CHAPTER 9

Rational Functions

Why does the lens on a camera need to move to focus on objects that are nearer or farther away? What is the relationship between the travel time for a plane and the velocity of the wind in which it is flying? How can you relate the amount of light from a source to the distance from the source? The mathematics behind all of these situations involves rational functions.

A simple rational function is used to relate distance, time, and speed. More complicated rational functions may be used in a business to model average costs of production or by a doctor to predict the amount of medication remaining in a patient's bloodstream.

In this chapter, you will explore a variety of rational functions. You have used the term *rational* before, with rational numbers and rational expressions, so what is a rational *function*?

Key Terms

rational function

point of discontinuity

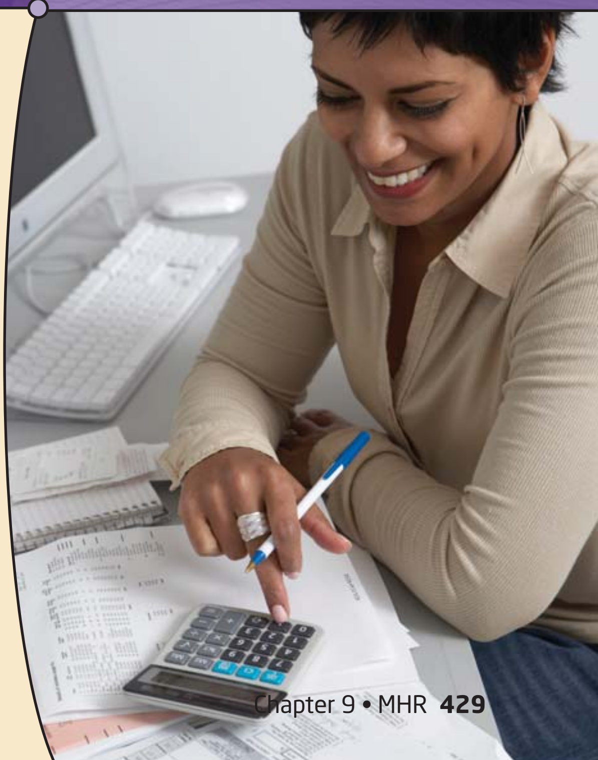
Career Link

To become a chartered accountant (CA), you need education, experience, and evaluation. A CA student first completes a bachelor's degree including several accounting courses. Then, the CA student works for a chartered firm while taking courses that lead to a final series of examinations that determine whether he or she meets the requirements of a CA.

Why should you do all this work? CAs are business, tax, and personal accounting specialists. Many CAs go on to careers in senior management because their skills are so valued.

Web Link

To learn more about a career as a chartered accountant, go to www.mcgrawhill.ca/school/learningcentres and follow the links.



9.1

Exploring Rational Functions Using Transformations

Focus on...

- graphing, analysing, and comparing rational functions using transformations and using technology
- examining the behaviour of the graphs of rational functions near non-permissible values

The Trans Canada Trail is a system of 22 000 km of linked trails that passes through every province and territory and connects the Pacific, Arctic, and Atlantic Oceans. When completely developed, it will be the world's longest network of trails. Millions of people walk, run, cycle, hike, canoe, horseback ride, snowmobile, and more on the trail.

If you cycle a 120-km section of the Trans Canada Trail, the time it takes is related to your average speed. Cycling more quickly means it takes less time; cycling more slowly means it takes more time. The relationship between the time and the average speed can be expressed mathematically with a **rational function**. What does the graph of this function look like?

rational function

- a function that can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x) \neq 0$
- some examples are $y = \frac{20}{x}$, $C(n) = \frac{100 + 2n}{n}$, and $f(x) = \frac{3x^2 + 4}{x - 5}$

Trans Canada Trail at Kettle Valley, British Columbia

Did You Know?

Some sections of the Trans Canada Trail are based on long-established routes of travel, such as the Dempster Highway in Northern Canada. This narrow gravel road is over 500 km long and connects Dawson City, Yukon Territory, with Inuvik, Norhwest Territories on the Mackenzie River delta. The route is based on an old First Nations trading route dating back to the last ice age. This corridor was ice-free during that period and is believed by many to be a route used by the first people in North America.

Web Link

To learn more about the Trans Canada Trail, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Investigate Rational Functions

A: Relate Time and Speed

Materials

- graphing technology

1. a) Copy and complete the table of values giving the time to cycle a 120-km stretch of the Trans Canada Trail for a variety of average speeds.

Average Speed (km/h)	1	2	3	4	5	6	8	10	12	15	20	24	30	40
Time (h)														

b) What happens to the time as the average speed gets smaller and smaller in value? larger and larger in value?

2. a) Write an equation to express the time, t, in hours, as a function of the average speed, v, in kilometres per hour.

b) Is the value zero a part of the domain or the range in this situation? Explain.

3. a) Graph the function.

b) How does the shape of the graph relate to your answer to step 2b)?

c) What does the graph show about the time to cycle a 120-km stretch of the Trans Canada Trail as the average speed gets closer to zero?

Reflect and Respond

4. a) How is the relationship between average speed and time connected to the shape of the graph?

b) Does the graph of this function have endpoints? Explain.

B: The Effect of the Parameters *a*, *h*, and *k* on the Function $y = \frac{a}{x - h} + k$

5. a) Graph the functions $y = \frac{1}{x}$, $y = \frac{4}{x}$, and $y = \frac{12}{x}$ using technology.

b) Describe the behaviour of these functions as x approaches zero.

c) What happens to the values of these functions as $|x|$ becomes larger and larger?

d) Compare the graphs. For any real value a, describe the relationship between the graphs of $y = \frac{1}{x}$ and $y = \frac{a}{x}$.

6. a) Graph the functions $y = \frac{1}{x}$ and $y = \frac{4}{x - 3} + 2$.

b) Compare the graphs. How do the numbers in the transformed function equation affect the shape and position of its graph relative to the graph of the base function $y = \frac{1}{x}$?

c) What function from step 5 has a graph that is congruent to the graph of $y = \frac{4}{x - 3} + 2$? Why do you think this is?

7. a) Predict the effects of the parameters in the function $y = -\frac{12}{x + 1} - 5$. Graph the function to check your predictions.

b) Which function from step 5 has a graph that is congruent to the graph of this function? How are the equations and graphs connected?

Reflect and Respond

8. Are the locations of the asymptotes of the function $y = \frac{1}{x}$ affected if a vertical stretch by a factor of a is applied to the function? Explain your thinking.

9. a) Can you tell from the equation of the transformed rational function $y = \frac{a}{x - h} + k$ where its graph has a vertical asymptote? Explain.

b) Describe how the values of a function change as x approaches a non-permissible value.

c) What non-translated function has a graph that is congruent to the graph of $y = \frac{8}{x - 7} + 6$? How might you graph $y = \frac{8}{x - 7} + 6$ without technology using this relationship?

Link the Ideas

Did You Know?

The equation of the function $y = \frac{a}{x}$ is equivalent to $xy = a$. The equation $xy = a$ shows that for any point on the graph, the product of the x- and y-coordinates is always equal to a.

The rational function that relates speed to time for a given distance is related to the base function $y = \frac{1}{x}$ by a vertical stretch.

The graph of a rational function of the form $y = \frac{a}{x}$ represents a vertical stretch by a factor of a of the graph of $y = \frac{1}{x}$, because $y = \frac{a}{x}$ can be written as $y = a\left(\frac{1}{x}\right)$.

Graphs of rational functions of the form $y = \frac{a}{x}$ have two separate branches that approach the asymptotes at $x = 0$ and $y = 0$.

Example 1

Graph a Rational Function Using a Table of Values

Analyse the function $y = \frac{10}{x}$ using a table of values and a graph. Identify characteristics of the graph, including the behaviour of the function for its non-permissible value.

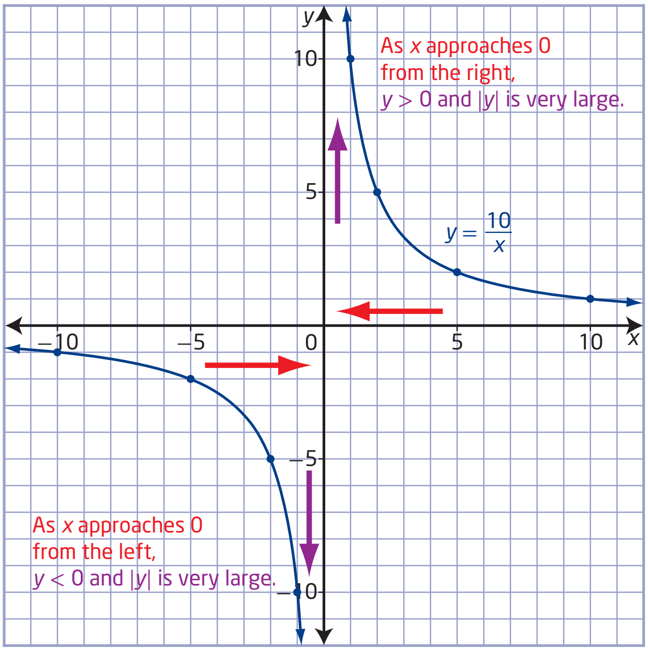
Solution

Select values of x that make it easy to calculate the corresponding values of y for $y = \frac{10}{x}$.

Why is the function undefined when x is zero?

The equation of the function can be rearranged to give $xy = 10$. How might this form be used to generate ordered pairs for the table and points on the graph?

x	y
−100	−0.1
−20	−0.5
−10	−1
−5	−2
−2	−5
−1	−10
−0.5	−20
−0.1	−100
0	undefined
0.1	100
0.5	20
1	10
2	5
5	2
10	1
20	0.5
100	0.1



Why does $|y|$ get larger as the values of x approach zero?

What happens to the values of y as $|x|$ becomes very large?

For this function, when 0 is substituted for the value of x, the denominator has a value of 0. Since division by 0 is undefined, 0 is a non-permissible value. This corresponds to the vertical asymptote in the graph at $x = 0$. As the values of x approach zero, the absolute value of y gets very large.

Summarize the characteristics of the function using a table.

Characteristic	$y = \frac{10}{x}$
Non-permissible value	$x = 0$
Behaviour near non-permissible value	As x approaches 0, $\lvert y \rvert$ becomes very large.
End behaviour	As $\lvert x \rvert$ becomes very large, y approaches 0.
Domain	$\{x \mid x \neq 0, x \in R\}$
Range	$\{y \mid y \neq 0, y \in R\}$
Equation of vertical asymptote	$x = 0$
Equation of horizontal asymptote	$y = 0$

Your Turn

Analyse the function $y = \frac{6}{x}$ using a table of values and a graph. Identify characteristics of the graph, including the behaviour of the function for its non-permissible value.

Did You Know?

In Pre-Calculus 11, you graphed and analysed rational functions that are reciprocals of linear or quadratic functions: $y = \frac{1}{f(x)}$, where $f(x) \neq 0$. In this chapter, you will explore rational functions with numerators and denominators that are monomials, binomials, or trinomials.

You can sometimes graph and analyse more complicated rational functions by considering how they are related by transformations to base rational functions.

To obtain the graph of a rational function of the form $y = \frac{a}{x - h} + k$ from the graph of $y = \frac{1}{x}$, apply a vertical stretch by a factor of a, followed by translations of h units horizontally and k units vertically.

- The graph has a vertical asymptote at $x = h$.
- The graph has a horizontal asymptote at $y = k$.
- Knowing the location of the asymptotes and drawing them first can help you graph and analyse the function.

Example 2

Graph a Rational Function Using Transformations

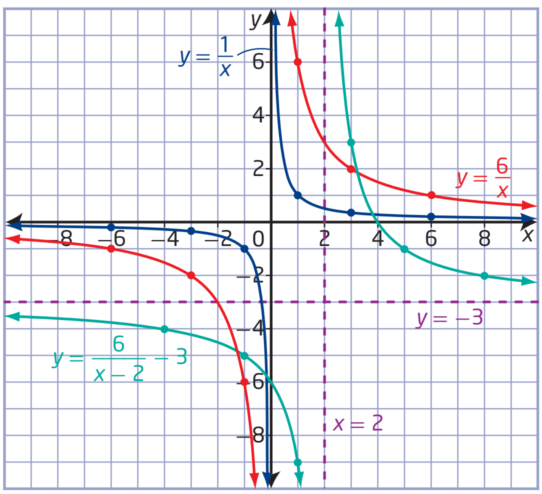
Sketch the graph of the function $y = \frac{6}{x - 2} - 3$ using transformations, and identify any important characteristics of the graph.

Solution

Compare the function $y = \frac{6}{x - 2} - 3$ to the form $y = \frac{a}{x - h} + k$ to determine the values of the parameters: $a = 6$, $h = 2$, and $k = -3$.

To obtain the graph of $y = \frac{6}{x - 2} - 3$ from the graph of $y = \frac{1}{x}$, apply a vertical stretch by a factor of 6, and then a translation of 2 units to the right and 3 units down.

The asymptotes of the graph of $y = \frac{6}{x - 2} - 3$ translate in the same way from their original locations of $x = 0$ and $y = 0$. Therefore, the vertical asymptote is located 2 units to the right at $x = 2$, and the horizontal asymptote is located 3 units down at $y = -3$.



How can you use the asymptotes to help you sketch the graph?

How might considering ordered pairs for $y = \frac{6}{x}$ help you graph $y = \frac{6}{x - 2} - 3$?

What happens to $|y|$ as the values of x approach 2?

What happens to the values of y as $|x|$ becomes very large?

Summarize the characteristics of the graph using a table:

Characteristic	$y = \frac{6}{x - 2} - 3$
Non-permissible value	$x = 2$
Behaviour near non-permissible value	As x approaches 2, $\lvert y \rvert$ becomes very large.
End behaviour	As $\lvert x \rvert$ becomes very large, y approaches -3.
Domain	$\{x \mid x \neq 2, x \in R\}$
Range	$\{y \mid y \neq -3, y \in R\}$
Equation of vertical asymptote	$x = 2$
Equation of horizontal asymptote	$y = -3$

Which of the characteristics listed are related to each other?

How is each of the function's characteristics related to the equation of the function?

Your Turn

Sketch the graph of the function $y = \frac{4}{x + 1} + 5$ using transformations, and identify the important characteristics of the graph.

Example 3

Graph a Rational Function With Linear Expressions in the Numerator and the Denominator

Graph the function $y = \frac{4x - 5}{x - 2}$. Identify any asymptotes and intercepts.

Solution

Method 1: Use Paper and Pencil

Determine the locations of the intercepts and asymptotes first, and then use them as a guide to sketch the graph.

Find the y-intercept of the function by substituting 0 for x.

$$y = \frac{4x - 5}{x - 2}$$
$$y = \frac{4(0) - 5}{0 - 2}$$
$$y = 2.5$$

The y-intercept occurs at (0, 2.5).

Find the x-intercept of the function by solving for x when $y = 0$.

$$y = \frac{4x - 5}{x - 2}$$
$$0 = \frac{4x - 5}{x - 2}$$
$$(x - 2)(0) = (x - 2)\left(\frac{4x - 5}{x - 2}\right)$$
$$0 = 4x - 5$$
$$x = 1.25$$

The x-intercept occurs at (1.25, 0).

Manipulate the equation of this function algebraically to obtain the form $y = \frac{a}{x - h} + k$, which reveals the location of both the vertical asymptote and the horizontal asymptote.

$$y = \frac{4x - 5}{x - 2}$$

$$y = \frac{4x - 8 + 8 - 5}{x - 2}$$

$$y = \frac{4(x - 2) + 3}{x - 2}$$

Why is it necessary to change the numerator so that it involves the expression $(x - 2)$?

$$y = \frac{4(x - 2)}{x - 2} + \frac{3}{x - 2}$$

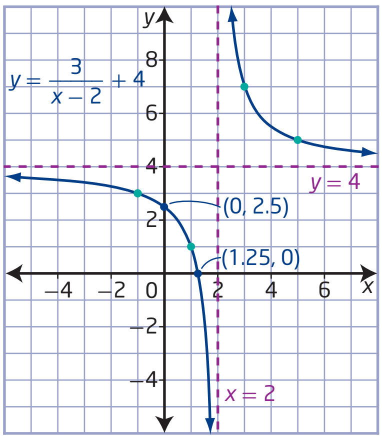
$$y = 4 + \frac{3}{x - 2}$$

How is this form related to polynomial division?

$$y = \frac{3}{x - 2} + 4$$

To obtain the graph of the function $y = \frac{3}{x - 2} + 4$ from the graph of $y = \frac{1}{x}$, apply a vertical stretch by a factor of 3, and then a translation of 2 units to the right and 4 units up.

Translate the asymptotes in the same way, to $x = 2$ and $y = 4$.

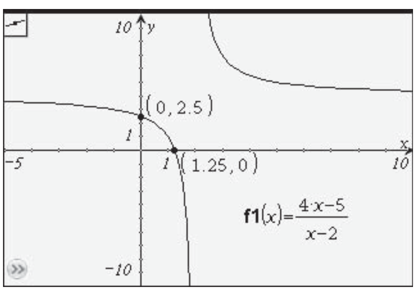


How can considering the pattern of ordered pairs for $y = \frac{3}{x}$ help you locate the points shown in green?

How are these four green points related to the symmetry in the graph?

Method 2: Use a Graphing Calculator

Graph the function $y = \frac{4x - 5}{x - 2}$ using a graphing calculator. Adjust the dimensions of the window so that all of the important features of the graph are visible.



Use the zero, value, trace and table features to verify the locations of the intercepts and asymptotes. The graph of the function has
- a y-intercept of 2.5
- an x-intercept of 1.25
- a vertical asymptote at $x = 2$
- a horizontal asymptote at $y = 4$

Your Turn

Graph the function $y = \frac{2x + 2}{x - 4}$. Identify any asymptotes and intercepts.

Example 4

Compare Rational Functions

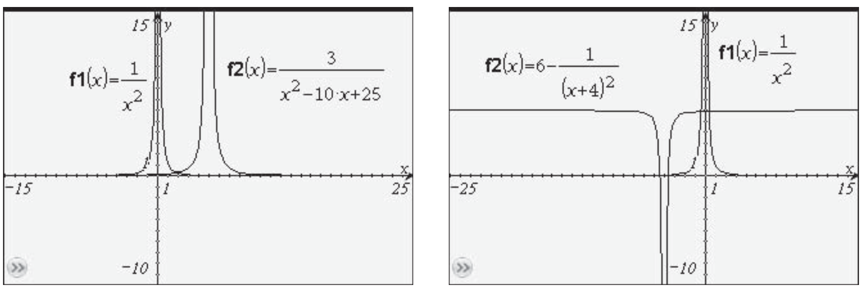
Consider the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{3}{x^2 - 10x + 25}$, and $h(x) = 6 - \frac{1}{(x + 4)^2}$.

Graph each pair of functions.
- $f(x)$ and $g(x)$
- $f(x)$ and $h(x)$

Compare the characteristics of the graphs of the functions.

Solution

Graph the functions using a graphing calculator. Set the window dimensions so that important features are visible.



Rewrite the functions $g(x)$ and $h(x)$ algebraically to reveal how they are related to the base function $f(x) = \frac{1}{x^2}$. Then, use transformations to explain some of the similarities in the graphs.

$$g(x) = \frac{3}{x^2 - 10x + 25}$$
$$g(x) = \frac{3}{(x - 5)^2}$$
$$g(x) = 3\left(\frac{1}{(x - 5)^2}\right)$$
$$g(x) = 3f(x - 5)$$

To obtain the graph of $g(x)$ from the graph of $f(x)$, apply a vertical stretch by a factor of 3 and a translation of 5 units to the right.

$$h(x) = 6 - \frac{1}{(x + 4)^2}$$
$$h(x) = -\frac{1}{(x + 4)^2} + 6$$
$$h(x) = -f(x + 4) + 6$$

To obtain the graph of $h(x)$ from the graph of $f(x)$, apply a reflection in the x-axis and a translation of 4 units to the left and 6 units up.

Use the appropriate graphing technology features to verify the locations of the asymptotes.

Characteristic	$f(x) = \frac{1}{x^2}$	$g(x) = \frac{3}{x^2 - 10x + 25}$	$h(x) = 6 - \frac{1}{(x + 4)^2}$
Non-permissible value	$x = 0$	$x = 5$	$x = -4$
Behaviour near non-permissible value	As x approaches 0, $\lvert y \rvert$ becomes very large.	As x approaches 5, $\lvert y \rvert$ becomes very large.	As x approaches -4, $\lvert y \rvert$ becomes very large.
End behaviour	As $\lvert x \rvert$ becomes very large, y approaches 0.	As $\lvert x \rvert$ becomes very large, y approaches 0.	As $\lvert x \rvert$ becomes very large, y approaches 6.
Domain	$\{x \mid x \neq 0, x \in R\}$	$\{x \mid x \neq 5, x \in R\}$	$\{x \mid x \neq -4, x \in R\}$
Range	$\{y \mid y > 0, y \in R\}$	$\{y \mid y > 0, y \in R\}$	$\{y \mid y < 6, y \in R\}$
Equation of vertical asymptote	$x = 0$	$x = 5$	$x = -4$
Equation of horizontal asymptote	$y = 0$	$y = 0$	$y = 6$

The graphs have the following in common:

- Each function has a single non-permissible value.
- Each has a vertical asymptote and a horizontal asymptote.
- The domain of each function consists of all real numbers except for a single value. The range of each function consists of a restricted set of the real numbers.
- $\lvert y \rvert$ becomes very large for each function when the values of x approach the non-permissible value for the function.

Your Turn

Graph the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{-1}{(x - 3)^2}$, and $h(x) = 2 + \frac{5}{x^2 + 2x + 1}$. Compare the characteristics of the graphs.

Example 5

Apply Rational Functions

A mobile phone service provider offers several different prepaid plans. One of the plans has a \$10 monthly fee and a rate of 10¢ per text message sent or minute of talk time. Another plan has a monthly fee of \$5 and a rate of 15¢ per text message sent or minute of talk time. Talk time is billed per whole minute.

a) Represent the average cost per text or minute of each plan with a rational function.

b) Graph the functions.

c) What do the graphs show about the average cost per text or minute for these two plans as the number of texts and minutes changes?

d) Which plan is the better choice?

Solution

a) Write a function to represent each plan.
Let f_1 and f_2 represent the average cost per text sent or minute used for the first and second plans, respectively.
Let x be the combined number of texts sent and minutes used. x is a whole number.

Calculate the average cost per text or minute of each plan as the quotient of the total cost and the combined number of texts and minutes.

Determine expressions for the total cost of each plan.

Total cost = monthly fee + (rate per text or minute)(combined number of texts and minutes)

Total cost = 10 + 0.1x for the first plan
or
Total cost = 5 + 0.15x for the second plan

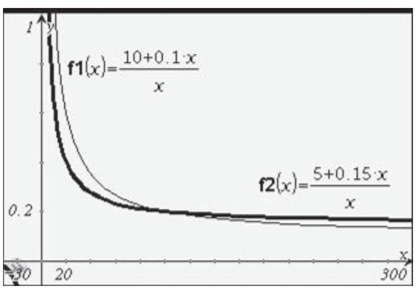
Substitute each expression into the following formula:

$$\text{Average cost} = \frac{\text{total cost}}{\text{combined number of texts and minutes}}$$

$$f_1(x) = \frac{10 + 0.1x}{x} \text{ and } f_2(x) = \frac{5 + 0.15x}{x}$$

Since x ≠ 0, the domain becomes the set of natural numbers.

b) Graph the two functions using technology.



Why does the graph show only quadrant I?

c) Both functions have a vertical asymptote at $x = 0$, corresponding to the non-permissible value of each function.

Although the data is discrete, the function that models it is continuous. Therefore, the average cost function is only valid in the domain $\{x \mid x \in \mathrm{N}\}$. The average cost per text or minute is undefined when x is exactly zero, but the average cost gets higher and higher as the combined number of texts and minutes approaches the non-permissible value of zero.

Both functions also appear to have a horizontal asymptote. The average cost for each plan decreases as the combined number of texts and minutes increases.

Rewrite the equations of the two functions in the form $y = \frac{a}{x - h} + k$ so you can analyse them using the locations of the asymptotes:

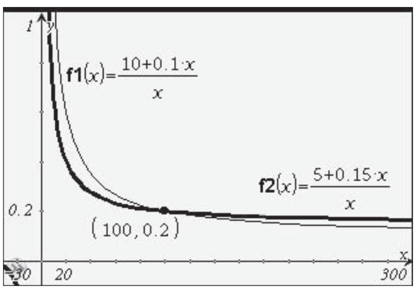
$$f_1(x) = \frac{10 + 0.1x}{x} \qquad f_2(x) = \frac{5 + 0.15x}{x}$$

$$f_1(x) = \frac{10}{x} + \frac{0.1x}{x} \qquad f_2(x) = \frac{5}{x} + \frac{0.15x}{x}$$

$$f_1(x) = \frac{10}{x} + 0.1 \qquad f_2(x) = \frac{5}{x} + 0.15$$

The horizontal asymptote of the function $f_1(x)$ is $y = 0.1$, and the horizontal asymptote of the function $f_2(x)$ is $y = 0.15$. The monthly fee is spread out over the combined number of texts and minutes used. The greater the combined number of texts and minutes becomes, the closer the average cost gets to the value of \$0.10 or \$0.15.

d) To decide how to make a choice between the two plans, determine when they have the same cost. The two functions intersect when $x = 100$. The plans have the same average cost for 100 combined texts and minutes. The first plan is better for more than 100 combined texts and minutes, while the second plan is better for fewer than 100 combined texts and minutes.



Your Turn

Marlysse is producing a tourism booklet for the town of Atlin, British Columbia, and its surrounding area. She is comparing the cost of printing from two different companies. The first company charges a \$50 setup fee and \$2.50 per booklet. The second charges \$80 for setup and \$2.10 per booklet.

a) Represent the average cost per booklet for each company as a function of the number of booklets printed.

b) Graph the two functions.

c) Explain how the characteristics of the graphs are related to the situation.

d) Give Marlysse advice about how she should choose a printing company.

Atlin, British Columbia

Did You Know?

Atlin is a remote but spectacularly beautiful community in the northwest corner of British Columbia on the eastern shore of Atlin Lake. The surrounding area has been used by the Taku River Tlingit First Nations people for many years, and the name *Atlin* comes from the Tlingit word *Aa Tlein*, meaning *big water*.

Key Ideas

- Rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x) \neq 0$.
- Rational functions where $p(x)$ and $q(x)$ have no common factor other than one have vertical asymptotes that correspond to the non-permissible values of the function, if there are any.
- You can sometimes use transformations to graph rational functions and explain common characteristics and differences between them.
- You can express the equations of some rational functions in an equivalent form and use it to analyse and graph functions without using technology.

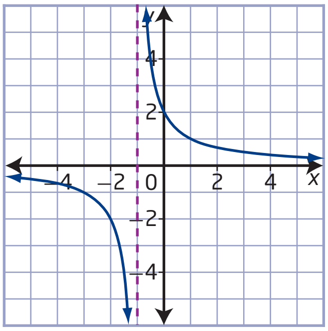
Practise

1. The equations and graphs of four rational functions are shown. Which graph matches which function? Give reason(s) for each choice.

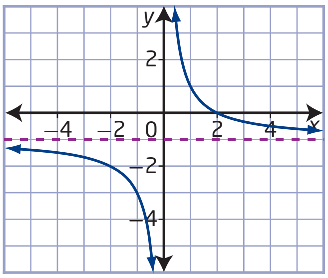
$A(x) = \frac{2}{x} - 1$ $\quad$ $B(x) = \frac{2}{x + 1}$

$C(x) = \frac{2}{x - 1}$ $\quad$ $D(x) = \frac{2}{x} + 1$

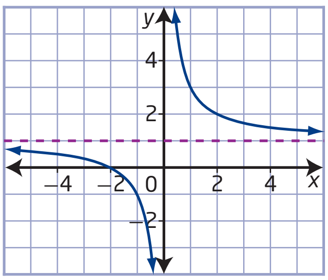
a)



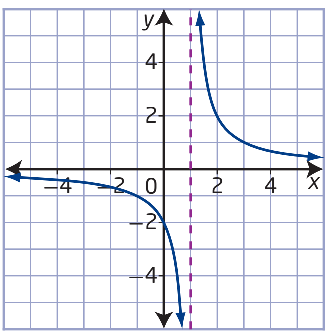
b)



c)



d)



2. Identify the appropriate base rational function, $y = \frac{1}{x}$ or $y = \frac{1}{x^2}$, and then use transformations of its graph to sketch the graph of each of the following functions. Identify the asymptotes.

a) $y = \frac{1}{x + 2}$ $\quad$ **b)** $y = \frac{1}{x - 3}$

c) $y = \frac{1}{(x + 1)^2}$ $\quad$ **d)** $y = \frac{1}{(x - 4)^2}$

3. Sketch the graph of each function using transformations. Identify the domain and range, intercepts, and asymptotes.

a) $y = \frac{6}{x + 1}$

b) $y = \frac{4}{x} + 1$

c) $y = \frac{2}{x - 4} - 5$

d) $y = -\frac{8}{x - 2} + 3$

4. Graph each function using technology and identify any asymptotes and intercepts.

a) $y = \frac{2x + 1}{x - 4}$

b) $y = \frac{3x - 2}{x + 1}$

c) $y = \frac{-4x + 3}{x + 2}$

d) $y = \frac{2 - 6x}{x - 5}$

5. Write each function in the form $y = \frac{a}{x - h} + k$. Determine the location of any asymptotes and intercepts. Then, confirm your answers by graphing with technology.

a) $y = \frac{11x + 12}{x}$

b) $y = \frac{x}{x + 8}$

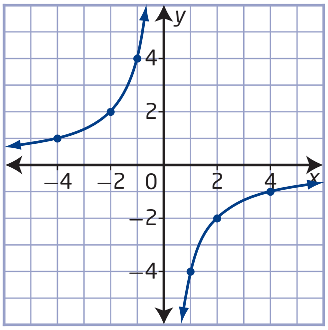
c) $y = \frac{-x - 2}{x + 6}$

6. Graph the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{-8}{(x + 6)^2}$, and $h(x) = \frac{4}{x^2 - 4x + 4} - 3$. Discuss the characteristics of the graphs and identify any common features.

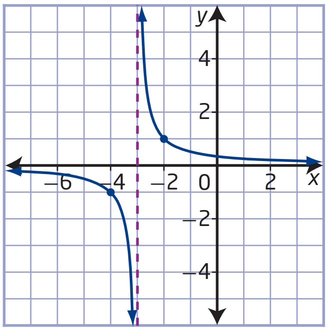
Apply

7. Write the equation of each function in the form $y = \frac{a}{x - h} + k$.

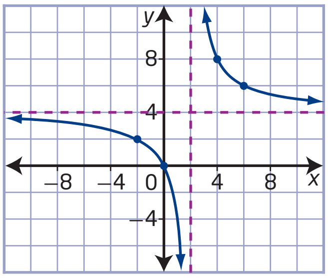
a)



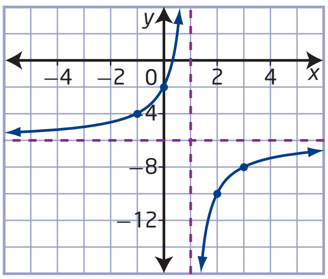
b)



c)



d)



8. The rational function $y = \frac{a}{x - 7} + k$ passes through the points (10, 1) and (2, 9).

a) Determine the values of a and k.

b) Graph the function.

9. a) Write a possible equation in the form $y = \frac{p(x)}{q(x)}$ that has asymptotes at $x = 2$ and $y = -3$.

b) Sketch its graph and identify its domain and range.

c) Is there only one possible function that meets these criteria? Explain.

10. Mira uses algebra to rewrite the function $y = \frac{2 - 3x}{x - 7}$ in an equivalent form that she can graph by hand.

$$y = \frac{2 - 3x}{x - 7}$$
$$y = \frac{-3x + 2}{x - 7}$$
$$y = \frac{-3x - 21 + 21 + 2}{x - 7}$$
$$y = \frac{-3(x - 7) + 23}{x - 7}$$
$$y = \frac{-3(x - 7)}{x - 7} + \frac{23}{x - 7}$$
$$y = -3 + \frac{23}{x - 7}$$
$$y = \frac{23}{x - 7} - 3$$

a) Identify and correct any errors in Mira's work.

b) How might Mira have discovered that she had made an error without using technology? How might she have done so with technology?

11. a) Write the function $y = \frac{x - 2}{2x + 4}$ in the form $y = \frac{a}{x - h} + k$.

b) Sketch the graph of the function using transformations.

12. Determine the locations of the intercepts of the function $y = \frac{3x - 5}{2x + 3}$. Use a graph of the function to help you determine the asymptotes.

13. The number, N, of buyers looking to buy a home in a particular city is related to the average price, p, of a home in that city by the function $N(p) = \frac{500\ 000}{p}$. Explain how the values of the function behave as the value of p changes and what the behaviour means in this situation.

14. A rectangle has a constant area of 24 cm².

a) Write an equation to represent the length, l, as a function of the width, w, for this rectangle. Graph the function.

b) Describe how the length changes as the width varies.

15. The student council at a large high school is having a fundraiser for a local charity. The council president suggests that they set a goal of raising $4000.

a) Let x represent the number of students who contribute. Let y represent the average amount required per student to meet the goal. What function y in terms of x represents this situation?

b) Graph the function.

c) Explain what the behaviour of the function for various values of x means in this context.

d) How would the equation and graph of the function change if the student council also received a $1000 donation from a local business?

16. Hanna is shopping for a new deep freezer and is deciding between two models. One model costs $500 and has an estimated electricity cost of $100/year. A second model that is more energy efficient costs $800 but has an estimated electricity cost of $60/year.

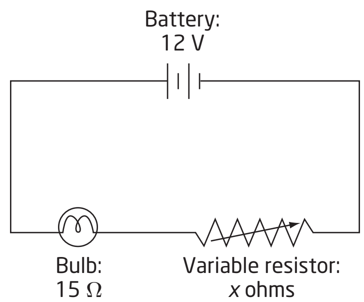
a) For each freezer, write an equation for the average cost per year as a function of the time, in years.

b) Graph the functions for a reasonable domain.

c) Identify important characteristics of each graph and explain what they show about the situation.

d) How can the graph help Hanna decide which model to choose?

17. Ohm's law relates the current, I, in amperes (A); the voltage, V, in volts (V); and the resistance, R, in ohms (Ω), in electrical circuits with the formula $I = \frac{V}{R}$. Consider the electrical circuit in the diagram.



A variable resistor is used to control the brightness of a small light bulb and can be set anywhere from 0 Ω to 100 Ω. The total resistance in the circuit is the sum of the resistances of the variable resistor and the bulb.

a) Write an equation for the current, I, in the circuit as a function of the resistance of the variable resistor, x.

b) What domain is appropriate for this situation? Does the graph of the function have a vertical asymptote? Explain.

c) Graph the function. What setting is needed on the variable resistor to produce a current of exactly 0.2 A?

d) How would the function change if the circuit consisted of only the battery and the variable resistor? Explain the significance of the vertical asymptote in this case.

18. Two stores rent bikes. One charges a fixed fee of \$20 plus \$4/h, and the other charges a fixed fee of \$10 plus \$5/h.

a) Write equations for the average cost per hour for each store as a function of the rental time, in hours. Graph the functions.

b) Identify key features of the graphs. What do the graphs show about how the average cost changes for different rental times?

c) Is one store always the better choice? Explain.

Extend

19. A truck leaves Regina and drives eastbound. Due to road construction, the truck takes 2 h to travel the first 80 km. Once it leaves the construction zone, the truck travels at 100 km/h for the rest of the trip.

a) Let v represent the average speed, in kilometres per hour, over the entire trip and t represent the time, in hours, since leaving the construction zone. Write an equation for v as a function of t.

b) Graph the function for an appropriate domain.

c) What are the equations of the asymptotes in this situation? Do they have meaning in this situation? Explain.

d) How long will the truck have to drive before its average speed is 80 km/h?

e) Suppose your job is to develop GPS technology. How could you use these types of calculations to help travellers save fuel?

20. Determine the equation of a rational function of the form $y = \frac{ax + b}{cx + d}$ that has a vertical asymptote at $x = 6$, a horizontal asymptote at $y = -4$, and an x-intercept of -1.

21. For each rational function given, determine the inverse function, $f^{-1}(x)$.

a) $f(x) = \frac{x - 3}{x + 1}$

b) $f(x) = \frac{2x}{x - 5} + 4$

22. State the characteristics of the graph of the function $y = \frac{x}{x + 2} + \frac{x - 4}{x - 2}$.

Create Connections

C1 Would you say that using transformations with rational functions is more difficult, easier, or no different than using transformations with other functions that you have studied? Give reasons for your answer using specific examples.

C2 The owners of a manufacturing plant are trying to eliminate harmful emissions. They use the function $C(p) = \frac{200\ 000p}{100 - p}$ to estimate the cost, C, in dollars, to eliminate p percent of the emissions from the plant.

a) What domain is appropriate in this situation? Why?

b) Graph the function. How is its shape related to the manufacturing context?

c) Does it cost twice as much to eliminate 80% as it does to eliminate 40%? Explain.

d) Is it possible to completely eliminate all of the emissions according to this model? Justify your answer in terms of the characteristics of the graph.

C3 What are the similarities and differences between graphing the functions $y = \frac{2}{x - 3} + 4$ and $y = 2\sqrt{x - 3} + 4$ without using technology?

9.2

Analysing Rational Functions

Focus on...

- graphing, analysing, and comparing rational functions
- determining whether graphs of rational functions have an asymptote or a point of discontinuity for a non-permissible value

Near McClusky Lake, Wind River, Yukon

The speed at which an airplane travels depends on the speed of the wind in which it is flying. A plane's *airspeed* is how fast it travels in relation to the air around it, but its *ground speed* is how fast it travels relative to the ground. A plane's ground speed is greater if it flies with a tailwind and less if it flies with a headwind.

Investigate Analysing Rational Functions

Materials

- graphing technology

1. Consider the function $y = \frac{x^2 - x - 2}{x - 2}$.

a) What value of x is important to consider when analysing this function? Predict the nature of the graph for this value of x.

b) Graph the function and display a table of values.

c) Are the pattern in the table and the shape of the graph what you expected? Explain.

2. a) What are the restrictions on the domain of this function?

b) How can you simplify the function? What function is it equivalent to?

c) Graph the simplified function and display a table of values. How do these compare to those of the original function?

d) How could you sketch the graphs of these two functions so that the difference between them is clear?

Reflect and Respond

3. a) How does the behaviour of the function $y = \frac{x^2 - x - 2}{x - 2}$ near its non-permissible value differ from the rational functions you have looked at previously?

b) What aspect of the equation of the original function do you think is the reason for this difference?

Link the Ideas

Graphs of rational functions can have a variety of shapes and different features—vertical asymptotes are one such feature. A vertical asymptote of the graph of a rational function corresponds to a non-permissible value in the equation of the function, but not all non-permissible values result in vertical asymptotes. Sometimes a non-permissible value instead results in a **point of discontinuity** in the graph.

point of discontinuity

- a point, described by an ordered pair, at which the graph of a function is not continuous
- occurs in a graph of a rational function when its function can be simplified by dividing the numerator and denominator by a common factor that includes a variable
- results in a single point missing from the graph, which is represented using an open circle
- sometimes referred to as a "hole in the graph"

Example 1

Graph a Rational Function With a Point of Discontinuity

Sketch the graph of the function $f(x) = \frac{x^2 - 5x + 6}{x - 3}$. Analyse its behaviour near its non-permissible value.

Solution

You can sometimes analyse and graph rational functions more easily by simplifying the equation of the function algebraically. To simplify the equation of $f(x)$, factor the numerator and the denominator:

$$f(x) = \frac{x^2 - 5x + 6}{x - 3}$$

$$f(x) = \frac{(x - 2)\overset{1}{\cancel{(x - 3)}}}{\underset{1}{\cancel{(x - 3)}}}$$

$$f(x) = x - 2,\ x \neq 3$$

As long as the restriction is included, this simplified equation represents the same function.

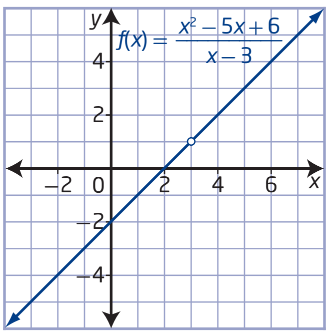
The graph of $f(x)$ is the same as the graph of $y = x - 2$, except that $f(x)$ has a point of discontinuity at $(3, y)$. To determine the y-coordinate of the point of discontinuity, substitute $x = 3$ into the simplified function equation.

$y = x - 2$
$y = 3 - 2$
$y = 1$

What happens when $x = 3$ is substituted into the original function?

The point of discontinuity occurs at (3, 1).

Graph a line with a y-intercept of -2 and a slope of 1. Plot an open circle on the graph at (3, 1) to indicate that the function does not exist at that point.



Why is the graph a straight line when its equation looks quite complex?

The function $f(x)$ has a point of discontinuity at (3, 1) because the numerator and the denominator have a common factor of $x - 3$. The common factor does not affect the values of the function except at $x = 3$, where $f(x)$ does not exist.

A table of values for the function shows the behaviour of the function near its non-permissible value $x = 3$:

x	2.5	2.8	2.9	2.99	2.999	3	3.001	3.01	3.1	3.2	3.5
$f(x)$	0.5	0.8	0.9	0.99	0.999	does not exist	1.001	1.01	1.1	1.2	1.5

From the table, it appears that the value of $f(x)$ gets closer and closer to 1 as x gets closer to 3 from either side even though the function does not exist when x is exactly 3.

Your Turn

Sketch the graph of the function $f(x) = \frac{x^2 + 2x - 3}{x - 1}$. Analyse its behaviour near its non-permissible value.

Example 2

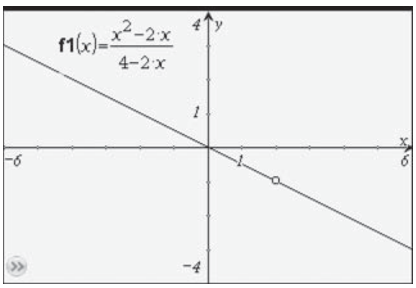
Rational Functions: Points of Discontinuity Versus Asymptotes

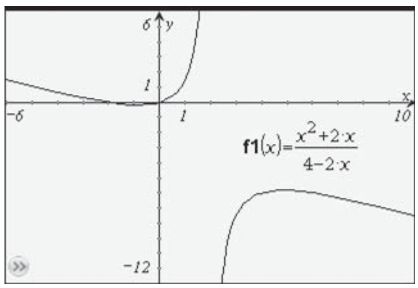
a) Compare the behaviour of the functions $f(x) = \frac{x^2 - 2x}{4 - 2x}$ and $g(x) = \frac{x^2 + 2x}{4 - 2x}$ near any non-permissible values.

b) Explain any differences.

Solution

a) Use a graphing calculator to graph the functions.





The non-permissible value for both functions is 2. However, the graph on the left does not exist at (2, −1), whereas the the graph on the right is undefined at $x = 2$.

Why are the graphs so different when the equations look so similar?

Did You Know?

In calculus, the term "undefined" is used for an asymptote, while the term "indeterminate" is used for a point of discontinuity.

- $\frac{n}{0}$ is undefined.
- $\frac{0}{0}$ is indeterminate.

Characteristic	$f(x) = \frac{x^2 - 2x}{4 - 2x}$	$g(x) = \frac{x^2 + 2x}{4 - 2x}$
Non-permissible value	$x = 2$	$x = 2$
Feature at non-permissible value	point of discontinuity	vertical asymptote
Behaviour near non-permissible value	As x approaches 2, y approaches -1.	As x approaches 2, $\|y\|$ becomes very large.

b) To explain the differences in the behaviour of the two functions near $x = 2$, factor the numerator and denominator of each function.

$$f(x) = \frac{x^2 - 2x}{4 - 2x}$$

$$f(x) = \frac{x\overset{1}{\cancel{(x - 2)}}}{-2\underset{1}{\cancel{(x - 2)}}}$$

$$f(x) = -\frac{1}{2}x,\ x \neq 2$$

$f(x)$ has a point of discontinuity at $(2, -1)$ because the numerator and denominator have a common factor of $x - 2$.

$$g(x) = \frac{x^2 + 2x}{4 - 2x}$$

$$g(x) = \frac{x(x + 2)}{-2(x - 2)},\ x \neq 2$$

$g(x)$ has a vertical asymptote at $x = 2$ because $x - 2$ is a factor of the denominator but not the numerator.

Your Turn

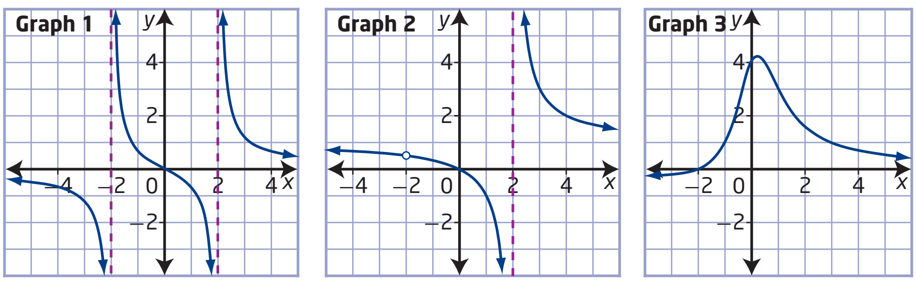
Compare the functions $f(x) = \frac{x^2 - 3x}{2x + 6}$ and $g(x) = \frac{x^2 + 3x}{2x + 6}$ and explain any differences.

Example 3

Match Graphs and Equations for Rational Functions

Match the equation of each rational function with the most appropriate graph. Give reasons for each choice.

$A(x) = \frac{x^2 + 2x}{x^2 - 4}$ $\quad B(x) = \frac{2x + 4}{x^2 + 1}$ $\quad C(x) = \frac{2x}{x^2 - 4}$



Solution

To match the equations of the functions with their graphs, use the locations of points of discontinuity, asymptotes, and intercepts. Write each function in factored form to determine the factors of the numerator and denominator and use them to predict the characteristics of each graph.

$$A(x) = \frac{x^2 + 2x}{x^2 - 4}$$
$$A(x) = \frac{x(x + 2)}{(x - 2)(x + 2)}$$

The graph of $A(x)$ has
- a vertical asymptote at $x = 2$
- a point of discontinuity at $\left(-2, \frac{1}{2}\right)$
- an x-intercept of 0

Therefore, graph 2 represents $A(x)$.

How do the factors in the equation reveal the features of the graph?

$$B(x) = \frac{2x + 4}{x^2 + 1}$$
$$B(x) = \frac{2(x + 2)}{x^2 + 1}$$

The graph of $B(x)$ has
- no vertical asymptotes or points of discontinuity
- an x-intercept of -2

Therefore, graph 3 represents $B(x)$.

How can you tell that the graph of $B(x)$ will have no points of discontinuity or vertical asymptotes?

$$C(x) = \frac{2x}{x^2 - 4}$$
$$C(x) = \frac{2x}{(x - 2)(x + 2)}$$

The graph of $C(x)$ has
- vertical asymptotes at $x = -2$ and $x = 2$
- no points of discontinuity
- an x-intercept of 0

Therefore, graph 1 represents $C(x)$.

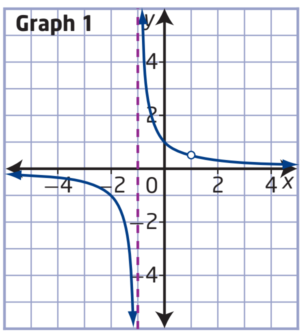
How can you tell that the graph of $C(x)$ will have two vertical asymptotes and one x-intercept, but no points of discontinuity?

Your Turn

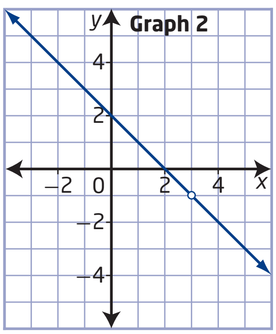
Match the equation of each rational function with the most appropriate graph. Explain your reasoning.

$K(x) = \dfrac{x^2 + 2}{x^2 - x - 2}$ $\quad L(x) = \dfrac{x - 1}{x^2 - 1}$ $\quad M(x) = \dfrac{x^2 - 5x + 6}{3 - x}$

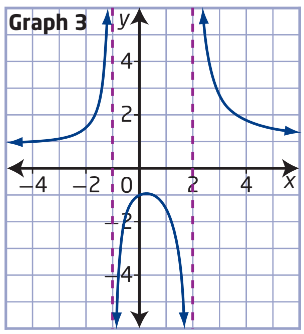
Graph 1



Graph 2



Graph 3



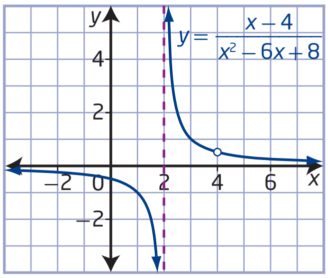
Key Ideas

- The graph of a rational function
  - has either a vertical asymptote or a point of discontinuity corresponding to each of its non-permissible values
  - has no vertical asymptotes or points of discontinuity
- To find any x-intercepts, points of discontinuity, and vertical asymptotes of a rational function, analyse the numerator and denominator.
  - A factor of only the numerator corresponds to an x-intercept.
  - A factor of only the denominator corresponds to a vertical asymptote.
  - A factor of both the numerator and the denominator corresponds to a point of discontinuity.
- To analyse the behaviour of a function near a non-permissible value, use a table of values or the graph, even though the function is undefined or does not exist at the non-permissible value itself.

Check Your Understanding

Practise

1. The graph of the rational function $y = \frac{x - 4}{x^2 - 6x + 8}$ is shown.



a) Copy and complete the table to summarize the characteristics of the function.

Characteristic	$y = \frac{x - 4}{x^2 - 6x + 8}$
Non-permissible value(s)	
Feature exhibited at each non-permissible value	
Behaviour near each non-permissible value	
Domain	
Range	

b) Explain the behaviour at each non-permissible value.

2. Create a table of values for each function for values near its non-permissible value. Explain how your table shows whether a point of discontinuity or an asymptote occurs in each case.

a) $y = \frac{x^2 - 3x}{x}$

b) $y = \frac{x^2 - 3x - 10}{x - 2}$

c) $y = \frac{3x^2 + 4x - 4}{x + 4}$

d) $y = \frac{5x^2 + 4x - 1}{5x - 1}$

3. a) Graph the functions $f(x) = \frac{x^2 - 2x - 3}{x + 3}$ and $g(x) = \frac{x^2 + 2x - 3}{x + 3}$ and analyse their characteristics.

b) Explain any differences in their behaviour near non-permissible values.

4. For each function, predict the locations of any vertical asymptotes, points of discontinuity, and intercepts. Then, graph the function to verify your predictions.

a) $y = \dfrac{x^2 + 4x}{x^2 + 9x + 20}$

b) $y = \dfrac{2x^2 - 5x - 3}{x^2 - 1}$

c) $y = \dfrac{x^2 + 2x - 8}{x^2 - 2x - 8}$

d) $y = \dfrac{2x^2 + 7x - 15}{9 - 4x^2}$

5. Which graph matches each rational function? Explain your choices.

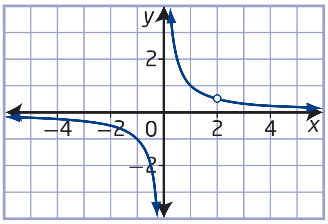
a) $A(x) = \dfrac{x^2 + 2x}{x^2 + 4}$

b) $B(x) = \dfrac{x - 2}{x^2 - 2x}$

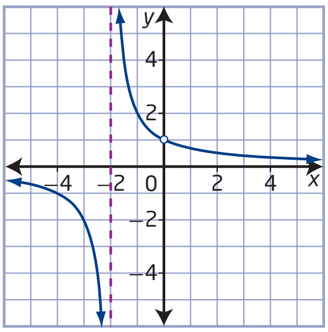
c) $C(x) = \dfrac{x + 2}{x^2 - 4}$

d) $D(x) = \dfrac{2x}{x^2 + 2x}$

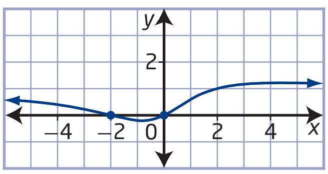
A



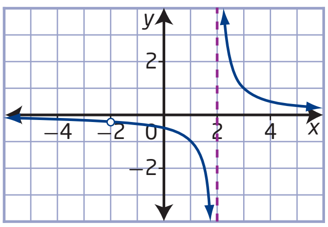
B



C

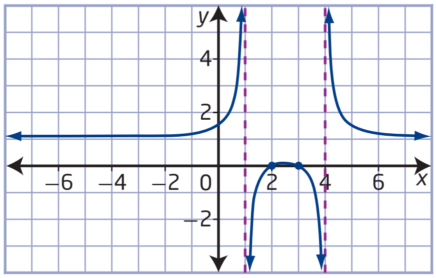


D

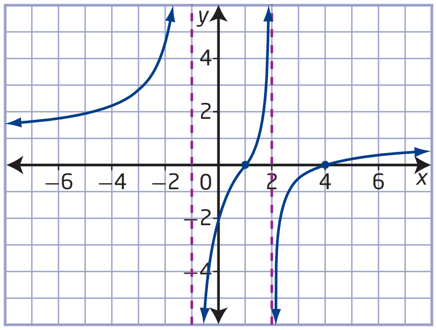


6. Match the graph of each rational function with the most appropriate equation. Give reasons for each choice.

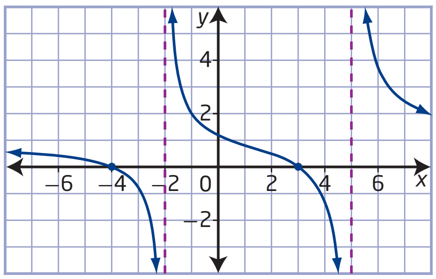
a)



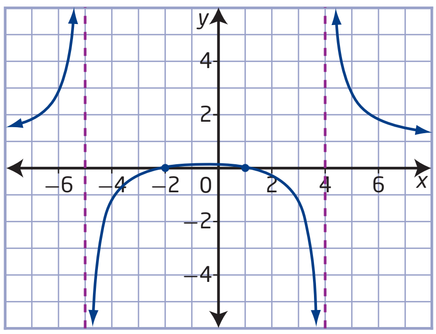
b)



c)



d)



A $f(x) = \dfrac{x^2 + x - 2}{x^2 + x - 20}$

B $g(x) = \dfrac{x^2 - 5x + 4}{x^2 - x - 2}$

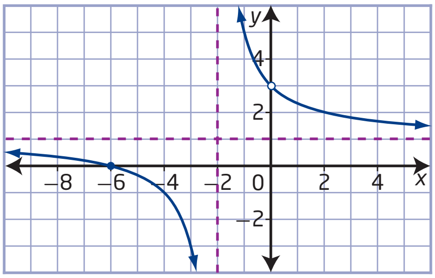
C $h(x) = \dfrac{x^2 - 5x + 6}{x^2 - 5x + 4}$

D $j(x) = \dfrac{x^2 + x - 12}{x^2 - 3x - 10}$

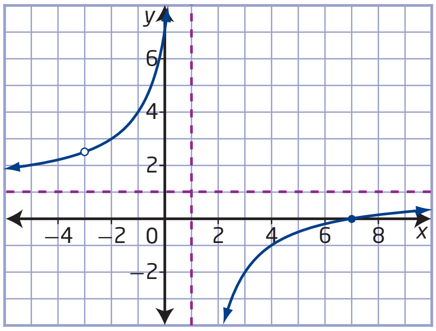
Apply

7. Write the equation for each rational function graphed below.

a)



b)



8. Write the equation of a possible rational function with each set of characteristics.

a) vertical asymptotes at $x = \pm 5$ and x-intercepts of -10 and 4

b) a vertical asymptote at $x = -4$, a point of discontinuity at $\left(-\frac{11}{2}, 9\right)$, and an x-intercept of 8

c) a point of discontinuity at $\left(-2, \frac{1}{5}\right)$, a vertical asymptote at $x = 3$, and an x-intercept of -1

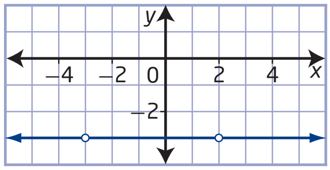
d) vertical asymptotes at $x = 3$ and $x = \frac{6}{7}$, and x-intercepts of $-\frac{1}{4}$ and 0

9. Sydney noticed that the functions $f(x) = \frac{x - 3}{x^2 - 5x - 6}$ and $g(x) = \frac{x - 3}{x^2 - 5x + 6}$ have equations that are very similar. She assumed that their graphs would also be very similar.

a) Predict whether or not Sydney is correct. Give reasons for your answer.

b) Graph the functions. Explain why your predictions were or were not accurate.

10. What rational function is shown in the graph?



11. a) Predict the shape of the graph of $y = \frac{2x^2 + 2}{x^2 - 1}$ and explain your reasoning.

b) Use graphing technology to confirm your prediction.

c) How would the graph of each of the following functions compare to the one in part a)? Check using graphing technology.

i) $y = \frac{2x^2 - 2}{x^2 - 1}$ **ii)** $y = \frac{2x^2 + 2}{x^2 + 1}$

12. A de Havilland Beaver is a small plane that is capable of an airspeed of about 250 km/h in still air. Consider a situation where this plane is flying 500 km from Lake Athabasca, Saskatchewan, to Great Slave Lake, Northwest Territories.

a) Let w represent the speed of the wind, in kilometres per hour, where w is positive for a tailwind and negative for a headwind, and t represent the time, in hours, it takes to fly. What equation represents t as a function of w? What is the non-permissible value for the function?

b) Graph the function for a domain that includes its non-permissible value.

c) Explain what the behaviour of the function for various values of w means in this context, including near its non-permissible value.

d) Which part(s) of your graph are actually realistic in this situation? Discuss this with a partner, and explain your thoughts.

Did You Know?

Bush planes like the de Havilland Beaver have been and still are critical to exploration and transportation in remote areas of Northern Canada where roads do not exist.

13. Ryan and Kandra are kayaking near Lowe Inlet Marine Provincial Park on Grenville Channel, British Columbia. The current can flow in either direction at up to 4 km/h depending on tidal conditions. Ryan and Kandra are capable of kayaking steadily at 4 km/h without the current.

a) What function relates the time, t, in hours, it will take them to travel 4 km along the channel as a function of the speed, w, in kilometres per hour, of the current? What domain is possible for w in this context?

b) Graph the function for an appropriate domain.

c) Explain the behaviour of the graph for values at and near its non-permissible value and what the behaviour means in this situation.

Did You Know?

The fastest navigable tidal currents in the world, which can have speeds of up to 30 km/h at their peak, occur in the Nakwakto Rapids, another narrow channel on British Columbia's coast. The name originates from the kwakwaka'wakw language meaning "trembling rock."

14. Paul is a humanitarian aid worker. He uses the function $C(p) = \frac{500p}{100 - p}$ to estimate the cost, C, in thousands of dollars, of vaccinating p percent of the population of the country in which he is working.

a) Predict the nature of the graph for its non-permissible value. Give a reason for your answer.

b) Graph the function for an appropriate domain. Explain what the graph shows about the situation.

c) Do you think this is a good model for the estimated cost of vaccinating the population? Explain.

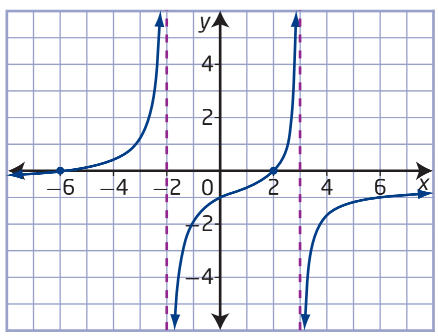
15. The function $h(v) = \frac{6378v^2}{125 - v^2}$ gives the maximum height, h, in kilometres, as a function of the initial velocity, v, in kilometres per second, for an object launched upward from Earth's surface, if the object gets no additional propulsion and air resistance is ignored.

a) Graph the function. What parts of the graph are applicable to this situation?

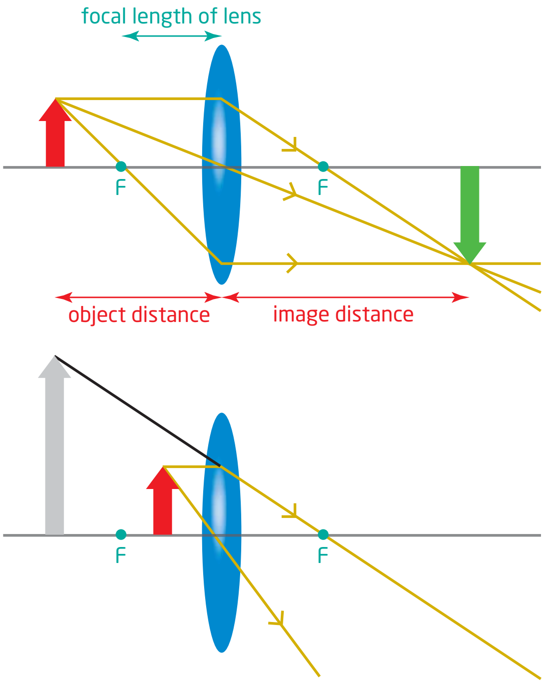
b) Explain what the graph indicates about how the maximum height is affected by the initial velocity.

c) The term *escape velocity* refers to the initial speed required to break free of a gravitational field. Describe the nature of the graph for its non-permissible value, and explain why it represents the escape velocity for the object.

16. Determine the equation of the rational function shown without using technology.



17. A convex lens focuses light rays from an object to create an image, as shown in the diagram. The image distance, I, is related to the object distance, b, by the function $I = \frac{fb}{b - f}$, where the focal length, f, is a constant for the particular lens used based on its specific curvature. When the object is placed closer to the lens than the focal length of the lens, an image is perceived to be behind the lens and is called a virtual image.



a) Graph I as a function of b for a lens with a focal length of 4 cm.

b) How does the location of the image change as the values of b change?

c) What type of behaviour does the graph exhibit for its non-permissible value? How is this connected to the situation?

Did You Know?

The image in your bathroom mirror is a virtual image—you perceive the image to be behind the mirror, even though the light rays do not actually travel or focus behind the mirror. You cannot project a virtual image on a screen. The study of images of objects using lenses and mirrors is part of a branch of physics called optics.

18. Consider the functions $f(x) = \frac{x + a}{x + b}$, $g(x) = \frac{x + a}{(x + b)(x + c)}$, and $h(x) = \frac{(x + a)(x + c)}{(x + b)(x + c)}$, where a, b, and c are different real numbers.

a) Which pair of functions do you think will have graphs that appear to be most similar to each other? Explain your choice.

b) What common characteristics will all three graphs have? Give reasons for your answer.

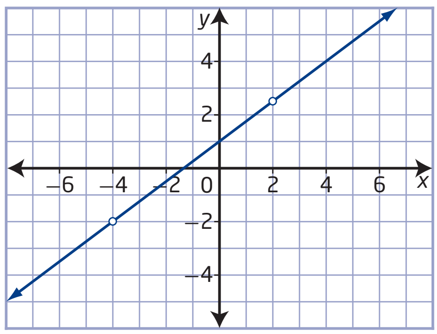
19. If the function $y = \frac{x^2 + bx + c}{4x^2 + 29x + c}$, where b and c are real numbers, has a point of discontinuity at $\left(-8, \frac{11}{35}\right)$, where does it have x-intercept(s) and vertical asymptote(s), if any?

Extend

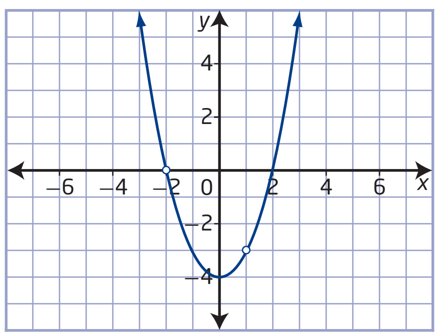
20. Given $f(x) = \frac{2x^2 - 4x}{x^2 + 3x - 28}$, what is the equation of $y = \frac{1}{4}f[-(x - 3)]$ in simplest form?

21. Write the equation of the rational function shown in each graph. Leave your answers in factored form.

a)



b)



22. The functions $f(x) = \frac{x^2 + 4}{x^2 - 4}$ and $g(x) = \frac{x^2 - 4}{x^2 + 4}$ are reciprocals of each other. That is, $f(x) = \frac{1}{g(x)}$ and vice versa. Graph the two functions on the same set of axes and explain how the shapes of the graphs support this fact.

23. Predict the location of any asymptotes and points of discontinuity for each function. Then, use technology to check your predictions.

a) $y = \frac{x + 2}{x^2 - 4} + \frac{5}{x + 2}$

b) $y = \frac{2x^3 - 7x^2 - 15x}{x^2 - x - 20}$

Create Connections

C1 Jeremy was absent the day his math class started learning about rational functions. His friend Rohan tells him that rational functions are functions that have asymptotes and points of discontinuity, but Jeremy is not sure what he means.

a) Jeremy takes Rohan's statement to mean that all rational functions have asymptotes and points of discontinuity. Is this statement true? Explain using several examples.

b) How would you elaborate on Rohan's explanation about what rational functions are to make it more clear for Jeremy?

C2 Consider the statement, "All polynomial functions are rational functions." Is this statement true? Explain your thinking.

C3 **MINI LAB** Graphs of rational functions can take on many shapes with a variety of features. Work with a partner to create your own classification system for rational functions.

Step 1 Use technology to create graphs for rational functions. Try creating graphs with as many different general shapes as you can by starting with a variety of types of equations. How many different general shapes can you create?

Step 2 Group the rational functions you have created into categories or classes. Consider the types of features, aspects, and symmetries that the various graphs exhibit.

Step 3 Create a descriptive name for each of your categories.

Step 4 For each category, describe an example function, including its equation and graph.

Project Corner — Visual Presentation

Create a video or slide presentation that demonstrates your understanding of a topic in Unit 4.

- Once you have chosen a topic, write a script for your movie or outline for your slide presentation.
- If you are making a video, choose your presenter and/or cast and the location. Prepare any materials needed and rehearse your presentation. Film your movie, edit it, add sound, and create the title and credits.
- If you are making a slide presentation, collect or make any digital images that you need; create title, contents, and credits slides; add sound or music; and test the presentation.

9.3

Connecting Graphs and Rational Equations

Focus on...

- relating the roots of rational equations to the x-intercepts of the graphs of rational functions
- determining approximate solutions to rational equations graphically

A wide range of illnesses and medical conditions can be effectively treated with various medications. Pharmacists, doctors, and other medical professionals need to understand how the level of medication in a patient's bloodstream changes after its administration. For example, they may need to know when the level will drop to a certain point. How might they predict when this will occur?

Did You Know?

Only a small fraction of the amount of many medications taken orally actually makes it into the bloodstream. The ratio of the amount of a medication in a patient's bloodstream to the amount given to the patient is called its *bioavailability*.

Investigate Solving Rational Equations

Materials

- graphing technology

Work with a partner.

A: Determine Medication Levels

1. The function $C(t) = \dfrac{40t}{1.1t^2 + 0.3}$ models the bloodstream concentration, C, in milligrams per decilitre (mg/dL), of a certain medication as a function of the time, t, in hours, since it was taken orally.

a) Graph the function for a reasonable domain.

b) What does the graph show about the situation?

2. A doctor needs to know when a patient's bloodstream concentration drops to 10 mg/dL.

a) Why might a doctor need to know this?

b) Brainstorm a list of possible methods you could use to determine the length of time it will take.

c) Use at least two of the methods you came up with to determine the length of time. Explain the steps required in each of your methods.

d) Share your solution methods with other pairs in your class. Are your methods similar? Explain.

Reflect and Respond

3. What are the strengths of each method you used? Which one do you prefer? Why?

B: Solve a Rational Equation Graphically and Algebraically

Consider the rational equation $\frac{x + 2}{x - 3} = x - 6$.

4. How can you solve the equation algebraically? Write a step-by-step algebraic solution, including an explanation of each step. Is there a restriction on the value of x?

5. How can you solve the equation graphically? Discuss possible methods with your partner, and then choose one and use it to solve the equation. Explain your process.

Reflect and Respond

6. Which method of solving this equation do you prefer, the algebraic approach or the graphical one? Give reasons for your choice.

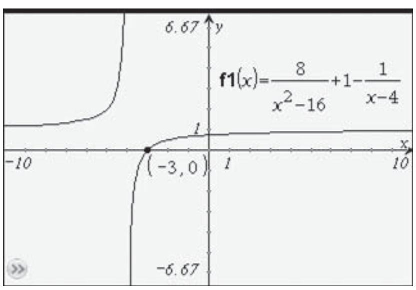
Link the Ideas

Just as with many other types of equations, rational equations can be solved algebraically or graphically. Solving rational equations using an algebraic approach will sometimes result in extraneous roots. For example, an algebraic solution to the equation $\frac{8}{x^2 - 16} + 1 = \frac{1}{x - 4}$ results in x-values of -3 and 4.

Why is $x = 4$ not a valid solution?

Solving a rational equation graphically involves using technology to graph the corresponding rational function and identify the x-intercepts of the graph. The x-intercepts of the graph of the corresponding function give the roots of the equation.

For example, a graphical solution to the equation $\frac{8}{x^2 - 16} + 1 = \frac{1}{x - 4}$ shows one solution, $x = -3$.



Note that the extraneous solution of $x = 4$, which was determined algebraically, is not observed when solving the equation graphically.

Example 1

Relate Roots and *x*-Intercepts

a) Determine the roots of the rational equation $x + \frac{6}{x + 2} - 5 = 0$ algebraically.

b) Graph the rational function $y = x + \frac{6}{x + 2} - 5$ and determine the x-intercepts.

c) What is the connection between the roots of the equation and the x-intercepts of the graph of the function?

Solution

a) Identify any restrictions on the variable before solving. The solution cannot be a non-permissible value. This equation has a single non-permissible value of −2.

To solve the rational equation algebraically, multiply each term in the equation by the lowest common denominator and then solve for x.

$$x + \frac{6}{x + 2} - 5 = 0$$

$$(x + 2)\left(x + \frac{6}{x + 2} - 5\right) = (x + 2)(0)$$

$$(x + 2)(x) + \cancel{(x + 2)}^{1}\left(\frac{6}{\cancel{x + 2}_{1}}\right) - (x + 2)(5) = 0$$

$$x^2 + 2x + 6 - 5x - 10 = 0$$

$$x^2 - 3x - 4 = 0$$

$$(x + 1)(x - 4) = 0$$

$x + 1 = 0$ or $x - 4 = 0$

$x = -1$ $\quad$ $x = 4$

Neither −1 nor 4 is a non-permissible value of the original equation.

Check:

For $x = -1$,

Left Side	Right Side
$x + \frac{6}{x + 2} - 5$	0
$= -1 + \frac{6}{-1 + 2} - 5$	
$= -1 + 6 - 5$	
$= 0$	

Left Side = Right Side

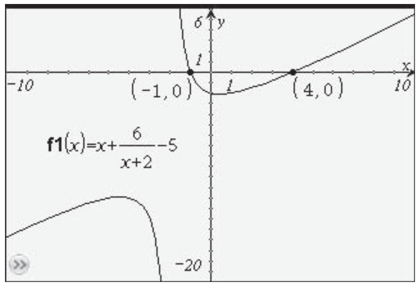
For $x = 4$,

Left Side	Right Side
$x + \frac{6}{x + 2} - 5$	0
$= 4 + \frac{6}{4 + 2} - 5$	
$= 4 + 1 - 5$	
$= 0$	

Left Side = Right Side

The equation has two roots or solutions, $x = -1$ and $x = 4$.

b) Use a graphing calculator to graph the function $y = x + \frac{6}{x + 2} - 5$ and determine the x-intercepts.



The function has x-intercepts at $(-1, 0)$ and $(4, 0)$.

c) The value of the function is 0 when the value of x is -1 or 4. The x-intercepts of the graph of the corresponding function are the roots of the equation.

Your Turn

a) Determine the roots of the equation $\frac{14}{x} - x + 5 = 0$ algebraically.

b) Determine the x-intercepts of the graph of the corresponding function $y = \frac{14}{x} - x + 5$.

c) Explain the connection between the roots of the equation and the x-intercepts of the graph of the corresponding function.

Example 2

Determine Approximate Solutions for Rational Equations

a) Solve the equation $\frac{x^2 - 3x - 7}{3 - 2x} = x - 1$ graphically. Express your answer to the nearest hundredth.

b) Verify your solution algebraically.

Solution

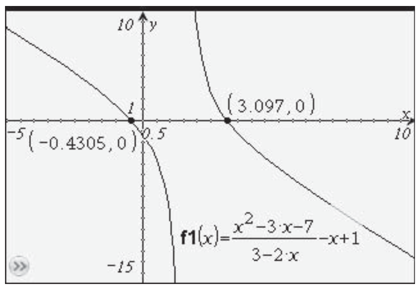
a) **Method 1: Use a Single Function**

Rearrange the rational equation so that one side is equal to zero:

$$\frac{x^2 - 3x - 7}{3 - 2x} = x - 1$$

$$\frac{x^2 - 3x - 7}{3 - 2x} - x + 1 = 0$$

Graph the corresponding function, $y = \frac{x^2 - 3x - 7}{3 - 2x} - x + 1$, and determine the x-intercept(s) of the graph.



The solution to the equation is $x \approx -0.43$ and $x \approx 3.10$.

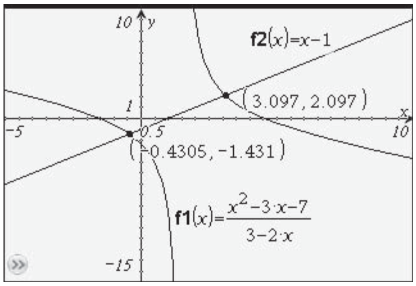
Method 2: Use a System of Two Functions

Write a function that corresponds to each side of the equation.

$y_1 = \frac{x^2 - 3x - 7}{3 - 2x}$

$y_2 = x - 1$

Use graphing technology to graph these functions and determine the value(s) of x at the point(s) of intersection, or where $y_1 = y_2$.



The solution to the equation is $x \approx -0.43$ and $x \approx 3.10$.

b) Determine any restrictions on the variable in this equation. To determine non-permissible values, set the denominator equal to zero and solve.

$$3 - 2x = 0$$
$$x = 1.5$$

The non-permissible value is $x = 1.5$.

Solve the equation by multiplying both sides by $3 - 2x$:

$$\frac{x^2 - 3x - 7}{3 - 2x} = x - 1$$
$$(3 - 2x)\frac{x^2 - 3x - 7}{3 - 2x} = (3 - 2x)(x - 1)$$
$$x^2 - 3x - 7 = 3x - 3 - 2x^2 + 2x$$
$$3x^2 - 8x - 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-4)}}{2(3)}$$
$$x = \frac{8 \pm \sqrt{112}}{6}$$
$$x = \frac{8 \pm 4\sqrt{7}}{6}$$
$$x = \frac{4 \pm 2\sqrt{7}}{3}$$

Why is the quadratic formula required here?

$x = \frac{4 - 2\sqrt{7}}{3}$ or $x = \frac{4 + 2\sqrt{7}}{3}$

$x = -0.4305\ldots$ $\quad$ $x = 3.0971\ldots$

$x \approx -0.43$ $\quad$ $x \approx 3.10$

How can you tell if either of these roots is extraneous?

The algebraic method gives an exact solution. The approximate values obtained algebraically, $x \approx -0.43$ and $x \approx 3.10$, are the same as the values obtained graphically.

Your Turn

a) Solve the equation $2 - \frac{3x}{2} = \frac{1 + 4x - x^2}{4x + 10}$ graphically. Express your answer to the nearest hundredth.

b) Verify your solution algebraically.

Example 3

Solve a Rational Equation With an Extraneous Root

a) Solve the equation $\frac{x}{2x + 5} + 2x = \frac{8x + 15}{4x + 10}$ algebraically and graphically.

b) Compare the solutions found using each method.

Solution

a) Factor the denominators to determine the non-permissible values.

$$\frac{x}{2x + 5} + 2x = \frac{8x + 15}{2(2x + 5)}$$

The equation has one non-permissible value of $-\frac{5}{2}$.

Multiply both sides of the equation by the lowest common denominator, $2(2x + 5)$.

$$2(2x + 5)\left(\frac{x}{2x + 5} + 2x\right) = 2(2x + 5)\left(\frac{8x + 15}{2(2x + 5)}\right)$$

$$2\cancel{(2x + 5)}^{1}\left(\frac{x}{\cancel{2x + 5}_{1}}\right) + 2(2x + 5)(2x) = \cancel{2(2x + 5)}^{1}\left(\frac{8x + 15}{\cancel{2(2x + 5)}_{1}}\right)$$

$$2x + 8x^2 + 20x = 8x + 15$$
$$8x^2 + 14x - 15 = 0$$
$$(2x + 5)(4x - 3) = 0$$

$2x + 5 = 0$ or $4x - 3 = 0$

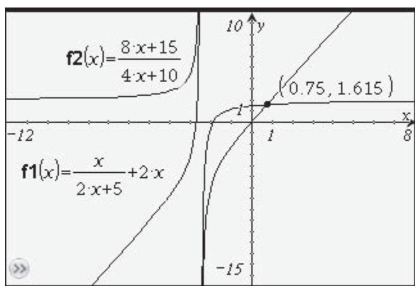
$x = -\frac{5}{2}$ $\qquad x = \frac{3}{4}$

However, $-\frac{5}{2}$ is a non-permissible value for the original equation. It is an extraneous root and must be rejected.

Therefore, the solution is $x = \frac{3}{4}$.

To solve the equation graphically, use two functions to represent the two sides of the equation.

$$y_1 = \frac{x}{2x + 5} + 2x$$
$$y_2 = \frac{8x + 15}{4x + 10}$$



The graphs of the two functions intersect when x is 0.75.

The solution to the equation is $x = 0.75$, or $\frac{3}{4}$.

The curves appear to meet at the top and bottom of the graphing calculator screen. Do these represent points of intersection? Explain.

b) The solutions obtained by both methods are the same. For this equation, the algebraic solution produced two values, one of which was rejected because it was extraneous. The graphical solution did not produce the extraneous root. There is only one point of intersection on the graph of the two functions.

Your Turn

a) Solve the equation $\frac{x + 3}{2x - 6} = 2x - \frac{x}{3 - x}$ algebraically and graphically.

b) Compare the solutions found using each method.

Example 4

Solve a Problem Using a Rational Equation

In basketball, a player's free-throw percentage is given by dividing the total number of successful free-throw baskets by the total number of attempts. So far this year, Larry has attempted 19 free-throws and has been successful on 12 of them. If he is successful on every attempt from now on, how many more free-throws does he need to attempt before his free-throw percentage is 80%?

Did You Know?

Basketball is one of the sports competitions included in the Canadian Francophone Games. The Canadian Francophone Games gives French speaking youth from across Canada a chance to demonstrate their talents in the areas of art, leadership, and sports.

Solution

Let x represent the number of free-throws Larry takes from now on.

Let P represent Larry's new free-throw percentage, as a decimal.

$$P = \frac{\text{successes}}{\text{attempts}}$$

$$P = \frac{12 + x}{19 + x}$$

Why is x used in both the numerator and the denominator?

Since the number of free-throws is discrete data, the continuous model is only valid in the domain $\{x \mid x \in W\}$.

Determine the value of x when P is 80%, or 0.8. Substitute 0.8 for P and solve the resulting equation.

$$P = \frac{12 + x}{19 + x}$$

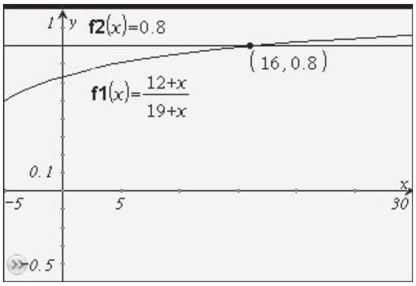
$$0.8 = \frac{12 + x}{19 + x}$$

Method 1: Solve Graphically

Graph two functions and determine the point of intersection.

$$y_1 = \frac{12 + x}{19 + x}$$

$$y_2 = 0.8$$



What domain is appropriate for this situation?

Larry will have a free-throw percentage of 80% after 16 more free-throw attempts if he is successful on all of them.

Method 2: Solve Algebraically

Multiply both sides of the equation by $19 + x$:

Is there a non-permissible value of x for this situation? Explain.

$$\begin{aligned} 0.8 &= \frac{12 + x}{19 + x} \\ 0.8(19 + x) &= \frac{12 + x}{\cancel{19 + x}_{1}}(\cancel{19 + x}^{1}) \\ 15.2 + 0.8x &= 12 + x \\ 3.2 &= 0.2x \\ x &= 16 \end{aligned}$$

Larry will have a free-throw percentage of 80% after 16 more free-throw attempts if he is successful on all of them.

Your Turn

Megan and her friends are organizing a fundraiser for the local children's hospital. They are asking local businesses to each donate a door prize. So far, they have asked nine businesses, but only one has donated a prize. Their goal was to have three quarters of the businesses donate. If they succeed in getting every business to donate a prize from now on, how many more businesses do they need to ask to reach their goal?

Key Ideas

- You can solve rational equations algebraically or graphically.
- The solutions or roots of a rational equation are equivalent to the x-intercepts of the graph of the corresponding rational function. You can use either of the following methods to solve rational equations graphically:
  - Manipulate the equation so that one side is equal to zero; then, graph the corresponding function and identify the value(s) of the x-intercept(s).
  - Graph a system of functions that corresponds to the expressions on both sides of the equal sign; then, identify the value(s) of x at the point(s) of intersection.
- When solving rational equations algebraically, remember to check for extraneous roots and to verify that the solution does not include any non-permissible values.

Check Your Understanding

Practise

1. Match each equation to the single function that can be used to solve it graphically.

a) $\frac{x}{x-2} + 6 = x$

b) $6 - x = \frac{x}{x-2} + 2$

c) $6 - \frac{x}{x-2} = x - 2$

d) $x + 6 = \frac{x}{x-2}$

A $y = \frac{x}{x-2} + x - 8$

B $y = \frac{x}{x-2} - x + 6$

C $y = \frac{x}{x-2} - x - 6$

D $y = \frac{x}{x-2} + x - 4$

2. a) Determine the roots of the rational equation $-\frac{2}{x} + x + 1 = 0$ algebraically.

b) Graph the rational function $y = -\frac{2}{x} + x + 1$ and determine the x-intercepts.

c) Explain the connection between the roots of the equation and the x-intercepts of the graph of the function.

3. Solve each equation algebraically.

a) $\frac{5x}{3x+4} = 7$

b) $2 = \frac{20-3x}{x}$

c) $\frac{x^2}{x-2} = x - 6$

d) $1 + \frac{2}{x} = \frac{x}{x+3}$

4. Use a graphical method to solve each equation. Then, use another method to verify your solution.

a) $\frac{8}{x} - 4 = x + 3$

b) $2x = \frac{10x}{2x-1}$

c) $\frac{3x^2 + 4x - 15}{x+3} = 2x - 1$

d) $\frac{3}{5x-7} + x = 1 + \frac{x^2 - 4x}{7 - 5x}$

5. Determine the approximate solution to each rational equation graphically, to the nearest hundredth. Then, solve the equation algebraically.

a) $\dfrac{x + 1}{2x} = x - 3$

b) $\dfrac{x^2 - 4x - 5}{2 - 5x} = x + 3$

c) $\dfrac{2}{x} = 3 - \dfrac{7x}{x - 2}$

d) $2 + \dfrac{5}{x + 3} = 1 - \dfrac{x + 1}{x}$

6. Solve each equation algebraically and graphically. Compare the solutions found using each method.

a) $\dfrac{3x}{x - 2} + 5x = \dfrac{x + 4}{x - 2}$

b) $2x + 3 = \dfrac{3x^2 + 14x + 8}{x + 4}$

c) $\dfrac{6x}{x - 3} + 3x = \dfrac{2x^2}{x - 3} - 5$

d) $\dfrac{2x - 1}{x^2 - x} + 4 = \dfrac{x}{x - 1}$

Apply

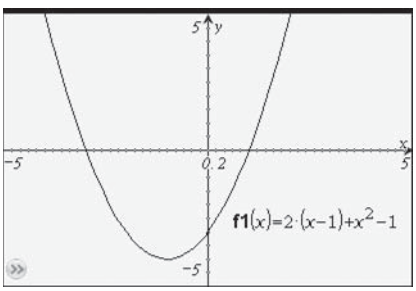
7. Yunah is solving the equation $2 + \dfrac{x^2}{x - 1} = \dfrac{1}{x - 1}$ graphically. She uses the following steps and then enters the resulting function into her graphing calculator.

$$2 + \frac{x^2}{x - 1} = \frac{1}{x - 1}$$

$$2(x - 1) + x^2 = 1$$

$$2(x - 1) + x^2 - 1 = 0$$

Enter $Y_1 = 2(x - 1) + x^2 - 1$ and find the x-intercepts.



Is her approach correct? Explain.

8. Determine the solution to the equation $\dfrac{2x + 1}{x - 1} = \dfrac{2}{x + 2} - \dfrac{3}{2}$ using two different methods.

9. Solve the equation $2 - \dfrac{1}{x + 2} = \dfrac{x}{x + 2} + 1$ algebraically and graphically, and explain how the results of both methods verify each other.

10. The intensity, I, of light, in watts per square metre (W/m^2), at a distance, d, in metres, from the point source is given by the formula $I = \dfrac{P}{4\pi d^2}$, where P is the average power of the source, in watts. How far away from a 500-W light source is intensity 5 W/m^2?

11. A researcher is studying the effects of caffeine on the body. As part of her research, she monitors the levels of caffeine in a person's bloodstream over time after drinking coffee. The function $C(t) = \dfrac{50t}{1.2t^2 + 5}$ models the level of caffeine in one particular person's bloodstream, where t is the time, in hours, since drinking the coffee and $C(t)$ is the person's bloodstream concentration of caffeine, in milligrams per litre. How long after drinking coffee has the person's level dropped to 2 mg/L?

12. The time it takes for two people working together to complete a job is given by the formula $T = \frac{ab}{a + b}$, where a and b are the times it takes for the two people to complete the same job individually. Sarah can set up the auditorium for an assembly in 30 min, but when she works with James they can set it up in 10 min. How long would it take James to set it up by himself?

13. In hockey, a player's shooting percentage is given by dividing the player's total goals scored by the player's total shots taken on goal. So far this season, Rachel has taken 28 shots on net but scored only 2 goals. She has set a target of achieving a 30% shooting percentage this season.

a) Write a function for Rachel's shooting percentage if x represents the number of shots she takes from now on and she scores on half of them.

b) How many more shots will it take for her to bring her shooting percentage up to her target?

14. The coefficient, C, in parts per million per kelvin, of thermal expansion for copper at a temperature, T, in kelvins, can be modelled with the function $C(T) = \frac{21.2T^2 - 877T + 9150}{T^2 + 23.6T + 760}$.

a) For what temperature is $C(T) = 15$ according to this model?

b) By how many kelvins does the temperature have to increase for copper's coefficient of thermal expansion to increase from 10 to 17?

Did You Know?

Most materials expand as the temperature increases, but not all in the same way. The *coefficient of thermal expansion* (CTE) for a given material is a measure of how much it will expand for each degree of temperature change as it is heated up. The higher the CTE, the more the material will expand per unit of temperature. A material's CTE value does not remain constant, but varies based on temperature. It is a ratio per unit of temperature, with units such as parts per million per kelvin or percent per degree Celsius. One part per million is equivalent to 0.0001%.

Extend

15. Solution A has a concentration of 0.05 g/mL and solution B has a concentration of 0.01 g/mL. You start with 200 mL of solution A, and pour in x millilitres of solution B.

a) Write an equation for the concentration, $C(x)$, of the solution after x millilitres have been added.

b) You need to make a solution with a concentration of 0.023 g/mL. How can you use your function equation to determine how many millilitres need to be added?

16. Solve the equation $\frac{x}{x + 2} - 3 = \frac{5x}{x^2 - 4} + x$ graphically and algebraically.

17. Solve each inequality.

a) $\frac{x - 18}{x - 1} \leq 5$

b) $\frac{5}{x - 2} \geq \frac{2x + 17}{x + 6}$

Create Connections

C1 Connor tells Brian that rational equations will always have at least one solution. Is this correct? Use a graphical approach and support your answer with examples.

C2 The rational equation $\frac{3x}{x + 2} = x - \frac{6}{x + 2}$ and the radical equation $x = \sqrt{x + 6}$ both have an extraneous root. Compare and contrast why they occur in each of these equations and how they can be identified when solving.

C3 Which method for solving a rational equation do you prefer to use: graphical, algebraic, or a combination of both? Discuss with a partner, and give reasons for your choice.

Chapter 9 Review

9.1 Exploring Rational Functions Using Transformations, pages 430–445

1. Sketch the graph of each function using transformations. Identify the domain and range, intercepts, and asymptotes.

   a) $y = \frac{8}{x - 1}$

   b) $y = \frac{3}{x} + 2$

   c) $y = -\frac{12}{x + 4} - 5$

2. Graph each function, and identify any asymptotes and intercepts.

   a) $y = \frac{x}{x + 2}$

   b) $y = \frac{2x + 5}{x - 1}$

   c) $y = \frac{-5x - 3}{x - 6}$

3. Graph the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{6}{(x - 3)^2} + 2$, and $h(x) = \frac{-4}{x^2 + 12x + 36}$. Compare the characteristics of the graphs and identify any common features.

4. A baseball league is planning to order new uniforms from a company that will charge \$500 plus \$35 per uniform.

   a) Represent the average cost per uniform for the company as a function of the number of uniforms ordered using an equation and a graph.

   b) Identify key features of the graph and explain what the graph shows about how the average cost changes for different numbers of uniforms ordered.

   c) The league needs to keep the cost per uniform at \$40. How many uniforms does it have to order?

9.2 Analysing Rational Functions, pages 446–456

5. Graph and analyse each function, including the behaviour near any non-permissible values.

   a) $y = \frac{x^2 + 2x}{x}$

   b) $y = \frac{x^2 - 16}{x - 4}$

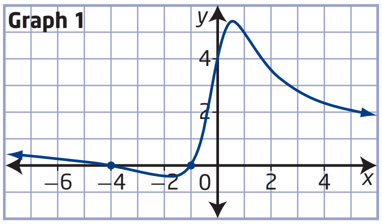
   c) $y = \frac{2x^2 - 3x - 5}{2x - 5}$

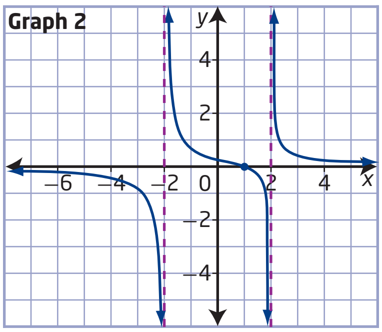
6. Which rational function matches each graph? Give reasons for your choices.

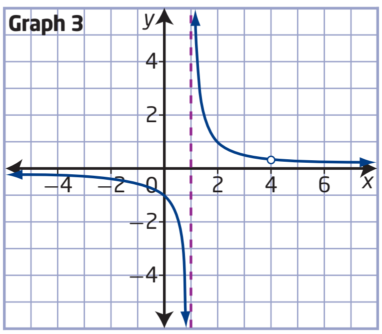
   $A(x) = \frac{x - 4}{x^2 - 5x + 4}$

   $B(x) = \frac{x^2 + 5x + 4}{x^2 + 1}$

   $C(x) = \frac{x - 1}{x^2 - 4}$







7. A company uses the function $C(p) = \frac{40\,000p}{100 - p}$ to estimate the cost of cleaning up a hazardous spill, where C is the cost, in dollars, and p is the percent of the spill that is cleaned up.

a) Graph the function for a domain applicable to the situation.

b) What does the shape of the graph show about the situation?

c) According to this model, is it possible to clean up the spill entirely? Justify your answer in terms of the features of the graph.

Did You Know?

Environment Canada responds to approximately 1000 hazardous spills each year. Given the location and quantity and type of substance spilled, the scientists from Environment Canada's National Spill Modelling Team run spill models. Spill modelling helps to minimize the environmental impact of spills involving hazardous substances and provides emergency teams with critical information.

9.3 Connecting Graphs and Rational Equations, pages 457–467

8. a) Determine the roots of the rational equation $x + \frac{4}{x - 2} - 7 = 0$ algebraically.

b) Graph the rational function $y = x + \frac{4}{x - 2} - 7$ and determine the x-intercepts.

c) Explain the connection between the roots of the equation and the x-intercepts of the graph of the function.

9. Determine the solution to each equation graphically, and then verify algebraically.

a) $x - 8 = \frac{33}{x}$

b) $\frac{x - 10}{x - 7} = x - 2$

c) $x = \frac{3x - 1}{x + 2} + 3$

d) $2x + 1 = \frac{13 - 4x}{x - 5}$

10. Solve each equation graphically, to the nearest hundredth.

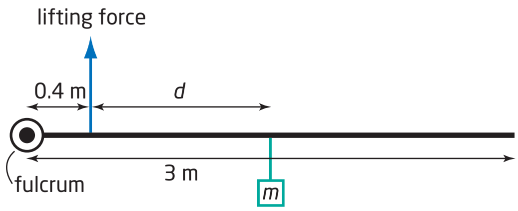
a) $\frac{x - 4}{5 - 2x} = 3$

b) $1.2x = \frac{x}{x + 6.7} + 3.9$

c) $3x + 2 = \frac{5x + 4}{x + 1}$

d) $\frac{x^2 - 2x - 8}{x + 2} = \frac{1}{4}x - 2$

11. The lever system shown is used to lift a mass of m kilograms on a cable with a constant force applied 0.4 m from the fulcrum. The maximum mass that can be lifted depends on the position, d, in metres, of the mass along the lever and is given by the formula $m = \frac{20}{d + 0.4}$.



a) What is the domain in this situation if the mass can be positioned at any point along the 3-m-long lever?

b) Graph the function. What does it show about this context?

c) Describe the behaviour of the function for its non-permissible value, and explain what the behaviour means in this situation.

d) How far from the fulcrum can the lever support a maximum possible mass of 17.5 kg?

Chapter 9 Practice Test

Multiple Choice

For #1 to #6, choose the best answer.

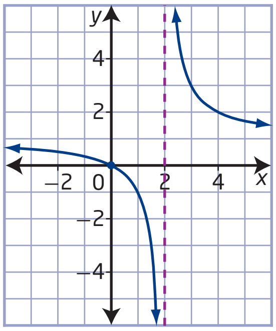
1. Which function has a vertical asymptote at $x = 2$?

A $y = \frac{x - 2}{x}$ **B** $y = \frac{x + 2}{x}$

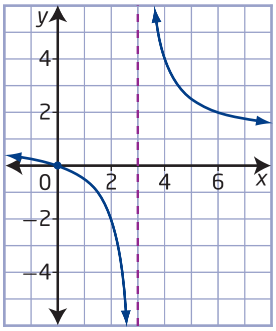
C $y = \frac{x}{x - 2}$ **D** $y = \frac{x}{x + 2}$

2. Which graph represents $y = \frac{x^2 - 2x}{x^2 - 5x + 6}$?

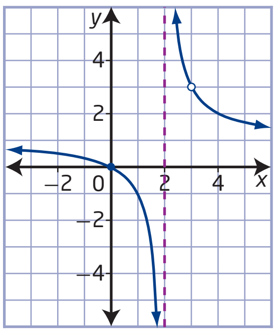
A



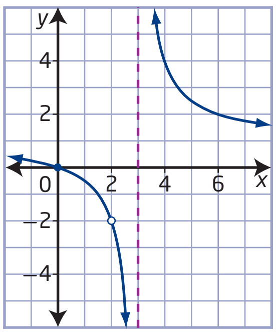
B



C



D



3. Which statement is true about the function $y = -\frac{4}{x - 2}$?

A As x approaches -2, $|y|$ becomes very large.

B As $|x|$ becomes very large, y approaches -4.

C As x approaches 2, $|y|$ becomes very large.

D As $|x|$ becomes very large, y approaches 4.

4. The roots of $5 - x = \frac{x + 2}{2x - 3}$ can be determined using which of the following?

A the x-coordinates of the points of intersection of $y = \frac{x + 2}{2x - 3}$ and $y = x - 5$

B the x-intercepts of the function $y = \frac{x + 2}{2x - 3} + x - 5$

C the x-coordinates of the points of intersection of $y = \frac{x + 2}{2x - 3}$ and $y = x + 5$

D the x-intercepts of the function $y = \frac{x + 2}{2x - 3} - x + 5$

5. Which function is equivalent to $y = \frac{6x - 5}{x + 7}$?

A $y = \frac{37}{x + 7} + 6$ **B** $y = -\frac{37}{x + 7} + 6$

C $y = \frac{47}{x + 7} + 6$ **D** $y = -\frac{47}{x + 7} + 6$

6. Which statement about the function $y = \frac{x}{x^2 - x}$ is true?

A It has an x-intercept of 0.

B It has a y-intercept of 0.

C It has a point of discontinuity at $(0, -1)$.

D It has a vertical asymptote at $x = 0$.

Short Answer

7. Solve the equation $x - 6 = \frac{x^2}{x + 1}$ graphically and algebraically.

8. a) Sketch the graph of the function $y = -\frac{6}{x + 4} - 3$.

b) Identify the domain and range, and state the locations of any asymptotes and intercepts.

9. Determine the approximate solution(s) to $\frac{3}{x} + 4 = \frac{2}{x + 3} - 1$ graphically. Give your answer(s) to two decimal places.

10. a) Sketch the graph of the function $y = \frac{x^2 - 2x - 8}{x - 4}$.

b) Explain the behaviour of the function for values of x near its non-permissible value.

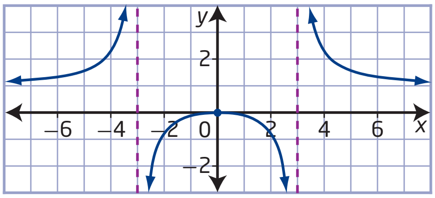
11. Predict the locations of any vertical asymptotes, points of discontinuity, and intercepts for the function $y = \frac{2x^2 + 7x - 4}{x^2 + x - 12}$. Give a reason for each prediction.

12. Match each rational function to its graph. Give reasons for each choice.

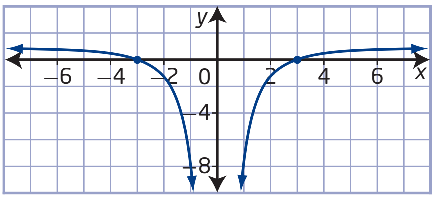
a) $A(x) = \frac{x^2 - 9x}{x}$ **b)** $B(x) = \frac{x^2}{x^2 - 9}$

c) $C(x) = \frac{x^2 - 9}{x^2}$ **d)** $D(x) = \frac{x^2}{x^2 - 9x}$

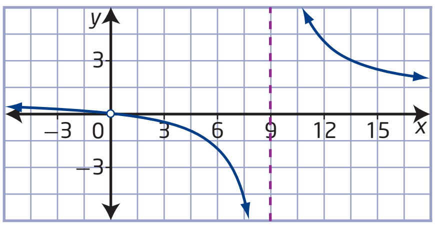
A



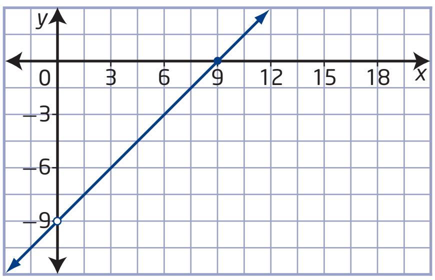
B



C



D



Extended Response

13. Compare the graphs of the functions $f(x) = \frac{2x - 3}{4x^2 - 9}$ and $g(x) = \frac{2x - 3}{4x^2 + 9}$. Explain any significant differences in the graphs by comparing the function equations.

14. Alex solved the equation $\frac{x^2}{x - 2} - 2 = \frac{3x - 2}{x - 2}$ algebraically and found a solution that consisted of two different values of x.

a) Solve the equation algebraically and explain what is incorrect about Alex's result.

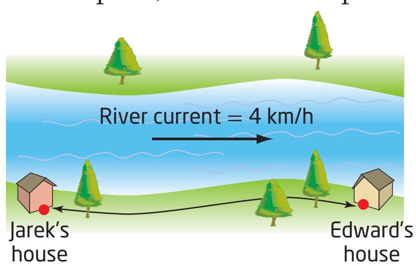
b) Use a graphical approach to solve the problem. Explain how this method can help you avoid the error that Alex made.

15. Jennifer is preparing for a golf tournament by practising her putting. So far she has been successful on 10 out of 31 putts.

a) If she tries x putts from now on and she is successful on half of them, what equation represents A, her overall average putting success rate as a function of x?

b) Use a graphical approach to determine how many putts it will take before her average is up to 40%.

16. Jarek lives 20 km upstream from his friend Edward on a river in which the water flows at 4 km/h. If Jarek travels by boat to Edward's house and back again, the total time, t, in hours, for the round trip is given by the function $t = \frac{40v}{v^2 - 16}$, where v is the boat's speed, in kilometres per hour.



a) What is the domain if Jarek makes the complete round trip? Explain.

b) Graph the function and explain what it shows about the situation.

c) Explain what the behaviour of the function near its non-permissible value represents in this context.

d) What boat speed does Jarek need to keep the trip to 45 min each way?