CHAPTER 3

Polynomial Functions

Polynomial functions can be used to model different real-world applications, from business profit and demand to construction and fabrication design. Many calculators use polynomial approximations to compute function key calculations. For example, the first four terms of the Taylor polynomial approximation for the square root function are

$$\sqrt{x} \approx 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3.$$

Try calculating $\sqrt{1.2}$ using this expression. How close is your answer to the one the square root key on a calculator gives you?

In this chapter, you will study polynomial functions and use them to solve a variety of problems.

Did You Know?

A Taylor polynomial is a partial sum of the Taylor series. Developed by British mathematician Brook Taylor (1685–1731), the Taylor series representation of a function has an infinite number of terms. The more terms included in the Taylor polynomial computation, the more accurate the answer.

Key Terms

- polynomial function
- end behaviour
- synthetic division
- remainder theorem
- factor theorem
- integral zero theorem
- multiplicity (of a zero)

Career Link

Computer engineers apply their knowledge of mathematics and science to solve practical and technical problems in a wide variety of industries. As computer technology is integrated into an increasing range of products and services, more and more designers, service workers, and technical specialists will be needed.

Web Link

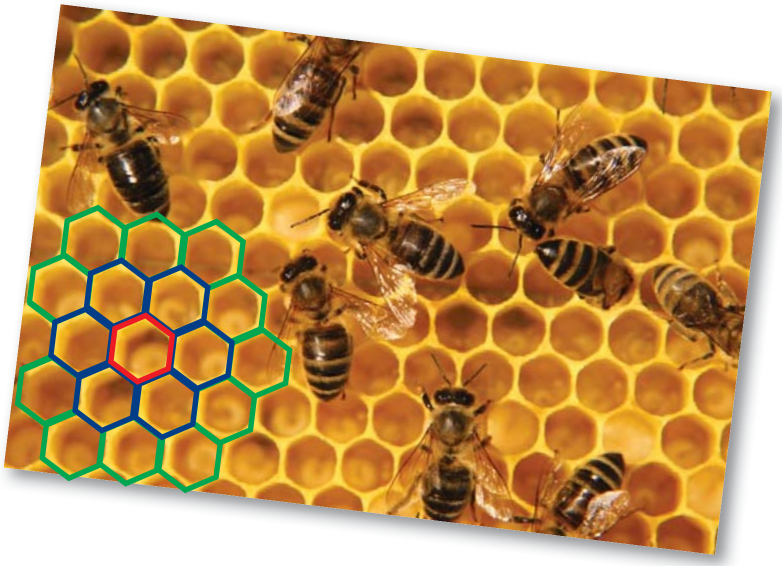
To learn more about a career in the field of computer engineering, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

3.1

Characteristics of Polynomial Functions

Focus on...

- identifying polynomial functions
- analysing polynomial functions



A cross-section of a honeycomb has a pattern with one hexagon surrounded by six more hexagons. Surrounding these is a third ring of 12 hexagons, and so on. The quadratic function $f(r)$ models the total number of hexagons in a honeycomb, where r is the number of rings. Then, you can use the graph of the function to solve questions about the honeycomb pattern.

A quadratic function that models this pattern will be discussed later in this section.

Did You Know?

Falher, Alberta is known as the "Honey Capital of Canada." The Falher Honey Festival is an annual event that celebrates beekeeping and francophone history in the region.

Investigate Graphs of Polynomial Functions

Materials

- graphing calculator or computer with graphing software

1. Graph each set of functions on a different set of coordinate axes using graphing technology. Sketch the results.

Type of Function	Set A	Set B	Set C	Set D
linear	$y = x$	$y = -3x$	$y = x + 1$	
quadratic	$y = x^2$	$y = -2x^2$	$y = x^2 - 3$	$y = x^2 - x - 2$
cubic	$y = x^3$	$y = -4x^3$	$y = x^3 - 4$	$y = x^3 + 4x^2 + x - 6$
quartic	$y = x^4$	$y = -2x^4$	$y = x^4 + 2$	$y = x^4 + 2x^3 - 7x^2 - 8x + 12$
quintic	$y = x^5$	$y = -x^5$	$y = x^5 - 1$	$y = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$

2. Compare the graphs *within* each set from step 1. Describe their similarities and differences in terms of
   - **end behaviour**
   - degree of the function in one variable, x
   - constant term
   - leading coefficient
   - number of x-intercepts

Recall that the degree of a polynomial is the greatest exponent of x.

end behaviour

- the behaviour of the y-values of a function as $|x|$ becomes very large

3. Compare the sets of graphs from step 1 to each other. Describe their similarities and differences as in step 2.

4. Consider the cubic, quartic, and quintic graphs from step 1. Which graphs are similar to the graph of
   - $y = x$?
   - $y = -x$?
   - $y = x^2$?
   - $y = -x^2$?

   Explain how they are similar.

Reflect and Respond

5. a) How are the graphs and equations of linear, cubic, and quintic functions similar?
   b) How are the graphs and equations of quadratic and quartic functions similar?
   c) Describe the relationship between the end behaviours of the graphs and the degree of the corresponding function.

6. What is the relationship between the sign of the leading coefficient of a function equation and the end behaviour of the graph of the function?

7. What is the relationship between the constant term in a function equation and the position of the graph of the function?

8. What is the relationship between the minimum and maximum numbers of x-intercepts of the graph of a function with the degree of the function?

Link the Ideas

The degree of a **polynomial function** in one variable, x, is n, the exponent of the greatest power of the variable x. The coefficient of the greatest power of x is the leading coefficient, a_n, and the term whose value is not affected by the variable is the constant term, a_0.

In this chapter, the coefficients a_n to a_1 and the constant a_0 are restricted to integral values.

What power of x is associated with a_0?

polynomial function

- a function of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$, where
  - n is a whole number
  - x is a variable
  - the coefficients a_n to a_0 are real numbers
- examples are $f(x) = 2x - 1$, $f(x) = x^2 + x - 6$, and $y = x^3 + 2x^2 - 5x - 6$

Example 1

Identify Polynomial Functions

Which functions are polynomials? Justify your answer. State the degree, the leading coefficient, and the constant term of each polynomial function.

a) $g(x) = \sqrt{x} + 5$

b) $f(x) = 3x^4$

c) $y = |x|$

d) $y = 2x^3 + 3x^2 - 4x - 1$

Solution

a) The function $g(x) = \sqrt{x} + 5$ is a radical function, not a polynomial function.
$\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, which has an exponent that is not a whole number.

b) The function $f(x) = 3x^4$ is a polynomial function of degree 4.
The leading coefficient is 3 and the constant term is 0.

c) The function $y = |x|$ is an absolute value function, not a polynomial function.
$|x|$ cannot be written directly as x^n.

d) $y = 2x^3 + 3x^2 - 4x - 1$ is a polynomial of degree 3.
The leading coefficient is 2 and the constant term is -1.

Your Turn

Identify whether each function is a polynomial function. Justify your answer. State the degree, the leading coefficient, and the constant term of each polynomial function.

a) $h(x) = \frac{1}{x}$

b) $y = 3x^2 - 2x^5 + 4$

c) $y = -4x^4 - 4x + 3$

d) $y = x^{\frac{1}{2}} - 7$

Characteristics of Polynomial Functions

Polynomial functions and their graphs can be analysed by identifying the degree, end behaviour, domain and range, and the number of x-intercepts.

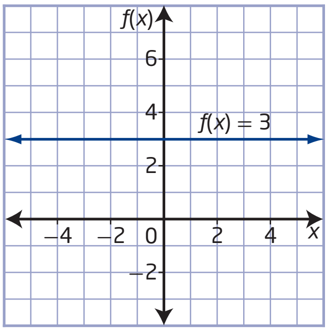
The chart shows the characteristics of polynomial functions with positive leading coefficients up to degree 5.

How would the characteristics of polynomial functions change if the leading coefficient were negative?

Degree 0: Constant Function

Even degree

Number of x-intercepts: 0 (for $f(x) \neq 0$)



Example: $f(x) = 3$

End behaviour: extends horizontally

Domain: $\{x \mid x \in \mathrm{R}\}$

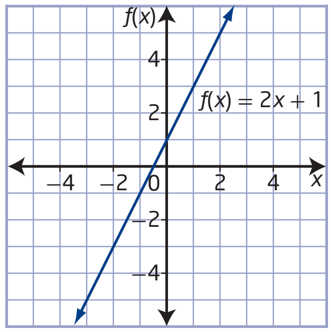
Range: $\{3\}$

Number of x-intercepts: 0

Degree 1: Linear Function

Odd degree

Number of x-intercepts: 1



Example: $f(x) = 2x + 1$

End behaviour: line extends down into quadrant III and up into quadrant I

Domain: $\{x \mid x \in \mathrm{R}\}$

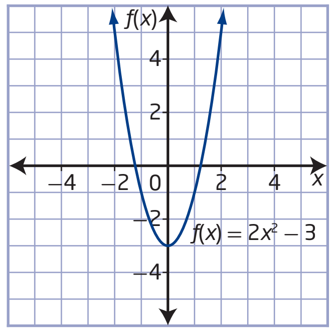
Range: $\{y \mid y \in \mathrm{R}\}$

Number of x-intercepts: 1

Degree 2: Quadratic Function

Even degree

Number of x-intercepts: 0, 1, or 2



Example: $f(x) = 2x^2 - 3$

End behaviour: curve extends up into quadrant II and up into quadrant I

Domain: $\{x \mid x \in \mathrm{R}\}$

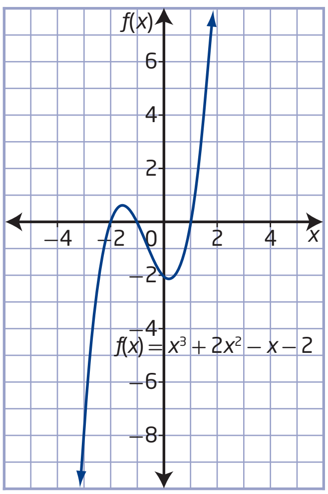
Range: $\{y \mid y \geq -2, y \in \mathrm{R}\}$

Number of x-intercepts: 2

Degree 3: Cubic Function

Odd degree

Number of x-intercepts: 1, 2, or 3



Example:

$f(x) = x^3 + 2x^2 - x - 2$

End behaviour: curve extends down into quadrant III and up into quadrant I

Domain: $\{x \mid x \in \mathrm{R}\}$

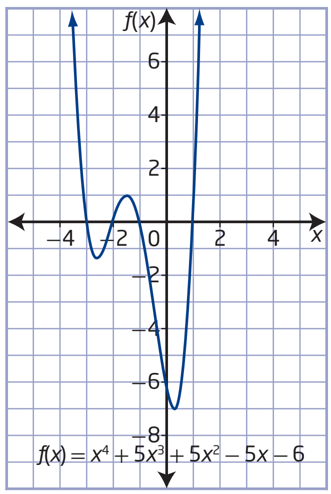
Range: $\{y \mid y \in \mathrm{R}\}$

Number of x-intercepts: 3

Degree 4: Quartic Function

Even degree

Number of x-intercepts: 0, 1, 2, 3, or 4



Example:

$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$

End behaviour: curve extends up into quadrant II and up into quadrant I

Domain: $\{x \mid x \in \mathrm{R}\}$

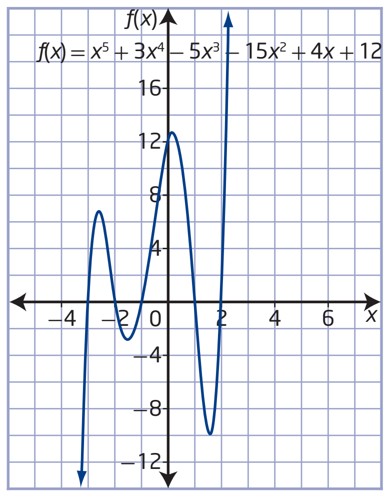
Range: $\{y \mid y \geq -6.91, y \in \mathrm{R}\}$

Number of x-intercepts: 4

Degree 5: Quintic Function

Odd degree

Number of x-intercepts: 1, 2, 3, 4, or 5



Example:

$f(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$

End behaviour: curve extends down into quadrant III and up into quadrant I

Domain: $\{x \mid x \in \mathrm{R}\}$

Range: $\{y \mid y \in \mathrm{R}\}$

Number of x-intercepts: 5

Example 2

Match a Polynomial Function With Its Graph

Identify the following characteristics of the graph of each polynomial function:

- the type of function and whether it is of even or odd degree
- the end behaviour of the graph of the function
- the number of possible x-intercepts
- whether the graph will have a maximum or minimum value
- the y-intercept

Then, match each function to its corresponding graph.

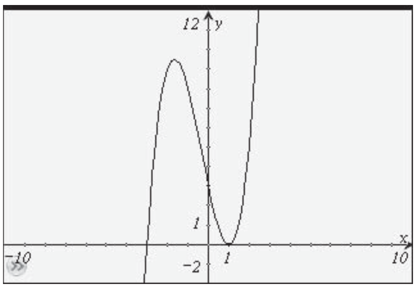
a) $g(x) = -x^4 + 10x^2 + 5x - 4$

b) $f(x) = x^3 + x^2 - 5x + 3$

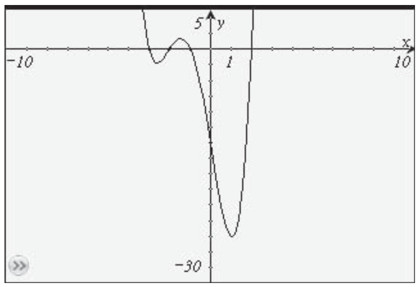
c) $p(x) = -2x^5 + 5x^3 - x$

d) $h(x) = x^4 + 4x^3 - x^2 - 16x - 12$

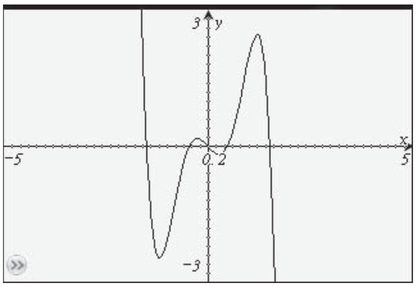
A



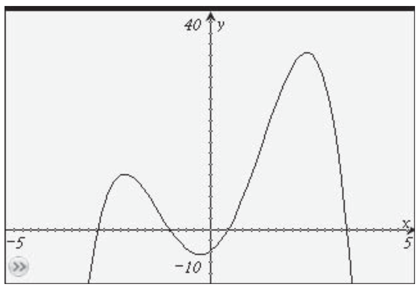
B



C



D



Solution

a) The function $g(x) = -x^4 + 10x^2 + 5x - 4$ is a quartic (degree 4), which is an even-degree polynomial function. Its graph has a maximum of four x-intercepts. Since the leading coefficient is negative, the graph of the function opens downward, extending down into quadrant III and down into quadrant IV (similar to $y = -x^2$), and has a maximum value. The graph has a y-intercept of $a_0 = -4$. This function corresponds to graph D.

b) The function $f(x) = x^3 + x^2 - 5x + 3$ is a cubic (degree 3), which is an odd-degree polynomial function. Its graph has at least one x-intercept and at most three x-intercepts. Since the leading coefficient is positive, the graph of the function extends down into quadrant III and up into quadrant I (similar to the line $y = x$). The graph has no maximum or minimum values. The graph has a y-intercept of $a_0 = 3$. This function corresponds to graph A.

c) The function $p(x) = -2x^5 + 5x^3 - x$ is a quintic (degree 5), which is an odd-degree polynomial function. Its graph has at least one x-intercept and at most five x-intercepts. Since the leading coefficient is negative, the graph of the function extends up into quadrant II and down into quadrant IV (similar to the line $y = -x$). The graph has no maximum or minimum values. The graph has a y-intercept of $a_0 = 0$. This function corresponds to graph C.

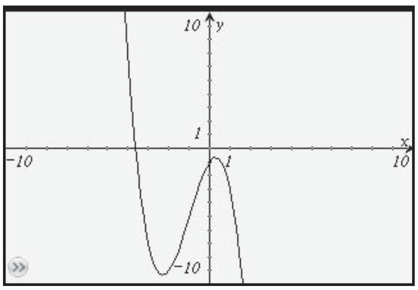
d) The function $h(x) = x^4 + 4x^3 - x^2 - 16x - 12$ is a quartic (degree 4), which is an even-degree polynomial function. Its graph has a maximum of four x-intercepts. Since the leading coefficient is positive, the graph of the function opens upward, extending up into quadrant II and up into quadrant I (similar to $y = x^2$), and has a minimum value. The graph has a y-intercept of $a_0 = -12$. This function corresponds to graph B.

Your Turn

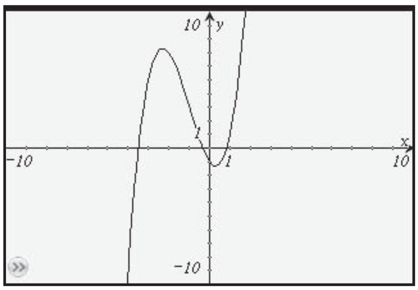
a) Describe the end behaviour of the graph of the function $f(x) = -x^3 - 3x^2 + 2x + 1$. State the possible number of x-intercepts, the y-intercept, and whether the graph has a maximum or minimum value.

b) Which of the following is the graph of the function?

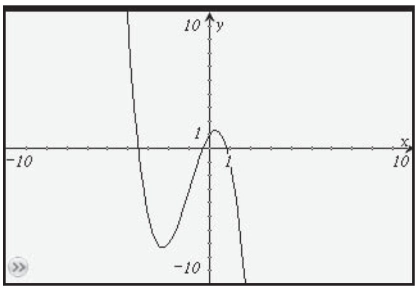
A



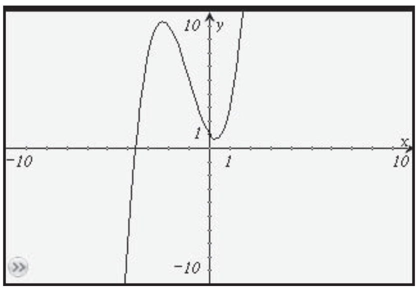
B



C



D



Example 3

Application of a Polynomial Function

A bank vault is built in the shape of a rectangular prism. Its volume, V, is related to the width, w, in metres, of the vault doorway by the function $V(w) = w^3 + 13w^2 + 54w + 72$.

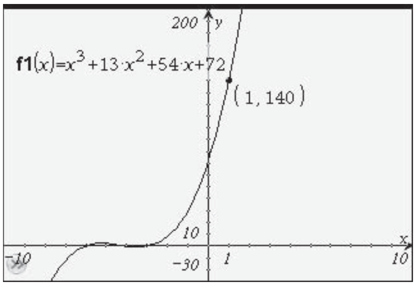
a) What is the volume, in cubic metres, of the vault if the door is 1 m wide?

b) What is the least volume of the vault? What is the width of the door for this volume? Why is this situation not realistic?

Solution

a) **Method 1: Graph the Polynomial Function**

Use a graphing calculator or computer with graphing software to graph the polynomial function. Then, use the trace feature to determine the value of V that corresponds to $w = 1$.



The volume of the vault is 140 m³.

Method 2: Substitute Into the Polynomial Function

Substitute $w = 1$ into the function and evaluate the result.

$V(w) = w^3 + 13w^2 + 54w + 72$
$V(1) = 1^3 + 13(1)^2 + 54(1) + 72$
$V(1) = 1 + 13 + 54 + 72$
$V(1) = 140$

The volume of the vault is 140 m³.

b) The least volume occurs when the width of the door is 0 m. This is the y-intercept of the graph of the function and is the constant term of the function, 72. The least volume of the vault is 72 m³. This situation is not realistic because the vault would not have a door.

What is the domain of the function in this situation?

Your Turn

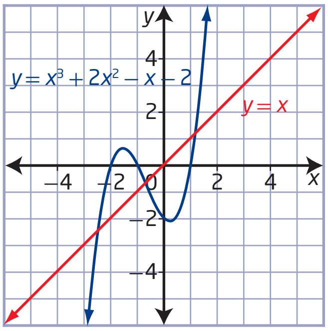
A toaster oven is built in the shape of a rectangular prism. Its volume, V, in cubic inches, is related to the height, h, in inches, of the oven door by the function $V(h) = h^3 + 10h^2 + 31h + 30$.

a) What is the volume, in cubic inches, of the toaster oven if the oven door height is 8 in.?

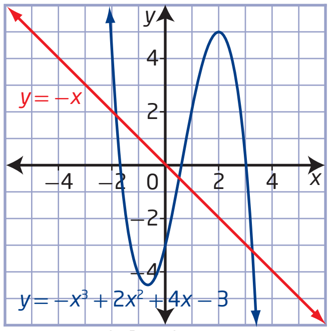
b) What is the height of the oven door for the least toaster oven volume? Explain.

Key Ideas

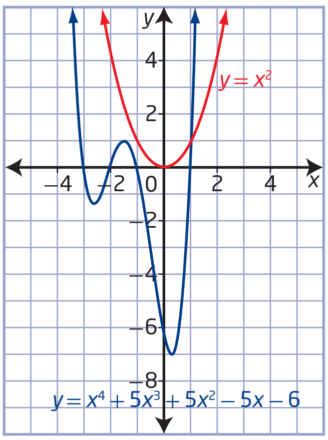
- A polynomial function has the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$, where a_n is the leading coefficient; a_0 is the constant; and the degree of the polynomial, n, is the exponent of the greatest power of the variable, x.
- Graphs of odd-degree polynomial functions have the following characteristics:
  - a graph that extends down into quadrant III and up into quadrant I (similar to the graph of $y = x$) when the leading coefficient is positive



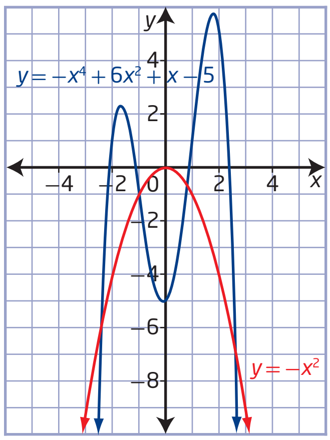
  - a graph that extends up into quadrant II and down into quadrant IV (similar to the graph of $y = -x$) when the leading coefficient is negative



  - a y-intercept that corresponds to the constant term of the function
  - at least one x-intercept and up to a maximum of n x-intercepts, where n is the degree of the function
  - a domain of $\{x \mid x \in R\}$ and a range of $\{y \mid y \in R\}$
  - no maximum or minimum points
- Graphs of even-degree polynomial functions have the following characteristics:
  - a graph that extends up into quadrant II and up into quadrant I (similar to the graph of $y = x^2$) when the leading coefficient is positive



  - a graph that extends down into quadrant III and down into quadrant IV (similar to the graph of $y = -x^2$) when the leading coefficient is negative



  - a y-intercept that corresponds to the constant term of the function
  - from zero to a maximum of n x-intercepts, where n is the degree of the function
  - a domain of $\{x \mid x \in R\}$ and a range that depends on the maximum or minimum value of the function

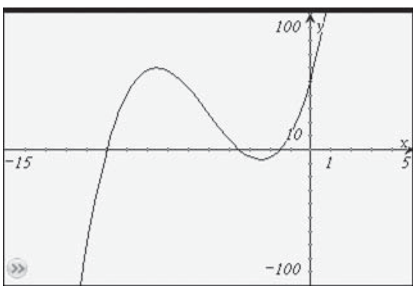
Check Your Understanding

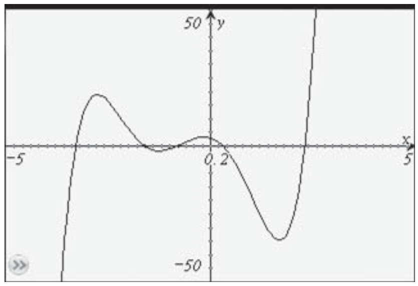
Practise

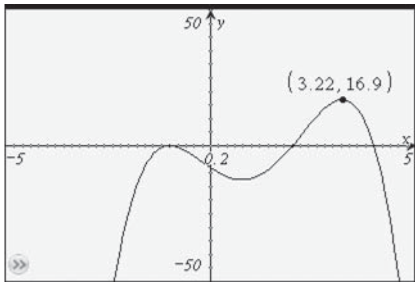
1. Identify whether each of the following is a polynomial function. Justify your answers.
   a) $h(x) = 2 - \sqrt{x}$
   b) $y = 3x + 1$
   c) $f(x) = 3^x$
   d) $g(x) = 3x^4 - 7$
   e) $p(x) = x^{-3} + x^2 + 3x$
   f) $y = -4x^3 + 2x + 5$

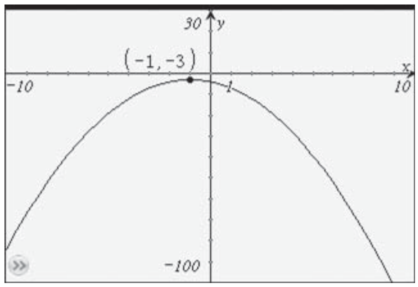
2. What are the degree, type, leading coefficient, and constant term of each polynomial function?
   a) $f(x) = -x + 3$
   b) $y = 9x^2$
   c) $g(x) = 3x^4 + 3x^2 - 2x + 1$
   d) $k(x) = 4 - 3x^3$
   e) $y = -2x^5 - 2x^3 + 9$
   f) $h(x) = -6$

3. For each of the following:
   - determine whether the graph represents an odd-degree or an even-degree polynomial function
   - determine whether the leading coefficient of the corresponding function is positive or negative
   - state the number of x-intercepts
   - state the domain and range

   a)


   b)


   c)


   d)


4. Use the degree and the sign of the leading coefficient of each function to describe the end behaviour of the corresponding graph. State the possible number of x-intercepts and the value of the y-intercept.
   a) $f(x) = x^2 + 3x - 1$
   b) $g(x) = -4x^3 + 2x^2 - x + 5$
   c) $h(x) = -7x^4 + 2x^3 - 3x^2 + 6x + 4$
   d) $q(x) = x^5 - 3x^2 + 9x$
   e) $p(x) = 4 - 2x$
   f) $v(x) = -x^3 + 2x^4 - 4x^2$

Apply

5. Jake claims that all graphs of polynomial functions of the form $y = ax^n + x + b$, where a, b, and n are even integers, extend from quadrant II to quadrant I. Do you agree? Use examples to explain your answer.

6. A snowboard manufacturer determines that its profit, P, in dollars, can be modelled by the function $P(x) = 1000x + x^4 - 3000$, where x represents the number, in hundreds, of snowboards sold.

a) What is the degree of the function $P(x)$?

b) What are the leading coefficient and constant of this function? What does the constant represent?

c) Describe the end behaviour of the graph of this function.

d) What are the restrictions on the domain of this function? Explain why you selected those restrictions.

e) What do the x-intercepts of the graph represent for this situation?

f) What is the profit from the sale of 1500 snowboards?

7. A medical researcher establishes that a patient's reaction time, r, in minutes, to a dose of a particular drug is $r(d) = -3d^3 + 3d^2$, where d is the amount of the drug, in millilitres, that is absorbed into the patient's blood.

a) What type of polynomial function is $r(d)$?

b) What are the leading coefficient and constant of this function?

c) Make a sketch of what you think the function will look like. Then, graph the function using technology. How does it compare to your sketch?

d) What are the restrictions on the domain of this function? Explain why you selected those restrictions.

8. Refer to the honeycomb example at the beginning of this section (page 106).

a) Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the correct total number of hexagons when $r = 1$, 2, and 3.

b) Determine the total number of hexagons in a honeycomb with 12 rings.

Did You Know?

Approximately 80% of Canadian honey production is concentrated in the three prairie provinces of Alberta, Saskatchewan, and Manitoba.

9. Populations in rural communities have declined in Western Canada, while populations in larger urban centres have increased. This is partly due to expanding agricultural operations and fewer traditional family farms. A demographer uses a polynomial function to predict the population, P, of a town t years from now. The function is $P(t) = t^4 - 20t^3 - 20t^2 + 1500t + 15\,000$. Assume this model can be used for the next 20 years.

a) What are the key features of the graph of this function?

b) What is the current population of this town?

c) What will the population of the town be 10 years from now?

d) When will the population of the town be approximately 24 000?

Did You Know?

A demographer uses statistics to study human populations. Demographers study the size, structure, and distribution of populations in response to birth, migration, aging, and death.

Extend

10. The volume, V, in cubic centimetres, of a collection of open-topped boxes can be modelled by $V(x) = 4x^3 - 220x^2 + 2800x$, where x is the height, in centimetres, of each box.

a) Use technology to graph $V(x)$. State the restrictions.

b) Fully factor $V(x)$. State the relationship between the factored form of the equation and the graph.

11. a) Graph each pair of even-degree functions. What do you notice? Provide an algebraic explanation for what you observe.

- $y = (-x)^2$ and $y = x^2$
- $y = (-x)^4$ and $y = x^4$
- $y = (-x)^6$ and $y = x^6$

b) Repeat part a) for each pair of odd-degree functions.

- $y = (-x)^3$ and $y = x^3$
- $y = (-x)^5$ and $y = x^5$
- $y = (-x)^7$ and $y = x^7$

c) Describe what you have learned about functions of the form $y = (-x)^n$, where n is a whole number. Support your answer with examples.

12. a) Describe the relationship between the graphs of $y = x^2$ and $y = 3(x - 4)^2 + 2$.

b) Predict the relationship between the graphs of $y = x^4$ and $y = 3(x - 4)^4 + 2$.

c) Verify the accuracy of your prediction in part b) by graphing using technology.

13. If a polynomial equation of degree n has exactly one real root, what can you conclude about the form of the corresponding polynomial function? Explain.

Create Connections

C1 Prepare a brief summary of the relationship between the degree of a polynomial function and the following features of the corresponding graph:

- the number of x-intercepts
- the maximum or minimum point
- the domain and range

C2 a) State a possible equation for a polynomial function whose graph extends

i) from quadrant III to quadrant I

ii) from quadrant II to quadrant I

iii) from quadrant II to quadrant IV

iv) from quadrant III to quadrant IV

b) Compare your answers to those of a classmate. Discuss what is similar and different between your answers.

C3 Describe to another student the similarities and differences between the line $y = x$ and polynomial functions with odd degree greater than one. Use graphs to support your answer.

C4 MINI LAB

Step 1 Graph each of the functions using technology. Copy and complete the table.

Step 2 For two functions with the same degree, how does the sign of the leading coefficient affect the end behaviour of the graph?

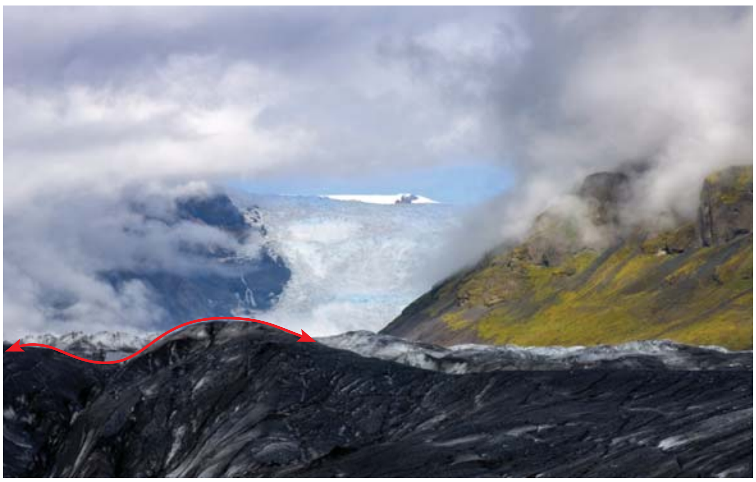
Step 3 How do the end behaviours of even-degree functions compare?

Step 4 How do the end behaviours of odd-degree functions compare?

Function	Degree	End Behaviour
$y = x + 2$		
$y = -3x + 1$		
$y = x^2 - 4$		
$y = -2x^2 - 2x + 4$		
$y = x^3 - 4x$		
$y = -x^3 + 3x - 2$		
$y = 2x^3 + 16$		
$y = -x^3 - 4x$		
$y = x^4 - 4x^2 + 5$		
$y = -x^4 + x^3 + 4x^2 - 4x$		
$y = x^4 + 2x^2 + 1$		
$y = x^5 - 2x^4 - 3x^3 + 5x^2 + 4x - 1$		
$y = x^5 - 1$		
$y = -x^5 + x^4 + 8x^3 + 8x^2 - 16x - 16$		
$y = x(x + 1)^2(x + 4)^2$		

Project Corner — Polynomials Abound

- Each image shows a portion of an object that can be modelled by a polynomial function. Describe the polynomial function that models each object.



3.2

The Remainder Theorem

Focus on…

- describing the relationship between polynomial long division and synthetic division
- dividing polynomials by binomials of the form $x - a$ using long division or synthetic division
- explaining the relationship between the remainder when a polynomial is divided by a binomial of the form $x - a$ and the value of the polynomial at $x = a$

Nested boxes or pots are featured in the teaching stories of many nations in many lands. A manufacturer of gift boxes receives an order for different-sized boxes that can be nested within each other. The box heights range from 6 cm to 16 cm. Based on cost calculations, the volume, V, in cubic centimetres, of each box can be modelled by the polynomial $V(x) = x^3 + 7x^2 + 14x + 8$, where x is a positive integer such that $5 \leq x \leq 15$. The height, h, of each box, in centimetres, is a linear function of x such that $h(x) = x + 1$. How can the box manufacturer use this information to determine the dimensions of the boxes in terms of polynomials?

Did You Know?

In Haida Gwaii, off the northwest coast of British Columbia, legends such as “Raven Steals the Light” are used to teach mathematical problem solving. This legend is about the trickster Raven who steals the light from three nested boxes to create the sun and stars. It is used to help students learn about surface area, perimeter, and volume.

Investigate Polynomial Division

A: Polynomial Long Division

1. Examine the two long-division statements.

a)
$$\begin{array}{r} 27 \\ 12\overline{)327} \\ \underline{24} \\ 87 \\ \underline{84} \\ 3 \end{array}$$

b)
$$\begin{array}{rr} & x + 4 \\ x + 3 & \overline{)x^2 + 7x + 17} \\ & \underline{x^2 + 3x} \\ & 4x + 17 \\ & \underline{4x + 12} \\ & 5 \end{array}$$

For statements a) and b), identify the value or expression that corresponds to

- the divisor
- the dividend
- the quotient
- the remainder

Reflect and Respond

2. a) Describe the long-division process used to divide the numbers in part a) of step 1.

   b) Describe the long-division process used to divide the polynomial by the binomial in part b) of step 1.

   c) What similarities and differences do you observe in the two processes?

3. Describe how you would check that the result of each long-division statement is correct.

Did You Know?

The ancient Greeks called the practical use of computing (adding, subtracting, multiplying, and dividing numbers) *logistic*. They considered arithmetic to be the study of abstract relationships connecting numbers—what we call number theory today.

B: Determine a Remainder

4. Copy the table. Identify the value of a in each binomial divisor of the form $x - a$. Then, substitute the value $x = a$ into the polynomial dividend and evaluate the result. Record these values in the last column of the table.

Polynomial Dividend	Binomial Divisor $x - a$	Value of a	Quotient	Remainder	Result of Substituting $x = a$ Into the Polynomial
$x^3 + 2x^2 - 5x - 6$	$x - 3$		$x^2 + 5x + 10$	24	
	$x - 2$		$x^2 + 4x + 3$	0	
	$x - 1$		$x^2 + 3x - 2$	-8	
	$x + 1$		$x^2 + x - 6$	0	
	$x + 2$		$x^2 - 5$	4	

5. Compare the values of each remainder from the long division to the value from substituting $x = a$ into the dividend. What do you notice?

Reflect and Respond

6. Make a conjecture about how to determine a remainder without using division.

7. a) Use your conjecture to predict the remainder when the polynomial $2x^3 - 4x^2 + 3x - 6$ is divided by each binomial.

   i) $x + 1$

   ii) $x + 3$

   iii) $x - 2$

   b) Verify your predictions using long division.

8. Describe the relationship between the remainder when a polynomial in x, $P(x)$, is divided by a binomial $x - a$, and the value of $P(a)$.

Link the Ideas

You can divide polynomials by other polynomials using the same long division process that you use to divide numbers.

> The result of the division of a polynomial in x, $P(x)$, by a binomial of the form $x - a$, $a \in I$, is $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$, where $Q(x)$ is the quotient and R is the remainder.

Check the division of a polynomial by multiplying the quotient, $Q(x)$, by the binomial divisor, $x - a$, and adding the remainder, R. The result should be equivalent to the polynomial dividend, $P(x)$:
$P(x) = (x - a)Q(x) + R$

Example 1

Divide a Polynomial by a Binomial of the Form $x - a$

a) Divide the polynomial $P(x) = 5x^3 + 10x - 13x^2 - 9$ by $x - 2$. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

d) Verify your answer.

Solution

a) Write the polynomial in order of descending powers:
$5x^3 - 13x^2 + 10x - 9$

$$
\begin{array}{r@{}l}
 & \;5x^2 - 3x + 4 \\
x - 2 \big) & \overline{\;5x^3 - 13x^2 + 10x - 9} \\
 & \;\underline{5x^3 - 10x^2} \\
 & \qquad\; -3x^2 + 10x \\
 & \qquad\; \underline{-3x^2 + \;\;6x} \\
 & \qquad\qquad\qquad 4x - 9 \\
 & \qquad\qquad\qquad \underline{4x - 8} \\
 & \qquad\qquad\qquad\quad\; -1
\end{array}
$$

Divide $5x^3$ by x to get $5x^2$.
Multiply $x - 2$ by $5x^2$ to get $5x^3 - 10x^2$.
Subtract. Bring down the next term, $10x$.
Then, divide $-3x^2$ by x to get $-3x$.
Multiply $x - 2$ by $-3x$ to get $-3x^2 + 6x$.
Subtract. Bring down the next term, -9.
Then, divide $4x$ by x to get 4.
Multiply $x - 2$ by 4 to get $4x - 8$.
Subtract. The remainder is -1.

$$\frac{5x^3 + 10x - 13x^2 - 9}{x - 2} = 5x^2 - 3x + 4 + \left(\frac{-1}{x - 2}\right)$$

b) Since division by zero is not defined, the divisor cannot be zero:
$x - 2 \neq 0$ or $x \neq 2$.

c) The corresponding statement that can be used to check the division is
$5x^3 + 10x - 13x^2 - 9 = (x - 2)(5x^2 - 3x + 4) - 1$.

d) To check, multiply the divisor by the quotient and add the remainder.

$$
\begin{aligned}
(x - 2)(5x^2 - 3x + 4) - 1 &= 5x^3 - 3x^2 + 4x - 10x^2 + 6x - 8 - 1 \\
&= 5x^3 - 13x^2 + 10x - 9 \\
&= 5x^3 + 10x - 13x^2 - 9
\end{aligned}
$$

Your Turn

a) Divide the polynomial $P(x) = x^4 - 2x^3 + x^2 - 3x + 4$ by $x - 1$.
Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.

b) Identify any restrictions on the variable.

c) Verify your answer.

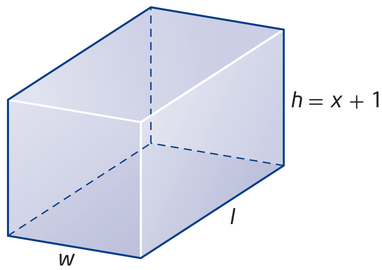
Example 2

Apply Polynomial Long Division to Solve a Problem

The volume, V, of the nested boxes in the introduction to this section, in cubic centimetres, is given by $V(x) = x^3 + 7x^2 + 14x + 8$. What are the possible dimensions of the boxes in terms of x if the height, h, in centimetres, is $x + 1$?

Solution

Divide the volume of the box by the height to obtain an expression for the area of the base of the box:
$\frac{V(x)}{h} = lw$, where lw is the area of the base.



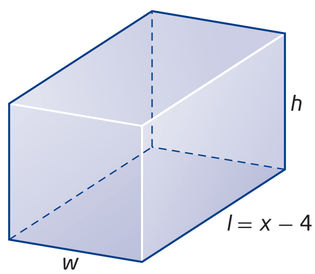
$$
\begin{array}{r}
x^2 + 6x + 8 \\
x + 1 \overline{)\, x^3 + 7x^2 + 14x + 8} \\
\underline{x^3 + x^2} \qquad\qquad \\
6x^2 + 14x \qquad \\
\underline{6x^2 + 6x} \qquad \\
8x + 8 \\
\underline{8x + 8} \\
0
\end{array}
$$

Since the remainder is zero, the volume $x^3 + 7x^2 + 14x + 8$ can be expressed as $(x + 1)(x^2 + 6x + 8)$. The quotient $x^2 + 6x + 8$ represents the area of the base. This expression can be factored as $(x + 2)(x + 4)$. The factors represent the possible width and length of the base of the box.

Expressions for the possible dimensions, in centimetres, are $x + 1$, $x + 2$, and $x + 4$.

Your Turn

The volume of a rectangular prism is given by $V(x) = x^3 + 3x^2 - 36x + 32$. Determine possible measures for w and h in terms of x if the length, l, is $x - 4$.



synthetic division

- a method of performing polynomial long division involving a binomial divisor that uses only the coefficients of the terms and fewer calculations

Did You Know?

Paolo Ruffini, an Italian mathematician, first described synthetic division in 1809.

Synthetic division is an alternate process for dividing a polynomial by a binomial of the form $x - a$. It allows you to calculate without writing variables and requires fewer calculations.

Example 3

Divide a Polynomial Using Synthetic Division

a) Use synthetic division to divide $2x^3 + 3x^2 - 4x + 15$ by $x + 3$.
b) Check the results using long division.

Solution

a) Write the terms of the dividend in order of descending power.
Use zero for the coefficient of any missing powers.
Write just the coefficients of the dividend. To the left, write the value of $+3$ from the factor $x + 3$. Below $+3$, place a "$-$" symbol to represent subtraction. Use the "$\times$" sign below the horizontal line to indicate multiplication of the divisor and the terms of the quotient.

$$\begin{array}{r|rrrr}
 & 2x^3 & 3x^2 & -4x & 15 \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
+3 & 2 & 3 & -4 & 15 \\
- & & & & \\
\hline
\times & & & &
\end{array}$$

To perform the synthetic division, bring down the first coefficient, 2, to the right of the $\times$ sign.

$$\begin{array}{r|rrrr}
+3 & 2 & 3 & -4 & 15 \\
- & \downarrow & 6 & -9 & 15 \\
\hline
\times & 2 & -3 & 5 & 0
\end{array}$$

remainder

Multiply $+3$ (top left) by 2 (right of $\times$ sign) to get 6.
Write 6 below 3 in the second column.
Subtract 6 from 3 to get -3.
Multiply $+3$ by -3 to get -9. Continue with $-4 - (-9) = 5$, $+3 \times 5 = 15$, and $15 - 15 = 0$.
2, -3, and 5 are the coefficients of the quotient, $2x^2 - 3x + 5$.

$(2x^3 + 3x^2 - 4x + 15) \div (x + 3) = 2x^2 - 3x + 5$
Restriction: $x + 3 \neq 0$ or $x \neq -3$

b) Long division check:

$$\begin{array}{r}
2x^2 - 3x + 5 \\
x + 3\overline{)\,2x^3 + 3x^2 - 4x + 15} \\
\underline{2x^3 + 6x^2}\phantom{{} - 4x + 15} \\
-3x^2 - 4x\phantom{{} + 15} \\
\underline{-3x^2 - 9x}\phantom{{} + 15} \\
5x + 15 \\
\underline{5x + 15} \\
0
\end{array}$$

The result of the long division is the same as that using synthetic division.

Your Turn

Use synthetic division to determine $\dfrac{x^3 + 7x^2 - 3x + 4}{x - 2}$.

The **remainder theorem** states that when a polynomial in x, $P(x)$, is divided by a binomial of the form $x - a$, the remainder is $P(a)$.

remainder theorem

- when a polynomial in x, $P(x)$, is divided by $x - a$, the remainder is $P(a)$

Example 4

Apply the Remainder Theorem

a) Use the remainder theorem to determine the remainder when $P(x) = x^3 - 10x + 6$ is divided by $x + 4$.

b) Verify your answer using synthetic division.

Solution

a) Since the binomial is $x + 4 = x - (-4)$, determine the remainder by evaluating $P(x)$ at $x = -4$, or $P(-4)$.

$$\begin{aligned} P(x) &= x^3 - 10x + 6 \\ P(-4) &= (-4)^3 - 10(-4) + 6 \\ P(-4) &= -64 + 40 + 6 \\ P(-4) &= -18 \end{aligned}$$

The remainder when $x^3 - 10x + 6$ is divided by $x + 4$ is -18.

b) To use synthetic division, first rewrite $P(x)$ as $P(x) = x^3 + 0x^2 - 10x + 6$.

Why is it important to rewrite the polynomial in this way?

$$\begin{array}{r|rrrr} +4 & 1 & 0 & -10 & 6 \\ - & & 4 & -16 & 24 \\ \hline \times & 1 & -4 & 6 & -18 \end{array}$$

remainder

The remainder when using synthetic division is -18.

Your Turn

What is the remainder when $11x - 4x^4 - 7$ is divided by $x - 3$? Verify your answer using either long or synthetic division.

Key Ideas

- Use long division to divide a polynomial by a binomial.
- Synthetic division is an alternate form of long division.
- The result of the division of a polynomial in x, $P(x)$, by a binomial of the form $x - a$ can be written as $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$ or $P(x) = (x - a)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.
- To check the result of a division, multiply the quotient, $Q(x)$, by the divisor, $x - a$, and add the remainder, R, to the product. The result should be the dividend, $P(x)$.
- The remainder theorem states that when a polynomial in x, $P(x)$, is divided by a binomial, $x - a$, the remainder is $P(a)$. A non-zero remainder means that the binomial is not a factor of $P(x)$.

Check Your Understanding

Practise

1. a) Use long division to divide $x^2 + 10x - 24$ by $x - 2$. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.
   b) Identify any restrictions on the variable.
   c) Write the corresponding statement that can be used to check the division.
   d) Verify your answer.

2. a) Divide the polynomial $3x^4 - 4x^3 - 6x^2 + 17x - 8$ by $x + 1$ using long division. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.
   b) Identify any restrictions on the variable.
   c) Write the corresponding statement that can be used to check the division.
   d) Verify your answer.

3. Determine each quotient, Q, using long division.
   a) $(x^3 + 3x^2 - 3x - 2) \div (x - 1)$
   b) $\frac{x^3 + 2x^2 - 7x - 2}{x - 2}$
   c) $(2w^3 + 3w^2 - 5w + 2) \div (w + 3)$
   d) $(9m^3 - 6m^2 + 3m + 2) \div (m - 1)$
   e) $\frac{t^4 + 6t^3 - 3t^2 - t + 8}{t + 1}$
   f) $(2y^4 - 3y^2 + 1) \div (y - 3)$

4. Determine each quotient, Q, using synthetic division.
   a) $(x^3 + x^2 + 3) \div (x + 4)$
   b) $\frac{m^4 - 2m^3 + m^2 + 12m - 6}{m - 2}$
   c) $(2 - x + x^2 - x^3 - x^4) \div (x + 2)$
   d) $(2s^3 + 3s^2 - 9s - 10) \div (s - 2)$
   e) $\frac{h^3 + 2h^2 - 3h + 9}{h + 3}$
   f) $(2x^3 + 7x^2 - x + 1) \div (x + 2)$

5. Perform each division. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$. Identify any restrictions on the variable.
   a) $(x^3 + 7x^2 - 3x + 4) \div (x + 2)$
   b) $\frac{11t - 4t^4 - 7}{t - 3}$
   c) $(x^3 + 3x^2 - 2x + 5) \div (x + 1)$
   d) $(4n^2 + 7n - 5) \div (n + 3)$
   e) $\frac{4n^3 - 15n + 2}{n - 3}$
   f) $(x^3 + 6x^2 - 4x + 1) \div (x + 2)$

6. Use the remainder theorem to determine the remainder when each polynomial is divided by $x + 2$.
   a) $x^3 + 3x^2 - 5x + 2$
   b) $2x^4 - 2x^3 + 5x$
   c) $x^4 + x^3 - 5x^2 + 2x - 7$
   d) $8x^3 + 4x^2 - 19$
   e) $3x^3 - 12x - 2$
   f) $2x^3 + 3x^2 - 5x + 2$

7. Determine the remainder resulting from each division.
   a) $(x^3 + 2x^2 - 3x + 9) \div (x + 3)$
   b) $\frac{2t - 4t^3 - 3t^2}{t - 2}$
   c) $(x^3 + 2x^2 - 3x + 5) \div (x - 3)$
   d) $\frac{n^4 - 3n^2 - 5n + 2}{n - 2}$

Apply

8. For each dividend, determine the value of k if the remainder is 3.
   a) $(x^3 + 4x^2 - x + k) \div (x - 1)$
   b) $(x^3 + x^2 + kx - 15) \div (x - 2)$
   c) $(x^3 + kx^2 + x + 5) \div (x + 2)$
   d) $(kx^3 + 3x + 1) \div (x + 2)$

9. For what value of c will the polynomial $P(x) = -2x^3 + cx^2 - 5x + 2$ have the same remainder when it is divided by $x - 2$ and by $x + 1$?

10. When $3x^2 + 6x - 10$ is divided by $x + k$, the remainder is 14. Determine the value(s) of k.

11. The area, $A(x)$, of a rectangle is represented by the polynomial $2x^2 - x - 6$.

a) If the height of the rectangle is $x - 2$, what is the width in terms of x?

b) If the height of the rectangle were changed to $x - 3$, what would the remainder of the quotient be? What does this remainder represent?

12. The product, $P(n)$, of two numbers is represented by the expression $2n^2 - 4n + 3$, where n is a real number.

a) If one of the numbers is represented by $n - 3$, what expression represents the other number?

b) What are the two numbers if $n = 1$?

13. A design team determines that a cost-efficient way of manufacturing cylindrical containers for their products is to have the volume, V, in cubic centimetres, modelled by $V(x) = 9\pi x^3 + 51\pi x^2 + 88\pi x + 48\pi$, where x is an integer such that $2 \le x \le 8$. The height, h, in centimetres, of each cylinder is a linear function given by $h(x) = x + 3$.

a) Determine the quotient $\dfrac{V(x)}{h(x)}$ and interpret this result.

b) Use your answer in part a) to express the volume of a container in the form $\pi r^2 h$.

c) What are the possible dimensions of the containers for the given values of x?

Extend

14. When the polynomial $mx^3 - 3x^2 + nx + 2$ is divided by $x + 3$, the remainder is -1. When it is divided by $x - 2$, the remainder is -4. What are the values of m and n?

15. When the polynomial $3x^3 + ax^2 + bx - 9$ is divided by $x - 2$, the remainder is -5. When it is divided by $x + 1$, the remainder is -16. What are the values of a and b?

16. Explain how to determine the remainder when $10x^4 - 11x^3 - 8x^2 + 7x + 9$ is divided by $2x - 3$ using synthetic division.

17. Write a polynomial that satisfies each set of conditions.

a) a quadratic polynomial that gives a remainder of -4 when it is divided by $x - 3$

b) a cubic polynomial that gives a remainder of 4 when it is divided by $x + 2$

c) a quartic polynomial that gives a remainder of 1 when it is divided by $2x - 1$

Create Connections

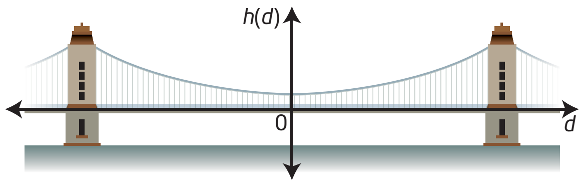
C1 How are numerical long division and polynomial long division similar, and how are they different?

C2 When the polynomial $bx^2 + cx + d$ is divided by $x - a$, the remainder is zero.

a) What can you conclude from this result?

b) Write an expression for the remainder in terms of a, b, c, and d.

C3 The support cable for a suspension bridge can be modelled by the function $h(d) = 0.0003d^2 + 2$, where $h(d)$ is the height, in metres, of the cable above the road, and d is the horizontal distance, in metres, from the lowest point on the cable.



a) What is the remainder when $0.0003d^2 + 2$ is divided by $d - 500$?

b) What is the remainder when $0.0003d^2 + 2$ is divided by $d + 500$?

c) Compare your results from parts a) and b). Use the graph of the function $h(d) = 0.0003d^2 + 2$ to explain your findings.

3.3

The Factor Theorem

Focus on...

- factoring polynomials
- explaining the relationship between the linear factors of a polynomial expression and the zeros of the corresponding function
- modelling and solving problems involving polynomial functions

Port of Vancouver

Each year, more than 1 million intermodal containers pass through the Port of Vancouver. The total volume of these containers is over 2 million twenty-foot equivalent units (TEU). Suppose the volume, in cubic feet, of a 1-TEU container can be approximated by the polynomial function $V(x) = x^3 + 7x^2 - 28x + 20$, where x is a positive real number. What dimensions, in terms of x, could the container have?

Did You Know?

An intermodal container is a standard-sized metal box that can be easily transferred between different modes of transportation, such as ships, trains, and trucks. A TEU represents the volume of a 20-ft intermodal container. Although container heights vary, the equivalent of 1 TEU is accepted as 1360 ft³.

Investigate Determining the Factors of a Polynomial

A: Remainder for a Factor of a Polynomial

1. a) Determine the remainder when $x^3 + 2x^2 - 5x - 6$ is divided by $x + 1$.

b) Determine the quotient $\dfrac{x^3 + 2x^2 - 5x - 6}{x + 1}$. Write the corresponding statement that can be used to check the division.

c) Factor the quadratic portion of the statement written in part b).

d) Write $x^3 + 2x^2 - 5x - 6$ as the product of its three factors.

e) What do you notice about the remainder when you divide $x^3 + 2x^2 - 5x - 6$ by any one of its three factors?

Reflect and Respond

2. What is the relationship between the remainder and the factors of a polynomial?

B: Determine Factors

3. Which of the following are factors of $P(x) = x^3 - 7x + 6$? Justify your reasoning.

a) $x + 1$ **b)** $x - 1$ **c)** $x + 2$

d) $x - 2$ **e)** $x + 3$ **f)** $x - 3$

Why would a factor such as $x - 5$ not be considered as a possible factor?

Reflect and Respond

4. Write a statement that describes the condition when a divisor $x - a$ is a factor of a polynomial $P(x)$.

5. What are the relationships between the factors of a polynomial expression, the zeros of the corresponding polynomial function, the x-intercepts of the graph of the corresponding polynomial function, and the remainder theorem?

6. a) Describe a method you could use to determine the factors of a polynomial.

b) Use your method to determine the factors of $f(x) = x^3 + 2x^2 - x - 2$.

c) Verify your answer.

Link the Ideas

The **factor theorem** states that $x - a$ is a factor of a polynomial in x, $P(x)$, if and only if $P(a) = 0$.

factor theorem

- a polynomial in x, $P(x)$, has a factor $x - a$ if and only if $P(a) = 0$

For example, given the polynomial $P(x) = x^3 - x^2 - 5x + 2$, determine if $x - 1$ and $x + 2$ are factors by calculating $P(1)$ and $P(-2)$, respectively.

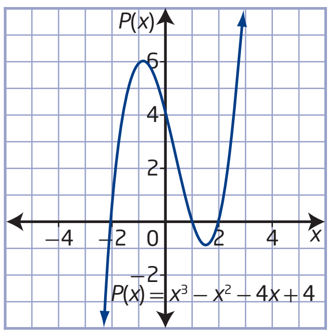
$$\begin{aligned} P(x) &= x^3 - x^2 - 5x + 2 \\ P(1) &= 1^3 - 1^2 - 5(1) + 2 \\ P(1) &= 1 - 1 - 5 + 2 \\ P(1) &= -3 \end{aligned}$$

$$\begin{aligned} P(x) &= x^3 - x^2 - 5x + 2 \\ P(-2) &= (-2)^3 - (-2)^2 - 5(-2) + 2 \\ P(-2) &= -8 - 4 + 10 + 2 \\ P(-2) &= 0 \end{aligned}$$

Since $P(1) = -3$, $P(x)$ is not divisible by $x - 1$. Therefore, $x - 1$ is not a factor of $P(x)$.

Since $P(-2) = 0$, $P(x)$ is divisible by $x + 2$. Therefore, $x + 2$ is a factor of $P(x)$.

The zeros of a polynomial function are related to the factors of the polynomial. The graph of $P(x) = x^3 - x^2 - 4x + 4$ shows that the zeros of the function, or the x-intercepts of the graph, are at $x = -2$, $x = 1$, and $x = 2$. The corresponding factors of the polynomial are $x + 2$, $x - 1$, and $x - 2$.



Did You Know?

"If and only if" is a term used in logic to say that the result works both ways.

So, the factor theorem means

- if $x - a$ is a factor of $P(x)$, then $P(a) = 0$
- if $P(a) = 0$, then $x - a$ is a factor of $P(x)$

Example 1

Use the Factor Theorem to Test for Factors of a Polynomial

Which binomials are factors of the polynomial $P(x) = x^3 - 3x^2 - x + 3$? Justify your answers.

a) $x - 1$
b) $x + 1$
c) $x + 3$
d) $x - 3$

Solution

a) Use the factor theorem to evaluate $P(a)$ given $x - a$.
For $x - 1$, substitute $x = 1$ into the polynomial expression.
$P(x) = x^3 - 3x^2 - x + 3$
$P(1) = 1^3 - 3(1)^2 - 1 + 3$
$P(1) = 1 - 3 - 1 + 3$
$P(1) = 0$
Since the remainder is zero, $x - 1$ is a factor of $P(x)$.

b) For $x + 1$, substitute $x = -1$ into the polynomial expression.
$P(x) = x^3 - 3x^2 - x + 3$
$P(-1) = (-1)^3 - 3(-1)^2 - (-1) + 3$
$P(-1) = -1 - 3 + 1 + 3$
$P(-1) = 0$
Since the remainder is zero, $x + 1$ is a factor of $P(x)$.

c) For $x + 3$, substitute $x = -3$ into the polynomial expression.
$P(x) = x^3 - 3x^2 - x + 3$
$P(-3) = (-3)^3 - 3(-3)^2 - (-3) + 3$
$P(-3) = -27 - 27 + 3 + 3$
$P(-3) = -48$
Since the remainder is not zero, $x + 3$ is not a factor of $P(x)$.

d) For $x - 3$, substitute $x = 3$ into the polynomial expression.
$P(x) = x^3 - 3x^2 - x + 3$
$P(3) = 3^3 - 3(3)^2 - 3 + 3$
$P(3) = 27 - 27 - 3 + 3$
$P(3) = 0$
Since the remainder is zero, $x - 3$ is a factor of $P(x)$.

Your Turn

Determine which of the following binomials are factors of the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.
$x - 1$, $x + 1$, $x - 2$, $x + 2$, $x - 3$, $x + 3$, $x - 6$, $x + 6$

Possible Factors of a Polynomial

When factoring a polynomial, $P(x)$, it is helpful to know which integer values of a to try when determining if $P(a) = 0$.

Consider the polynomial $P(x) = x^3 - 7x^2 + 14x - 8$. If $x = a$ satisfies $P(a) = 0$, then $a^3 - 7a^2 + 14a - 8 = 0$, or $a^3 - 7a^2 + 14a = 8$. Factoring out the common factor on the left side of the equation gives the product $a(a^2 - 7a + 14) = 8$. Then, the possible integer values for the factors in the product on the left side are the factors of 8. They are ±1, ±2, ±4, and ±8.

The relationship between the factors of a polynomial and the constant term of the polynomial is stated in the **integral zero theorem**.

The integral zero theorem states that if $x - a$ is a factor of a polynomial function $P(x)$ with integral coefficients, then a is a factor of the constant term of $P(x)$.

integral zero theorem

- if $x = a$ is an integral zero of a polynomial, $P(x)$, with integral coefficients, then a is a factor of the constant term of $P(x)$

Example 2

Factor Using the Integral Zero Theorem

a) Factor $2x^3 - 5x^2 - 4x + 3$ fully.

b) Describe how to use the factors of the polynomial expression to determine the zeros of the corresponding polynomial function.

Solution

a) Let $P(x) = 2x^3 - 5x^2 - 4x + 3$. Find a factor by evaluating $P(x)$ for values of x that are factors of 3: ±1 and ±3.

Test the values.

$P(x) = 2x^3 - 5x^2 - 4x + 3$
$P(1) = 2(1)^3 - 5(1)^2 - 4(1) + 3$
$P(1) = 2 - 5 - 4 + 3$
$P(1) = -4$

Since $P(1) \neq 0$, $x - 1$ is not a factor of $2x^3 - 5x^2 - 4x + 3$.

$P(x) = 2x^3 - 5x^2 - 4x + 3$
$P(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$
$P(-1) = -2 - 5 + 4 + 3$
$P(-1) = 0$

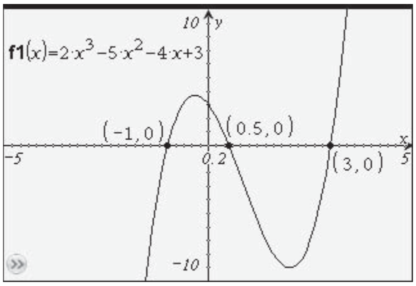
Since $P(-1) = 0$, $x + 1$ is a factor of $2x^3 - 5x^2 - 4x + 3$.

Use synthetic or long division to find the other factors.

$$\begin{array}{c|rrrr} +1 & 2 & -5 & -4 & 3 \\ - & & 2 & -7 & 3 \\ \hline \times & 2 & -7 & 3 & 0 \end{array}$$

The remaining factor is $2x^2 - 7x + 3$.
So, $2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3)$.
Factoring $2x^2 - 7x + 3$ gives $(2x - 1)(x - 3)$.
Therefore, $2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x - 1)(x - 3)$.

b) Since the factors of $2x^3 - 5x^2 - 4x + 3$ are $x + 1$, $2x - 1$, and $x - 3$, the corresponding zeros of the function are -1, $\frac{1}{2}$, and 3. Confirm the zeros by graphing $P(x)$ and using the trace or zero feature of a graphing calculator.



Your Turn

What is the factored form of $x^3 - 4x^2 - 11x + 30$? How can you use the graph of the corresponding polynomial function to simplify your search for integral roots?

Example 3

Factor Higher-Degree Polynomials

Fully factor $x^4 - 5x^3 + 2x^2 + 20x - 24$.

Solution

Let $P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24$.
Find a factor by testing factors of -24: ±1, ±2, ±3, ±4, ±6, ±8, ±12, and ±24

$$P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24$$
$$P(1) = 1^4 - 5(1)^3 + 2(1)^2 + 20(1) - 24$$
$$P(1) = 1 - 5 + 2 + 20 - 24$$
$$P(1) = -6$$

When should you stop testing possible factors?

$$P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24$$
$$P(-1) = (-1)^4 - 5(-1)^3 + 2(-1)^2 + 20(-1) - 24$$
$$P(-1) = 1 + 5 + 2 - 20 - 24$$
$$P(-1) = -36$$

$$P(x) = x^4 - 5x^3 + 2x^2 + 20x - 24$$
$$P(2) = 2^4 - 5(2)^3 + 2(2)^2 + 20(2) - 24$$
$$P(2) = 16 - 40 + 8 + 40 - 24$$
$$P(2) = 0$$

Since $P(2) = 0$, $x - 2$ is a factor of $x^4 - 5x^3 + 2x^2 + 20x - 24$.

Use division to find the other factors.

−2	1	−5	2	20	−24
−		−2	6	8	−24
×	1	−3	−4	12	0

The remaining factor is $x^3 - 3x^2 - 4x + 12$.

Method 1: Apply the Factor Theorem Again
Let $f(x) = x^3 - 3x^2 - 4x + 12$.

Since $f(2) = 0$, $x - 2$ is a second factor.

Use division to determine that the other factor is $x^2 - x - 6$.

-2	1	-3	-4	12
$-$		-2	2	12
$\times$	1	-1	-6	0

Factoring $x^2 - x - 6$ gives $(x + 2)(x - 3)$.

Therefore,
$$\begin{aligned} x^4 - 5x^3 + 2x^2 + 20x - 24 &= (x - 2)(x - 2)(x + 2)(x - 3) \\ &= (x - 2)^2(x + 2)(x - 3) \end{aligned}$$

Method 2: Factor by Grouping

$x^3 - 3x^2 - 4x + 12 = x^2(x - 3) - 4(x - 3)$ Group the first two terms and factor out x^2. Then, group the second two terms and factor out -4.

$= (x - 3)(x^2 - 4)$ Factor out $x - 3$.

$= (x - 3)(x - 2)(x + 2)$ Factor the difference of squares $x^2 - 4$.

Therefore,
$$\begin{aligned} &x^4 - 5x^3 + 2x^2 + 20x - 24 \\ &= (x - 2)(x - 3)(x - 2)(x + 2) \\ &= (x - 2)^2(x + 2)(x - 3) \end{aligned}$$

Your Turn

What is the fully factored form of $x^4 - 3x^3 - 7x^2 + 15x + 18$?

Example 4

Solve Problems Involving Polynomial Expressions

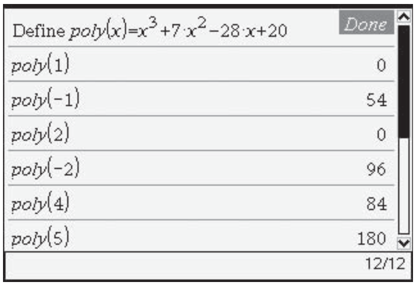
An intermodal container that has the shape of a rectangular prism has a volume, in cubic feet, represented by the polynomial function $V(x) = x^3 + 7x^2 - 28x + 20$, where x is a positive real number.

What are the factors that represent possible dimensions, in terms of x, of the container?

Dockside at Port of Vancouver

Solution

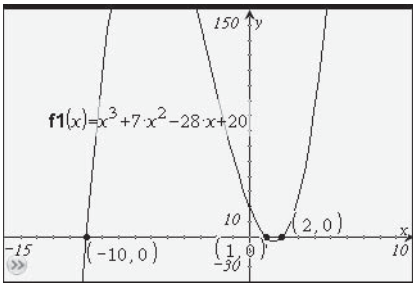
Method 1: Use Factoring
The possible integral factors correspond to the factors of the constant term of the polynomial, 20: ±1, ±2, ±4, ±5, ±10, and ±20. Use the factor theorem to determine which of these values correspond to the factors of the polynomial. Use a graphing calculator or spreadsheet to help with the multiple calculations.



x	P(x)
1	0
-1	54
2	0
-2	96
4	84
-4	180
5	180
-5	210
10	1440
-10	0
20	10260
-20	-4620

The values of x that result in a remainder of zero are -10, 1, and 2. The factors that correspond to these values are $x + 10$, $x - 1$, and $x - 2$. The factors represent the possible dimensions, in terms of x, of the container.

Method 2: Use Graphing
Since the zeros of the polynomial function correspond to the factors of the polynomial expression, use the graph of the function to determine the factors.



For this example, what are the restrictions on the domain?

The trace or zero feature of a graphing calculator shows that the zeros of the function are $x = -10$, $x = 1$, and $x = 2$. These correspond to the factors $x + 10$, $x - 1$, and $x - 2$. The factors represent the possible dimensions, in terms of x, of the container.

Your Turn

A form that is used to make large rectangular blocks of ice comes in different dimensions such that the volume, V, in cubic centimetres, of each block can be modelled by $V(x) = x^3 + 7x^2 + 16x + 12$, where x is in centimetres. Determine the possible dimensions, in terms of x, that result in this volume.

Key Ideas

- The factor theorem states that $x - a$ is a factor of a polynomial $P(x)$ if and only if $P(a) = 0$.
- The integral zero theorem states that if $x - a$ is a factor of a polynomial function $P(x)$ with integral coefficients, then a is a factor of the constant term of $P(x)$.
- You can use the factor theorem and the integral zero theorem to factor some polynomial functions.
  - Use the integral zero theorem to list possible integer values for the zeros.
  - Next, apply the factor theorem to determine one factor.
  - Then, use division to determine the remaining factor.
  - Repeat the above steps until all factors are found or the remaining factor is a trinomial which can be factored.

Check Your Understanding

Practise

1. What is the corresponding binomial factor of a polynomial, $P(x)$, given the value of the zero?

a) $P(1) = 0$

b) $P(-3) = 0$

c) $P(4) = 0$

d) $P(a) = 0$

2. Determine whether $x - 1$ is a factor of each polynomial.

a) $x^3 - 3x^2 + 4x - 2$

b) $2x^3 - x^2 - 3x - 2$

c) $3x^3 - x - 3$

d) $2x^3 + 4x^2 - 5x - 1$

e) $x^4 - 3x^3 + 2x^2 - x + 1$

f) $4x^4 - 2x^3 + 3x^2 - 2x + 1$

3. State whether each polynomial has $x + 2$ as a factor.

a) $5x^2 + 2x + 6$

b) $2x^3 - x^2 - 5x - 8$

c) $2x^3 + 2x^2 - x - 6$

d) $x^4 - 2x^2 + 3x - 4$

e) $x^4 + 3x^3 - x^2 - 3x + 6$

f) $3x^4 + 5x^3 + x - 2$

4. What are the possible integral zeros of each polynomial?

a) $P(x) = x^3 + 3x^2 - 6x - 8$

b) $P(s) = s^3 + 4s^2 - 15s - 18$

c) $P(n) = n^3 - 3n^2 - 10n + 24$

d) $P(p) = p^4 - 2p^3 - 8p^2 + 3p - 4$

e) $P(z) = z^4 + 5z^3 + 2z^2 + 7z - 15$

f) $P(y) = y^4 - 5y^3 - 7y^2 + 21y + 4$

5. Factor fully.

a) $P(x) = x^3 - 6x^2 + 11x - 6$

b) $P(x) = x^3 + 2x^2 - x - 2$

c) $P(v) = v^3 + v^2 - 16v - 16$

d) $P(x) = x^4 + 4x^3 - 7x^2 - 34x - 24$

e) $P(k) = k^5 + 3k^4 - 5k^3 - 15k^2 + 4k + 12$

6. Factor fully.

a) $x^3 - 2x^2 - 9x + 18$

b) $t^3 + t^2 - 22t - 40$

c) $h^3 - 27h + 10$

d) $x^5 + 8x^3 + 2x - 15$

e) $q^4 + 2q^3 + 2q^2 - 2q - 3$

Apply

7. Determine the value(s) of k so that the binomial is a factor of the polynomial.

a) $x^2 - x + k$, $x - 2$

b) $x^2 - 6x - 7$, $x + k$

c) $x^3 + 4x^2 + x + k$, $x + 2$

d) $x^2 + kx - 16$, $x - 2$

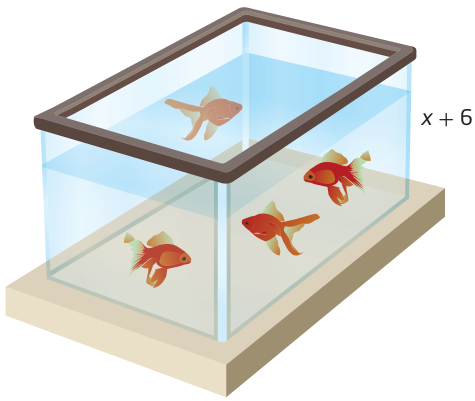
8. The volume, $V(h)$, of a bookcase can be represented by the expression $h^3 - 2h^2 + h$, where h is the height of the bookcase. What are the possible dimensions of the bookcase in terms of h?

9. A racquetball court has a volume that can be represented by the polynomial $V(l) = l^3 - 2l^2 - 15l$, where l is the length of the side walls. Factor the expression to determine the possible width and height of the court in terms of l.

10. Mikisiti Saila (1939–2008), an Inuit artist from Cape Dorset, Nunavut, was the son of famous soapstone carver Pauta Saila. Mikisita's preferred theme was wildlife presented in a minimal but graceful and elegant style. Suppose a carving is created from a rectangular block of soapstone whose volume, V, in cubic centimetres, can be modelled by $V(x) = x^3 + 5x^2 - 2x - 24$. What are the possible dimensions of the block, in centimetres, in terms of binomials of x?

Walrus created in 1996 by Mikisiti Saila

11. The volume of water in a rectangular fish tank can be modelled by the polynomial $V(x) = x^3 + 14x^2 + 63x + 90$. If the depth of the tank is given by the polynomial $x + 6$, what polynomials represent the possible length and width of the fish tank?



12. When a certain type of plastic is cut into sections, the length of each section determines its relative strength. The function $f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100$ describes the relative strength of a section of length x feet. After testing the plastic, engineers discovered that 5-ft sections were extremely weak.

a) Why is $x - 5$ a possible factor when $x = 5$ is the length of the pipe? Show that $x - 5$ is a factor of the polynomial function.

b) Are there other lengths of plastic that are extremely weak? Explain your reasoning.

Did You Know?

The strength of a material can be affected by its mechanical resonance. Mechanical resonance is the tendency of a mechanical system to absorb more energy when it oscillates at the system's natural frequency of vibration. It may cause intense swaying motions and even catastrophic failure in improperly constructed structures including bridges, buildings, and airplanes. The collapse of the Tacoma Narrows Bridge into Puget Sound on November 7, 1940, was due in part to the effects of mechanical resonance.

13. The product of four integers is $x^4 + 6x^3 + 11x^2 + 6x$, where x is one of the integers. What are possible expressions for the other three integers?

Extend

14. Consider the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a + b + c + d + e = 0$. Show that this polynomial is divisible by $x - 1$.

15. Determine the values of m and n so that the polynomials $2x^3 + mx^2 + nx - 3$ and $x^3 - 3mx^2 + 2nx + 4$ are both divisible by $x - 2$.

16. **a)** Factor each polynomial.

i) $x^3 - 1$

ii) $x^3 - 27$

iii) $x^3 + 1$

iv) $x^3 + 64$

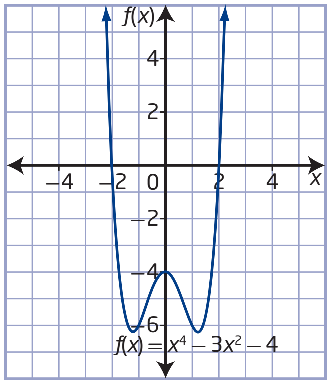
b) Use the results from part a) to decide whether $x + y$ or $x - y$ is a factor of $x^3 + y^3$. State the other factor(s).

c) Use the results from part a) to decide whether $x + y$ or $x - y$ is a factor of $x^3 - y^3$. State the other factor(s).

d) Use your findings to factor $x^6 + y^6$.

Create Connections

C1 Explain to a classmate how to use the graph of $f(x) = x^4 - 3x^2 - 4$ to determine at least one binomial factor of the polynomial. What are all of the factors of the polynomial?



C2 Identify the possible factors of the expression $x^4 - x^3 + 2x^2 - 5$. Explain your reasoning in more than one way.

C3 How can the factor theorem, the integral zero theorem, the quadratic formula, and synthetic division be used together to factor a polynomial of degree greater than or equal to three?

3.4

Equations and Graphs of Polynomial Functions

Focus on...

- describing the relationship between zeros, roots, and x-intercepts of polynomial functions and equations
- sketching the graph of a polynomial function without technology
- modelling and solving problems involving polynomial functions

On an airplane, carry-on baggage must fit into the overhead compartment or under the seat in front of you. As a result, the dimensions of carry-on baggage for some airlines are restricted so that the width of the carry-on is 17 cm less than the height, and the length is no more than 15 cm greater than the height. The maximum volume, V, in cubic centimetres, of carry-on bags can be represented by the polynomial function $V(h) = h^3 - 2h^2 - 255h$, where h is the height, in centimetres, of the bag. If the maximum volume of the overhead compartment is 50 600 cm^3, how could you determine the maximum dimensions of the carry-on bags?

In this section, you will use polynomial functions to model real-life situations such as this one. You will also sketch graphs of polynomial functions to help you solve problems.

Did You Know?

In 1973, Rosella Bjornson became the first female pilot in Canada to be hired by an airline. In 1990, she became the first female captain.

Investigate Sketching the Graph of a Polynomial Function

Materials

- graphing calculator or computer with graphing software

A: The Relationship Among the Roots, x-Intercepts, and Zeros of a Function

1. a) Graph the function $f(x) = x^4 + x^3 - 10x^2 - 4x + 24$ using graphing technology.

b) Determine the x-intercepts from the graph.

c) Factor $f(x)$. Then, use the factors to determine the zeros of $f(x)$.

What are the possible integral factors of this polynomial?

2. a) Set the polynomial function $f(x) = x^4 + x^3 - 10x^2 - 4x + 24$ equal to 0. Solve the equation $x^4 + x^3 - 10x^2 - 4x + 24 = 0$ to determine the roots.

b) What do you notice about the roots of the equation and the x-intercepts of the graph of the function?

Reflect and Respond

3. What is the relationship between the zeros of a function, the x-intercepts of the corresponding graph, and the roots of the polynomial equation?

B: Determine When a Function Is Positive and When It Is Negative

4. Refer to the graph you made in step 1. The x-intercepts divide the x-axis into four intervals. Copy and complete the table by writing in the intervals and indicating whether the function is positive (above the x-axis) or negative (below the x-axis) for each interval.

Interval	$x < -3$			
Sign of $f(x)$	positive			

Reflect and Respond

5. a) What happens to the sign of $f(x)$ if the graph crosses from one side of the x-axis to the other?

b) How does the graph behave if there are two identical zeros?

C: Sketch the Graph of a Polynomial Function

Materials

- grid paper

6. Without using a graphing calculator, determine the following characteristics of the function $f(x) = -x^3 - 5x^2 - 3x + 9$:
- the degree of the polynomial
- the sign of the leading coefficient
- the zeros of the function
- the y-intercept
- the interval(s) where the function is positive
- the interval(s) where the function is negative

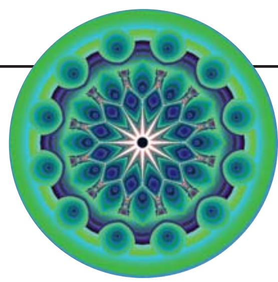
7. Use the characteristics you determined in step 6 to sketch the graph of the function. Graph the function using technology and compare the result to your hand-drawn sketch.

Reflect and Respond

8. Describe how to sketch the graph of a polynomial using the x-intercepts, the y-intercept, the sign of the leading coefficient, and the degree of the function.

Did You Know?

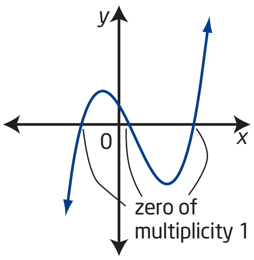
Polynomiography is a fusion of art, mathematics, and computer science. It creates a visualization of the approximation of zeros of polynomial functions.

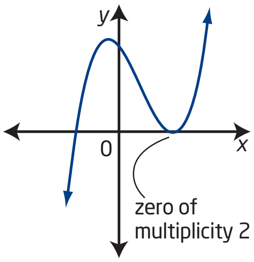


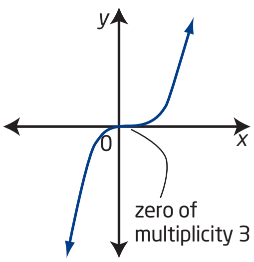
Link the Ideas

As is the case with quadratic functions, the zeros of any polynomial function $y = f(x)$ correspond to the x-intercepts of the graph and to the roots of the corresponding equation, $f(x) = 0$. For example, the function $f(x) = (x - 1)(x - 1)(x + 2)$ has two identical zeros at $x = 1$ and a third zero at $x = -2$. These are the roots of the equation $(x - 1)(x - 1)(x + 2) = 0$.

multiplicity (of a zero)
- the number of times a zero of a polynomial function occurs
- the shape of the graph of a function close to a zero depends on its multiplicity

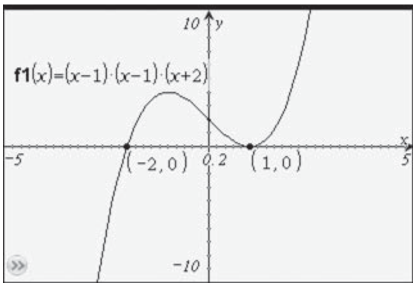






If a polynomial has a factor $x - a$ that is repeated n times, then $x = a$ is a zero of **multiplicity**, n. The function $f(x) = (x - 1)^2(x + 2)$ has a zero of multiplicity 2 at $x = 1$ and the equation $(x - 1)^2(x + 2) = 0$ has a root of multiplicity 2 at $x = 1$.

Consider the graph of the function $f(x) = (x - 1)(x - 1)(x + 2)$.



At $x = -2$ (zero of odd multiplicity), the sign of the function changes.

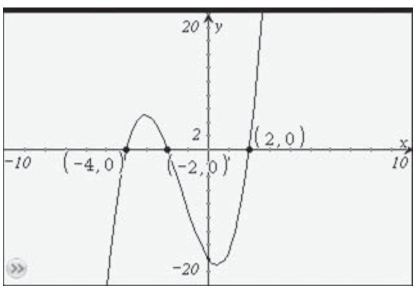
At $x = 1$ (zero of even multiplicity), the sign of the function does not change.

Example 1

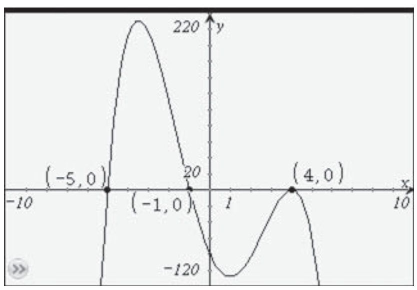
Analyse Graphs of Polynomial Functions

For each graph of a polynomial function, determine
- the least possible degree
- the sign of the leading coefficient
- the x-intercepts and the factors of the function with least possible degree
- the intervals where the function is positive and the intervals where it is negative

a)



b)



Did You Know?

The multiplicity of a zero or root can also be referred to as the *order* of the zero or root.

Solution

a) • The graph of the polynomial function crosses the x-axis (negative to positive or positive to negative) at all three x-intercepts. The three x-intercepts are of odd multiplicity. The least possible multiplicity of each x-intercept is 1, so the least possible degree is 3.

Could the multiplicity of each x-intercept be something other than 1?

• The graph extends down into quadrant III and up into quadrant I, so the leading coefficient is positive.

• The x-intercepts are -4, -2, and 2. The factors are $x + 4$, $x + 2$, and $x - 2$.

• The function is positive for values of x in the intervals $-4 < x < -2$ and $x > 2$. The function is negative for values of x in the intervals $x < -4$ and $-2 < x < 2$.

b) • The graph of the polynomial function crosses the x-axis at two of the x-intercepts and touches the x-axis at one of the x-intercepts. The least possible multiplicities of these x-intercepts are, respectively, 1 and 2, so the least possible degree is 4.

Could the multiplicities of the x-intercepts be something other than 1 or 2?

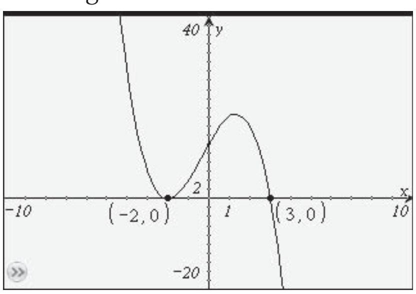
• The graph extends down into quadrant III and down into quadrant IV, so the leading coefficient is negative.

• The x-intercepts are -5, -1, and 4 (multiplicity 2). The factors are $x + 5$, $x + 1$, and $(x - 4)^2$.

• The function is positive for values of x in the interval $-5 < x < -1$. The function is negative for values of x in the intervals $x < -5$, $-1 < x < 4$, and $x > 4$.

Your Turn

For the graph of the polynomial function shown, determine
- the least possible degree
- the sign of the leading coefficient
- the x-intercepts and the factors of the function of least possible degree
- the intervals where the function is positive and the intervals where it is negative



Example 2

Analyse Equations to Sketch Graphs of Polynomial Functions

Sketch the graph of each polynomial function.

a) $y = (x - 1)(x + 2)(x + 3)$

b) $f(x) = -(x + 2)^3(x - 4)$

c) $y = -2x^3 + 6x - 4$

Solution

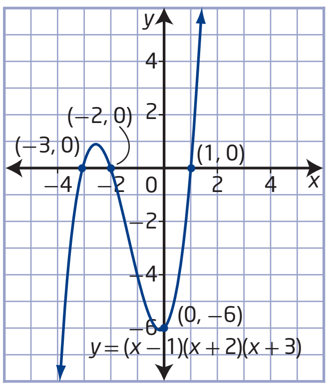
a) The function $y = (x - 1)(x + 2)(x + 3)$ is in factored form.

Use a table to organize information about the function. Then, use the information to sketch the graph.

Degree	3
Leading Coefficient	1
End Behaviour	extends down into quadrant III and up into quadrant I
Zeros/x-Intercepts	-3, -2, and 1
y-Intercept	$(0 - 1)(0 + 2)(0 + 3) = -6$
Interval(s) Where the Function is Positive or Negative	positive values of $f(x)$ in the intervals $-3 < x < -2$ and $x > 1$ negative values of $f(x)$ in the intervals $x < -3$ and $-2 < x < 1$

To check whether the function is positive or negative, test values within the interval, rather than close to either side of the interval.

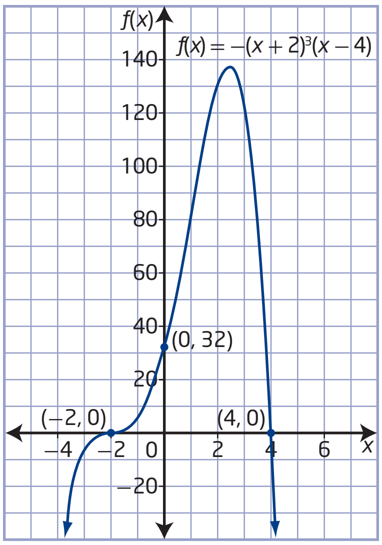
Mark the intercepts. Since the multiplicity of each zero is 1, the graph crosses the x-axis at each x-intercept. Beginning in quadrant III, sketch the graph so that it passes through $x = -3$ to above the x-axis, back down through $x = -2$ to below the x-axis, through the y-intercept -6, up through $x = 1$, and upward in quadrant I.



b) The function $f(x) = -(x + 2)^3(x - 4)$ is in factored form.

Degree	When the function is expanded, the exponent of the highest-degree term is 4. The function is of degree 4.
Leading Coefficient	When the function is expanded, the leading coefficient is $(-1)(1^3)(1)$ or -1.
End Behaviour	extends down into quadrant III and down into quadrant IV
Zeros/*x*-Intercepts	-2 (multiplicity 3) and 4
***y*-Intercept**	$-(0 + 2)^3(0 - 4) = 32$
Interval(s) Where the Function Is Positive or Negative	positive values of $f(x)$ in the interval $-2 < x < 4$ negative values of $f(x)$ in the intervals $x < -2$ and $x > 4$

Mark the intercepts. Since the multiplicity of each zero is odd, the graph crosses the x-axis at both x-intercepts. Beginning in quadrant III, sketch the graph so that it passes through $x = -2$ to above the x-axis through the y-intercept 32, continuing upward, and then back down to pass through $x = 4$, and then downward in quadrant IV. In the neighbourhood of $x = -2$, the graph behaves like the cubic curve $y = (x + 2)^3$.



Why is it useful to evaluate the function for values such as $x = 2$ and $x = 3$?

How are the multiplicity of the zero of -2 and the shape of the graph at this x-intercept related?

c) First factor out the common factor.

$y = -2x^3 + 6x - 4$

$y = -2(x^3 - 3x + 2)$

How does factoring out the common factor help?

Next, use the integral zero theorem and the factor theorem to determine the factors of the polynomial expression $x^3 - 3x + 2$. Test possible factors of 2, that is, ± 1 and ± 2.

Substitute $x = 1$.

$$\begin{aligned} & x^3 - 3x + 2 \\ &= 1^3 - 3(1) + 2 \\ &= 1 - 3 + 2 \\ &= 0 \end{aligned}$$

Therefore, $x - 1$ is a factor.

Divide the polynomial expression $x^3 - 3x + 2$ by $x - 1$ to get the factor $x^2 + x - 2$.

−1	1	0	−3	2
−		−1	−1	2
×	1	1	−2	0

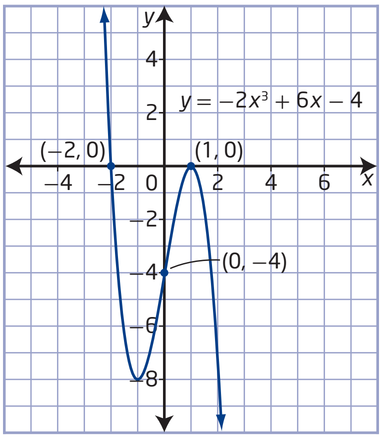
Why is one of the coefficients 0?

Then, factor $x^2 + x - 2$ to give $(x + 2)(x - 1)$. So, the factored form of $y = -2x^3 + 6x - 4$ is $y = -2(x - 1)^2(x + 2)$.

How can you check that the factored form is equivalent to the original polynomial?

Degree	3
Leading Coefficient	-2
End Behaviour	extends up into quadrant II and down into quadrant IV
Zeros/*x*-Intercepts	-2 and 1 (multiplicity 2)
***y*-Intercept**	-4
Interval(s) Where the Function Is Positive or Negative	positive values of $f(x)$ in the interval $x < -2$ negative values of $f(x)$ in the intervals $-2 < x < 1$ and $x > 1$

Mark the intercepts. The graph crosses the x-axis at $x = -2$ (multiplicity 1) and touches the x-axis at $x = 1$ (multiplicity 2). Beginning in quadrant II, sketch the graph so that it passes through $x = -2$ to below the x-axis, up through the y-intercept -4 to touch the x-axis at $x = 1$, and then downward in quadrant IV.



Your Turn

Sketch a graph of each polynomial function by hand. State the characteristics of the polynomial functions that you used to sketch the graphs.

a) $g(x) = (x - 2)^3(x + 1)$

b) $f(x) = -x^3 + 13x + 12$

Graphing Polynomial Functions using Transformations

The graph of a function of the form $y = a(b(x - h))^n + k$ is obtained by applying transformations to the graph of the general polynomial function $y = x^n$, where $n \in \mathrm{N}$. The effects of changing parameters in polynomial functions are the same as the effects of changing parameters in other types of functions.

Parameter	Transformation
k	• Vertical translation up or down • $(x, y) \rightarrow (x, y + k)$
h	• Horizontal translation left or right • $(x, y) \rightarrow (x + h, y)$
a	• Vertical stretch about the x-axis by a factor of $\lvert a \rvert$ • For $a < 0$, the graph is also reflected in the x-axis. • $(x, y) \rightarrow (x, ay)$
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ • For $b < 0$, the graph is also reflected in the y-axis. • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$

To obtain an accurate sketch of a transformed graph, apply the transformations represented by a and b (reflections and stretches) before the transformations represented by h and k (translations).

Example 3

Apply Transformations to Sketch a Graph

The graph of $y = x^3$ is transformed to obtain the graph of $y = -2(4(x - 1))^3 + 3$.

a) State the parameters and describe the corresponding transformations.

b) Copy and complete the table to show what happens to the given points under each transformation.

$y = x^3$	$y = (4x)^3$	$y = -2(4x)^3$	$y = -2(4(x - 1))^3 + 3$
$(-2, -8)$			
$(-1, -1)$			
$(0, 0)$			
$(1, 1)$			
$(2, 8)$			

c) Sketch the graph of $y = -2(4(x - 1))^3 + 3$.

Solution

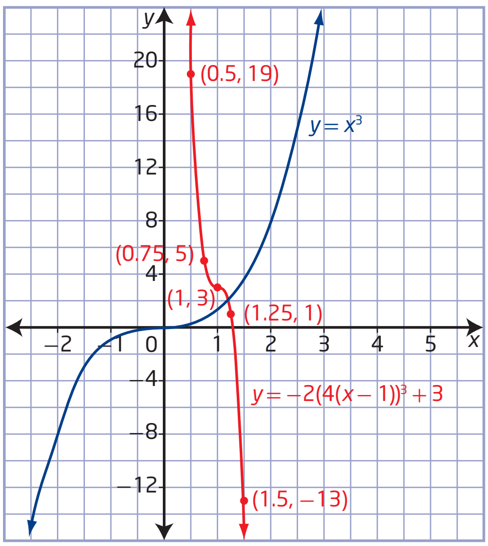
a) Compare the functions $y = -2(4(x - 1))^3 + 3$ and $y = a(b(x - h))^n + k$ to determine the values of the parameters.

- $b = 4$ corresponds to a horizontal stretch of factor $\frac{1}{4}$. Multiply the x-coordinates of the points in column 1 by $\frac{1}{4}$.
- $a = -2$ corresponds to a vertical stretch of factor 2 and a reflection in the x-axis. Multiply the y-coordinates of the points in column 2 by -2.
- $h = 1$ corresponds to a translation of 1 unit to the right and $k = 3$ corresponds to a translation of 3 units up. Add 1 to the x-coordinates and 3 to the y-coordinates of the points in column 3.

b)

$y = x^3$	$y = (4x)^3$	$y = -2(4x)^3$	$y = -2(4(x - 1))^3 + 3$
$(-2, -8)$	$(-0.5, -8)$	$(-0.5, 16)$	$(0.5, 19)$
$(-1, -1)$	$(-0.25, -1)$	$(-0.25, 2)$	$(0.75, 5)$
$(0, 0)$	$(0, 0)$	$(0, 0)$	$(1, 3)$
$(1, 1)$	$(0.25, 1)$	$(0.25, -2)$	$(1.25, 1)$
$(2, 8)$	$(0.5, 8)$	$(0.5, -16)$	$(1.5, -13)$

c) To sketch the graph, plot the points from column 4 and draw a smooth curve through them.



Your Turn

Transform the graph of $y = x^3$ to sketch the graph of $y = -4(2(x + 2))^3 - 5$.

Example 4

Model and Solve Problems Involving Polynomial Functions

Bill is preparing to make an ice sculpture. He has a block of ice that is 3 ft wide, 4 ft high, and 5 ft long. Bill wants to reduce the size of the block of ice by removing the same amount from each of the three dimensions. He wants to reduce the volume of the ice block to 24 ft³.

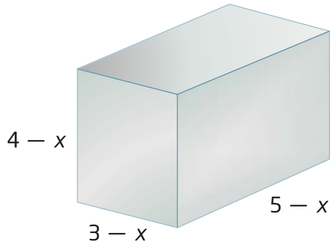
a) Write a polynomial function to model this situation.

b) How much should he remove from each dimension?

Solution

a) Let x represent the amount to be removed from each dimension.

Then, the new dimensions are length $= 5 - x$, width $= 3 - x$, and height $= 4 - x$.



The volume of the ice block is

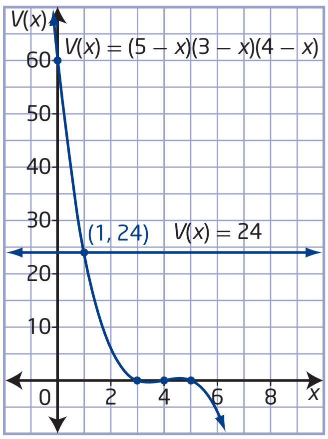
$V(x) = lwh$

$V(x) = (5 - x)(3 - x)(4 - x)$

b) **Method 1: Intersecting Graphs**

Sketch the graphs of $V(x) = (5 - x)(3 - x)(4 - x)$ and $V(x) = 24$ on the same set of coordinate axes. The point of intersection of the two graphs gives the value of x that will result in a volume of 24 ft³.

Degree	3
Leading Coefficient	−1
End Behaviour	extends up into quadrant II and down into quadrant IV
Zeros/x-Intercepts	3, 4, and 5
y-Intercept	60
Interval(s) Where the Function Is Positive or Negative	positive values of $V(x)$ in the intervals $x < 3$ and $4 < x < 5$ negative values of $V(x)$ in the intervals $3 < x < 4$ and $x > 5$



Since the point of intersection is (1, 24), 1 ft should be removed from each dimension.

Method 2: Factoring

Since the volume of the reduced block of ice is 24 ft³, substitute this value into the function.

$$\begin{aligned} V(x) &= (5 - x)(3 - x)(4 - x) \\ 24 &= (5 - x)(3 - x)(4 - x) \\ 24 &= -x^3 + 12x^2 - 47x + 60 && \text{Expand the right side.} \\ 0 &= -x^3 + 12x^2 - 47x + 36 && \text{Collect like terms.} \\ 0 &= -(x^3 - 12x^2 + 47x - 36) \end{aligned}$$

The possible integral factors of the constant term of the polynomial expression $x^3 - 12x^2 + 47x - 36$ are ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, and ±36.

Test $x = 1$.

$$\begin{aligned} &x^3 - 12x^2 + 47x - 36 \\ &= 1^3 - 12(1)^2 + 47(1) - 36 \\ &= 1 - 12 + 47 - 36 \\ &= 0 \end{aligned}$$

Therefore, $x - 1$ is a factor.

Divide the polynomial expression $x^3 - 12x^2 + 47x - 36$ by this factor.

$$\frac{x^3 - 12x^2 + 47x - 36}{x - 1} = x^2 - 11x + 36$$

The remaining factor, $x^2 - 11x + 36$, cannot be factored further.

Then, the roots of the equation are the solutions to $x - 1 = 0$ and $x^2 - 11x + 36 = 0$.

Use the quadratic formula with $a = 1$, $b = -11$, and $c = 36$ to check for other real roots.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-11) \pm \sqrt{(-11)^2 - 4(1)(36)}}{2(1)} \\ x &= \frac{11 \pm \sqrt{121 - 144}}{2} \\ x &= \frac{11 \pm \sqrt{-23}}{2} && \text{Since the square root of a negative number is not a real number, there are no real roots.} \end{aligned}$$

So, the only real root of $0 = -(x^3 - 12x^2 + 47x - 36)$ is $x = 1$. Bill needs to remove 1 ft from each dimension to get a volume of 24 ft³.

Your Turn

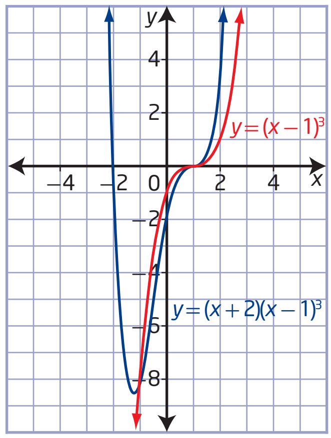
Three consecutive integers have a product of -210.

a) Write a polynomial function to model this situation.

b) What are the three integers?

Key Ideas

- You can sketch the graph of a polynomial function using the x-intercepts, the y-intercept, the degree of the function, and the sign of the leading coefficient.
- The x-intercepts of the graph of a polynomial function are the roots of the corresponding polynomial equation.
- When a polynomial function is in factored form, you can determine the zeros from the factors. When it is not in factored form, you can use the factor theorem and the integral zero theorem to determine the factors.
- When a factor is repeated n times, the corresponding zero has multiplicity, n.
- The shape of a graph close to a zero of $x = a$ (multiplicity n) is similar to the shape of the graph of a function with degree equal to n of the form $y = (x - a)^n$. For example, the graph of a function with a zero of $x = 1$ (multiplicity 3) will look like the graph of the cubic function (degree 3) $y = (x - 1)^3$ in the region close to $x = 1$.



- Polynomial functions change sign at x-intercepts that correspond to zeros of odd multiplicity. The graph crosses over the x-axis at these intercepts.
- Polynomial functions do not change sign at x-intercepts that correspond to zeros of even multiplicity. The graph touches, but does not cross, the x-axis at these intercepts.
- The graph of a polynomial function of the form $y = a(b(x - h))^n + k$ [or $y - k = a(b(x - h))^n$] can be sketched by applying transformations to the graph of $y = x^n$, where $n \in \mathrm{N}$. The transformations represented by a and b may be applied in any order before the transformations represented by h and k.

Check Your Understanding

Practise

1. Solve.

a) $x(x + 3)(x - 4) = 0$

b) $(x - 3)(x - 5)(x + 1) = 0$

c) $(2x + 4)(x - 3) = 0$

2. Solve.

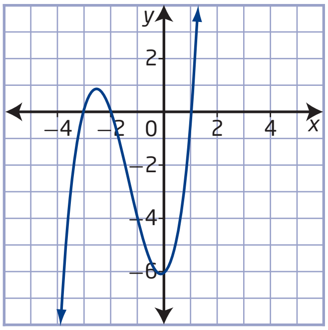
a) $(x + 1)^2(x + 2) = 0$

b) $x^3 - 1 = 0$

c) $(x + 4)^3(x + 2)^2 = 0$

3. Use the graph of the given function to write the corresponding polynomial possible equation. State the roots of the equation. The roots are all integral values.

a)



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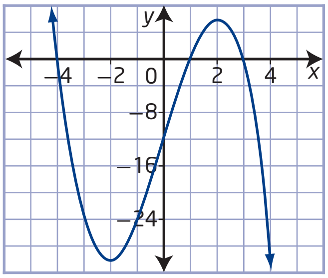
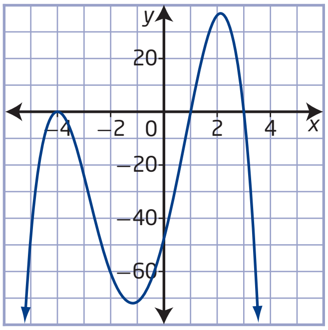


c)



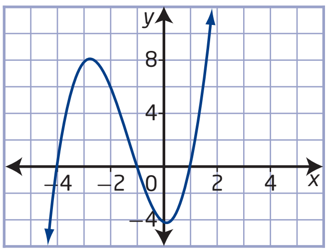
4. For each graph,

i) state the x-intercepts

ii) state the intervals where the function is positive and the intervals where it is negative

iii) explain whether the graph might represent a polynomial that has zero(s) of multiplicity 1, 2, or 3

a)



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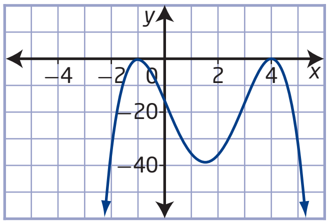

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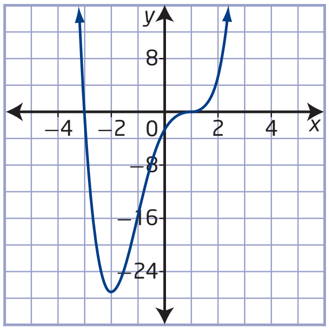

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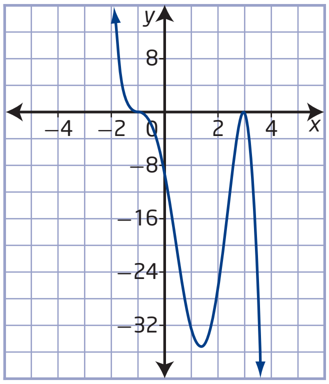
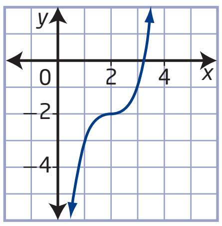
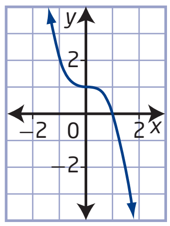


5. Without using technology, match each graph with the corresponding function. Justify your choice.

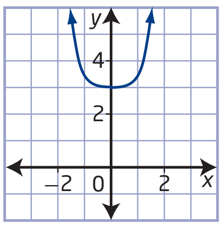
a)



b)



c)



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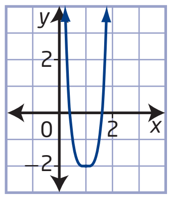


A $y = (2(x - 1))^4 - 2$

B $y = (x - 2)^3 - 2$

C $y = 0.5x^4 + 3$

D $y = (-x)^3 + 1$

6. The graph of $y = x^3$ is transformed to obtain the graph of $y = 0.5(-3(x - 1))^3 + 4$.

a) What are the parameters and corresponding transformations?

b) Copy and complete the table. Use the headings $y = (-3x)^3$, $y = 0.5(-3x)^3$, and $y = 0.5(-3(x - 1))^3 + 4$ for columns two, three, and four, respectively.

$y = x^3$			
(−2, −8)			
(−1, −1)			
(0, 0)			
(1, 1)			
(2, 8)			

c) Sketch the graph of $y = 0.5(-3(x - 1))^3 + 4$.

7. For each function, determine

i) the x-intercepts of the graph

ii) the degree and end behaviour of the graph

iii) the zeros and their multiplicity

iv) the y-intercept of the graph

v) the intervals where the function is positive and the intervals where it is negative

a) $y = x^3 - 4x^2 - 45x$

b) $f(x) = x^4 - 81x^2$

c) $h(x) = x^3 + 3x^2 - x - 3$

d) $k(x) = -x^4 - 2x^3 + 7x^2 + 8x - 12$

8. Sketch the graph of each function in #7.

9. Without using technology, sketch the graph of each function. Label all intercepts.

a) $f(x) = x^4 - 4x^3 + x^2 + 6x$

b) $y = x^3 + 3x^2 - 6x - 8$

c) $y = x^3 - 4x^2 + x + 6$

d) $h(x) = -x^3 + 5x^2 - 7x + 3$

e) $g(x) = (x - 1)(x + 2)^2(x + 3)^2$

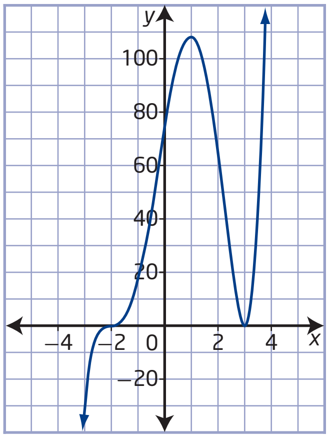
f) $f(x) = -x^4 - 2x^3 + 3x^2 + 4x - 4$

Apply

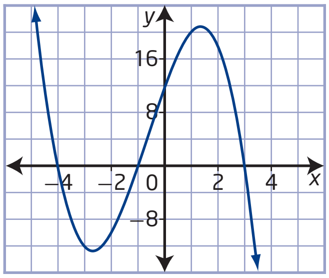
10. For each graph of a polynomial function shown, determine

- the sign of the leading coefficient
- the x-intercepts
- the intervals where the function is positive and the intervals where it is negative
- the equation for the polynomial function

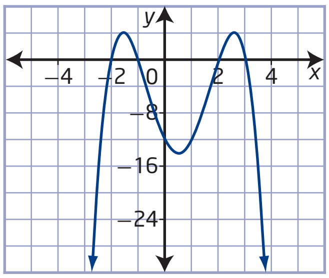
a)



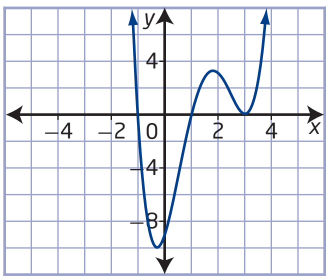
b)



c)



d)



11. a) Given the function $y = x^3$, list the parameters of the transformed polynomial function $y = \left(\frac{1}{2}(x - 2)\right)^3 - 3$.

b) Describe how each parameter in part a) transforms the graph of the function $y = x^3$.

c) Determine the domain and range for the transformed function.

12. The competition swimming pool at Saanich Commonwealth Place is in the shape of a rectangular prism and has a volume of 2100 m^3. The dimensions of the pool are x metres deep by $25x$ metres long by $10x + 1$ metres wide. What are the actual dimensions of the pool?

Did You Know?

Forty-four aquatic events in diving and swimming were held at the Saanich Commonwealth Pool during the 1994 Commonwealth Games held in Victoria, British Columbia. Canada won 32 medals in aquatics.

13. A boardwalk that is x feet wide is built around a rectangular pond. The pond is 30 ft wide and 40 ft long. The combined surface area of the pond and the boardwalk is 2000 ft^2. What is the width of the boardwalk?

14. Determine the equation with least degree for each polynomial function. Sketch a graph of each.

a) a cubic function with zeros -3 (multiplicity 2) and 2 and y-intercept -18

b) a quintic function with zeros -1 (multiplicity 3) and 2 (multiplicity 2) and y-intercept 4

c) a quartic function with a negative leading coefficient, zeros -2 (multiplicity 2) and 3 (multiplicity 2), and a constant term of -6

15. The width of a rectangular prism is w centimetres. The height is 2 cm less than the width. The length is 4 cm more than the width. If the magnitude of the volume of the prism is 8 times the measure of the length, what are the dimensions of the prism?

16. Three consecutive odd integers have a product of -105. What are the three integers?

17. A monument consists of two cubical blocks of limestone. The smaller block rests on the larger. The total height of the monument is 5 m and the area of exposed surface is 61 m^2. Determine the dimensions of the blocks.

Did You Know?

A type of limestone called Tyndall stone has been quarried in Garson, Manitoba, since the 1890s. You can see this stone in structures such as the Parliament Buildings in Ottawa, Ontario, the Saskatchewan Legislative Building in Regina, Saskatchewan, and the Manitoba Legislative Building in Winnipeg, Manitoba.

18. Olutie is learning from her grandmother how to make traditional Inuit wall hangings from stroud and felt. She plans to make a square border for her square wall hanging. The dimensions of the wall hanging with its border are shown. Olutie needs 144 in.^{2} of felt for the border.

a) Write a polynomial expression to model the area of the border.

b) What are the dimensions of her wall hanging, in inches?

c) What are the dimensions of the border, in inches?

Did You Know?

Stroud is a coarse woollen cloth traditionally used to make wall hangings.

Inuit wall hanging, artist unknown.

19. Four consecutive integers have a product of 840. What are the four integers?

Extend

20. Write a cubic function with x-intercepts of $\sqrt{3}$, $-\sqrt{3}$, and 1 and a y-intercept of -1.

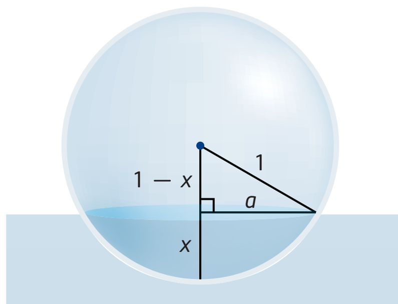
21. The roots of the equation $2x^3 + 3x^2 - 23x - 12 = 0$ are represented by a, b, and c (from least to greatest). Determine the equation with roots $a + b$, $\frac{a}{b}$, and ab.

22. a) Predict the relationship between the graphs of $y = x^3 - x^2$ and $y = (x - 2)^3 - (x - 2)^2$.

b) Graph each function using technology to verify your prediction.

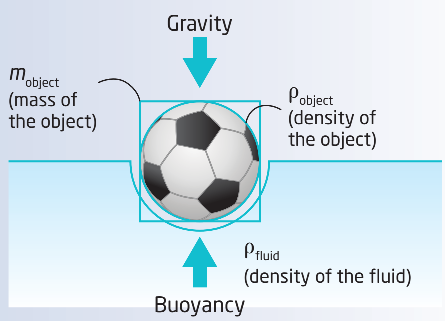
c) Factor each function in part a) to determine the x-intercepts.

23. Suppose a spherical floating buoy has radius 1 m and density $\frac{1}{4}$ that of sea water. Given that the formula for the volume of a spherical cap is $V_{cap} = \frac{\pi x}{6}(3a^2 + x^2)$, to what depth does the buoy sink in sea water?



Did You Know?

Archimedes of Syracuse (287–212 B.C.E.) was a Greek mathematician, physicist, engineer, inventor, and astronomer. He developed what is now known as Archimedes' principle: Any floating object displaces its own weight of fluid.



Create Connections

C1 Why is it useful to express a polynomial in factored form? Explain with examples.

C2 Describe what is meant by a root, a zero, and an x-intercept. How are they related?

C3 How can you tell from a graph if the multiplicity of a zero is 1, an even number, or an odd number greater than 1?

C4 **MINI LAB**

Materials

- graphing calculator or computer with graphing software

Apply your prior knowledge of transformations to predict the effects of translations, stretches, and reflections on polynomial functions of the form $y = a(b(x - h))^n + k$ and the associated graphs.

Step 1 Graph each set of functions on one set of coordinate axes. Sketch the graphs in your notebook.

Set A	**Set B**
i) $y = x^3$	**i)** $y = x^4$
ii) $y = x^3 + 2$	**ii)** $y = (x + 2)^4$
iii) $y = x^3 - 2$	**iii)** $y = (x - 2)^4$

a) Compare the graphs in set A. For any constant k, describe the relationship between the graphs of $y = x^3$ and $y = x^3 + k$.

b) Compare the graphs in set B. For any constant h, describe the relationship between the graphs of $y = x^4$ and $y = (x - h)^4$.

Step 2 Describe the roles of the parameters h and k in functions of the form $y = a(b(x - h))^n + k$.

Step 3 Graph each set of functions on one set of coordinate axes. Sketch the graphs in your notebook.

Set C	**Set D**
i) $y = x^3$	**i)** $y = x^4$
ii) $y = 3x^3$	**ii)** $y = \frac{1}{3}x^4$
iii) $y = -3x^3$	**iii)** $y = -\frac{1}{3}x^4$

a) Compare the graphs in set C. For any integer value a, describe the relationship between the graphs of $y = x^3$ and $y = ax^3$.

b) Compare the graphs in set D. For any rational value a such that $-1 < a < 0$ or $0 < a < 1$, describe the relationship between the graphs of $y = x^4$ and $y = ax^4$.

Step 4 Graph each set of functions on one set of coordinate axes. Sketch the graphs in your notebook.

Set E	**Set F**
i) $y = x^3$	**i)** $y = x^4$
ii) $y = (3x)^3$	**ii)** $y = \left(\frac{1}{3}x\right)^4$
iii) $y = (-3x)^3$	**iii)** $y = \left(-\frac{1}{3}x\right)^4$

a) Compare the graphs in set E. For any integer value b, describe the relationship between the graphs of $y = x^3$ and $y = (bx)^3$.

b) Compare the graphs in set F. For any rational value b such that $-1 < b < 0$ or $0 < b < 1$, describe the relationship between the graphs of $y = x^4$ and $y = (bx)^4$.

Step 5 Describe the roles of the parameters a and b in functions of the form $y = a(b(x - h))^n + k$.

Chapter 3 Review

3.1 Characteristics of Polynomial Functions, pages 106–117

1. Which of the following are polynomial functions? Justify your answer.

a) $y = \sqrt{x + 1}$

b) $f(x) = 3x^4$

c) $g(x) = -3x^3 - 2x^2 + x$

d) $y = \frac{1}{2}x + 7$

2. Use the degree and the sign of the leading coefficient of each function to describe the end behaviour of its corresponding graph. State the possible number of x-intercepts and the value of the y-intercept.

a) $s(x) = x^4 - 3x^2 + 5x$

b) $p(x) = -x^3 + 5x^2 - x + 4$

c) $y = 3x - 2$

d) $y = 2x^2 - 4$

e) $y = 2x^5 - 3x^3 + 1$

3. A parachutist jumps from a plane 11 500 ft above the ground. The height, h, in feet, of the parachutist above the ground t seconds after the jump can be modelled by the function $h(t) = 11\,500 - 16t^2$.

a) What type of function is $h(t)$?

b) What will the parachutist's height above the ground be after 12 s?

c) When will the parachutist be 1500 ft above the ground?

d) Approximately how long will it take the parachutist to reach the ground?

3.2 The Remainder Theorem, pages 118–125

4. Use the remainder theorem to determine the remainder for each division. Then, perform each division using the indicated method. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$ and identify any restrictions on the variable.

a) $x^3 + 9x^2 - 5x + 3$ divided by $x - 2$ using long division

b) $2x^3 + x^2 - 2x + 1$ divided by $x + 1$ using long division

c) $12x^3 + 13x^2 - 23x + 7$ divided by $x - 1$ using synthetic division

d) $-8x^4 - 4x + 10x^3 + 15$ divided by $x + 1$ using synthetic division

5. a) Determine the value of k such that when $f(x) = x^4 + kx^3 - 3x - 5$ is divided by $x - 3$, the remainder is -14.

b) Using your value from part a), determine the remainder when $f(x)$ is divided by $x + 3$.

6. For what value of b will the polynomial $P(x) = 4x^3 - 3x^2 + bx + 6$ have the same remainder when it is divided by both $x - 1$ and $x + 3$?

3.3 The Factor Theorem, pages 126–135

7. Which binomials are factors of the polynomial $P(x) = x^3 - x^2 - 16x + 16$? Justify your answers.

a) $x - 1$

b) $x + 1$

c) $x + 4$

d) $x - 4$

8. Factor fully.

a) $x^3 - 4x^2 + x + 6$

b) $-4x^3 - 4x^2 + 16x + 16$

c) $x^4 - 4x^3 - x^2 + 16x - 12$

d) $x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12$

9. Rectangular blocks of granite are to be cut and used to build the front entrance of a new hotel. The volume, V, in cubic metres, of each block can be modelled by the function $V(x) = 2x^3 + 7x^2 + 2x - 3$, where x is in metres.

a) What are the possible dimensions of the blocks in terms of x?

b) What are the possible dimensions of the blocks when $x = 1$?

10. Determine the value of k so that $x + 3$ is a factor of $x^3 + 4x^2 - 2kx + 3$.

3.4 Equations and Graphs of Polynomial Functions, pages 136–152

11. For each function, determine

- the x-intercepts of the graph
- the degree and end behaviour of the graph
- the zeros and their multiplicity
- the y-intercept of the graph
- the interval(s) where the function is positive and the interval(s) where it is negative

Then, sketch the graph.

a) $y = (x + 1)(x - 2)(x + 3)$

b) $y = (x - 3)(x + 2)^2$

c) $g(x) = x^4 - 16x^2$

d) $g(x) = -x^5 + 16x$

12. The graph of $y = x^3$ is transformed to obtain the graph of $y = 2(-4(x - 1))^3 + 3$.

a) What are the parameters and corresponding transformations?

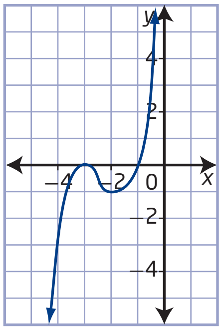
b) Copy and complete the table.

Transformation	Parameter Value	Equation
horizontal stretch/ reflection in y-axis		$y =$
vertical stretch/ reflection in x-axis		$y =$
translation left/right		$y =$
translation up/down		$y =$

c) Sketch the graph of $y = 2(-4(x - 1))^3 + 3$.

13. Determine the equation of the polynomial function that corresponds to each graph.

a)



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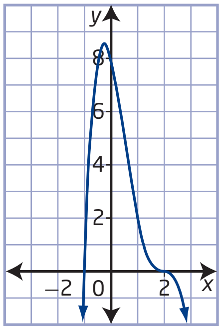


14. The zeros of a quartic function are -2, -1, and 3 (multiplicity 2).

a) Determine equations for two functions that satisfy this condition.

b) Determine the equation of the function that satisfies this condition and passes through the point (2, 24).

15. The specifications for a cardboard box state that the width must be 5 cm less than the length, and the height must be double the length.

a) Write the equation for the volume of the box.

b) What are the dimensions of a box with a volume of 384 cm^3?

Chapter 3 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. Which statement is true?

A Some odd-degree polynomial functions have no x-intercepts.

B Even-degree polynomial functions always have an even number of x-intercepts.

C All odd-degree polynomial functions have at least one x-intercept.

D All even-degree polynomial functions have at least one x-intercept.

2. Which statement is true for $P(x) = 3x^3 + 4x^2 + 2x - 9$?

A When $P(x)$ is divided by $x + 1$, the remainder is 6.

B $x - 1$ is a factor of $P(x)$.

C $P(3) = 36$

D $P(x) = (x + 3)(3x^2 - 5x + 17) + 42$

3. Which set of values for x should be tested to determine the possible zeros of $x^4 - 2x^3 - 7x^2 - 8x + 12$?

A $\pm1, \pm2, \pm4, \pm12$

B $\pm1, \pm2, \pm3, \pm4, \pm6$

C $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8$

D $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

4. Which of the following is a factor of $2x^3 - 5x^2 - 9x + 18$?

A $x - 1$

B $x + 2$

C $x + 3$

D $x - 6$

5. Which statement describes how to transform the function $y = x^3$ into $y = 3\left(\frac{1}{4}(x - 5)\right)^3 - 2$?

A stretch horizontally by a factor of 3, stretch vertically by a factor of $\frac{1}{4}$, and translate 5 units to the left and 2 units up

B stretch horizontally by a factor of 3, stretch vertically by a factor of $\frac{1}{4}$, and translate 2 units to the right and 5 units down

C stretch horizontally by a factor of 4, stretch vertically by a factor of 3, and translate 5 units to the right and 2 units down

D stretch horizontally by a factor of 4, stretch vertically by a factor of 3, and translate 2 units to the left and 5 units up

Short Answer

6. Determine the real roots of each equation.

a) $(x + 4)^2(x - 3) = 0$

b) $(x - 3)^2(x + 1)^2 = 0$

c) $(4x^2 - 16)(x^2 - 3x - 10) = 0$

d) $(9x^2 - 81)(x^2 - 9) = 0$

7. Factor each polynomial in x.

a) $P(x) = x^3 + 4x^2 + 5x + 2$

b) $P(x) = x^3 - 13x^2 + 12$

c) $P(x) = -x^3 + 6x^2 - 9x$

d) $P(x) = x^3 - 3x^2 + x + 5$

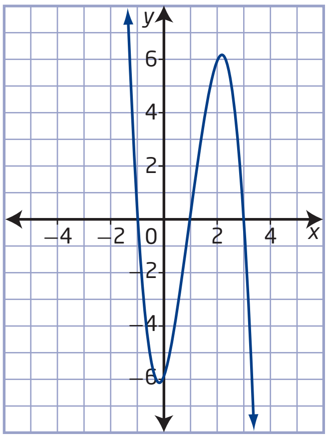
8. Match each equation with the corresponding graph of a polynomial function. Justify your choices.

a) $y = x^4 + 3x^3 - 3x^2 - 7x + 6$

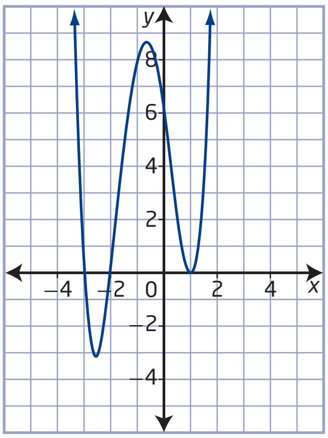
b) $y = x^3 - 4x^2 + 4x$

c) $y = -2x^3 + 6x^2 + 2x - 6$

A



B



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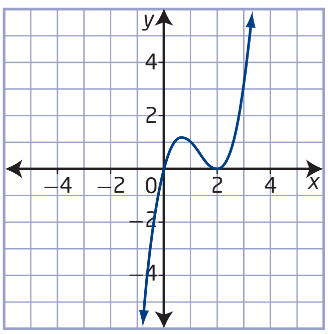
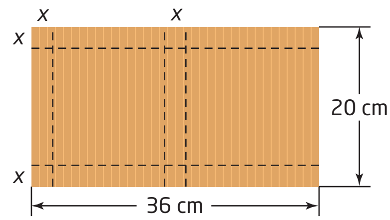


Extended Response

9. Boxes for candies are to be constructed from cardboard sheets that measure 36 cm by 20 cm. Each box is formed by folding a sheet along the dotted lines, as shown.



a) What is the volume of the box as a function of x?

b) What are the possible whole-number dimensions of the box if the volume is to be 512 cm^3?

10. a) Identify the parameters a, b, h, and k in the polynomial $y = \frac{1}{3}(x + 3)^3 - 2$. Describe how each parameter transforms the base function $y = x^3$.

b) State the domain and range of the transformed function.

c) Sketch graphs of the base function and the transformed function on the same set of axes.