_1}{A_0}\right)$

$A_2 = 15A_1$

$M_2 = ?$

$10^{6.4} = \frac{A_1}{A_0}$

$A_1 = A_0 \cdot 10^{6.4}$

$\rightarrow A_2 = 15 \cdot A_0 \cdot 10^{6.4}$

$\rightarrow M_2 = \log \frac{A_2}{A_0}$

$M_2 = \log\left(\frac{15\cancel{A_0}\cdot 10^{6.4}}{\cancel{A_0}}\right) = \log(15 \cdot 10^{6.4}) \simeq \boxed{7.6}$

Hwk : p 364 # 9 – 14 p 381 # 19 p 401 # 13, 16 p 412 # 10 – 13, 15 – 18.

VI – geometric sequences and series

A. Sequences

A **geometric sequence** is a list of values, where each value is obtained by multiplying the previous value by a fixed number (called common ratio).

Examples : $2, 6, 18, 54, \dots$
$1, -5, 25, -125, \dots$
$18, 6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots$

To determine the common ratio of the sequence (in case it's not obvious) you just need to divide a term of the sequence by the previous one.

To know if a sequence is geometric, you need to divide each term by the previous one and see if you always get the same result.

General Term of a geometric sequence: (it's the formula that gives the value of any term knowing its position)

position of the nth term in the sequence.

$$\boxed{t_n = t_1 \bullet r^{n-1}}$$

value of the nth term

Value of the 1st term

common ratio

Examples : using the previous examples, we get:

$t_n = 2 \times 3^{n-1}$ in particular : $t_5 = 2 \times 3^4 = 162$

$t_n = (-5)^{n-1}$ in particular : $t_5 = (-5)^4 = 625$

$t_n = 18 \times \left(\frac{1}{3}\right)^{n-1}$ in particular : $t_7 = 18 \times \left(\frac{1}{3}\right)^6 = \frac{2}{81}$

We notice that we get a geometric sequence if we restrict the domain of an exponential function to the whole numbers only… (That's why it's in this chapter!)

Applications:

i) **Determining the general term**

Bacteria reproduce themselves by dividing into 2 bacteria. If you have a sample of 10 bacteria, how many bacteria will you have on the nth generation?

$10 \times 2^{n-1}$

ii) **Determining a specific term**

A photocopy machine can reduce an original 67% of its size. If you reduce a square picture with side 25cm five times in a row, what is the size of the picture obtained?

$25 \times 0.67^5 \simeq 3.38$ cm

iii) **Determining t_1 and r**

The third term of a geometric sequence is 54 and the sixth term is -1458.

Determine the values of the 3 first terms of this sequence.

$t_3 = 54$

$t_6 = -1458$

$t_6 = t_3 \times r^3$

$-1458 = 54 \times r^3$

$-27 = r^3$

$r = \sqrt[3]{-27}$

$r = -3$

- $t_3 = t_1 \times r^2$

$54 = t_1 \times (-3)^2$

$\frac{54}{9} = t_1$ $\boxed{t_1 = 6}$

- $t_2 = t_1 \times r$

$t_2 = 6 \times (-3)$ $\boxed{t_2 = -18}$

iv) **Determining the number of terms of a sequence**

Determine the number of terms of the following geometric sequence : 2, 1, $\frac{1}{2}$, …, $\frac{1}{512}$.

$t_1 = 2$

$r = \frac{1}{2}$

$t_n = \frac{1}{512}$

?

$\frac{1}{512} = 2 \times \left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{1024} = \left(\frac{1}{2}\right)^{n-1}$

$\left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{n-1}$

$n - 1 = 10$

$\boxed{n = 11}$

Hwk : worksheet p 39 # 1, 2, 3ab, 4 – 6, 8 – 10, 12, 14 – 16, 20, 25 + Extra practice

B. Series and sigma notation

A **geometric series** is the sum of the terms of a geometric sequence.

$S_n = t_1 + t_2 + \ldots + t_n$.

Example : The sequence 5, 15, 45, 135, … is geometric with common ration 3 and first term 5.

$S_4 = 5 + 15 + 45 + 135 = 200$.

General formulas to calculate these sums :

value of the 1st term

number of terms

- value of the sum of the n terms $\quad S_n = \dfrac{t_1(1-r^n)}{1-r}$ common ratio

This formula is useful if we know the number of terms but not necessarily the value of the last term…

value of the 1st term

- value of the sum of the n terms $\quad S_n = \dfrac{t_1 - r t_n}{1-r}$ value of the last term. common ratio

This formula is useful if we know the value of the last term, but not necessarily the number of terms…

Examples :

i) Determine the sum of the first 10 terms of each series :

$4 + 12 + 36 + \cdots$

$$S_{10} = \frac{4(1-3^{10})}{1-3}$$

$$S_{10} = 118\,096$$

$t_1 = 5$ et $r = \frac{1}{2}$

$$S_{10} = \frac{5\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$$

$$S_{10} = \frac{5115}{512}$$

ii) Determine the following sums :

$\frac{1}{27} + \frac{1}{9} + \frac{1}{3} + \cdots + 729$

$$S = \frac{\frac{1}{27} - 3(729)}{1-3}$$

$$S = \frac{29524}{27}$$

$4 - 16 + 64 - \cdots - 65536$

$$S = \frac{4-(-4)(-65536)}{1-(-4)}$$

$$S = -52428$$

Hwk: worksheet p 53 # 2ac, 3ac, 4ac, 5 – 7, 10, 13, 14, 16, 18, 19, 22.

Sigma Notation :
When we need to add many values that depend on a variable that takes all the integer values on a interval, we can use the following notation: $\sum$

Examples :

i) $\sum_{k=1}^{10} k =$ $1+2+3+4+5+6+7+8+9+10$

ii) $\sum_{k=1}^{10} 2 \times 3^k =$ $2\times3 + 2\times3^2 + 2\times3^3 + \dots + 2\times3^{10}$

iii) $\sum_{k=1}^{10} 2k - 3 =$ $(2\times1-3) + (2\times2-3) + (2\times3-3) + \dots + (2\times10-3)$

$= -1 + 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$

iv) $5 + 5 \times 2^1 + 5 \times 2^2 + 5 \times 2^3 + 5 \times 2^4 + 5 \times 2^5 + 5 \times 2^6 =$ $\sum_{k=0}^{6} 5 \times 2^k$

v) $1 + 3 + 9 + 27 + \dots + 59049 =$ $\sum_{k=0}^{10} 3^k$

vi) $2 + 4 + +6 + 8 + \dots + 50 =$ $\sum_{k=1}^{25} 2k$

C. Infinite Series

An infinite series is a series that doesn't have a last term (it keeps going forever). It has an infinite number of terms.

Definitions:

- We say that a **series** is **convergent** if S_n approaches a fixed value when n becomes very big.

  Example: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$
  The more terms we add, the closer the sum gets to the value 2.
  We write $S_\infty = 2$

- We say that a series is **divergent** if S_n doesn't approach any fixed value as n becomes really big…

  Example: $1 + 2 + 4 + 8 + \cdots$
  The ore terms we add, the larger the sum gets. It goes towards infinity…
  We can't use the notation S_∞ !

Property: Geometric series with ratios strictly between -1 and 1 are convergent. The others are divergent.

CONVERGENT SERIES iff $-1 < r < 1$

General formula **for convergent series only**!!:

$$S_\infty = \frac{t_1}{1-r}$$

value of the 1st term

limit value of the infinite sum

common ratio

Examples:

a) Determine if each series is convergent or divergent and calculate the infinite sum when possible:

$1 - \frac{1}{3} + \frac{1}{9} - \cdots$

$r = -\frac{1}{3}$

since $-1 < r < 1$, then it is convergent.

$S_\infty = \frac{1}{1-(-1/3)} = \boxed{\frac{3}{4}}$

$2 - 4 + 8 - \cdots$

$r = -2$

since $r < -1$ then it is divergent.

b) For which values of the variable are these following series convergent?

$1 + 2x + 4x^2 + 8x^3 + \cdots$

$r = 2x$

$-1 < 2x < 1$

$\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

$2 + \frac{2x}{5} + \frac{2x^2}{25} + \frac{2x^3}{125} + \cdots$

$r = \frac{x}{5}$

$-1 < \frac{x}{5} < 1$

$\boxed{-5 < x < 5}$

Hwk : worksheet p 63 # 1 – 5, 6 – 12, 15 – 17, 19, 22