McGraw-Hill Ryerson

Pre-Calculus 12

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Contents

A Tour of Your Textbook

Unit Opener

Each unit begins with a two-page spread. The first page of the **Unit Opener** introduces what you will learn in the unit. The **Unit Project** is introduced on the second page. Each Unit Project helps you connect the math in the unit to real life using experiences that may interest you.

Project Corner boxes throughout the chapters help you gather information for your project. Some **Project Corner** boxes include questions to help you to begin thinking about and discussing your project.

The **Unit Projects** in Units 1, 3, and 4 provide an opportunity for you to choose a single **Project Wrap-Up** at the end of the unit.

The **Unit Project** in Unit 2 is designed for you to complete in pieces, chapter by chapter, throughout the unit. At the end of the unit, a **Project Wrap-Up** allows you to consolidate your work in a meaningful presentation.

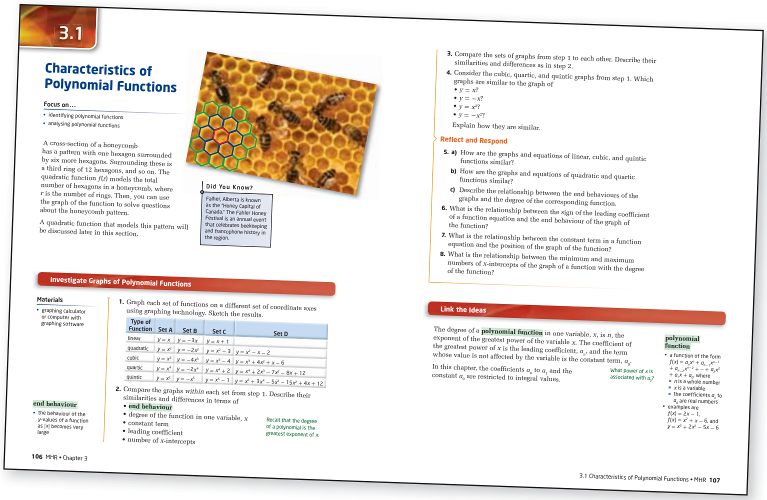
Chapter Opener

Each chapter begins with a two-page spread that introduces you to what you will learn in the chapter.

The opener includes information about a career that uses the skills covered in the chapter. A Web Link allows you to learn more about this career and how it involves the mathematics you are learning.

Visuals on the chapter opener spread show other ways the skills and concepts from the chapter are used in daily life.

Three-Part Lesson



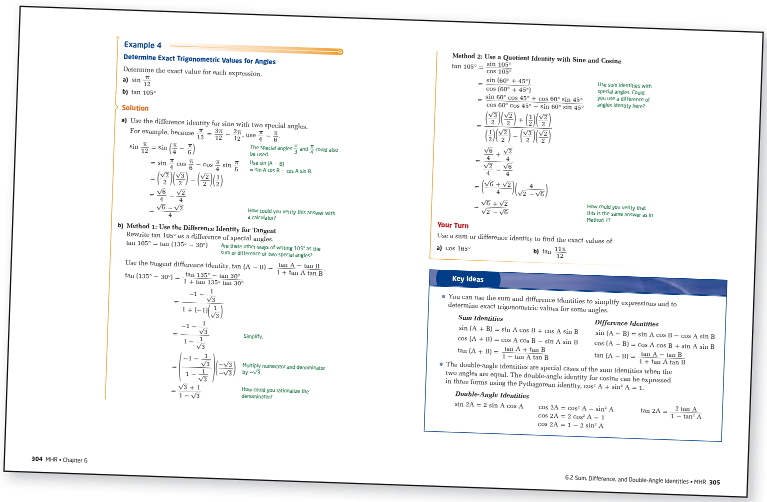
Each numbered section is organized in a three-part lesson: Investigate, Link the Ideas, and Check Your Understanding.

Investigate

- The **Investigate** consists of short steps often accompanied by illustrations. It is designed to help you build your own understanding of the new concept.
- The **Reflect and Respond** questions help you to analyse and communicate what you are learning and draw conclusions.

Link the Ideas

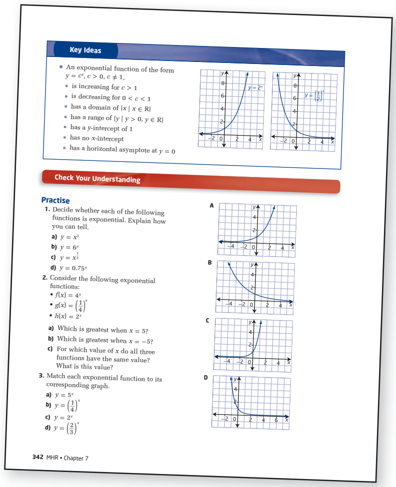
- The explanations in this section help you connect the concepts explored in the **Investigate** to the **Examples**.
- The **Examples** and worked **Solutions** show how to use the concepts. The Examples include several tools to help you understand the work.
  - Words in green font help you think through the steps.
  - Different methods of solving the same problem are sometimes shown. One method may make more sense to you than the others. Or, you may develop another method that means more to you.



- Each Example is followed by a **Your Turn**. The Your Turn allows you to explore your understanding of the skills covered in the Example.
- After all the Examples are presented, the **Key Ideas** summarize the main new concepts.

Check Your Understanding

- **Practise:** These questions allow you to check your understanding of the concepts. You can often do the first few questions by checking the Link the Ideas notes or by following one of the worked Examples.
- **Apply:** These questions ask you to apply what you have learned to solve problems. You can choose your own methods of solving a variety of problem types.
- **Extend:** These questions may be more challenging. Many connect to other concepts or lessons. They also allow you to choose your own methods of solving a variety of problem types.
- **Create Connections:** These questions focus your thinking on the Key Ideas and also encourage communication. Many of these questions also connect to other subject areas or other topics within mathematics.
- **Mini-Labs**: These questions provide hands-on activities that encourage you to further explore the concept you are learning.



Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$, $c \neq 1$,
  - is increasing for $c > 1$
  - is decreasing for $0 < c < 1$
  - has a domain of $\{x \mid x \in R\}$
  - has a range of $\{y \mid y > 0, y \in R\}$
  - has a y-intercept of 1
  - has no x-intercept
  - has a horizontal asymptote at $y = 0$

Check Your Understanding

Practise

1. Decide whether each of the following functions is exponential. Explain how you can tell.
   a) $y = x^3$
   b) $y = 6^x$
   c) $y = x^{\frac{1}{2}}$
   d) $y = 0.75^x$
2. Consider the following exponential functions:
   - $f(x) = 4^x$
   - $g(x) = \left(\frac{1}{4}\right)^x$
   - $h(x) = 2^x$

   a) Which is greatest when $x = 5$?
   b) Which is greatest when $x = -5$?
   c) For which value of x do all three functions have the same value? What is this value?
3. Match each exponential function to its corresponding graph.
   a) $y = 5^x$
   b) $y = \left(\frac{1}{4}\right)^x$
   c) $y = 2^x$
   d) $y = \left(\frac{2}{3}\right)^x$

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Other Features

Key Terms are listed on the Chapter Opener pages. You may already know the meaning of some of them. If not, watch for these terms the first time they are used in the chapter. The meaning is given in the margin. Many definitions include visuals that help clarify the term.

Key Terms
logarithmic function
logarithm
common logarithm
logarithmic equation

Some **Did You Know?** boxes provide additional information about the meaning of words that are not Key Terms. Other boxes contain interesting facts related to the math you are learning.

Did You Know?

The SI unit used to measure radioactivity is the becquerel (Bq), which is one particle emitted per second from a radioactive source. Commonly used multiples are kilobecquerel (kBq), for 10^3 Bq, and megabecquerel (MBq), for 10^6 Bq.

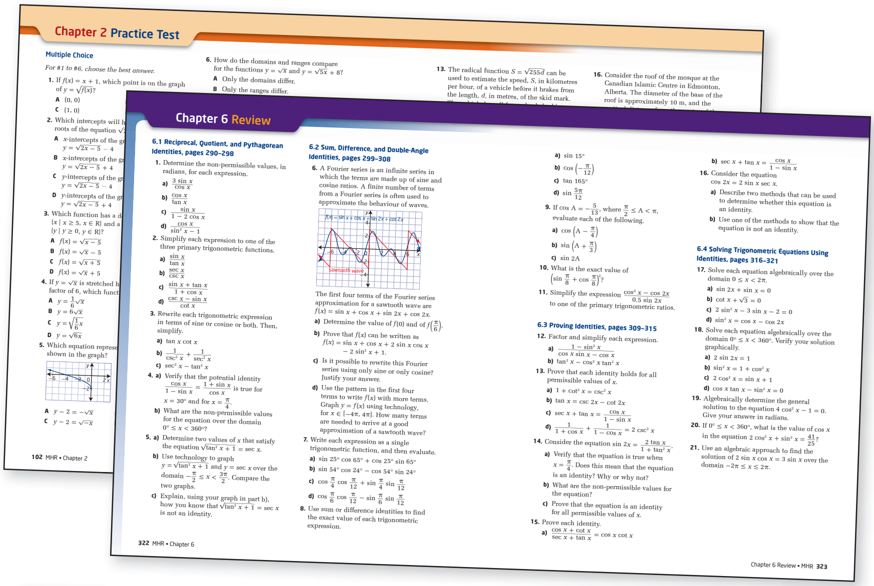
Opportunities are provided to use a variety of **Technology** tools. You can use technology to explore patterns and relationships, test predictions, and solve problems. A technology approach is usually provided as only one of a variety of approaches and tools to be used to help you develop your understanding.

Web Links provide Internet information related to some topics. Log on to www.mcgrawhill.ca/school/learningcentres and you will be able to link to recommended Web sites.

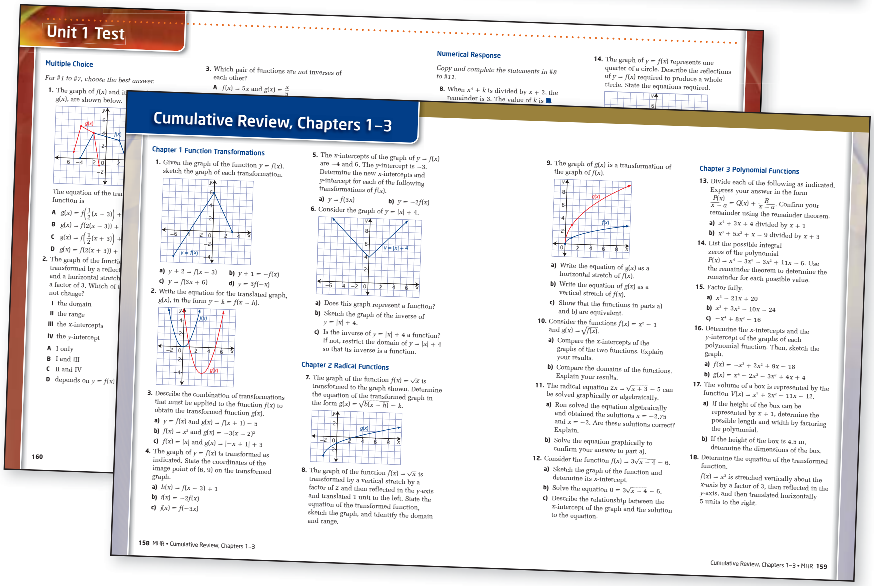
Web Link

To learn more about a career in radiology, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

A **Chapter Review** and a **Practice Test** appear at the end of each chapter. The review is organized by section number so you can look back if you need help with a question. The test includes multiple choice, short answer, and extended response questions.



A **Cumulative Review** and a **Unit Test** appear at the end of each unit. The review is organized by chapter. The test includes multiple choice, numerical response, and written response questions.



Answers are provided for the Practise, Apply, Extend, Create Connections, Chapter Review, Practice Test, Cumulative Review, and Unit Test questions. Sample answers are provided for questions that have a variety of possible answers or that involve communication. If you need help with a question like this, read the sample and then try to give an alternative response.

Refer to the illustrated **Glossary** at the back of the student resource if you need to check the exact meaning of mathematical terms.

If you want to find a particular math topic in *Pre-Calculus 12*, look it up in the **Index**, which is at the back of the student resource. The index provides page references that may help you review that topic.

Unit 1

Transformations and Functions

Functions help you make sense of the world around you. Many ordinary measuring devices are based on mathematical functions:

- Car odometer: The odometer reading is a function of the number of rotations of the car's transmission drive shaft.
- Display on a barcode reader: When the screen displays the data about the object, the reader performs an inverse function by decoding the barcode image.

Many natural occurrences can be modelled by mathematical functions:

- Ripples created by a water droplet in a pond: You can model the area spanned by the ripples by a polynomial function.
- Explosion of a supernova: You can model the time the explosion takes to affect a volume of space by a radical function.

In this unit, you will expand your knowledge of transformations while exploring radical and polynomial functions. These functions and associated transformations are useful in a variety of applications within mathematics.

Looking Ahead

In this unit, you will solve problems involving…

- transformations of functions
- inverses of functions
- radical functions and equations
- polynomial functions and equations

Unit 1 Project — The Art of Mathematics

Simone McLeod, a Cree-Ojibway originally from Winnipeg, Manitoba, now lives in Saskatchewan and is a member of the James Smith Cree Nation. Simone began painting later in life.

"I really believed that I had to wait until I could find something that had a lot of meaning to me. Each painting contains a piece of my soul. I have a strong faith in humankind and my paintings are silent prayers of hope for the future…."

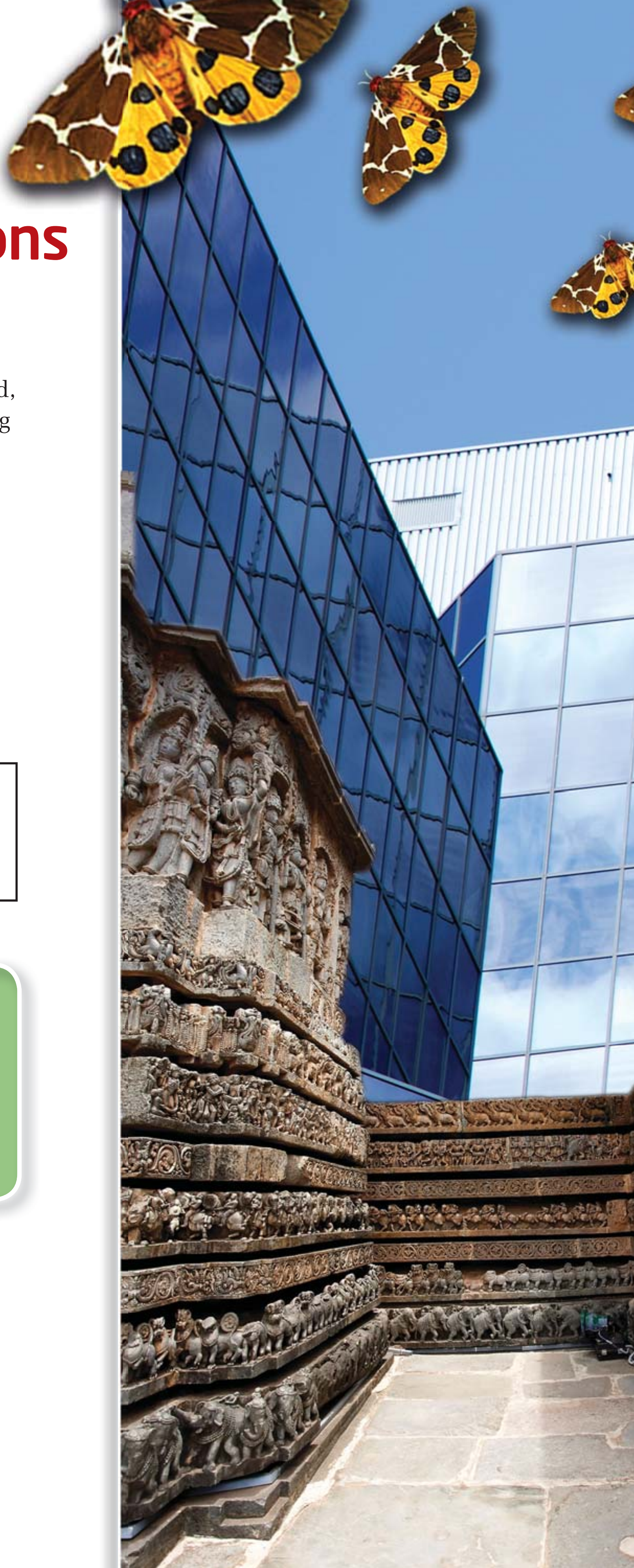
"My Indian name is Earth Blanket (all that covers the earth such as grass, flowers, and trees). The sun, the blankets, and the flowers/rocks are all the same colours to show how all things are equal."

Simone's work is collected all over the world, including Europe, India, Asia, South Africa, and New Zealand.

In this project, you will search for mathematical functions in art, nature, and manufactured objects. You will determine equations for the functions or partial functions you find. You will justify your equations and display them superimposed on the image you have selected.

CHAPTER 1

Function Transformations



Mathematical shapes are found in architecture, bridges, containers, jewellery, games, decorations, art, and nature. Designs that are repeated, reflected, stretched, or transformed in some way are pleasing to the eye and capture our imagination.

In this chapter, you will explore the mathematical relationship between a function and its transformed graph. Throughout the chapter, you will explore how functions are transformed and develop strategies for relating complex functions to simpler functions.

Did You Know?

Albert Einstein (1879–1955) is often regarded as the father of modern physics. He won the Nobel Prize for Physics in 1921 for "his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect." The Lorentz transformations are an important part of Einstein's theory of relativity.

Key Terms

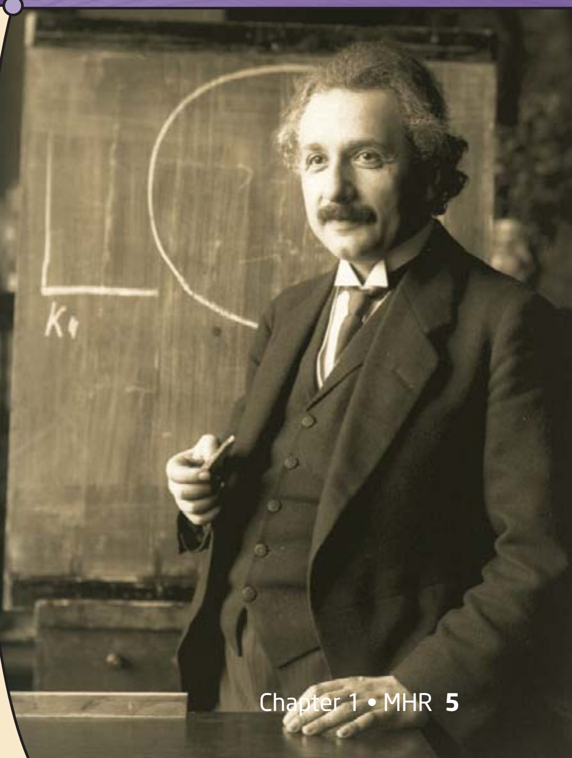
transformation
mapping
translation
image point
reflection
invariant point
stretch
inverse of a function
horizontal line test

Career Link

A physicist is a scientist who studies the natural world, from sub-atomic particles to matters of the universe. Some physicists focus on theoretical areas, while others apply their knowledge of physics to practical areas, such as the development of advanced materials and electronic and optical devices. Some physicists observe, measure, interpret, and develop theories to explain celestial and physical phenomena using mathematics. Physicists use mathematical functions to make numerical and algebraic computations easier.

Web Link

To find out more about the career of a physicist, go to www.mcgrawhill.ca/school/learningcentres and follow the links.



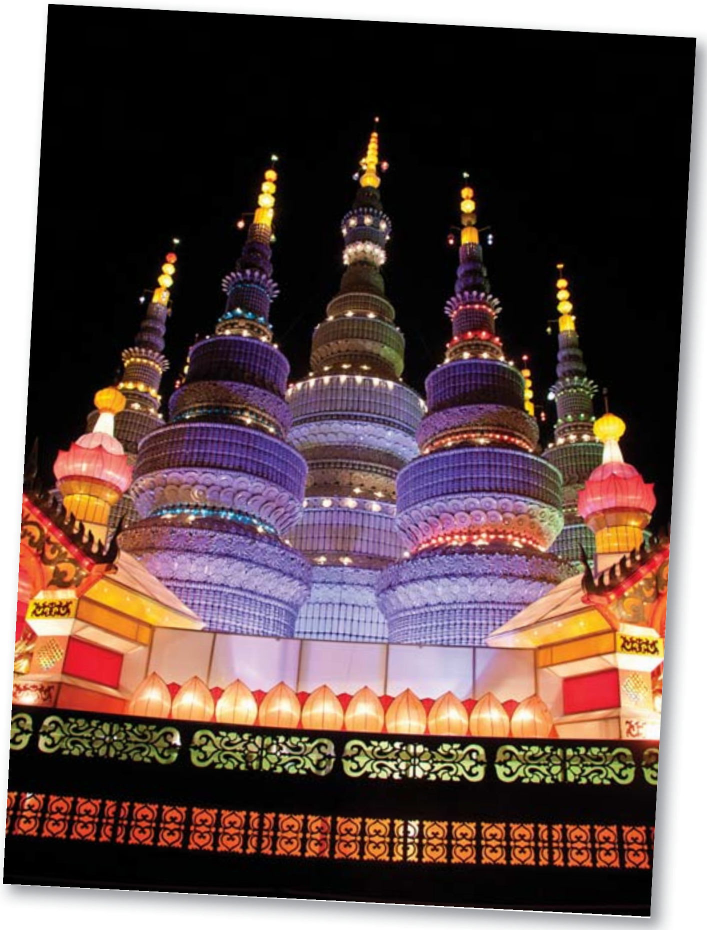
1.1

Horizontal and Vertical Translations

Focus on…

- determining the effects of h and k in $y - k = f(x - h)$ on the graph of $y = f(x)$
- sketching the graph of $y - k = f(x - h)$ for given values of h and k, given the graph of $y = f(x)$
- writing the equation of a function whose graph is a vertical and/or horizontal translation of the graph of $y = f(x)$

A linear frieze pattern is a decorative pattern in which a section of the pattern repeats along a straight line. These patterns often occur in border decorations and textiles. Frieze patterns are also used by artists, craftspeople, musicians, choreographers, and mathematicians. Can you think of places where you have seen a frieze pattern?



Lantern Festival in China

Investigate Vertical and Horizontal Translations

Materials

- grid paper

A: Compare the Graphs of $y = f(x)$ and $y - k = f(x)$

1. Consider the function $f(x) = |x|$.

  a) Use a table of values to compare the output values for $y = f(x)$, $y = f(x) + 3$, and $y = f(x) - 3$ given input values of -3, -2, -1, 0, 1, 2, and 3.

  b) Graph the functions on the same set of coordinate axes.

2. a) Describe how the graphs of $y = f(x) + 3$ and $y = f(x) - 3$ compare to the graph of $y = f(x)$.

  b) Relative to the graph of $y = f(x)$, what information about the graph of $y = f(x) + k$ does k provide?

3. Would the relationship between the graphs of $y = f(x)$ and $y = f(x) + k$ change if $f(x) = x$ or $f(x) = x^2$? Explain.

B: Compare the Graphs of $y = f(x)$ and $y = f(x - h)$

4. Consider the function $f(x) = |x|$.

a) Use a table of values to compare the output values for $y = f(x)$, $y = f(x + 3)$, and $y = f(x - 3)$ given input values of -9, -6, -3, 0, 3, 6, and 9.

b) Graph the functions on the same set of coordinate axes.

5. a) Describe how the graphs of $y = f(x + 3)$ and $y = f(x - 3)$ compare to the graph of $y = f(x)$.

b) Relative to the graph of $y = f(x)$, what information about the graph of $y = f(x - h)$ does h provide?

6. Would the relationship between the graphs of $y = f(x)$ and $y = f(x - h)$ change if $f(x) = x$ or $f(x) = x^2$? Explain.

Reflect and Respond

7. How is the graph of a function $y = f(x)$ related to the graph of $y = f(x) + k$ when $k > 0$? when $k < 0$?

8. How is the graph of a function $y = f(x)$ related to the graph of $y = f(x - h)$ when $h > 0$? when $h < 0$?

9. Describe how the parameters h and k affect the properties of the graph of a function. Consider such things as shape, orientation, x-intercepts and y-intercept, domain, and range.

Link the Ideas

A **transformation** of a function alters the equation and any combination of the location, shape, and orientation of the graph.

Points on the original graph correspond to points on the transformed, or image, graph. The relationship between these sets of points can be called a **mapping**.

Mapping notation can be used to show a relationship between the coordinates of a set of points, (x, y), and the coordinates of a corresponding set of points, $(x, y + 3)$, for example, as $(x, y) \rightarrow (x, y + 3)$.

Did You Know?

Mapping notation is an alternate notation for function notation. For example, $f(x) = 3x + 4$ can be written as $f\colon x \rightarrow 3x + 4$. This is read as "$f$ is a function that maps x to $3x + 4$."

transformation

- a change made to a figure or a relation such that the figure or the graph of the relation is shifted or changed in shape

mapping

- the relating of one set of points to another set of points so that each point in the original set corresponds to exactly one point in the image set

translation

- a slide transformation that results in a shift of a graph without changing its shape or orientation
- vertical and horizontal translations are types of transformations with equations of the forms $y - k = f(x)$ and $y = f(x - h)$, respectively
- a translated graph is congruent to the original graph

One type of transformation is a **translation**. A translation can move the graph of a function up, down, left, or right. A translation occurs when the location of a graph changes but not its shape or orientation.

Example 1

Graph Translations of the Form $y - k = f(x)$ and $y = f(x - h)$

a) Graph the functions $y = x^2$, $y - 2 = x^2$, and $y = (x - 5)^2$ on the same set of coordinate axes.

b) Describe how the graphs of $y - 2 = x^2$ and $y = (x - 5)^2$ compare to the graph of $y = x^2$.

Solution

a) The notation $y - k = f(x)$ is often used instead of $y = f(x) + k$ to emphasize that this is a transformation on y. In this case, the base function is $f(x) = x^2$ and the value of k is 2.

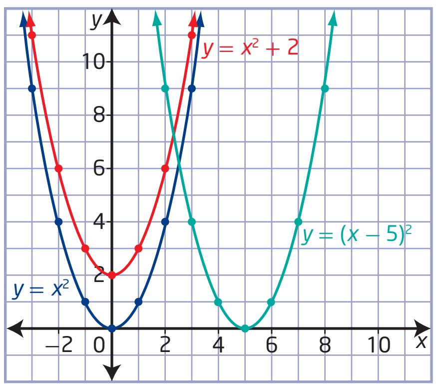
The notation $y = f(x - h)$ shows that this is a transformation on x. In this case, the base function is $f(x) = x^2$ and the value of h is 5.

Rearrange equations as needed and use tables of values to help you graph the functions.

x	$y = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

x	$y = x^2 + 2$
−3	11
−2	6
−1	3
0	2
1	3
2	6
3	11

x	$y = (x - 5)^2$
2	9
3	4
4	1
5	0
6	1
7	4
8	9



For $y = x^2 + 2$, the input values are the same but the output values change. Each point (x, y) on the graph of $y = x^2$ is transformed to $(x, y + 2)$.

For $y = (x - 5)^2$, to maintain the same output values as the base function table, the input values are different. Every point (x, y) on the graph of $y = x^2$ is transformed to $(x + 5, y)$. How do the input changes relate to the translation direction?

b) The transformed graphs are congruent to the graph of $y = x^2$.

Each point (x, y) on the graph of $y = x^2$ is transformed to become the point $(x, y + 2)$ on the graph of $y - 2 = x^2$. Using mapping notation, $(x, y) \rightarrow (x, y + 2)$.

Therefore, the graph of $y - 2 = x^2$ is the graph of $y = x^2$ translated vertically 2 units up.

Each point (x, y) on the graph of $y = x^2$ is transformed to become the point $(x + 5, y)$ on the graph of $y = (x - 5)^2$. In mapping notation, $(x, y) \rightarrow (x + 5, y)$.

Therefore, the graph of $y = (x - 5)^2$ is the graph of $y = x^2$ translated horizontally 5 units to the right.

Your Turn

How do the graphs of $y + 1 = x^2$ and $y = (x + 3)^2$ compare to the graph of $y = x^2$? Justify your reasoning.

Example 2

Horizontal and Vertical Translations

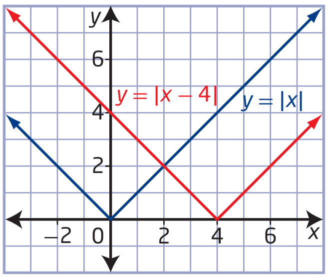
Sketch the graph of $y = |x - 4| + 3$.

Solution

For $y = |x - 4| + 3$, $h = 4$ and $k = -3$.

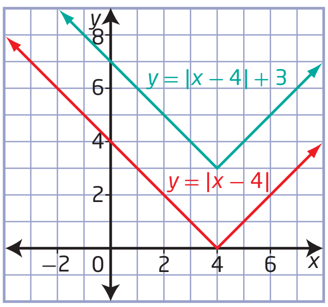
- Start with a sketch of the graph of the base function $y = |x|$, using key points.
- Apply the horizontal translation of 4 units to the right to obtain the graph of $y = |x - 4|$.

To ensure an accurate sketch of a transformed function, translate key points on the base function first.



- Apply the vertical translation of 3 units up to $y = |x - 4|$ to obtain the graph of $y = |x - 4| + 3$.

Would the graph be in the correct location if the order of the translations were reversed?



Did You Know?

Key points are points on a graph that give important information, such as the *x*-intercepts, the *y*-intercept, the maximum, and the minimum.

The point (0, 0) on the function $y = |x|$ is transformed to become the point (4, 3). In general, the transformation can be described as $(x, y) \rightarrow (x + 4, y + 3)$.

Your Turn

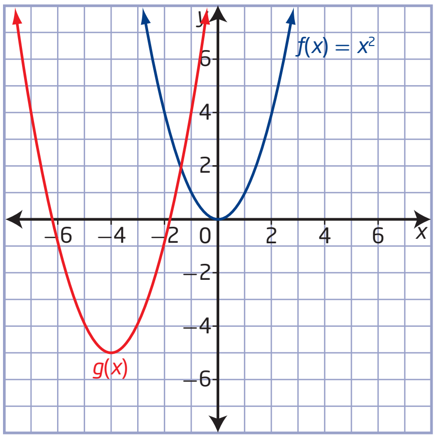
Sketch the graph of $y = (x + 5)^2 - 2$.

Example 3

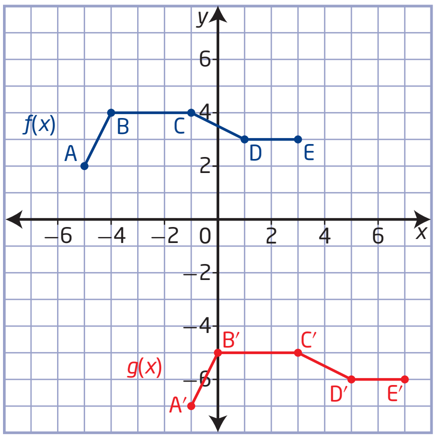
Determine the Equation of a Translated Function

Describe the translation that has been applied to the graph of $f(x)$ to obtain the graph of $g(x)$. Determine the equation of the translated function in the form $y - k = f(x - h)$.

a)



b)



It is a common convention to use a prime (′) next to each letter representing an image point.

Solution

image point
- the point that is the result of a transformation of a point on the original graph

a) The base function is $f(x) = x^2$. Choose key points on the graph of $f(x) = x^2$ and locate the corresponding **image points** on the graph of $g(x)$.

$f(x)$		$g(x)$
(0, 0)	→	(−4, −5)
(−1, 1)	→	(−5, −4)
(1, 1)	→	(−3, −4)
(−2, 4)	→	(−6, −1)
(2, 4)	→	(−2, −1)
(x, y)	→	(x − 4, y − 5)

For a horizontal translation and a vertical translation where every point (x, y) on the graph of $y = f(x)$ is transformed to $(x + h, y + k)$, the equation of the transformed graph is of the form $y - k = f(x - h)$.

To obtain the graph of $g(x)$, the graph of $f(x) = x^2$ has been translated 4 units to the left and 5 units down. So, $h = -4$ and $k = -5$.

To write the equation in the form $y - k = f(x - h)$, substitute -4 for h and -5 for k.

$y + 5 = f(x + 4)$

b) Begin with key points on the graph of $f(x)$. Locate the corresponding image points.

$f(x)$		$g(x)$
A(−5, 2)	→	A′(−1, −7)
B(−4, 4)	→	B′(0, −5)
C(−1, 4)	→	C′(3, −5)
D(1, 3)	→	D′(5, −6)
E(3, 3)	→	E′(7, −6)
(x, y)	→	$(x + 4, y - 9)$

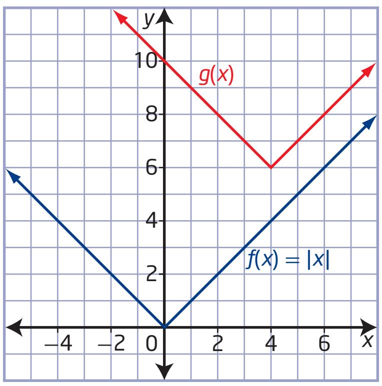
To obtain the graph of $g(x)$, the graph of $f(x)$ has been translated 4 units to the right and 9 units down. Substitute $h = 4$ and $k = -9$ into the equation of the form $y - k = f(x - h)$:

$y + 9 = f(x - 4)$

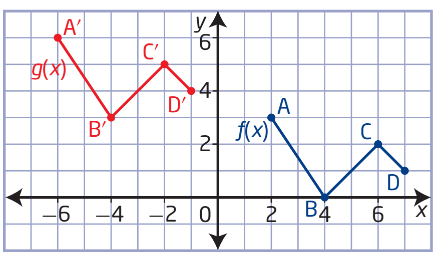
Your Turn

Describe the translation that has been applied to the graph of $f(x)$ to obtain the graph of $g(x)$. Determine the equation of the translated function in the form $y - k = f(x - h)$.

a)



b)

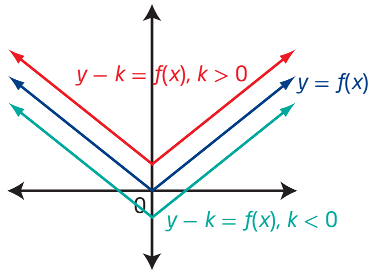
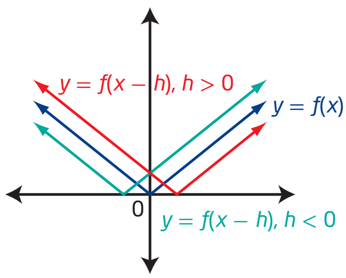


Did You Know?

In Pre-Calculus 11, you graphed quadratic functions of the form $y = (x - p)^2 + q$ by considering transformations from the graph of $y = x^2$. In $y = (x - p)^2 + q$, the parameter p determines the horizontal translation and the parameter q determines the vertical translation of the graph. In this unit, the parameters for horizontal and vertical translations are represented by h and k, respectively.

Key Ideas

- Translations are transformations that shift all points on the graph of a function up, down, left, and right without changing the shape or orientation of the graph.
- The table summarizes translations of the function $y = f(x)$.

Function	Transformation from $y = f(x)$	Mapping	Example
$y - k = f(x)$ or $y = f(x) + k$	A vertical translation If $k > 0$, the translation is up. If $k < 0$, the translation is down.	$(x, y) \rightarrow (x, y + k)$	
$y = f(x - h)$	A horizontal translation If $h > 0$, the translation is to the right. If $h < 0$, the translation is to the left.	$(x, y) \rightarrow (x + h, y)$	

- A sketch of the graph of $y - k = f(x - h)$, or $y = f(x - h) + k$, can be created by translating key points on the graph of the base function $y = f(x)$.

Check Your Understanding

Practise

1. For each function, state the values of h and k, the parameters that represent the horizontal and vertical translations applied to $y = f(x)$.

a) $y - 5 = f(x)$

b) $y = f(x) - 4$

c) $y = f(x + 1)$

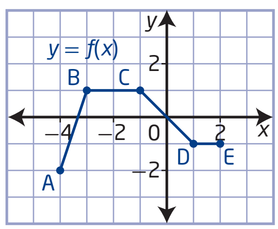
d) $y + 3 = f(x - 7)$

e) $y = f(x + 2) + 4$

2. Given the graph of $y = f(x)$ and each of the following transformations,
- state the coordinates of the image points A′, B′, C′, D′ and E′
- sketch the graph of the transformed function

a) $g(x) = f(x) + 3$ **b)** $h(x) = f(x - 2)$

c) $s(x) = f(x + 4)$ **d)** $t(x) = f(x) - 2$



3. Describe, using mapping notation, how the graphs of the following functions can be obtained from the graph of $y = f(x)$.

a) $y = f(x + 10)$

b) $y + 6 = f(x)$

c) $y = f(x - 7) + 4$

d) $y - 3 = f(x - 1)$

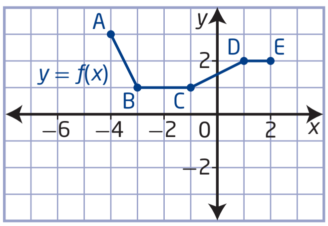
4. Given the graph of $y = f(x)$, sketch the graph of the transformed function. Describe the transformation that can be applied to the graph of $f(x)$ to obtain the graph of the transformed function. Then, write the transformation using mapping notation.

a) $r(x) = f(x + 4) - 3$

b) $s(x) = f(x - 2) - 4$

c) $t(x) = f(x - 2) + 5$

d) $v(x) = f(x + 3) + 2$



Apply

5. For each transformation, identify the values of h and k. Then, write the equation of the transformed function in the form $y - k = f(x - h)$.

a) $f(x) = \frac{1}{x}$, translated 5 units to the left and 4 units up

b) $f(x) = x^2$, translated 8 units to the right and 6 units up

c) $f(x) = |x|$, translated 10 units to the right and 8 units down

d) $y = f(x)$, translated 7 units to the left and 12 units down

6. What vertical translation is applied to $y = x^2$ if the transformed graph passes through the point (4, 19)?

7. What horizontal translation is applied to $y = x^2$ if the translation image graph passes through the point (5, 16)?

8. Copy and complete the table.

Translation	Transformed Function	Transformation of Points
vertical	$y = f(x) + 5$	$(x, y) \rightarrow (x, y + 5)$
	$y = f(x + 7)$	$(x, y) \rightarrow (x - 7, y)$
	$y = f(x - 3)$	
	$y = f(x) - 6$	
horizontal and vertical	$y + 9 = f(x + 4)$	
horizontal and vertical		$(x, y) \rightarrow (x + 4, y - 6)$
		$(x, y) \rightarrow (x - 2, y + 3)$
horizontal and vertical	$y = f(x - h) + k$	

9. The graph of the function $y = x^2$ is translated 4 units to the left and 5 units up to form the transformed function $y = g(x)$.

a) Determine the equation of the function $y = g(x)$.

b) What are the domain and range of the image function?

c) How could you use the description of the translation of the function $y = x^2$ to determine the domain and range of the image function?

10. The graph of $f(x) = |x|$ is transformed to the graph of $g(x) = f(x - 9) + 5$.

a) Determine the equation of the function $g(x)$.

b) Compare the graph of $g(x)$ to the graph of the base function $f(x)$.

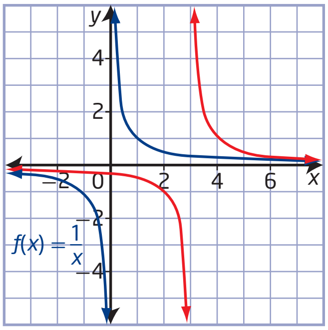
c) Determine three points on the graph of $f(x)$. Write the coordinates of the image points if you perform the horizontal translation first and then the vertical translation.

d) Using the same original points from part c), write the coordinates of the image points if you perform the vertical translation first and then the horizontal translation.

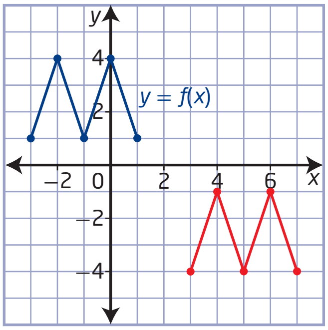
e) What do you notice about the coordinates of the image points from parts c) and d)? Is the order of the translations important?

11. The graph of the function drawn in red is a translation of the original function drawn in blue. Write the equation of the translated function in the form $y - k = f(x - h)$.

a)



b)

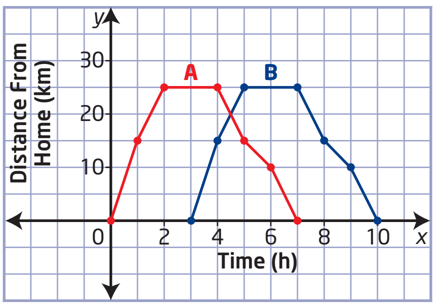


12. Janine is an avid cyclist. After cycling to a lake and back home, she graphs her distance versus time (graph A).

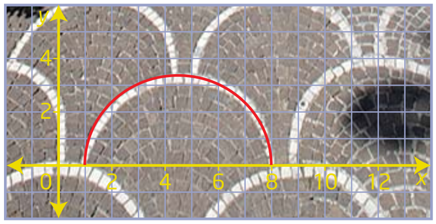
a) If she left her house at 12 noon, briefly describe a possible scenario for Janine's trip.

b) Describe the differences it would make to Janine's cycling trip if the graph of the function were translated, as shown in graph B.

c) The equation for graph A could be written as $y = f(x)$. Write the equation for graph B.



13. Architects and designers often use translations in their designs. The image shown is from an Italian roadway.



a) Use the coordinate plane overlay with the base semicircle shown to describe the approximate transformations of the semicircles.

b) If the semicircle at the bottom left of the image is defined by the function $y = f(x)$, state the approximate equations of three other semicircles.

14. This Pow Wow belt shows a frieze pattern where a particular image has been translated throughout the length of the belt.

a) With or without technology, create a design using a pattern that is a function. Use a minimum of four horizontal translations of your function to create your own frieze pattern.

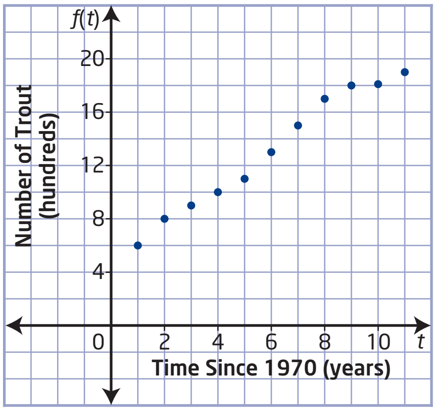
b) Describe the translation of your design in words and in an equation of the form $y = f(x - h)$.

Did You Know?

In First Nations communities today, Pow Wows have evolved into multi-tribal festivals. Traditional dances are performed by men, women, and children. The dancers wear traditional regalia specific to their dance style and nation of origin.

15. Michele Lake and Coral Lake, located near the Columbia Ice Fields, are the only two lakes in Alberta in which rare golden trout live.

Suppose the graph represents the number of golden trout in Michelle Lake in the years since 1970.



Let the function $f(t)$ represent the number of fish in Michelle Lake since 1970.

Describe an event or a situation for the fish population that would result in the following transformations of the graph. Then, use function notation to represent the transformation.

a) a vertical translation of 2 units up

b) a horizontal translation of 3 units to the right

16. Paul is an interior house painter. He determines that the function $n = f(A)$ gives the number of gallons, n, of paint needed to cover an area, A, in square metres. Interpret $n = f(A) + 10$ and $n = f(A + 10)$ in this context.

Extend

17. The graph of the function $y = x^2$ is translated to an image parabola with zeros 7 and 1.

a) Determine the equation of the image function.

b) Describe the translations on the graph of $y = x^2$.

c) Determine the y-intercept of the translated function.

18. Use translations to describe how the graph of $y = \frac{1}{x}$ compares to the graph of each function.

a) $y - 4 = \frac{1}{x}$

b) $y = \frac{1}{x + 2}$

c) $y - 3 = \frac{1}{x - 5}$

d) $y = \frac{1}{x + 3} - 4$

19. a) Predict the relationship between the graph of $y = x^3 - x^2$ and the graph of $y + 3 = (x - 2)^3 - (x - 2)^2$.

b) Graph each function to verify your prediction.

Create Connections

C1 The graph of the function $y = f(x)$ is transformed to the graph of $y = f(x - h) + k$.

a) Show that the order in which you apply translations does not matter. Explain why this is true.

b) How are the domain and range affected by the parameters h and k?

C2 Complete the square and explain how to transform the graph of $y = x^2$ to the graph of each function.

a) $f(x) = x^2 + 2x + 1$

b) $g(x) = x^2 - 4x + 3$

C3 The roots of the quadratic equation $x^2 - x - 12 = 0$ are -3 and 4. Determine the roots of the equation $(x - 5)^2 - (x - 5) - 12 = 0$.

C4 The function $f(x) = x + 4$ could be a vertical translation of 4 units up or a horizontal translation of 4 units to the left. Explain why.

1.2

Reflections and Stretches

Focus on…

- developing an understanding of the effects of reflections on the graphs of functions and their related equations
- developing an understanding of the effects of vertical and horizontal stretches on the graphs of functions and their related equations

Reflections, symmetry, as well as horizontal and vertical stretches, appear in architecture, textiles, science, and works of art. When something is symmetrical or stretched in the geometric sense, its parts have a one-to-one correspondence. How does this relate to the study of functions?

Ndebele artist, South Africa

Investigate Reflections and Stretches of Functions

Materials

- grid paper
- graphing technology

A: Graph Reflections in the *x*-Axis and the *y*-Axis

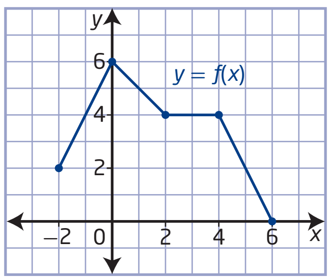
1. a) Draw a set of coordinate axes on grid paper. In quadrant I, plot a point A. Label point A with its coordinates.

b) Use the x-axis as a mirror line, or line of reflection, and plot point A′, the mirror image of point A in the x-axis.

c) How are the coordinates of points A and A′ related?

d) If point A is initially located in any of the other quadrants, does the relationship in part c) still hold true?

2. Consider the graph of the function $y = f(x)$.



a) Explain how you could graph the mirror image of the function in the x-axis.

b) Make a conjecture about how the equation of $f(x)$ changes to graph the mirror image.

3. Use graphing technology to graph the function $y = x^2 + 2x$, $-5 \leq x \leq 5$, and its mirror image in the x-axis. What equation did you enter to graph the mirror image?

4. Repeat steps 1 to 3 for a mirror image in the y-axis.

Reflect and Respond

5. Copy and complete the table to record your observations. Write concluding statements summarizing the effects of reflections in the axes.

	Reflection in	Verbal Description	Mapping	Equation of Transformed Function
Function $y = f(x)$	x-axis		$(x, y) \rightarrow (\quad , \quad)$	
	y-axis		$(x, y) \rightarrow (\quad , \quad)$	

B: Graph Vertical and Horizontal Stretches

6. a) Plot a point A on a coordinate grid and label it with its coordinates.
   b) Plot and label a point A′ with the same x-coordinate as point A, but with the y-coordinate equal to 2 times the y-coordinate of point A.
   c) Plot and label a point A″ with the same x-coordinate as point A, but with the y-coordinate equal to $\frac{1}{2}$ the y-coordinate of point A.
   d) Compare the location of points A′ and A″ to the location of the original point A. Describe how multiplying the y-coordinate by a factor of 2 or a factor of $\frac{1}{2}$ affects the position of the image point.

Has the distance to the x-axis or the y-axis changed?

7. Consider the graph of the function $y = f(x)$ in step 2. Sketch the graph of the function when the y-values have been
   a) multiplied by 2
   b) multiplied by $\frac{1}{2}$

8. What are the equations of the transformed functions in step 7 in the form $y = af(x)$?

9. For step 7a), the graph has been vertically stretched about the x-axis by a factor of 2. Explain the statement. How would you describe the graph in step 7b)?

10. Consider the graph of the function $y = f(x)$ in step 2.
    a) If the x-values were multiplied by 2 or multiplied by $\frac{1}{2}$, describe what would happen to the graph of the function $y = f(x)$.
    b) Determine the equations of the transformed functions in part a) in the form $y = f(bx)$.

Reflect and Respond

11. Copy and complete the table to record your observations. Write concluding statements summarizing the effects of stretches about the axes.

	Stretch About	Verbal Description	Mapping	Equation of Transformed Function
Function $y = f(x)$	x-axis		$(x, y) \rightarrow (\ ,\ )$	
	y-axis		$(x, y) \rightarrow (\ ,\ )$	

Link the Ideas

reflection

- a transformation where each point of the original graph has an image point resulting from a reflection in a line
- may result in a change of orientation of a graph while preserving its shape

A **reflection** of a graph creates a mirror image in a line called the line of reflection. Reflections, like translations, do not change the shape of the graph. However, unlike translations, reflections may change the orientation of the graph.

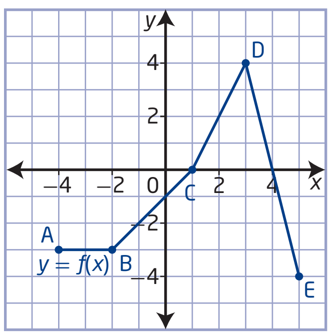
- When the output of a function $y = f(x)$ is multiplied by -1, the result, $y = -f(x)$, is a reflection of the graph in the x-axis.
- When the input of a function $y = f(x)$ is multiplied by -1, the result, $y = f(-x)$, is a reflection of the graph in the y-axis.

Example 1

Compare the Graphs of $y = f(x)$, $y = -f(x)$, and $y = f(-x)$

a) Given the graph of $y = f(x)$, graph the functions $y = -f(x)$ and $y = f(-x)$.

b) How are the graphs of $y = -f(x)$ and $y = f(-x)$ related to the graph of $y = f(x)$?



Solution

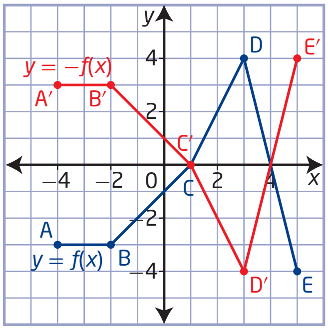
a) Use key points on the graph of $y = f(x)$ to create tables of values.

- The image points on the graph of $y = -f(x)$ have the same x-coordinates but different y-coordinates. Multiply the y-coordinates of points on the graph of $y = f(x)$ by -1.

The negative sign can be interpreted as a change in sign of one of the coordinates.

	x	$y = f(x)$
A	−4	−3
B	−2	−3
C	1	0
D	3	4
E	5	−4

	x	$y = -f(x)$
A′	−4	−1(−3) = 3
B′	−2	−1(−3) = 3
C′	1	−1(0) = 0
D′	3	−1(4) = −4
E′	5	−1(−4) = 4

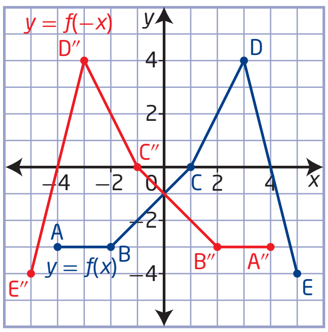


Each image point is the same distance from the line of reflection as the corresponding key point. A line drawn perpendicular to the line of reflection contains both the key point and its image point.

- The image points on the graph of $y = f(-x)$ have the same y-coordinates but different x-coordinates. Multiply the x-coordinates of points on the graph of $y = f(x)$ by -1.

	x	$y = f(x)$
A	−4	−3
B	−2	−3
C	1	0
D	3	4
E	5	−4

	x	$y = f(-x)$
A″	−1(−4) = 4	−3
B″	−1(−2) = 2	−3
C″	−1(1) = −1	0
D″	−1(3) = −3	4
E″	−1(5) = −5	−4



b) The transformed graphs are congruent to the graph of $y = f(x)$.

The points on the graph of $y = f(x)$ relate to the points on the graph of $y = -f(x)$ by the mapping $(x, y) \rightarrow (x, -y)$. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

Notice that the point C(1, 0) maps to itself, C′(1, 0). This point is an **invariant point**.

What is another invariant point?

invariant point

- a point on a graph that remains unchanged after a transformation is applied to it
- any point on a curve that lies on the line of reflection is an invariant point

The points on the graph of $y = f(x)$ relate to the points on the graph of $y = f(-x)$ by the mapping $(x, y) \rightarrow (-x, y)$. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

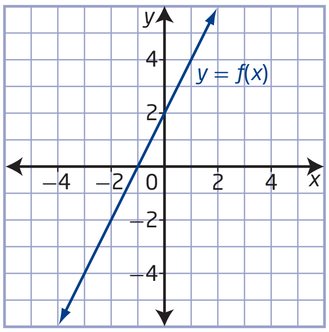
The point $(0, -1)$ is an invariant point.

Your Turn

a) Given the graph of $y = f(x)$, graph the functions $y = -f(x)$ and $y = f(-x)$.

b) Show the mapping of key points on the graph of $y = f(x)$ to image points on the graphs of $y = -f(x)$ and $y = f(-x)$.

c) Describe how the graphs of $y = -f(x)$ and $y = f(-x)$ are related to the graph of $y = f(x)$. State any invariant points.



Vertical and Horizontal Stretches

A **stretch**, unlike a translation or a reflection, changes the shape of the graph. However, like translations, stretches do not change the orientation of the graph.

stretch

- a transformation in which the distance of each x-coordinate or y-coordinate from the line of reflection is multiplied by some scale factor
- scale factors between 0 and 1 result in the point moving closer to the line of reflection; scale factors greater than 1 result in the point moving farther away from the line of reflection

- When the output of a function $y = f(x)$ is multiplied by a non-zero constant a, the result, $y = af(x)$ or $\frac{y}{a} = f(x)$, is a vertical stretch of the graph about the x-axis by a factor of $|a|$. If $a < 0$, then the graph is also reflected in the x-axis.
- When the input of a function $y = f(x)$ is multiplied by a non-zero constant b, the result, $y = f(bx)$, is a horizontal stretch of the graph about the y-axis by a factor of $\frac{1}{|b|}$. If $b < 0$, then the graph is also reflected in the y-axis.

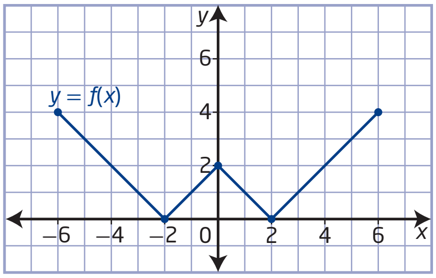
Example 2

Graph $y = af(x)$

Given the graph of $y = f(x)$,
- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions

a) $g(x) = 2f(x)$

b) $g(x) = \frac{1}{2}f(x)$

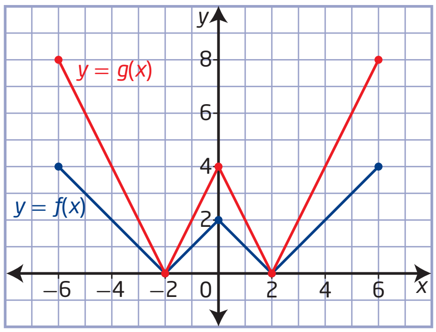


Solution

a) Use key points on the graph of $y = f(x)$ to create a table of values.

The image points on the graph of $g(x) = 2f(x)$ have the same x-coordinates but different y-coordinates. Multiply the y-coordinates of points on the graph of $y = f(x)$ by 2.

x	$y = f(x)$	$y = g(x) = 2f(x)$
–6	4	8
–2	0	0
0	2	4
2	0	0
6	4	8



The vertical distances of the transformed graph have been changed by a factor of a, where $|a| > 1$. The points on the graph of $y = af(x)$ are farther away from the x-axis than the corresponding points of the graph of $y = f(x)$.

Since $a = 2$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow (x, 2y)$. Therefore, each point on the graph of $g(x)$ is twice as far from the x-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = 2f(x)$ is a vertical stretch of the graph of $y = f(x)$ about the x-axis by a factor of 2.

The invariant points are $(-2, 0)$ and $(2, 0)$.

What is unique about the invariant points?

For $f(x)$, the domain is $\{x \mid -6 \le x \le 6, x \in \mathrm{R}\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

How can you determine the range of the new function, $g(x)$, using the range of $f(x)$ and the parameter a?

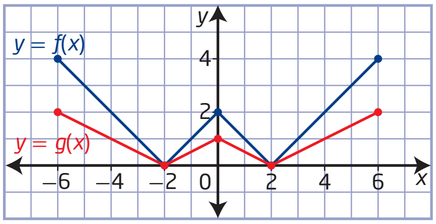
For $g(x)$, the domain is $\{x \mid -6 \le x \le 6, x \in \mathrm{R}\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \le y \le 8, y \in \mathrm{R}\}$, or $[0, 8]$.

Did You Know?

There are several ways to express the domain and range of a function. For example, you can use words, a number line, set notation, or interval notation.

b) The image points on the graph of $g(x) = \frac{1}{2}f(x)$ have the same x-coordinates but different y-coordinates. Multiply the y-coordinates of points on the graph of $y = f(x)$ by $\frac{1}{2}$.

x	$y = f(x)$	$y = g(x) = \frac{1}{2}f(x)$
−6	4	2
−2	0	0
0	2	1
2	0	0
6	4	2



The vertical distances of the transformed graph have been changed by a factor a, where $0 < |a| < 1$. The points on the graph of $y = af(x)$ are closer to the x-axis than the corresponding points of the graph of $y = f(x)$.

Did You Know?

Translations and reflections are called *rigid* transformations because the shape of the graph does not change. Stretches are called *non-rigid* because the shape of the graph can change.

Since $a = \frac{1}{2}$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$. Therefore, each point on the graph of $g(x)$ is one half as far from the x-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = \frac{1}{2}f(x)$ is a vertical stretch of the graph of $y = f(x)$ about the x-axis by a factor of $\frac{1}{2}$.

The invariant points are $(-2, 0)$ and $(2, 0)$.

What conclusion can you make about the invariant points after a vertical stretch?

For $f(x)$, the domain is $\{x \mid -6 \leq x \leq 6, x \in R\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \leq y \leq 4, y \in R\}$, or $[0, 4]$.

For $g(x)$, the domain is $\{x \mid -6 \leq x \leq 6, x \in R\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \leq y \leq 2, y \in R\}$, or $[0, 2]$.

Your Turn

Given the function $f(x) = x^2$,
- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions

a) $g(x) = 4f(x)$

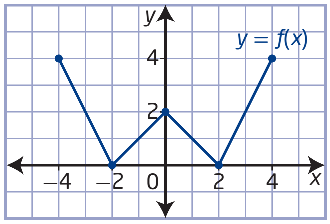
b) $g(x) = \frac{1}{3}f(x)$

Example 3

Graph $y = f(bx)$

Given the graph of $y = f(x)$,

- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions



a) $g(x) = f(2x)$

b) $g(x) = f\left(\frac{1}{2}x\right)$

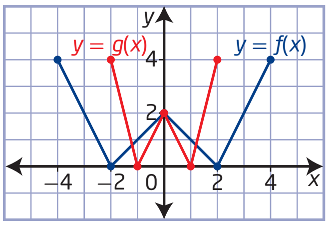
Solution

a) Use key points on the graph of $y = f(x)$ to create a table of values.

The image points on the graph of $g(x) = f(2x)$ have the same y-coordinates but different x-coordinates. Multiply the x-coordinates of points on the graph of $y = f(x)$ by $\frac{1}{2}$.

x	$y = f(x)$
−4	4
−2	0
0	2
2	0
4	4

x	$y = g(x) = f(2x)$
−2	4
−1	0
0	2
1	0
2	4



The horizontal distances of the transformed graph have been changed by a factor of $\frac{1}{b}$, where $|b| > 1$. The points on the graph of $y = f(bx)$ are closer to the y-axis than the corresponding points of the graph of $y = f(x)$.

Since $b = 2$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$. Therefore, each point on the graph of $g(x)$ is one half as far from the y-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = f(2x)$ is a horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ of the graph of $f(x)$.

The invariant point is (0, 2).

For $f(x)$, the domain is $\{x \mid -4 \le x \le 4, x \in \mathrm{R}\}$, or $[-4, 4]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

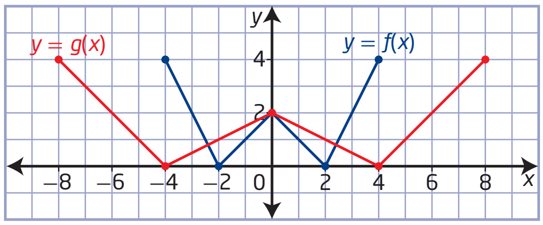
For $g(x)$, the domain is $\{x \mid -2 \le x \le 2, x \in \mathrm{R}\}$, or $[-2, 2]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

How can you determine the domain of the new function, $g(x)$, using the domain of $f(x)$ and the parameter b?

b) The image points on the graph of $g(x) = f\left(\frac{1}{2}x\right)$ have the same y-coordinates but different x-coordinates. Multiply the x-coordinates of points on the graph of $y = f(x)$ by 2.

x	$y = f(x)$
−4	4
−2	0
0	2
2	0
4	4

x	$y = g(x) = f\left(\frac{1}{2}x\right)$
−8	4
−4	0
0	2
4	0
8	4



The horizontal distances of the transformed graph have been changed by a factor $\frac{1}{b}$, where $0 < |b| < 1$. The points on the graph of $y = f(bx)$ are farther away from the y-axis than the corresponding points of the graph of $y = f(x)$.

Since $b = \frac{1}{2}$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow (2x, y)$. Therefore, each point on the graph of $g(x)$ is twice as far from the y-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = f\left(\frac{1}{2}x\right)$ is a horizontal stretch about the y-axis by a factor of 2 of the graph of $f(x)$.

The invariant point is $(0, 2)$.

How do you know which points will be invariant points after a horizontal stretch?

For $f(x)$, the domain is $\{x \mid -4 \le x \le 4, x \in \mathrm{R}\}$, or $[-4, 4]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

For $g(x)$, the domain is $\{x \mid -8 \le x \le 8, x \in \mathrm{R}\}$, or $[-8, 8]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

Your Turn

Given the function $f(x) = x^2$,
- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions

a) $g(x) = f(3x)$

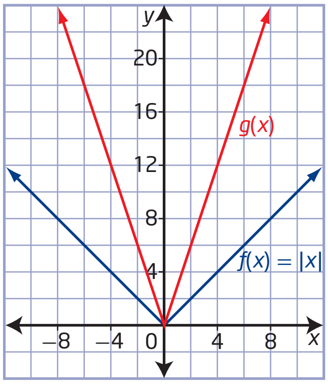
b) $g(x) = f\left(\frac{1}{4}x\right)$

Example 4

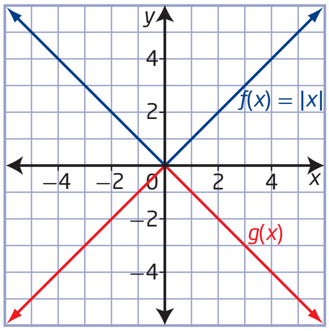
Write the Equation of a Transformed Function

The graph of the function $y = f(x)$ has been transformed by either a stretch or a reflection. Write the equation of the transformed graph, $g(x)$.

a)



b)



Solution

a) Notice that the V-shape has changed, so the graph has been transformed by a stretch.

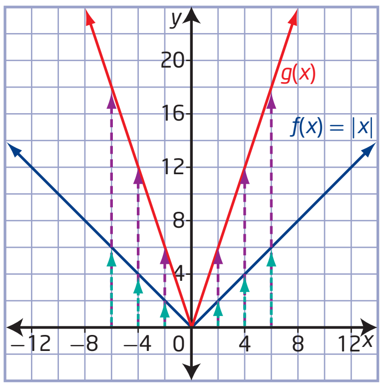
Since the original function is $f(x) = |x|$, a stretch can be described in two ways.

Why is this the case?

Choose key points on the graph of $y = f(x)$ and determine their image points on the graph of the transformed function, $g(x)$.

Case 1
Check for a pattern in the y-coordinates.

x	$y = f(x)$	$y = g(x)$
–6	6	18
–4	4	12
–2	2	6
0	0	0
2	2	6
4	4	12
6	6	18



A vertical stretch results when the vertical distances of the transformed graph are a constant multiple of those of the original graph with respect to the x-axis.

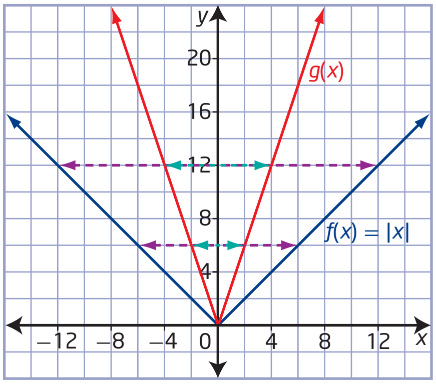
The transformation can be described by the mapping $(x, y) \rightarrow (x, 3y)$. This is of the form $y = af(x)$, indicating that there is a vertical stretch about the x-axis by a factor of 3. The equation of the transformed function is $g(x) = 3f(x)$ or $g(x) = 3|x|$.

Case 2

Check for a pattern in the x-coordinates.

x	$y = f(x)$
−12	12
−6	6
0	0
6	6
12	12

x	$y = g(x)$
−4	12
−2	6
0	0
2	6
4	12



A horizontal stretch results when the horizontal distances of the transformed graph are a constant multiple of those of the original graph with respect to the y-axis.

The transformation can be described by the mapping $(x, y) \rightarrow \left(\frac{1}{3}x, y\right)$. This is of the form $y = f(bx)$, indicating that there is a horizontal stretch about the y-axis by a factor of $\frac{1}{3}$. The equation of the transformed function is $g(x) = f(3x)$ or $g(x) = |3x|$.

b) Notice that the shape of the graph has not changed, so the graph has been transformed by a reflection.

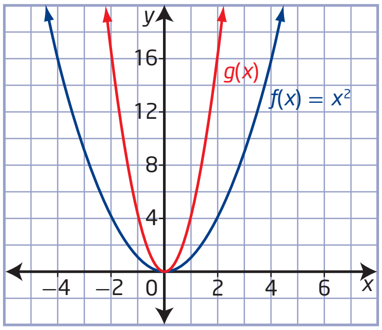
Choose key points on the graph of $f(x) = |x|$ and determine their image points on the graph of the transformed function, $g(x)$.

x	$y = f(x)$	$y = g(x)$
−4	4	−4
−2	2	−2
0	0	0
2	2	−2
4	4	−4

The transformation can be described by the mapping $(x, y) \rightarrow (x, -y)$. This is of the form $y = -f(x)$, indicating a reflection in the x-axis. The equation of the transformed function is $g(x) = -|x|$.

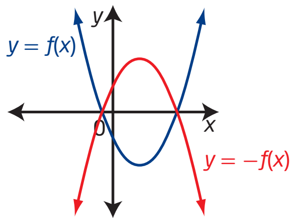
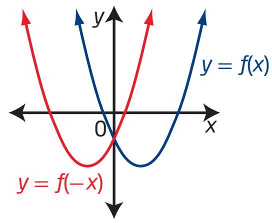
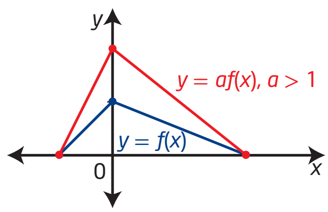
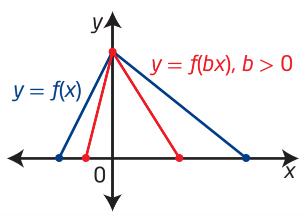
Your Turn

The graph of the function $y = f(x)$ has been transformed. Write the equation of the transformed graph, $g(x)$.



Key Ideas

- Any point on a line of reflection is an invariant point.

Function	Transformation from $y = f(x)$	Mapping	Example
$y = -f(x)$	A reflection in the x-axis	$(x, y) \rightarrow (x, -y)$	
$y = f(-x)$	A reflection in the y-axis	$(x, y) \rightarrow (-x, y)$	
$y = af(x)$	A vertical stretch about the x-axis by a factor of $\lvert a \rvert$; if $a < 0$, then the graph is also reflected in the x-axis	$(x, y) \rightarrow (x, ay)$	
$y = f(bx)$	A horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$; if $b < 0$, then the graph is also reflected in the y-axis	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	

Check Your Understanding

Practise

1. a) Copy and complete the table of values for the given functions.

x	$f(x) = 2x + 1$	$g(x) = -f(x)$	$h(x) = f(-x)$
−4			
−2			
0			
2			
4			

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) Explain how the points on the graphs of $g(x)$ and $h(x)$ relate to the transformation of the function $f(x) = 2x + 1$. List any invariant points.

d) How is each function related to the graph of $f(x) = 2x + 1$?

2. a) Copy and complete the table of values for the given functions.

x	$f(x) = x^2$	$g(x) = 3f(x)$	$h(x) = \frac{1}{3}f(x)$
−6			
−3			
0			
3			
6			

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

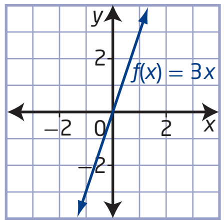
c) Explain how the points on the graphs of $g(x)$ and $h(x)$ relate to the transformation of the function $f(x) = x^2$. List any invariant points.

d) How is each function related to the graph of $f(x) = x^2$?

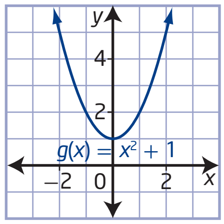
3. Consider each graph of a function.

- Copy the graph of the function and sketch its reflection in the x-axis on the same set of axes.
- State the equation of the reflected function in simplified form.
- State the domain and range of each function.

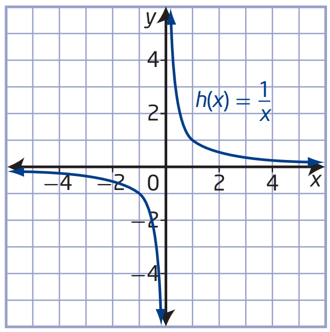
a)



b)



c)



4. Consider each function in #3.

- Copy the graph of the function and sketch its reflection in the y-axis on the same set of axes.
- State the equation of the reflected function.
- State the domain and range for each function.

5. Use words and mapping notation to describe how the graph of each function can be found from the graph of the function $y = f(x)$.

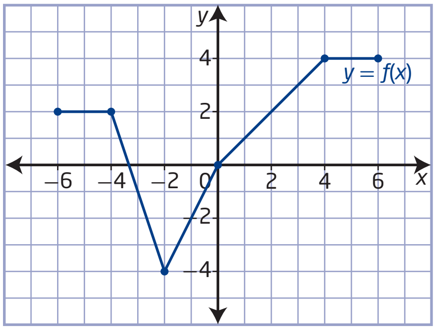
a) $y = 4f(x)$

b) $y = f(3x)$

c) $y = -f(x)$

d) $y = f(-x)$

6. The graph of the function $y = f(x)$ is vertically stretched about the x-axis by a factor of 2.

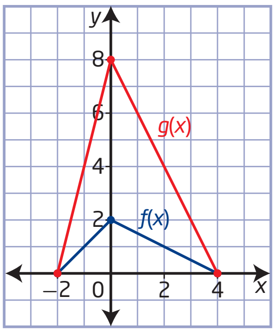


a) Determine the domain and range of the transformed function.

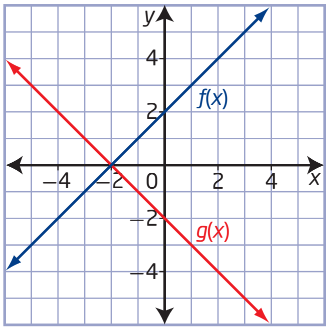
b) Explain the effect that a vertical stretch has on the domain and range of a function.

7. Describe the transformation that must be applied to the graph of $f(x)$ to obtain the graph of $g(x)$. Then, determine the equation of $g(x)$ in the form $y = af(bx)$.

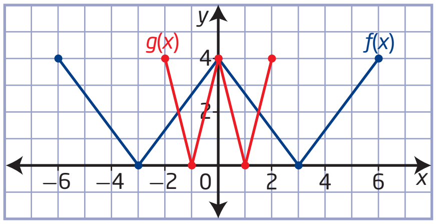
a)



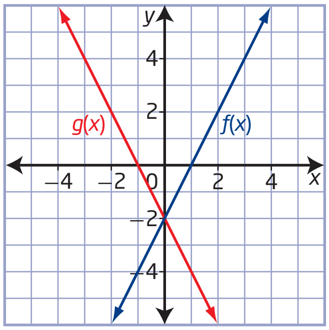
b)



c)

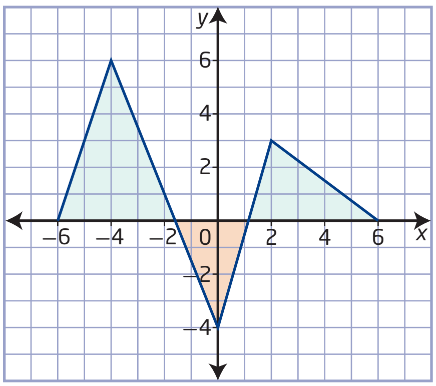


d)



Apply

8. A weaver sets up a pattern on a computer using the graph shown. A new line of merchandise calls for the design to be altered to $y = f(0.5x)$. Sketch the graph of the new design.



9. Describe what happens to the graph of a function $y = f(x)$ after the following changes are made to its equation.

a) Replace x with $4x$.

b) Replace x with $\frac{1}{4}x$.

c) Replace y with $2y$.

d) Replace y with $\frac{1}{4}y$.

e) Replace x with $-3x$.

f) Replace y with $-\frac{1}{3}y$.

10. Thomas and Sharyn discuss the order of the transformations of the graph of $y = -3|x|$ compared to the graph of $y = |x|$. Thomas states that the reflection must be applied first. Sharyn claims that the vertical stretch should be applied first.

a) Sketch the graph of $y = -3|x|$ by applying the reflection first.

b) Sketch the graph of $y = -3|x|$ by applying the stretch first.

c) Explain your conclusions. Who is correct?

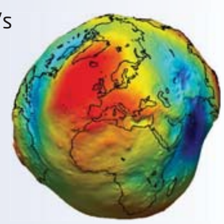
11. An object falling in a vacuum is affected only by the gravitational force. An equation that can model a free-falling object on Earth is $d = -4.9t^2$, where d is the distance travelled, in metres, and t is the time, in seconds. An object free falling on the moon can be modelled by the equation $d = -1.6t^2$.

a) Sketch the graph of each function.

b) Compare each function equation to the base function $d = t^2$.

Did You Know?

The actual strength of Earth's gravity varies depending on location.



On March 17, 2009, the European Space Agency launched a gravity-mapping satellite called Gravity and Ocean Circulation Explorer (GOCE). The data transmitted from GOCE are being used to build a model of Earth's shape and a gravity map of the planet.

12. Explain the differences that occur in transforming the graph of the function $y = f(x)$ to the graph of the function $y = f(bx)$ as compared to transforming $y = f(x)$ to $y = af(x)$.

13. The speed of a vehicle the moment the brakes are applied can be determined by its skid marks. The length, D, in feet, of the skid mark is related to the speed, S, in miles per hour, of the vehicle before braking by the function $D = \frac{1}{30fn}S^2$, where f is the drag factor of the road surface and n is the braking efficiency as a decimal. Suppose the braking efficiency is 100% or 1.

a) Sketch the graph of the length of the skid mark as a function of speed for a drag factor of 1, or $D = \frac{1}{30}S^2$.

b) The drag factor for asphalt is 0.9, for gravel is 0.8, for snow is 0.55, and for ice is 0.25. Compare the graphs of the functions for these drag factors to the graph in part a).

Did You Know?

A technical accident investigator or reconstructionist is a specially trained police officer who investigates serious traffic accidents. These officers use photography, measurements of skid patterns, and other information to determine the cause of the collision and if any charges should be laid.

Extend

14. Consider the function $f(x) = (x + 4)(x - 3)$. Without graphing, determine the zeros of the function after each transformation.

a) $y = 4f(x)$

b) $y = f(-x)$

c) $y = f\left(\frac{1}{2}x\right)$

d) $y = f(2x)$

15. The graph of a function $y = f(x)$ is contained completely in the fourth quadrant. Copy and complete each statement.

a) If $y = f(x)$ is transformed to $y = -f(x)$, it will be in quadrant ■.

b) If $y = f(x)$ is transformed to $y = f(-x)$, it will be in quadrant ■.

c) If $y = f(x)$ is transformed to $y = 4f(x)$, it will be in quadrant ■.

d) If $y = f(x)$ is transformed to $y = f\left(\frac{1}{4}x\right)$, it will be in quadrant ■.

16. Sketch the graph of $f(x) = |x|$ reflected in each line.

a) $x = 3$

b) $y = -2$

Create Connections

C1 Explain why the graph of $g(x) = f(bx)$ is a horizontal stretch about the y-axis by a factor of $\frac{1}{b}$, for $b > 0$, rather than a factor of b.

C2 Describe a transformation that results in each situation. Is there more than one possibility?

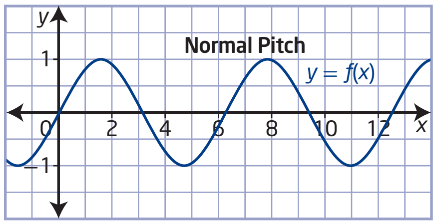
a) The x-intercepts are invariant points.

b) The y-intercepts are invariant points.

C3 A point on the function $f(x)$ is mapped onto the image point on the function $g(x)$. Copy and complete the table by describing a possible transformation of $f(x)$ to obtain $g(x)$ for each mapping.

$f(x)$	$g(x)$	Transformation
(5, 6)	(5, −6)	
(4, 8)	(−4, 8)	
(2, 3)	(2, 12)	
(4, −12)	(2, −6)	

C4 Sound is a form of energy produced and transmitted by vibrating matter that travels in waves. Pitch is the measure of how high or how low a sound is. The graph of $f(x)$ demonstrates a normal pitch. Copy the graph, then sketch the graphs of $y = f(3x)$, indicating a higher pitch, and $y = f\left(\frac{1}{2}x\right)$, for a lower pitch.



Did You Know?

The *pitch* of a sound wave is directly related to its *frequency*. A high-pitched sound has a high frequency (a mosquito). A low-pitched sound has a low frequency (a fog-horn).

A healthy human ear can hear frequencies in the range of 20 Hz to 20 000 Hz.

C5 a) Write the equation for the general term of the sequence $-10, -6, -2, 2, 6, \ldots$.

b) Write the equation for the general term of the sequence $10, 6, 2, -2, -6, \ldots$.

c) How are the graphs of the two sequences related?

1.3

Combining Transformations

Focus on...

- sketching the graph of a transformed function by applying translations, reflections, and stretches
- writing the equation of a function that has been transformed from the function $y = f(x)$

Architects, artists, and craftspeople use transformations in their work. Towers that stretch the limits of architectural technologies, paintings that create futuristic landscapes from ordinary objects, and quilt designs that transform a single shape to create a more complex image are examples of these transformations.

In this section, you will apply a combination of transformations to base functions to create more complex functions.

National-Nederlanden Building in Prague, Czech Republic

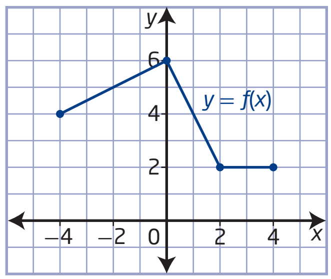
Investigate the Order of Transformations

Materials

- grid paper

New graphs can be created by vertical or horizontal translations, vertical or horizontal stretches, or reflections in an axis. When vertical and horizontal translations are applied to the graph of a function, the order in which they occur does not affect the position of the final image.

Explore whether order matters when other combinations of transformations are applied. Consider the graph of $y = f(x)$.



A: Stretches

1. a) Copy the graph of $y = f(x)$.

b) Sketch the transformed graph after the following two stretches are performed in order. Write the resulting function equation after each transformation.

- Stretch vertically about the x-axis by a factor of 2.
- Stretch horizontally about the y-axis by a factor of 3.

c) Sketch the transformed graph after the same two stretches are performed in reverse order. Write the resulting function equation after each transformation.
   - Stretch horizontally about the y-axis by a factor of 3.
   - Stretch vertically about the x-axis by a factor of 2.

2. Compare the final graphs and equations from step 1b) and c). Did reversing the order of the stretches change the final result?

B: Combining Reflections and Translations

3. a) Copy the graph of $y = f(x)$.

   b) Sketch the transformed graph after the following two transformations are performed in order. Write the resulting function equation after each transformation.
   - Reflect in the x-axis.
   - Translate vertically 4 units up.

   c) Sketch the transformed graph after the same two transformations are performed in reverse order. Write the resulting function equation after each transformation.
   - Translate vertically 4 units up.
   - Reflect in the x-axis.

4. Compare the final graphs and equations from step 3b) and c). Did reversing the order of the transformations change the final result? Explain.

5. a) Copy the graph of $y = f(x)$.

   b) Sketch the transformed graph after the following two transformations are performed in order. Write the resulting function equation after each transformation.
   - Reflect in the y-axis.
   - Translate horizontally 4 units to the right.

   c) Sketch the transformed graph after the same two transformations are performed in reverse order. Write the resulting function equation after each transformation.
   - Translate horizontally 4 units to the right.
   - Reflect in the y-axis.

6. Compare the final graphs and equations from step 5b) and c). Did reversing the order of the transformations change the final result? Explain.

Reflect and Respond

7. a) What do you think would happen if the graph of a function were transformed by a vertical stretch about the x-axis and a vertical translation? Would the order of the transformations matter?

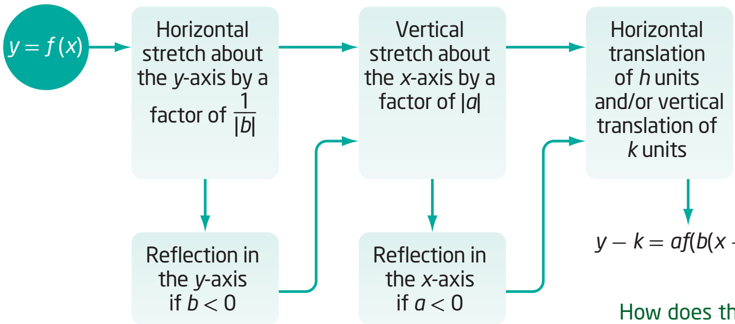
   b) Use the graph of $y = |x|$ to test your prediction.

8. In which order do you think transformations should be performed to produce the correct graph? Explain.

Link the Ideas

Multiple transformations can be applied to a function using the general transformation model $y - k = af(b(x - h))$ or $y = af(b(x - h)) + k$.

To accurately sketch the graph of a function of the form $y - k = af(b(x - h))$, the stretches and reflections (values of a and b) should occur before the translations (h-value and k-value). The diagram shows one recommended sequence for the order of transformations.



How does this compare to the usual order of operations?

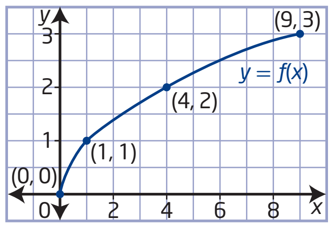
Example 1

Graph a Transformed Function

Describe the combination of transformations that must be applied to the function $y = f(x)$ to obtain the transformed function. Sketch the graph, showing each step of the transformation.

a) $y = 3f(2x)$

b) $y = f(3x + 6)$

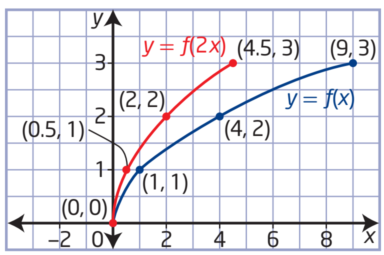


Solution

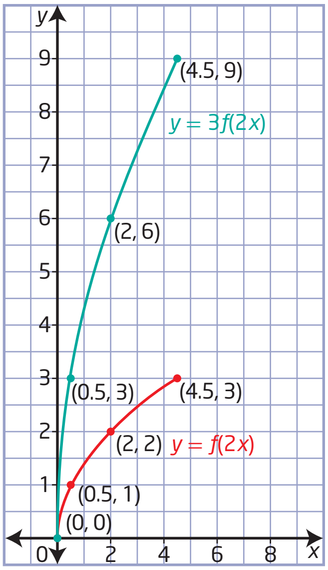
a) Compare the function to $y = af(b(x - h)) + k$. For $y = 3f(2x)$, $a = 3$, $b = 2$, $h = 0$, and $k = 0$.

The graph of $y = f(x)$ is horizontally stretched about the y-axis by a factor of $\frac{1}{2}$ and then vertically stretched about the x-axis by a factor of 3.

- Apply the horizontal stretch by a factor of $\frac{1}{2}$ to obtain the graph of $y = f(2x)$.



- Apply the vertical stretch by a factor of 3 to $y = f(2x)$ to obtain the graph of $y = 3f(2x)$.



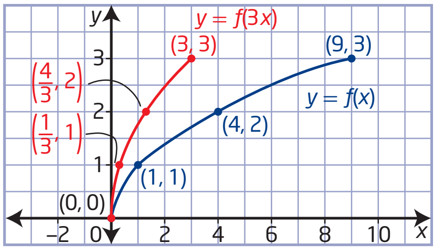
Would performing the stretches in reverse order change the final result?

b) First, rewrite $y = f(3x + 6)$ in the form $y = af(b(x - h)) + k$. This makes it easier to identify specific transformations.

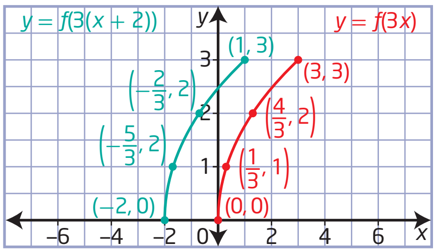
$y = f(3x + 6)$
$y = f(3(x + 2))$ Factor out the coefficient of x.
For $y = f(3(x + 2))$, $a = 1$, $b = 3$, $h = -2$, and $k = 0$.

The graph of $y = f(x)$ is horizontally stretched about the y-axis by a factor of $\frac{1}{3}$ and then horizontally translated 2 units to the left.

- Apply the horizontal stretch by a factor of $\frac{1}{3}$ to obtain the graph of $y = f(3x)$.

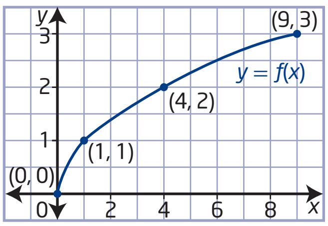


- Apply the horizontal translation of 2 units to the left to $y = f(3x)$ to obtain the graph of $y = f(3(x + 2))$.



Your Turn

Describe the combination of transformations that must be applied to the function $y = f(x)$ to obtain the transformed function. Sketch the graph, showing each step of the transformation.



a) $y = 2f(x) - 3$ **b)** $y = f\left(\frac{1}{2}x - 2\right)$

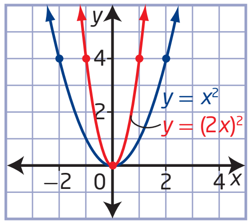
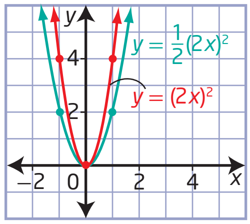
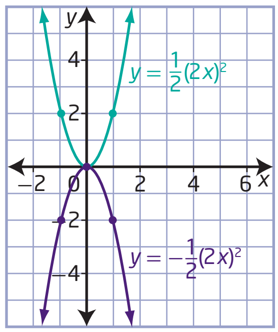
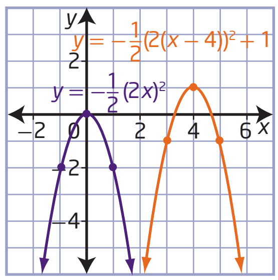
Example 2

Combination of Transformations

Show the combination of transformations that should be applied to the graph of the function $f(x) = x^2$ in order to obtain the graph of the transformed function $g(x) = -\frac{1}{2}f(2(x - 4)) + 1$. Write the corresponding equation for $g(x)$.

Solution

For $g(x) = -\frac{1}{2}f(2(x - 4)) + 1$, $a = -\frac{1}{2}$, $b = 2$, $h = 4$, and $k = 1$.

Description	Mapping	Graph
Horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ $y = (2x)^2$	$(-2, 4) \rightarrow (-1, 4)$ $(0, 0) \rightarrow (0, 0)$ $(2, 4) \rightarrow (1, 4)$ $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$	
Vertical stretch about the x-axis by a factor of $\frac{1}{2}$ $y = \frac{1}{2}(2x)^2$	$(-1, 4) \rightarrow (-1, 2)$ $(0, 0) \rightarrow (0, 0)$ $(1, 4) \rightarrow (1, 2)$ $\left(\frac{1}{2}x, y\right) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$	
Reflection in the x-axis $y = -\frac{1}{2}(2x)^2$	$(-1, 2) \rightarrow (-1, -2)$ $(0, 0) \rightarrow (0, 0)$ $(1, 2) \rightarrow (1, -2)$ $\left(\frac{1}{2}x, \frac{1}{2}y\right) \rightarrow \left(\frac{1}{2}x, -\frac{1}{2}y\right)$	
Translation of 4 units to the right and 1 unit up $y = -\frac{1}{2}(2(x - 4))^2 + 1$	$(-1, -2) \rightarrow (3, -1)$ $(0, 0) \rightarrow (4, 1)$ $(1, -2) \rightarrow (5, -1)$ $\left(\frac{1}{2}x, -\frac{1}{2}y\right) \rightarrow \left(\frac{1}{2}x + 4, -\frac{1}{2}y + 1\right)$	

The equation of the transformed function is $g(x) = -\frac{1}{2}(2(x - 4))^2 + 1$.

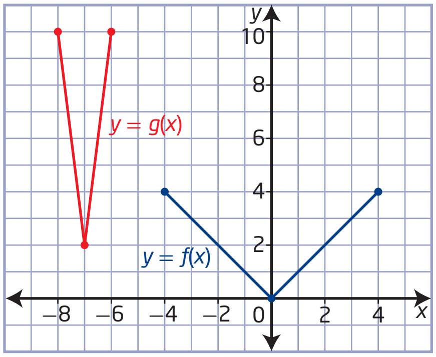
Your Turn

Describe the combination of transformations that should be applied to the function $f(x) = x^2$ in order to obtain the transformed function $g(x) = -2f\left(\frac{1}{2}(x + 8)\right) - 3$. Write the corresponding equation and sketch the graph of $g(x)$.

Example 3

Write the Equation of a Transformed Function Graph

The graph of the function $y = g(x)$ represents a transformation of the graph of $y = f(x)$. Determine the equation of $g(x)$ in the form $y = af(b(x - h)) + k$. Explain your answer.



Solution

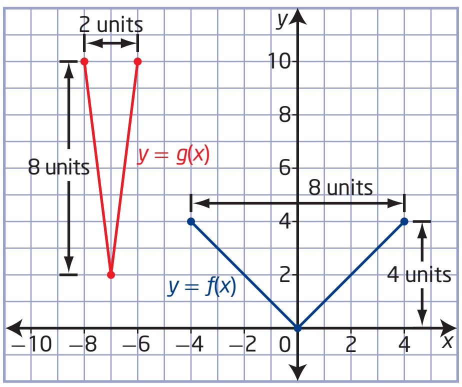
Locate key points on the graph of $f(x)$ and their image points on the graph of $g(x)$.

$(-4, 4) \rightarrow (-8, 10)$
$(0, 0) \rightarrow (-7, 2)$
$(4, 4) \rightarrow (-6, 10)$

The point (0, 0) on the graph of $f(x)$ is not affected by any stretch, either horizontal or vertical, or any reflection so it can be used to determine the vertical and horizontal translations. The graph of $g(x)$ has been translated 7 units to the left and 2 units up.

$h = -7$ and $k = 2$
There is no reflection.

Compare the distances between key points. In the vertical direction, 4 units becomes 8 units. There is a vertical stretch by a factor of 2. In the horizontal direction, 8 units becomes 2 units. There is also a horizontal stretch by a factor of $\frac{1}{4}$.



$a = 2$ and $b = 4$

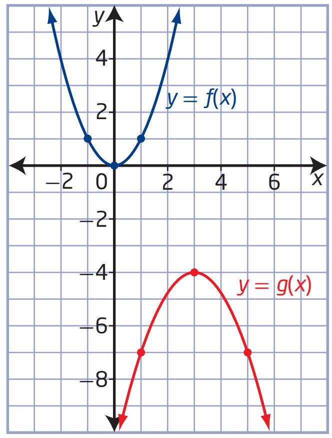
Substitute the values of a, b, h, and k into $y = af(b(x - h)) + k$.

The equation of the transformed function is $g(x) = 2f(4(x + 7)) + 2$.

How could you use the mapping $(x, y) \rightarrow \left(\frac{1}{b}x + h, ay + k\right)$ to verify this equation?

Your Turn

The graph of the function $y = g(x)$ represents a transformation of the graph of $y = f(x)$. State the equation of the transformed function. Explain your answer.



Key Ideas

- Write the function in the form $y = af(b(x - h)) + k$ to better identify the transformations.
- Stretches and reflections may be performed in any order before translations.
- The parameters a, b, h, and k in the function $y = af(b(x - h)) + k$ correspond to the following transformations:
  - a corresponds to a vertical stretch about the x-axis by a factor of $|a|$. If $a < 0$, then the function is reflected in the x-axis.
  - b corresponds to a horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$. If $b < 0$, then the function is reflected in the y-axis.
  - h corresponds to a horizontal translation.
  - k corresponds to a vertical translation.

Check Your Understanding

Practise

1. The function $y = x^2$ has been transformed to $y = af(bx)$. Determine the equation of each transformed function.

a) Its graph is stretched horizontally about the y-axis by a factor of 2 and then reflected in the x-axis.

b) Its graph is stretched horizontally about the y-axis by a factor of $\frac{1}{4}$, reflected in the y-axis, and then stretched vertically about the x-axis by a factor of $\frac{1}{4}$.

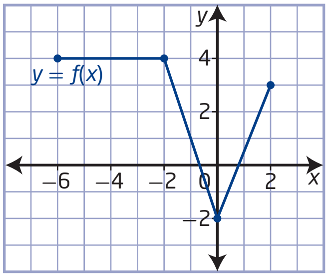
2. The function $y = f(x)$ is transformed to the function $g(x) = -3f(4x - 16) - 10$. Copy and complete the following statements by filling in the blanks.

The function $f(x)$ is transformed to the function $g(x)$ by a horizontal stretch about the ■ by a factor of ■. It is vertically stretched about the ■ by a factor of ■. It is reflected in the ■, and then translated ■ units to the right and ■ units down.

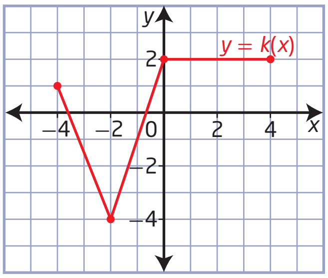
3. Copy and complete the table by describing the transformations of the given functions, compared to the function $y = f(x)$.

Function	Reflections	Vertical Stretch Factor	Horizontal Stretch Factor	Vertical Translation	Horizontal Translation
$y - 4 = f(x - 5)$					
$y + 5 = 2f(3x)$					
$y = \frac{1}{2}f\left(\frac{1}{2}(x - 4)\right)$					
$y + 2 = -3f(2(x + 2))$					

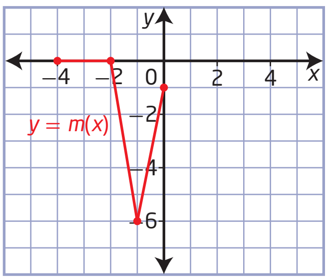
4. Using the graph of $y = f(x)$, write the equation of each transformed graph in the form $y = af(b(x - h)) + k$.



a)

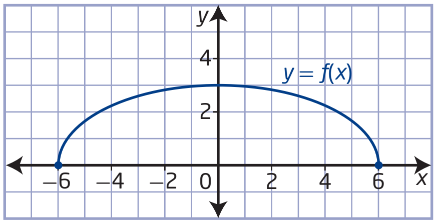


b)



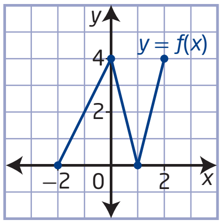
5. For each graph of $y = f(x)$, sketch the graph of the combined transformations. Show each transformation in the sequence.

a)



- vertical stretch about the x-axis by a factor of 2
- horizontal stretch about the y-axis by a factor of $\frac{1}{3}$
- translation of 5 units to the left and 3 units up

b)



- vertical stretch about the x-axis by a factor of $\frac{3}{4}$
- horizontal stretch about the y-axis by a factor of 3
- translation of 3 units to the right and 4 units down

6. The key point $(-12, 18)$ is on the graph of $y = f(x)$. What is its image point under each transformation of the graph of $f(x)$?

a) $y + 6 = f(x - 4)$

b) $y = 4f(3x)$

c) $y = -2f(x - 6) + 4$

d) $y = -2f\left(-\frac{2}{3}x - 6\right) + 4$

e) $y + 3 = -\frac{1}{3}f(2(x + 6))$

Apply

7. Describe, using an appropriate order, how to obtain the graph of each function from the graph of $y = f(x)$. Then, give the mapping for the transformation.

a) $y = 2f(x - 3) + 4$

b) $y = -f(3x) - 2$

c) $y = -\frac{1}{4}f(-(x + 2))$

d) $y - 3 = -f(4(x - 2))$

e) $y = -\frac{2}{3}f\left(-\frac{3}{4}x\right)$

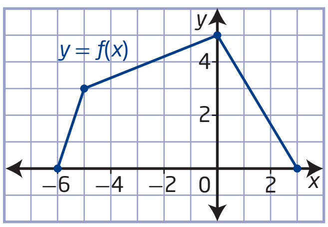
f) $3y - 6 = f(-2x + 12)$

8. Given the function $y = f(x)$, write the equation of the form $y - k = af(b(x - h))$ that would result from each combination of transformations.

a) a vertical stretch about the x-axis by a factor of 3, a reflection in the x-axis, a horizontal translation of 4 units to the left, and a vertical translation of 5 units down

b) a horizontal stretch about the y-axis by a factor of $\frac{1}{3}$, a vertical stretch about the x-axis by a factor of $\frac{3}{4}$, a reflection in both the x-axis and the y-axis, and a translation of 6 units to the right and 2 units up

9. The graph of $y = f(x)$ is given. Sketch the graph of each of the following functions.



a) $y + 2 = f(x - 3)$

b) $y = -f(-x)$

c) $y = f(3(x - 2)) + 1$

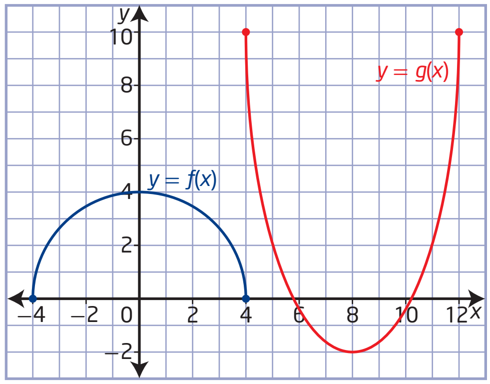
d) $y = 3f\left(\frac{1}{3}x\right)$

e) $y + 2 = -3f(x + 4)$

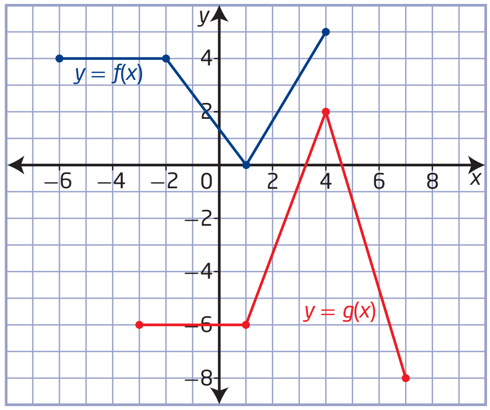
f) $y = \frac{1}{2}f\left(-\frac{1}{2}(x + 2)\right) - 1$

10. The graph of the function $y = g(x)$ represents a transformation of the graph of $y = f(x)$. Determine the equation of $g(x)$ in the form $y = af(b(x - h)) + k$.

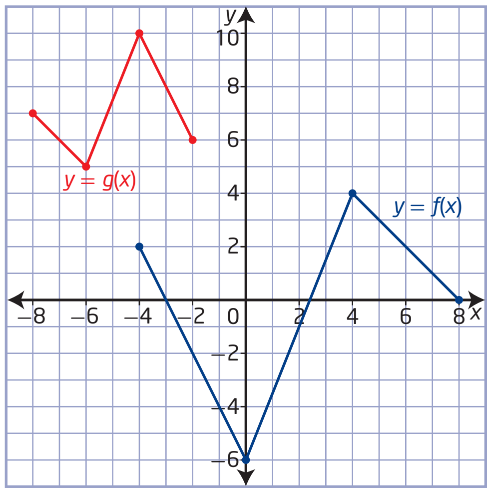
a)



b)



c)



11. Given the function $f(x)$, sketch the graph of the transformed function $g(x)$.

a) $f(x) = x^2$, $g(x) = -2f(4(x + 2)) - 2$

b) $f(x) = |x|$, $g(x) = -2f(-3x + 6) + 4$

c) $f(x) = x$, $g(x) = -\frac{1}{3}f(-2(x + 3)) - 2$

12. Alison often sketches her quilt designs on a coordinate grid. The coordinates for a section of one her designs are A(−4, 6), B(−2, −2), C(0, 0), D(1, −1), and E(3, 6). She wants to transform the original design by a horizontal stretch about the y-axis by a factor of 2, a reflection in the x-axis, and a translation of 4 units up and 3 units to the left.

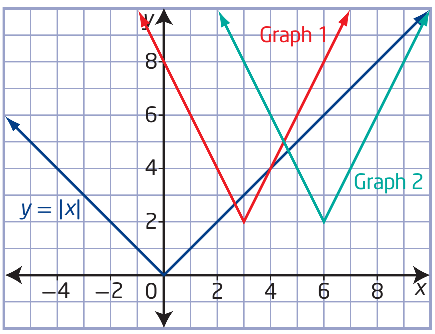
a) Determine the coordinates of the image points, A′, B′, C′, D′, and E′.

b) If the original design was defined by the function $y = f(x)$, determine the equation of the design resulting from the transformations.

13. Gil is asked to translate the graph of $y = |x|$ according to the equation $y = |2x - 6| + 2$. He decides to do the horizontal translation of 3 units to the right first, then the stretch about the y-axis by a factor of $\frac{1}{2}$, and lastly the translation of 2 units up. This gives him Graph 1. To check his work, he decides to apply the horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ first, and then the horizontal translation of 6 units to the right and the vertical translation of 2 units up. This results in Graph 2.

a) Explain why the two graphs are in different locations.

b) How could Gil have rewritten the equation so that the order in which he did the transformations for Graph 2 resulted in the same position as Graph 1?



14. Two parabolic arches are being built. The first arch can be modelled by the function $y = -x^2 + 9$, with a range of $0 \leq y \leq 9$. The second arch must span twice the distance and be translated 6 units to the left and 3 units down.

a) Sketch the graph of both arches.

b) Determine the equation of the second arch.

Extend

15. If the x-intercept of the graph of $y = f(x)$ is located at $(a, 0)$ and the y-intercept is located at $(0, b)$, determine the x-intercept and y-intercept after the following transformations of the graph of $y = f(x)$.

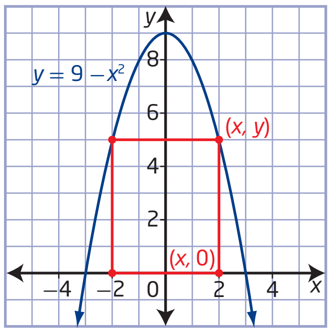
a) $y = -f(-x)$

b) $y = 2f\left(\frac{1}{2}x\right)$

c) $y + 3 = f(x - 4)$

d) $y + 3 = \frac{1}{2}f\left(\frac{1}{4}(x - 4)\right)$

16. A rectangle is inscribed between the x-axis and the parabola $y = 9 - x^2$ with one side along the x-axis, as shown.



a) Write the equation for the area of the rectangle as a function of x.

b) Suppose a horizontal stretch by a factor of 4 is applied to the parabola. What is the equation for the area of the transformed rectangle?

c) Suppose the point (2, 5) is the vertex of the rectangle on the original parabola. Use this point to verify your equations from parts a) and b).

17. The graph of the function $y = 2x^2 + x + 1$ is stretched vertically about the x-axis by a factor of 2, stretched horizontally about the y-axis by a factor of $\frac{1}{3}$, and translated 2 units to the right and 4 units down. Write the equation of the transformed function.

18. This section deals with transformations in a specific order. Give one or more examples of transformations in which the order does not matter. Show how you know that order does not matter.

Create Connections

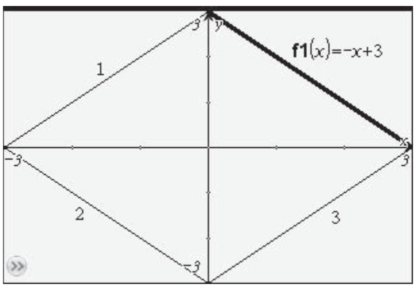
C1 **MINI LAB** Many designs, such as this Moroccan carpet, are based on transformations.

Materials

- grid paper
- graphing calculator

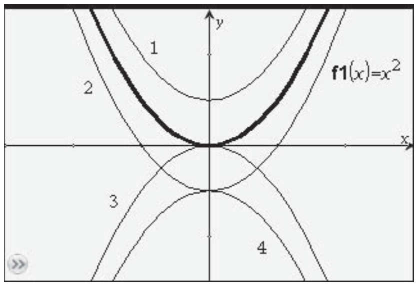
Work with a partner. Use transformations of functions to create designs on a graphing calculator.

Step 1 The graph shows the function $f(x) = -x + 3$ and transformations 1, 2, and 3.



- Recreate the diagram on a graphing calculator. Use the window settings x: [−3, 3, 1] y: [−3, 3, 1].
- Describe the transformations necessary to create the image.
- Write the equations necessary to transform the original function.

Step 2 The graph shows the function $f(x) = x^2$ and transformations 1, 2, 3, and 4.



- Recreate the diagram on a graphing calculator. Use the window settings x: [−3, 3, 1] y: [−3, 3, 1].
- Describe the transformations necessary to create the image.
- Write the equations necessary to transform the original function.

C2 Kokitusi`aki (Diana Passmore) and Siksmissi (Kathy Anderson) make and sell beaded bracelets such as the one shown representing the bear and the wolf.

If they make b bracelets per week at a cost of $f(b)$, what do the following expressions represent? How do they relate to transformations?

a) $f(b + 12)$

b) $f(b) + 12$

c) $3f(b)$

d) $f(2b)$

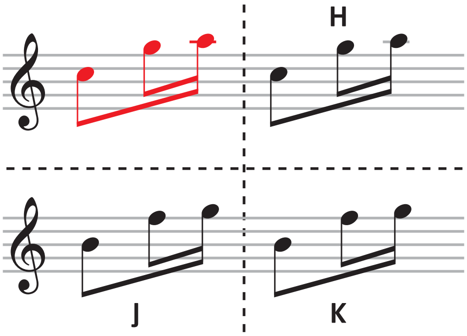
Did You Know?

Sisters Diana Passmore and Kathy Anderson are descendants of the Little Dog Clan of the Piegan (Pikuni'l') Nation of the Blackfoot Confederacy.

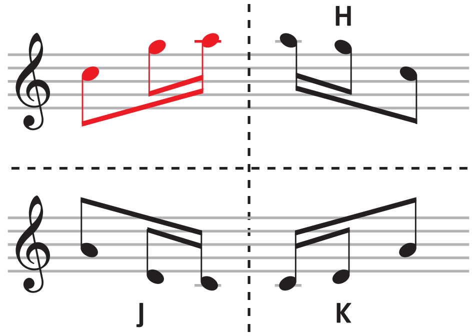
C3 Express the function $y = 2x^2 - 12x + 19$ in the form $y = a(x - h)^2 + k$. Use that form to describe how the graph of $y = x^2$ can be transformed to the graph of $y = 2x^2 - 12x + 19$.

C4 Musical notes can be repeated (translated horizontally), transposed (translated vertically), inverted (horizontal mirror), in retrograde (vertical mirror), or in retrograde inversion (180° rotation). If the musical pattern being transformed is the pattern in red, describe a possible transformation to arrive at the patterns H, J, and K.

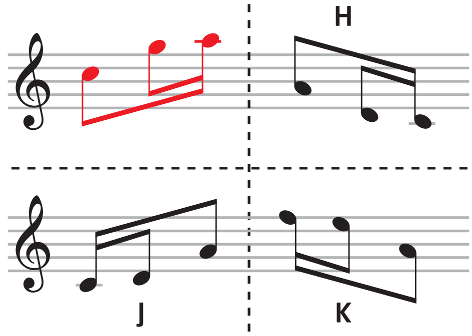
a)



b)

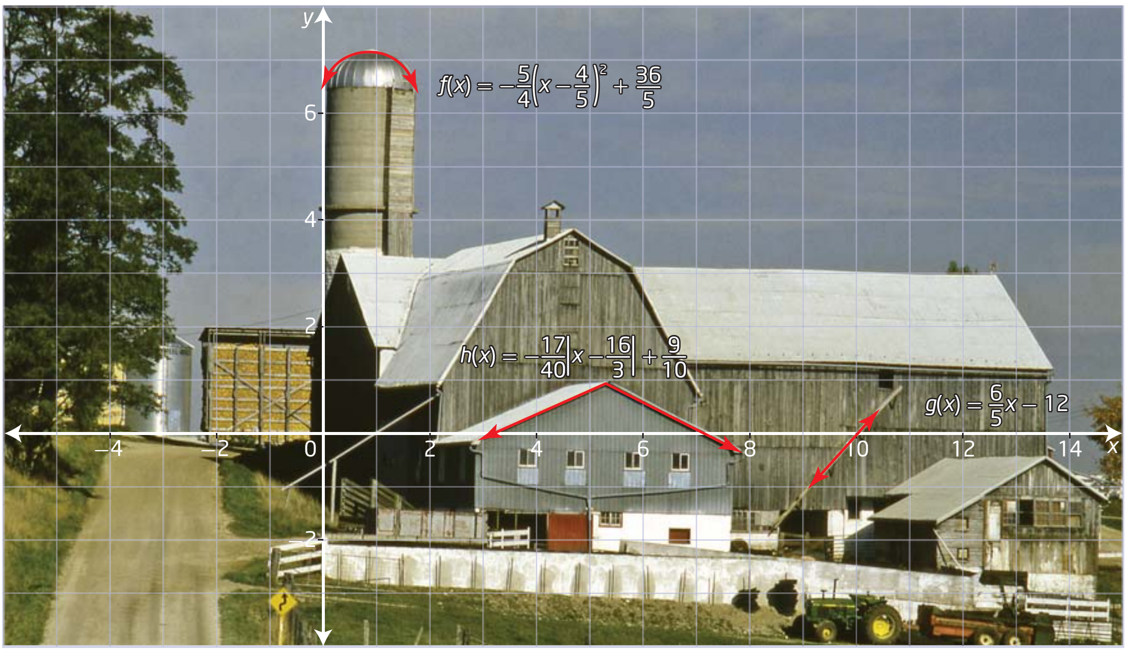


c)



Project Corner — Transformations Around You

- What type(s) of function(s) do you see in the image?
- Describe how each base function has been transformed.



1.4

Inverse of a Relation

Focus on...

- sketching the graph of the inverse of a relation
- determining if a relation and its inverse are functions
- determining the equation of an inverse

An inverse is often thought of as "undoing" or "reversing" a position, order, or effect. Whenever you undo something that you or someone else did, you are using an inverse, whether it is unwrapping a gift that someone else wrapped or closing a door that has just been opened, or deciphering a secret code.

For example, when sending a secret message, a key is used to encode the information. Then, the receiver uses the key to decode the information.

Let each letter in the alphabet be mapped to the numbers 0 to 25.

Plain Text	I	N	V	E	R	S	E
Numeric Values, x	8	13	21	4	17	18	4
Cipher, $x - 2$	6	11	19	2	15	16	2
Cipher Text	G	L	T	C	P	Q	C

Decrypting is the inverse of encrypting. What decryption function would you use on GLTCPQC? What other examples of inverses can you think of?

Investigate the Inverse of a Function

Materials

- grid paper

1. Consider the function $f(x) = \frac{1}{4}x - 5$.

a) Copy the table. In the first column, enter the ordered pairs of five points on the graph of $f(x)$. To complete the second column of the table, interchange the x-coordinates and y-coordinates of the points in the first column.

Key Points on the Graph of $f(x)$	Image Points on the Graph of $g(x)$

b) Plot the points for the function $f(x)$ and draw a line through them.

c) Plot the points for the relation $g(x)$ on the same set of axes and draw a line through them.

2. a) Draw the graph of $y = x$ on the same set of axes as in step 1.

b) How do the distances from the line $y = x$ for key points and corresponding image points compare?

c) What type of transformation occurs in order for $f(x)$ to become $g(x)$?

3. a) What observation can you make about the relationship of the coordinates of your ordered pairs between the graphs of $f(x)$ and g(x)?

b) Determine the equation of $g(x)$. How is this equation related to $f(x) = \frac{1}{4}x - 5$?

c) The relation $g(x)$ is considered to be the inverse of $f(x)$. Is the inverse of $f(x)$ a function? Explain.

Reflect and Respond

4. Describe a way to draw the graph of the **inverse of a function** using reflections.

5. Do you think all inverses of functions are functions? What factors did you base your decision on?

6. a) State a hypothesis for writing the equation of the inverse of a linear function.

b) Test your hypothesis. Write the equation of the inverse of $y = 3x + 2$. Check by graphing.

7. Determine the equation of the inverse of $y = mx + b$, $m \neq 0$.

a) Make a conjecture about the relationship between the slope of the inverse function and the slope of the original function.

b) Make a conjecture about the relationship between the x-intercepts and the y-intercept of the original function and those of the inverse function.

8. Describe how you could determine if two relations are inverses of each other.

inverse of a function

- if f is a function with domain A and range B, the inverse function, if it exists, is denoted by f^{-1} and has domain B and range A
- f^{-1} maps y to x if and only if f maps x to y

Link the Ideas

The inverse of a relation is found by interchanging the x-coordinates and y-coordinates of the ordered pairs of the relation. In other words, for every ordered pair (x, y) of a relation, there is an ordered pair (y, x) on the inverse of the relation. This means that the graphs of a relation and its inverse are reflections of each other in the line $y = x$.

$(x, y) \rightarrow (y, x)$

Did You Know?

The –1 in $f^{-1}(x)$ does not represent an exponent; that is $f^{-1}(x) \neq \frac{1}{f(x)}$.

The inverse of a function $y = f(x)$ may be written in the form $x = f(y)$. The inverse of a function is not necessarily a function. When the inverse of f is itself a function, it is denoted as f^{-1} and read as "f inverse." When the inverse of a function is not a function, it may be possible to restrict the domain to obtain an inverse function for a portion of the original function.

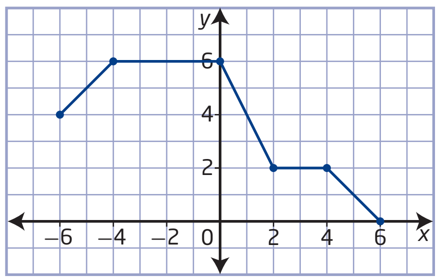
The inverse of a function reverses the processes represented by that function. Functions $f(x)$ and $g(x)$ are inverses of each other if the operations of $f(x)$ reverse all the operations of $g(x)$ in the opposite order and the operations of $g(x)$ reverse all the operations of $f(x)$ in the opposite order.

For example, $f(x) = 2x + 1$ multiplies the input value by 2 and then adds 1. The inverse function subtracts 1 from the input value and then divides by 2. The inverse function is $f^{-1}(x) = \frac{x - 1}{2}$.

Example 1

Graph an Inverse

Consider the graph of the relation shown.



a) Sketch the graph of the inverse relation.

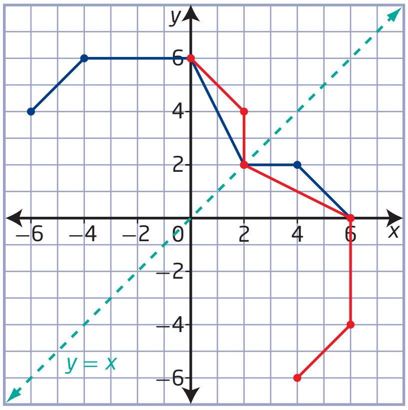
b) State the domain and range of the relation and its inverse.

c) Determine whether the relation and its inverse are functions.

Solution

a) To graph the inverse relation, interchange the x-coordinates and y-coordinates of key points on the graph of the relation.

Points on the Relation	Points on the Inverse Relation
(–6, 4)	(4, –6)
(–4, 6)	(6, –4)
(0, 6)	(6, 0)
(2, 2)	(2, 2)
(4, 2)	(2, 4)
(6, 0)	(0, 6)



The graphs are reflections of each other in the line $y = x$. The points on the graph of the relation are related to the points on the graph of the inverse relation by the mapping $(x, y) \rightarrow (y, x)$.

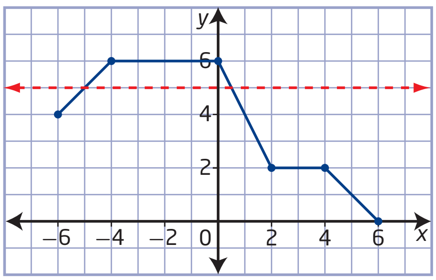
What points are invariant after a reflection in the line $y = x$?

b) The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

	Domain	Range
Relation	$\{x \mid -6 \leq x \leq 6, x \in R\}$	$\{y \mid 0 \leq y \leq 6, y \in R\}$
Inverse Relation	$\{x \mid 0 \leq x \leq 6, x \in R\}$	$\{y \mid -6 \leq y \leq 6, y \in R\}$

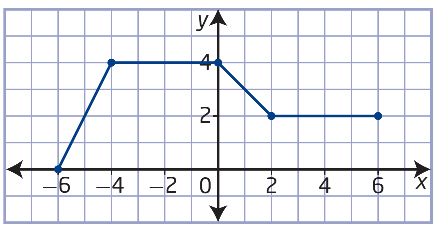
c) The relation is a function of x because there is only one value of y in the range for each value of x in the domain. In other words, the graph of the relation passes the vertical line test.

The inverse relation is not a function of x because it fails the vertical line test. There is more than one value of y in the range for at least one value of x in the domain. You can confirm this by using the **horizontal line test** on the graph of the original relation.



Your Turn

Consider the graph of the relation shown.



a) Determine whether the relation and its inverse are functions.
b) Sketch the graph of the inverse relation.
c) State the domain, range, and intercepts for the relation and the inverse relation.
d) State any invariant points.

Did You Know?

A *one-to-one function* is a function for which every element in the range corresponds to exactly one element in the domain. The graph of a relation is a function if it passes the vertical line test. If, in addition, it passes the horizontal line test, it is a one-to-one function.

horizontal line test

- a test used to determine if the graph of an inverse relation will be a function
- if it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function

Example 2

Restrict the Domain

Consider the function $f(x) = x^2 - 2$.

a) Graph the function $f(x)$. Is the inverse of $f(x)$ a function?

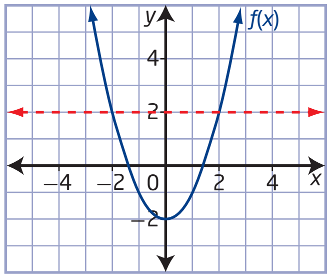
b) Graph the inverse of $f(x)$ on the same set of coordinate axes.

c) Describe how the domain of $f(x)$ could be restricted so that the inverse of $f(x)$ is a function.

Solution

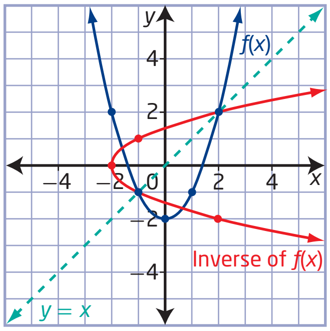
a) The graph of $f(x) = x^2 - 2$ is a translation of the graph of $y = x^2$ by 2 units down.

Since the graph of the function fails the horizontal line test, the inverse of $f(x)$ is not a function.



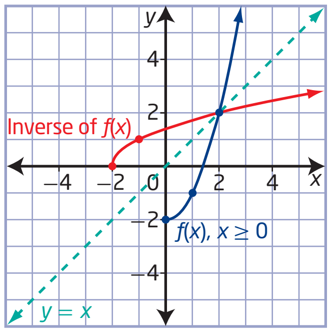
b) Use key points on the graph of $f(x)$ to help you sketch the graph of the inverse of $f(x)$.

Notice that the graph of the inverse of $f(x)$ does not pass the vertical line test. The inverse of $f(x)$ is not a function.



c) The inverse of $f(x)$ is a function if the graph of $f(x)$ passes the horizontal line test.

One possibility is to restrict the domain of $f(x)$ so that the resulting graph is only one half of the parabola. Since the equation of the axis of symmetry is $x = 0$, restrict the domain to $\{x \mid x \geq 0, x \in \mathrm{R}\}$.



How else could the domain of $f(x)$ be restricted?

Your Turn

Consider the function $f(x) = (x + 2)^2$.

a) Graph the function $f(x)$. Is the inverse of $f(x)$ a function?

b) Graph the inverse of $f(x)$ on the same set of coordinate axes.

c) Describe how the domain of $f(x)$ could be restricted so that the inverse of $f(x)$ is a function.

Example 3

Determine the Equation of the Inverse

Algebraically determine the equation of the inverse of each function. Verify graphically that the relations are inverses of each other.

a) $f(x) = 3x + 6$

b) $f(x) = x^2 - 4$

Solution

a) Let $y = f(x)$. To find the equation of the inverse, $x = f(y)$, interchange x and y, and then solve for y.

$f(x) = 3x + 6$

$y = 3x + 6$ — Replace $f(x)$ with y.

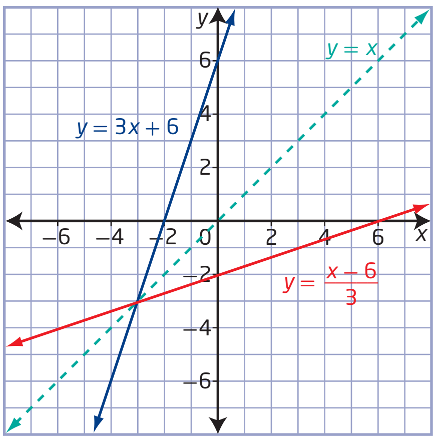
$x = 3y + 6$ — Interchange x and y to determine the inverse.

$x - 6 = 3y$ — Solve for y.

$\frac{x-6}{3} = y$

$f^{-1}(x) = \frac{x-6}{3}$ — Replace y with $f^{-1}(x)$, since the inverse of a linear function is also a function.

Graph $y = 3x + 6$ and $y = \frac{x-6}{3}$ on the same set of coordinate axes.



Notice that the x-intercept and y-intercept of $y = 3x + 6$ become the y-intercept and x-intercept, respectively, of $y = \frac{x-6}{3}$. Since the functions are reflections of each other in the line $y = x$, the functions are inverses of each other.

b) The same method applies to quadratic functions.

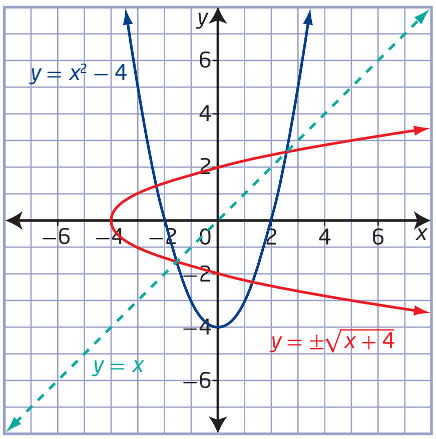
$$f(x) = x^2 - 4$$
$$y = x^2 - 4$$ Replace $f(x)$ with y.
$$x = y^2 - 4$$ Interchange x and y to determine the inverse.
$$x + 4 = y^2$$ Solve for y.
$$\pm\sqrt{x + 4} = y$$
$$y = \pm\sqrt{x + 4}$$

Why is this y not replaced with $f^{-1}(x)$? What could be done so that $f^{-1}(x)$ could be used?

Graph $y = x^2 - 4$ and $y = \pm\sqrt{x + 4}$ on the same set of coordinate axes.

x	$y = x^2 - 4$
−3	5
−2	0
−1	−3
0	−4
1	−3
2	0
3	5

x	$y = \pm\sqrt{x + 4}$
5	±3
0	±2
−3	±1
−4	0



How could you use the tables of values to verify that the relations are inverses of each other?

Notice that the x-intercepts and y-intercept of $y = x^2 - 4$ become the y-intercepts and x-intercept, respectively, of $y = \pm\sqrt{x + 4}$. The relations are reflections of each other in the line $y = x$. While the relations are inverses of each other, $y = \pm\sqrt{x + 4}$ is not a function.

Your Turn

Write the equation for the inverse of the function $f(x) = \frac{x + 8}{3}$. Verify your answer graphically.

Key Ideas

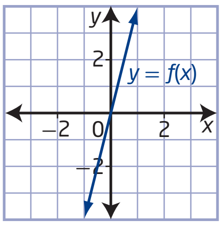
- You can find the inverse of a relation by interchanging the x-coordinates and y-coordinates of the graph.
- The graph of the inverse of a relation is the graph of the relation reflected in the line $y = x$.
- The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.
- Use the horizontal line test to determine if an inverse will be a function.
- You can create an inverse that is a function over a specified interval by restricting the domain of a function.
- When the inverse of a function $f(x)$ is itself a function, it is denoted by $f^{-1}(x)$.
- You can verify graphically whether two functions are inverses of each other.

Check Your Understanding

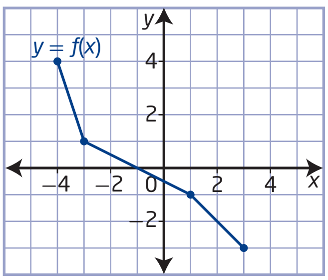
Practise

1. Copy each graph. Use the reflection line $y = x$ to sketch the graph of $x = f(y)$ on the same set of axes.

a)

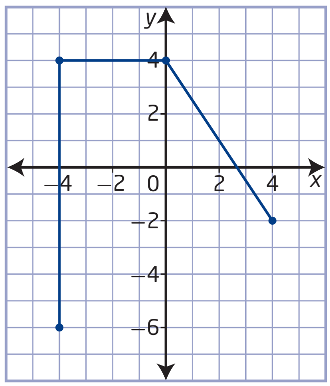


b)

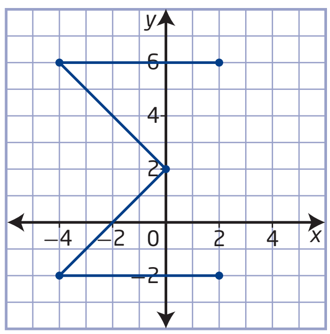


2. Copy the graph of each relation and sketch the graph of its inverse relation.

a)

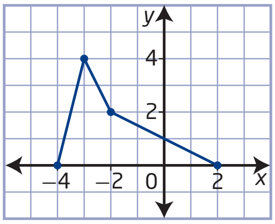


b)

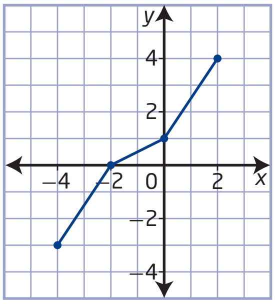


3. State whether or not the graph of the relation is a function. Then, use the horizontal line test to determine whether the inverse relation will be a function.

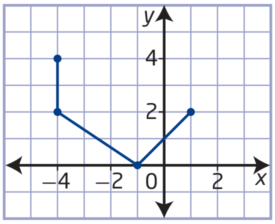
a)



b)

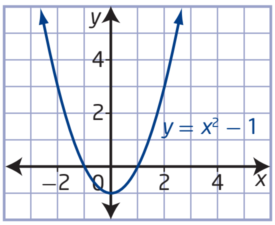


c)

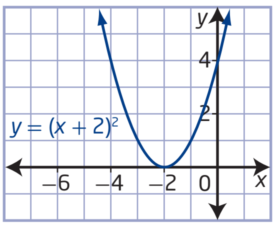


4. For each graph, identify a restricted domain for which the function has an inverse that is also a function.

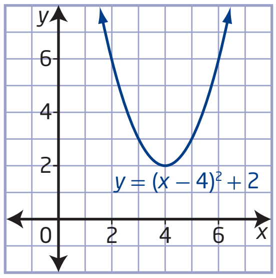
a)



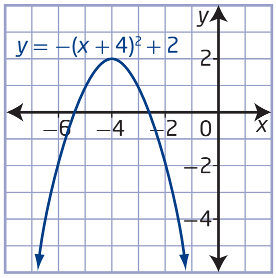
b)



c)



d)



5. Algebraically determine the equation of the inverse of each function.

a) $f(x) = 7x$

b) $f(x) = -3x + 4$

c) $f(x) = \dfrac{x + 4}{3}$

d) $f(x) = \dfrac{x}{3} - 5$

e) $f(x) = 5 - 2x$

f) $f(x) = \dfrac{1}{2}(x + 6)$

6. Match the function with its inverse.

Function

a) $y = 2x + 5$

b) $y = \dfrac{1}{2}x - 4$

c) $y = 6 - 3x$

d) $y = x^2 - 12, x \geq 0$

e) $y = \dfrac{1}{2}(x + 1)^2, x \leq -1$

Inverse

A $y = \sqrt{x + 12}$

B $y = \dfrac{6 - x}{3}$

C $y = 2x + 8$

D $y = -\sqrt{2x} - 1$

E $y = \dfrac{x - 5}{2}$

Apply

7. For each table, plot the ordered pairs (x, y) and the ordered pairs (y, x). State the domain of the function and its inverse.

a)

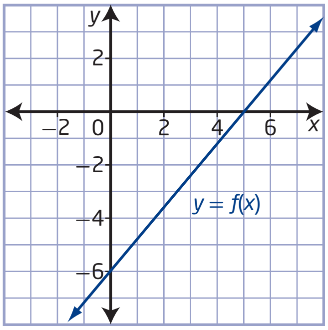
x	y
−2	−2
−1	1
0	4
1	7
2	10

b)

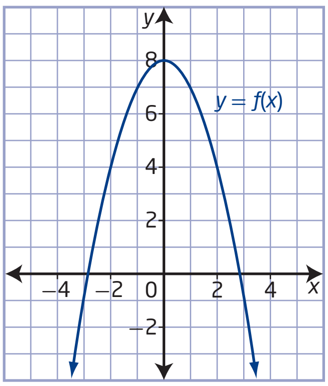
x	y
−6	2
−4	4
−1	5
2	5
5	3

8. Copy each graph of $y = f(x)$ and then sketch the graph of its inverse. Determine if the inverse is a function. Give a reason for your answer.

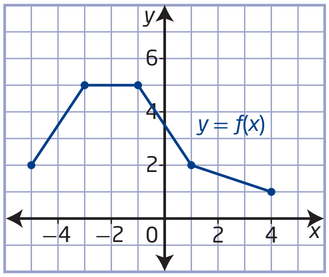
a)



b)



c)



9. For each of the following functions,
- determine the equation for the inverse, $f^{-1}(x)$
- graph $f(x)$ and $f^{-1}(x)$
- determine the domain and range of $f(x)$ and $f^{-1}(x)$

a) $f(x) = 3x + 2$

b) $f(x) = 4 - 2x$

c) $f(x) = \frac{1}{2}x - 6$

d) $f(x) = x^2 + 2,\ x \leq 0$

e) $f(x) = 2 - x^2,\ x \geq 0$

10. For each function $f(x)$,

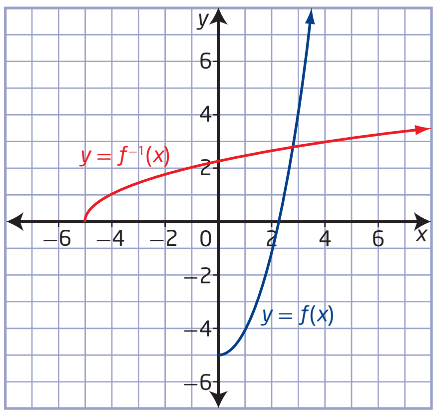
i) determine the equation of the inverse of $f(x)$ by first rewriting the function in the form $y = a(x - h)^2 + k$

ii) graph $f(x)$ and the inverse of $f(x)$

a) $f(x) = x^2 + 8x + 12$

b) $f(x) = x^2 - 4x + 2$

11. Jocelyn and Gerry determine that the inverse of the function $f(x) = x^2 - 5,\ x \geq 0$, is $f^{-1}(x) = \sqrt{x + 5}$. Does the graph verify that these functions are inverses of each other? Explain why.



12. For each of the following functions,
- determine the equation of the inverse
- graph $f(x)$ and the inverse of $f(x)$
- restrict the domain of $f(x)$ so that the inverse of $f(x)$ is a function
- with the domain of $f(x)$ restricted, sketch the graphs of $f(x)$ and $f^{-1}(x)$

a) $f(x) = x^2 + 3$

b) $f(x) = \frac{1}{2}x^2$

c) $f(x) = -2x^2$

d) $f(x) = (x + 1)^2$

e) $f(x) = -(x - 3)^2$

f) $f(x) = (x - 1)^2 - 2$

13. Determine graphically whether the functions in each pair are inverses of each other.

a) $f(x) = x - 4$ and $g(x) = x + 4$

b) $f(x) = 3x + 5$ and $g(x) = \frac{x - 5}{3}$

c) $f(x) = x - 7$ and $g(x) = 7 - x$

d) $f(x) = \frac{x - 2}{2}$ and $g(x) = 2x + 2$

e) $f(x) = \frac{8}{x - 7}$ and $g(x) = \frac{8}{x + 7}$

14. For each function, state two ways to restrict the domain so that the inverse is a function.

a) $f(x) = x^2 + 4$

b) $f(x) = 2 - x^2$

c) $f(x) = (x - 3)^2$

d) $f(x) = (x + 2)^2 - 4$

15. Given the function $f(x) = 4x - 2$, determine each of the following.

a) $f^{-1}(4)$

b) $f^{-1}(-2)$

c) $f^{-1}(8)$

d) $f^{-1}(0)$

16. The function for converting the temperature from degrees Fahrenheit, x, to degrees Celsius, y, is $y = \frac{5}{9}(x - 32)$.

a) Determine the equivalent temperature in degrees Celsius for 90 °F.

b) Determine the inverse of this function. What does it represent? What do the variables represent?

c) Determine the equivalent temperature in degrees Fahrenheit for 32 °C.

d) Graph both functions. What does the invariant point represent in this situation?

17. A forensic specialist can estimate the height of a person from the lengths of their bones. One function relates the length, x, of the femur to the height, y, of the person, both in centimetres.

For a male: $y = 2.32x + 65.53$
For a female: $y = 2.47x + 54.13$

a) Determine the height of a male and of a female with a femur length of 45.47 cm.

b) Use inverse functions to determine the femur length of

i) a male whose height is 187.9 cm

ii) a female whose height is 175.26 cm

18. In Canada, ring sizes are specified using a numerical scale. The numerical ring size, y, is approximately related to finger circumference, x, in millimetres, by $y = \frac{x - 36.5}{2.55}$.

a) What whole-number ring size corresponds to a finger circumference of 49.3 mm?

b) Determine an equation for the inverse of the function. What do the variables represent?

c) What finger circumferences correspond to ring sizes of 6, 7, and 9?

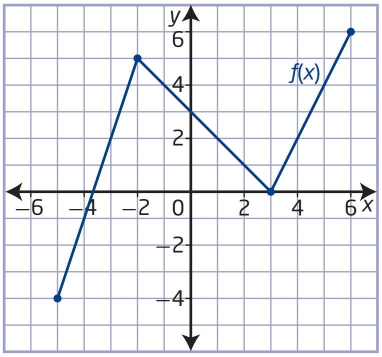
Extend

19. When a function is constantly increasing or decreasing, its inverse is a function. For each graph of $f(x)$,

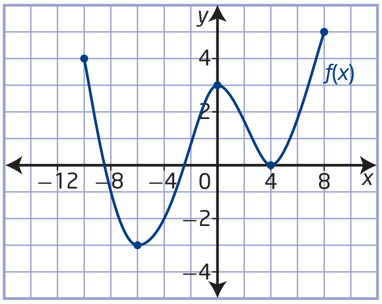
i) choose an interval over which the function is increasing and sketch the inverse of the function when it is restricted to that domain

ii) choose an interval over which the function is decreasing and sketch the inverse of the function when it is restricted to that domain

a)



b)



20. Suppose a function $f(x)$ has an inverse function, $f^{-1}(x)$.

a) Determine $f^{-1}(5)$ if $f(17) = 5$.

b) Determine $f(-2)$ if $f^{-1}(\sqrt{3}) = -2$.

c) Determine the value of a if $f^{-1}(a) = 1$ and $f(x) = 2x^2 + 5x + 3$, $x \geq -1.25$.

21. If the point (10, 8) is on the graph of the function $y = f(x)$, what point must be on the graph of each of the following?

a) $y = f^{-1}(x + 2)$

b) $y = 2f^{-1}(x) + 3$

c) $y = -f^{-1}(-x) + 1$

Create Connections

C1 Describe the inverse sequence of operations for each of the following.

a) $f(x) = 6x + 12$

b) $f(x) = (x + 3)^2 - 1$

C2 **a)** Sketch the graphs of the function $f(x) = -x + 3$ and its inverse, $f^{-1}(x)$.

b) Explain why $f(x) = f^{-1}(x)$.

c) If a function and its inverse are the same, how are they related to the line $y = x$?

C3 Two students are arguing about whether or not a given relation and its inverse are functions. Explain how the students could verify who is correct.

C4 **MINI LAB** Two functions, $f(x) = \dfrac{x + 5}{3}$ and $g(x) = 3x - 5$, are inverses of each other.

Step 1 Evaluate output values for $f(x)$ for $x = 1$, $x = 4$, $x = -8$, and $x = a$. Use the results as input values for $g(x)$. What do you notice about the output values for $g(x)$? Explain why this happens. State a hypothesis that could be used to verify whether or not two functions are inverses of each other.

Step 2 Reverse the order in which you used the functions. Start with using the input values for $g(x)$, and then use the outputs in $f(x)$. What conclusion can you make about inverse functions?

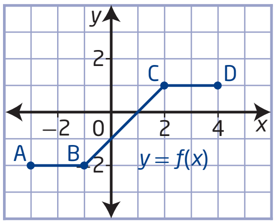
Step 3 Test your conclusions and hypothesis by selecting two functions of your own.

Step 4 Explain how your results relate to the statement "if $f(a) = b$ and $f^{-1}(b) = a$, then the two functions are inverses of each other." Note that this must also be true when the function roles are switched.

Chapter 1 Review

1.1 Horizontal and Vertical Translations, pages 6–15

1. Given the graph of the function $y = f(x)$, sketch the graph of each transformed function.

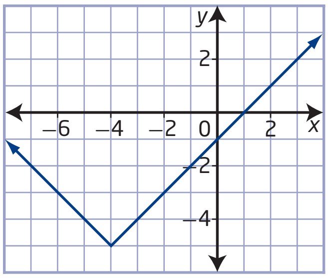


a) $y - 3 = f(x)$

b) $h(x) = f(x + 1)$

c) $y + 1 = f(x - 2)$

2. Describe how to translate the graph of $y = |x|$ to obtain the graph of the function shown. Write the equation of the transformed function in the form $y - k = |x - h|$.



3. The range of the function $y = f(x)$ is $\{y \mid -2 \leq y \leq 5, y \in \mathrm{R}\}$. What is the range of the function $y = f(x - 2) + 4$?

4. James wants to explain vertical and horizontal translations by describing the effect of the translation on the coordinates of a point on the graph of a function. He says, "If the point (a, b) is on the graph of $y = f(x)$, then the point $(a - 5, b + 4)$ is the image point on the graph of $y + 4 = f(x - 5)$." Do you agree with James? Explain your reasoning.

1.2 Reflections and Stretches, pages 16–31

5. Name the line of reflection when the graph of $y = f(x)$ is transformed as indicated. Then, state the coordinates of the image point of (3, 5) on the graph of each reflection.

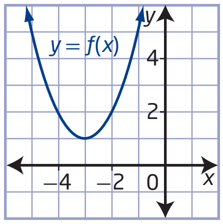
a) $y = -f(x)$

b) $y = f(-x)$

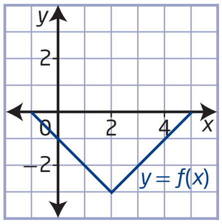
6. Copy each graph of $y = f(x)$. Then,

- sketch the reflection indicated
- state the domain and range of the transformed function
- list any invariant points

a) $y = f(-x)$



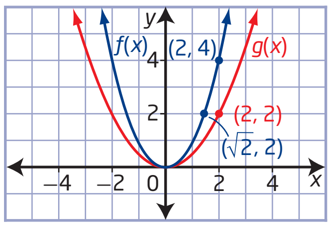
b) $y = -f(x)$



7. a) Sketch the graphs of the functions $f(x) = x^2$, $g(x) = f(2x)$, and $h(x) = f\left(\frac{1}{2}x\right)$ on the same set of coordinate axes.

b) Describe how the value of the coefficient of x for $g(x)$ and $h(x)$ affects the graph of the function $f(x) = x^2$.

8. Consider the graphs of the functions $f(x)$ and $g(x)$.

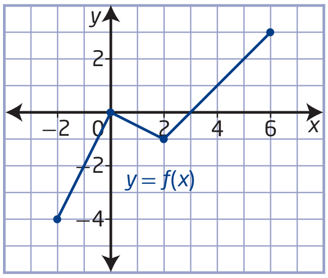


a) Is the graph of $g(x)$ a horizontal or a vertical stretch of the graph of $f(x)$? Explain your reasoning.

b) Write the equation that models the graph of $g(x)$ as a transformation of the graph of $f(x)$.

1.3 Combining Transformations, pages 32–43

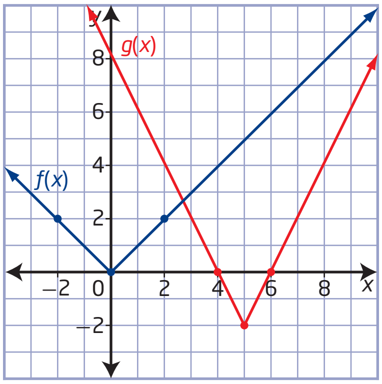
9. Given the graph of $y = f(x)$, sketch the graph of each transformed function.



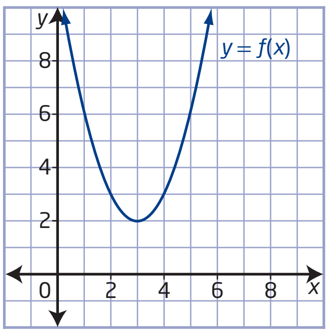
a) $y = 2f\left(\frac{1}{2}x\right)$ **b)** $y = \frac{1}{2}f(3x)$

10. Explain how the transformations described by $y = f(4(x + 1))$ and $y = f(4x + 1)$ are similar and how they are different.

11. Write the equation for the graph of $g(x)$ as a transformation of the equation for the graph of $f(x)$.



12. Consider the graph of $y = f(x)$. Sketch the graph of each transformation.



a) $y = \frac{1}{2}f(-(x + 2))$

b) $y - 2 = -f(2(x - 3))$

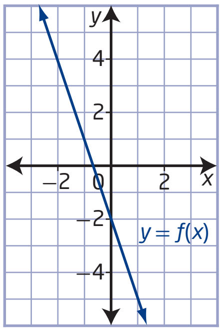
c) $y - 1 = 3f(2x + 4)$

1.4 Inverse of a Relation, pages 44–55

13. a) Copy the graph of $y = f(x)$ and sketch the graph of $x = f(y)$.

b) Name the line of reflection and list any invariant points.

c) State the domain and range of the two functions.

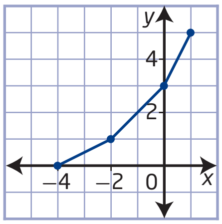


14. Copy and complete the table.

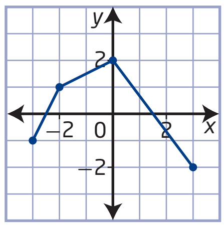
$y = f(x)$		$y = f^{-1}(x)$	
x	**y**	**x**	**y**
−3	7		
		4	2
10		−12	

15. Sketch the graph of the inverse relation for each graph. State whether the relation and its inverse are functions.

a)



b)



16. Algebraically determine the equation of the inverse of the function $y = (x - 3)^2 + 1$. Determine a restriction on the domain of the function in order for its inverse to be a function. Show your thinking.

17. Graphically determine if the functions are inverses of each other.

a) $f(x) = -6x + 5$ and $g(x) = \dfrac{x + 5}{6}$

b) $f(x) = \dfrac{x - 3}{8}$ and $g(x) = 8x + 3$

Chapter 1 Practice Test

Multiple Choice

For #1 to #7, choose the best answer.

1. What is the effect on the graph of the function $y = x^2$ when the equation is changed to $y = (x + 1)^2$?

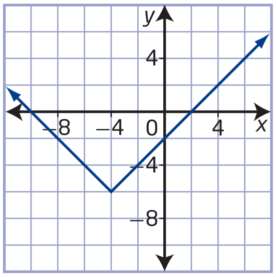
A The graph is stretched vertically.

B The graph is stretched horizontally.

C The graph is the same shape but translated up.

D The graph is the same shape but translated to the left.

2. The graph shows a transformation of the graph of $y = |x|$. Which equation models the graph?



A $y + 4 = |x - 6|$

B $y - 6 = |x - 4|$

C $y - 4 = |x + 6|$

D $y + 6 = |x + 4|$

3. If (a, b) is a point on the graph of $y = f(x)$, which of the following points is on the graph of $y = f(x + 2)$?

A $(a + 2, b)$

B $(a - 2, b)$

C $(a, b + 2)$

D $(a, b - 2)$

4. Which equation represents the image of $y = x^2 + 2$ after a reflection in the y-axis?

A $y = -x^2 - 2$

B $y = x^2 + 2$

C $y = -x^2 + 2$

D $y = x^2 - 2$

5. The effect on the graph of $y = f(x)$ if it is transformed to $y = \frac{1}{4}f(3x)$ is

A a vertical stretch by a factor of $\frac{1}{4}$ and a horizontal stretch by a factor of 3

B a vertical stretch by a factor of $\frac{1}{4}$ and a horizontal stretch by a factor of $\frac{1}{3}$

C a vertical stretch by a factor of 4 and a horizontal stretch by a factor of 3

D a vertical stretch by a factor of 4 and a horizontal stretch by a factor of $\frac{1}{3}$

6. Which of the following transformations of $f(x)$ produces a graph that has the same y-intercept as $f(x)$? Assume that $(0, 0)$ is not a point on $f(x)$.

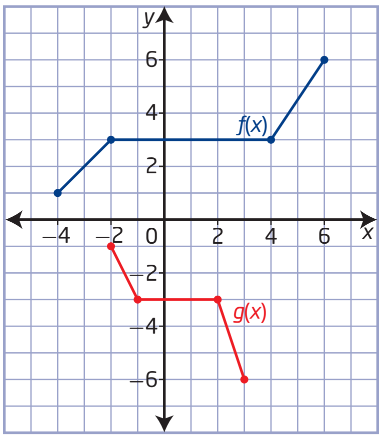
A $-9f(x)$

B $f(x) - 9$

C $f(-9x)$

D $f(x - 9)$

7. Given the graphs of $y = f(x)$ and $y = g(x)$, what is the equation for $g(x)$ in terms of $f(x)$?



A $g(x) = f\left(-\frac{1}{2}x\right)$

B $g(x) = f(-2x)$

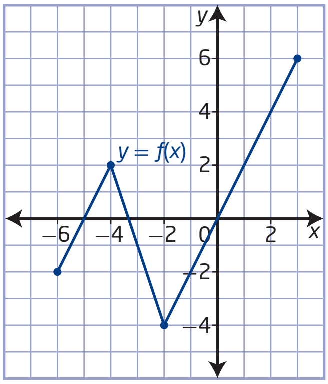
C $g(x) = -f(2x)$

D $g(x) = -f\left(\frac{1}{2}x\right)$

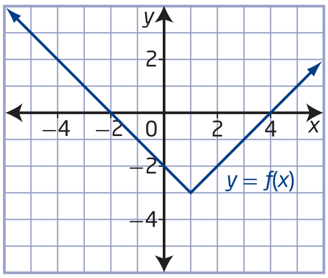
Short Answer

8. The domain of the function $y = f(x)$ is $\{x \mid -3 \le x \le 4, x \in R\}$. What is the domain of the function $y = f(x + 2) - 1$?

9. Given the graph of $y = f(x)$, sketch the graph of $y - 4 = -\frac{1}{4}f\left(\frac{1}{2}(x + 3)\right)$.



10. Consider the graph of the function $y = f(x)$.



a) Sketch the graph of the inverse.

b) Explain how the coordinates of key points are transformed.

c) State any invariant points.

11. Write the equation of the inverse function of $y = 5x + 2$. Verify graphically that the functions are inverses of each other.

12. A transformation of the graph of $y = f(x)$ results in a horizontal stretch about the y-axis by a factor of 2, a horizontal reflection in the y-axis, a vertical stretch about the x-axis by a factor of 3, and a horizontal translation of 2 units to the right. Write the equation for the transformed function.

Extended Response

13. The graph of the function $f(x) = |x|$ is transformed to the graph of $g(x) = f(x + 2) - 7$.

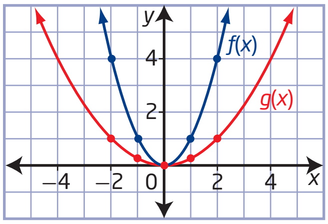
a) Describe the transformation.

b) Write the equation of the function $g(x)$.

c) Determine the minimum value of $g(x)$.

d) The domain of the function $f(x)$ is the set of real numbers. The domain of the function $g(x)$ is also the set of real numbers. Does this imply that all of the points are invariant? Explain your answer.

14. The function $g(x)$ is a transformation of the function $f(x)$.



a) Write the equation of the function $f(x)$.

b) Write the equation of the function $g(x)$ in the form $g(x) = af(x)$, and describe the transformation.

c) Write the equation of the function $g(x)$ in the form $g(x) = f(bx)$, and describe the transformation.

d) Algebraically prove that the two equations from parts b) and c) are equivalent.

15. Consider the function $h(x) = -(x + 3)^2 - 5$.

a) Explain how you can determine whether or not the inverse of $h(x)$ is a function.

b) Write the equation of the inverse relation in simplified form.

c) What restrictions could be placed on the domain of the function so that the inverse is also a function?

CHAPTER 2

Radical Functions

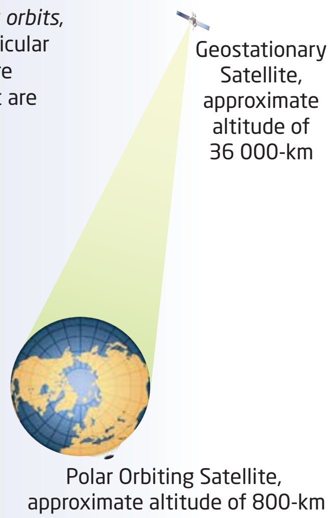
How far can you see from the top of a hill? What range of vision does a submarine's periscope have? How much fertilizer is required for a particular crop? How much of Earth's surface can a satellite "see"? You can model each of these situations using a radical function. The functions can range from simple square root functions to more complex radical functions of higher orders.

In this chapter, you will explore a variety of square root functions and work with radical functions used by an aerospace engineer when relating the distance to the horizon for a satellite above Earth. Would you expect this to be a simple or a complex radical function?

Did You Know?

Some satellites are put into *polar orbits*, where they follow paths perpendicular to the equator. Other satellites are put into *geostationary orbits* that are parallel to the equator.

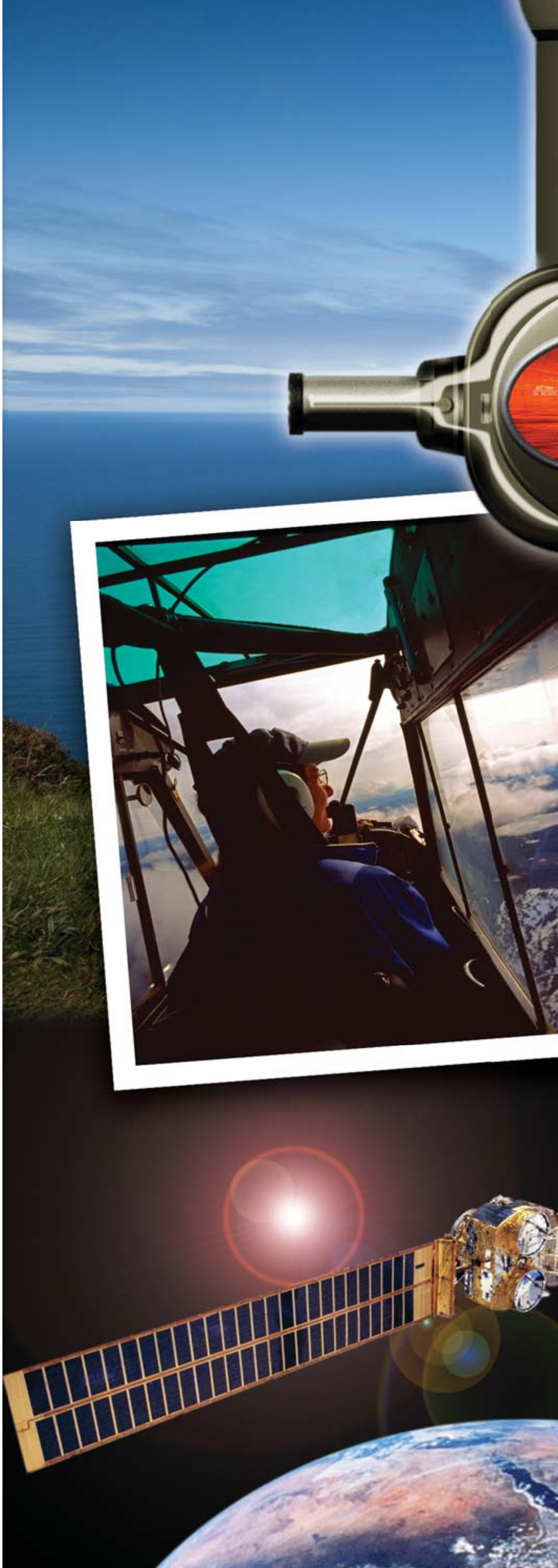
Polar orbiting satellites are useful for taking high-resolution photographs. Geostationary satellites allow for weather monitoring and communications for a specific country or continent.



Key Terms

radical function

square root of a function

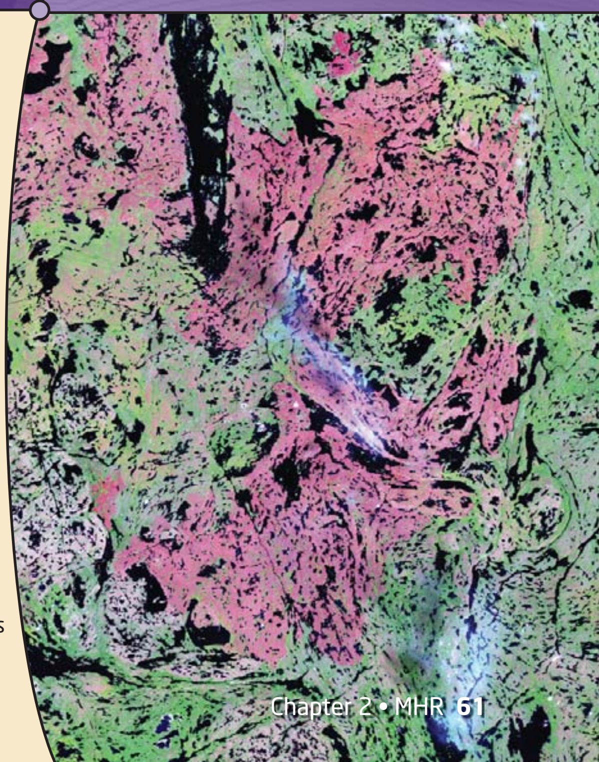


Career Link

Scientists and engineers use remote sensing to create satellite images. They use instruments and satellites to produce information that is used to manage resources, investigate environmental issues, and produce sophisticated maps.

Web Link

To learn more about a career or educational opportunities involving remote sensing, go to www.mcgrawhill.ca/school/learningcentres and follow the links.



Yellowknife Wetlands

2.1

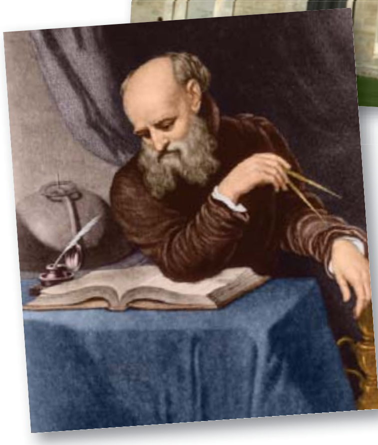
Radical Functions and Transformations

Focus on…

- investigating the function $y = \sqrt{x}$ using a table of values and a graph
- graphing radical functions using transformations
- identifying the domain and range of radical functions

Does a feather fall more slowly than a rock? Galileo Galilei, a mathematician and scientist, pondered this question more than 400 years ago. He theorized that the rate of falling objects depends on air resistance, not on mass. It is believed that he tested his idea by dropping spheres of different masses but the same diameter from the top of the Leaning Tower of Pisa in what is now Italy. The result was exactly as he predicted—they fell at the same rate.

In 1971, during the Apollo 15 lunar landing, Commander David Scott performed a similar demonstration on live television. Because the surface of the moon is essentially a vacuum, a hammer and a feather fell at the same rate.



Web Link

For more information about Galileo or the Apollo 15 mission, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Investigate a Radical Function

Materials

- grid paper
- graphing technology (optional)

For objects falling near the surface of Earth, the function $d = 5t^2$ approximately models the time, t, in seconds, for an object to fall a distance, d, in metres, if the resistance caused by air can be ignored.

1. a) Identify any restrictions on the domain of this function. Why are these restrictions necessary? What is the range of the function?

b) Create a table of values and a graph showing the distance fallen as a function of time.

2. Express time in terms of distance for the distance-time function from step 1. Represent the new function graphically and using a table of values.

3. For each representation, how is the equation of the new function from step 2 related to the original function?

Reflect and Respond

4. **a)** The original function is a distance-time function. What would you call the new function? Under what circumstances would you use each function?

   b) What is the shape of the graph of the original function? Describe the shape of the graph of the new function.

Link the Ideas

The function that gives the predicted fall time for an object under the influence of gravity is an example of a **radical function**. Radical functions have restricted domains if the index of the radical is an even number. Like many types of functions, you can represent radical functions in a variety of ways, including tables, graphs, and equations. You can create graphs of radical functions using tables of values or technology, or by transforming the base radical function, $y = \sqrt{x}$.

radical function

- a function that involves a radical with a variable in the radicand
- $y = \sqrt{3x}$ and $y = 4\sqrt[3]{5 + x}$ are radical functions.

Example 1

Graph Radical Functions Using Tables of Values

Use a table of values to sketch the graph of each function. Then, state the domain and range of each function.

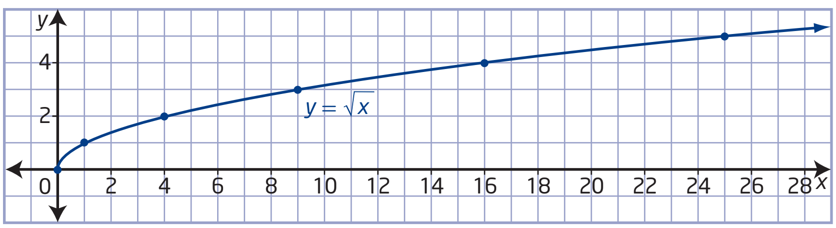
a) $y = \sqrt{x}$ **b)** $y = \sqrt{x - 2}$ **c)** $y = \sqrt{x} - 3$

Solution

a) For the function $y = \sqrt{x}$, the radicand x must be greater than or equal to zero, $x \geq 0$.

x	y
0	0
1	1
4	2
9	3
16	4
25	5

How can you choose values of x that allow you to complete the table without using a calculator?



The graph has an endpoint at (0, 0) and continues up and to the right. The domain is $\{x \mid x \geq 0, x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

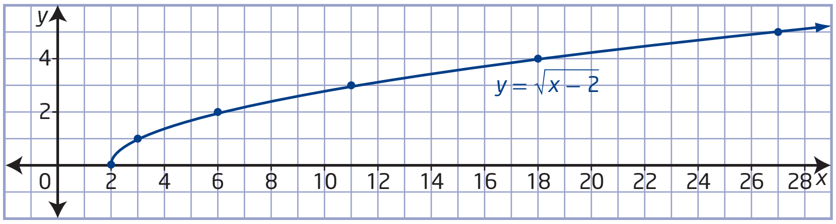
b) For the function $y = \sqrt{x - 2}$, the value of the radicand must be greater than or equal to zero.

$x - 2 \geq 0$
$x \geq 2$

x	y
2	0
3	1
6	2
11	3
18	4
27	5

How is this table related to the table for $y = \sqrt{x}$ in part a)?

How does the graph of $y = \sqrt{x - 2}$ compare to the graph of $y = \sqrt{x}$?

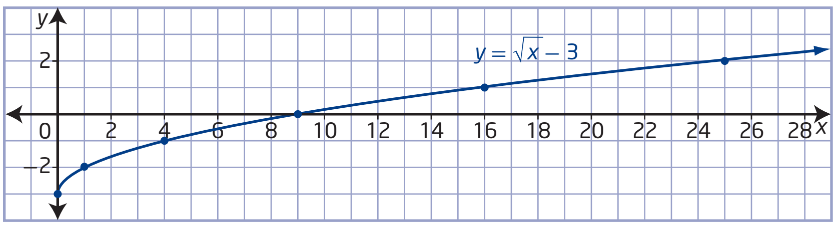


The domain is $\{x \mid x \geq 2, x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

c) The radicand of $y = \sqrt{x} - 3$ must be non-negative.
$x \geq 0$

x	y
0	−3
1	−2
4	−1
9	0
16	1
25	2

How does the graph of $y = \sqrt{x} - 3$ compare to the graph of $y = \sqrt{x}$?



The domain is $\{x \mid x \geq 0, x \in \mathrm{R}\}$ and the range is $\{y \mid y \geq -3, y \in \mathrm{R}\}$.

Your Turn

Sketch the graph of the function $y = \sqrt{x + 5}$ using a table of values. State the domain and the range.

Graphing Radical Functions Using Transformations

You can graph a radical function of the form $y = a\sqrt{b(x - h)} + k$ by transforming the graph of $y = \sqrt{x}$ based on the values of a, b, h, and k. The effects of changing parameters in radical functions are the same as the effects of changing parameters in other types of functions.

- Parameter a results in a vertical stretch of the graph of $y = \sqrt{x}$ by a factor of $|a|$. If $a < 0$, the graph of $y = \sqrt{x}$ is reflected in the x-axis.
- Parameter b results in a horizontal stretch of the graph of $y = \sqrt{x}$ by a factor of $\frac{1}{|b|}$. If $b < 0$, the graph of $y = \sqrt{x}$ is reflected in the y-axis.
- Parameter h determines the horizontal translation. If $h > 0$, the graph of $y = \sqrt{x}$ is translated to the right h units. If $h < 0$, the graph is translated to the left $|h|$ units.
- Parameter k determines the vertical translation. If $k > 0$, the graph of $y = \sqrt{x}$ is translated up k units. If $k < 0$, the graph is translated down $|k|$ units.

Example 2

Graph Radical Functions Using Transformations

Sketch the graph of each function using transformations. Compare the domain and range to those of $y = \sqrt{x}$ and identify any changes.

a) $y = 3\sqrt{-(x - 1)}$ **b)** $y - 3 = -\sqrt{2x}$

Solution

a) The function $y = 3\sqrt{-(x - 1)}$ is expressed in the form $y = a\sqrt{b(x - h)} + k$. Identify the value of each parameter and how it will transform the graph of $y = \sqrt{x}$.

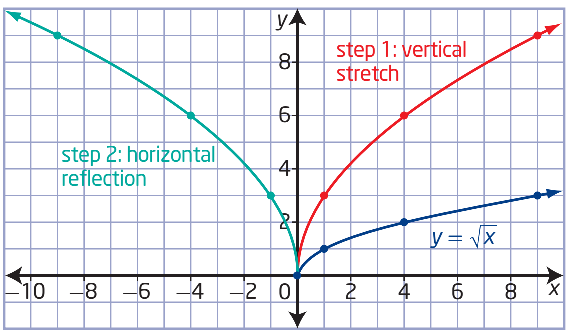
Why is it acceptable to have a negative sign under a square root sign?

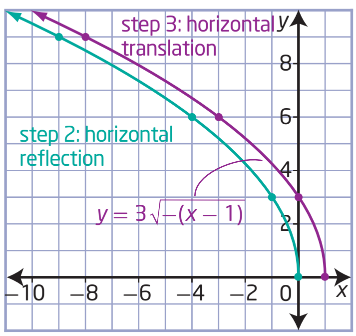
- $a = 3$ results in a vertical stretch by a factor of 3 (step 1).
- $b = -1$ results in a reflection in the y-axis (step 2).
- $h = 1$ results in a horizontal translation of 1 unit to the right (step 3).
- $k = 0$, so the graph has no vertical translation.

Method 1: Transform the Graph Directly

Start with a sketch of $y = \sqrt{x}$ and apply the transformations one at a time.

In what order do transformations need to be performed?





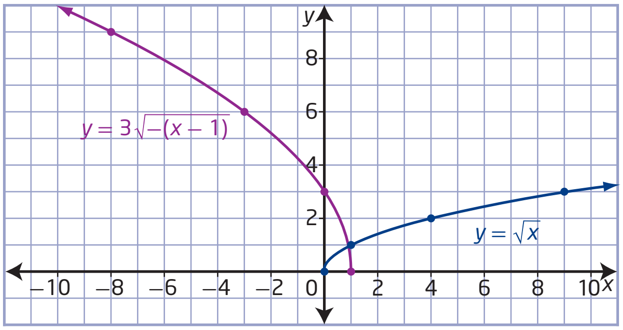
Method 2: Map Individual Points

Choose key points on the graph of $y = \sqrt{x}$ and map them for each transformation.

How can you use mapping notation to express each transformation step?

Transformation of $y = \sqrt{x}$	Mapping
Vertical stretch by a factor of 3	$(0, 0) \rightarrow (0, 0)$ $(1, 1) \rightarrow (1, 3)$ $(4, 2) \rightarrow (4, 6)$ $(9, 3) \rightarrow (9, 9)$
Horizontal reflection in the y-axis	$(0, 0) \rightarrow (0, 0)$ $(1, 3) \rightarrow (-1, 3)$ $(4, 6) \rightarrow (-4, 6)$ $(9, 9) \rightarrow (-9, 9)$
Horizontal translation of 1 unit to the right	$(0, 0) \rightarrow (1, 0)$ $(-1, 3) \rightarrow (0, 3)$ $(-4, 6) \rightarrow (-3, 6)$ $(-9, 9) \rightarrow (-8, 9)$

Plot the image points to create the transformed graph.



The function $y = \sqrt{x}$ is reflected horizontally, stretched vertically by a factor of 3, and then translated 1 unit right. So, the graph of $y = 3\sqrt{-(x - 1)}$ extends to the left from $x = 1$ and its domain is $\{x \mid x \leq 1, x \in \mathrm{R}\}$.

Since the function is not reflected vertically or translated vertically, the graph of $y = 3\sqrt{-(x - 1)}$ extends up from $y = 0$, similar to the graph of $y = \sqrt{x}$. The range, $\{y \mid y \geq 0, y \in \mathrm{R}\}$, is unchanged by the transformations.

b) Express the function $y - 3 = -\sqrt{2x}$ in the form $y = a\sqrt{b(x - h)} + k$ to identify the value of each parameter.

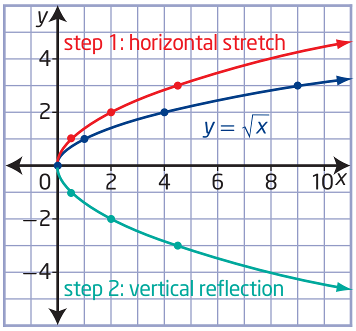
$$y - 3 = -\sqrt{2x}$$
$$y = -\sqrt{2x} + 3$$

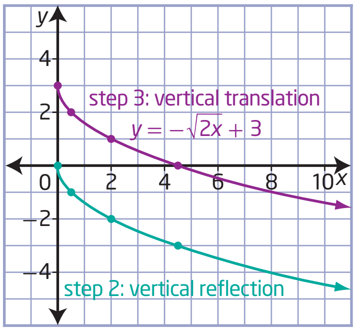
- $b = 2$ results in horizontal stretch by a factor of $\frac{1}{2}$ (step 1).
- $a = -1$ results in a reflection in the x-axis (step 2).
- $h = 0$, so the graph is not translated horizontally.
- $k = 3$ results in a vertical translation of 3 units up (step 3).

Apply these transformations either directly to the graph of $y = \sqrt{x}$ or to key points, and then sketch the transformed graph.

Method 1: Transform the Graph Directly

Use a sketch of $y = \sqrt{x}$ and apply the transformations to the curve one at a time.





Method 2: Use Mapping Notation

Apply each transformation to the point (x, y) to determine a general mapping notation for the transformed function.

Transformation of $y = \sqrt{x}$	Mapping
Horizontal stretch by a factor of $\frac{1}{2}$	$(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$
Reflection in the x-axis	$\left(\frac{1}{2}x, y\right) \rightarrow \left(\frac{1}{2}x, -y\right)$
Vertical translation of 3 units up	$\left(\frac{1}{2}x, -y\right) \rightarrow \left(\frac{1}{2}x, -y + 3\right)$

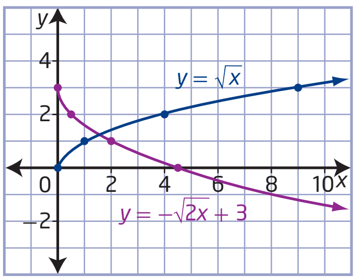
Choose key points on the graph of $y = \sqrt{x}$ and use the general mapping notation $(x, y) \rightarrow \left(\frac{1}{2}x, -y + 3\right)$ to determine their image points on the function $y - 3 = -\sqrt{2x}$.

$(0, 0) \rightarrow (0, 3)$

$(1, 1) \rightarrow (0.5, 2)$

$(4, 2) \rightarrow (2, 1)$

$(9, 3) \rightarrow (4.5, 0)$



Since there are no horizontal reflections or translations, the graph still extends to the right from $x = 0$. The domain, $\{x \mid x \geq 0, x \in \mathrm{R}\}$, is unchanged by the transformations as compared with $y = \sqrt{x}$.

The function is reflected vertically and then translated 3 units up, so the graph extends down from $y = 3$. The range is $\{y \mid y \leq 3, y \in \mathrm{R}\}$, which has changed as compared to $y = \sqrt{x}$.

Your Turn

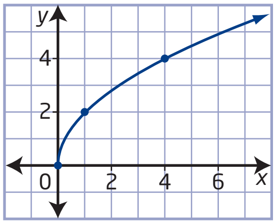
a) Sketch the graph of the function $y = -2\sqrt{x + 3} - 1$ by transforming the graph of $y = \sqrt{x}$.

b) Identify the domain and range of $y = \sqrt{x}$ and describe how they are affected by the transformations.

Example 3

Determine a Radical Function From a Graph

Mayleen is designing a symmetrical pattern. She sketches the curve shown and wants to determine its equation and the equation of its reflection in each quadrant. The graph is a transformation of the graph of $y = \sqrt{x}$. What are the equations of the four functions Mayleen needs to work with?

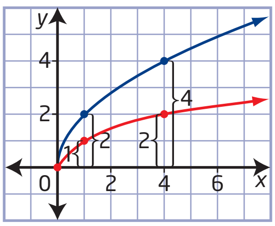


Solution

The base function $y = \sqrt{x}$ is not reflected or translated, but it is stretched. A radical function that involves a stretch can be obtained from either a vertical stretch or a horizontal stretch. Use an equation of the form $y = a\sqrt{x}$ or $y = \sqrt{bx}$ to represent the image function for each type of stretch.

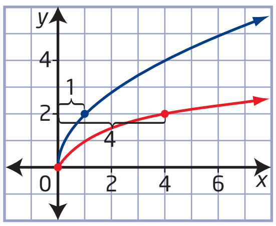
Method 1: Compare Vertical or Horizontal Distances
Superimpose the graph of $y = \sqrt{x}$ and compare corresponding distances to determine the factor by which the function has been stretched.

View as a Vertical Stretch ($y = a\sqrt{x}$)
Each vertical distance is 2 times the corresponding distance for $y = \sqrt{x}$.



This represents a vertical stretch by a factor of 2, which means $a = 2$. The equation $y = 2\sqrt{x}$ represents the function.

View as a Horizontal Stretch ($y = \sqrt{bx}$)
Each horizontal distance is $\frac{1}{4}$ the corresponding distance for $y = \sqrt{x}$.



This represents a horizontal stretch by a factor of $\frac{1}{4}$, which means $b = 4$. The equation $y = \sqrt{4x}$ represents the function.

Express the equation of the function as either $y = 2\sqrt{x}$ or $y = \sqrt{4x}$.

Method 2: Substitute Coordinates of a Point

Use the coordinates of one point on the function, such as (1, 2), to determine the stretch factor.

View as a Vertical Stretch	View as a Horizontal Stretch
Substitute 1 for x and 2 for y in the equation $y = a\sqrt{x}$. Then, solve for a. $y = a\sqrt{x}$ $2 = a\sqrt{1}$ $2 = a(1)$ $2 = a$ The equation of the function is $y = 2\sqrt{x}$.	Substitute the coordinates (1, 2) in the equation $y = \sqrt{bx}$ and solve for b. $y = \sqrt{bx}$ $2 = \sqrt{b(1)}$ $2 = \sqrt{b}$ $2^2 = (\sqrt{b})^2$ $4 = b$ The equation can also be expressed as $y = \sqrt{4x}$.

Represent the function in simplest form by $y = 2\sqrt{x}$ or by $y = \sqrt{4x}$.

Determine the equations of the reflected curves using $y = 2\sqrt{x}$.

- A reflection in the y-axis results in the function $y = 2\sqrt{-x}$, since $b = -1$.
- A reflection in the x-axis results in $y = -2\sqrt{x}$, since $a = -1$.

Reflecting these graphs into the third quadrant results in the function $y = -2\sqrt{-x}$.

Mayleen needs to use the equations $y = 2\sqrt{x}$, $y = 2\sqrt{-x}$, $y = -2\sqrt{x}$, and $y = -2\sqrt{-x}$. Similarly, she could use the equations $y = \sqrt{4x}$, $y = \sqrt{-4x}$, $y = -\sqrt{4x}$, and $y = -\sqrt{-4x}$.

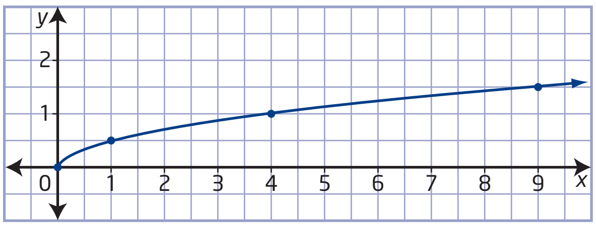
Are the restrictions on the domain in each function consistent with the quadrant in which the curve lies?

Your Turn

a) Determine two forms of the equation for the function shown. The function is a transformation of the function $y = \sqrt{x}$.

b) Show algebraically that the two equations are equivalent.

c) What is the equation of the curve reflected in each quadrant?



Example 4

Model the Speed of Sound

Justin's physics textbook states that the speed, s, in metres per second, of sound in dry air is related to the air temperature, T, in degrees Celsius, by the function $s = 331.3\sqrt{1 + \frac{T}{273.15}}$.

a) Determine the domain and range in this context.

b) On the Internet, Justin finds another formula for the speed of sound, $s = 20\sqrt{T + 273}$. Use algebra to show that the two functions are approximately equivalent.

c) How is the graph of this function related to the graph of the base square root function? Which transformation do you predict will be the most noticeable on a graph?

d) Graph the function $s = 331.3\sqrt{1 + \frac{T}{273.15}}$ using technology.

e) Determine the speed of sound, to the nearest metre per second, at each of the following temperatures.

- **i)** 20 °C (normal room temperature)
- **ii)** 0 °C (freezing point of water)
- **iii)** −63 °C (coldest temperature ever recorded in Canada)
- **iv)** −89 °C (coldest temperature ever recorded on Earth)

Solution

a) Use the following inequality to determine the domain:

$$\begin{aligned} \text{radicand} &\geq 0 \\ 1 + \frac{T}{273.15} &\geq 0 \\ \frac{T}{273.15} &\geq -1 \\ T &\geq -273.15 \end{aligned}$$

The domain is $\{T \mid T \geq -273.15, T \in \mathrm{R}\}$. This means that the temperature must be greater than or equal to −273.15 °C, which is the lowest temperature possible and is referred to as absolute zero.

The range is $\{s \mid s \geq 0, s \in \mathrm{R}\}$, which means that the speed of sound is a non-negative value.

b) Rewrite the function from the textbook in simplest form.

$$s = 331.3\sqrt{1 + \frac{T}{273.15}}$$

$$s = 331.3\sqrt{\frac{273.15}{273.15} + \frac{T}{273.15}}$$

$$s = 331.3\sqrt{\frac{273.15 + T}{273.15}}$$

$$s = 331.3\frac{\sqrt{273.15 + T}}{\sqrt{273.15}}$$

$$s \approx 20\sqrt{T + 273}$$

How could you verify that these expressions are approximately equivalent?

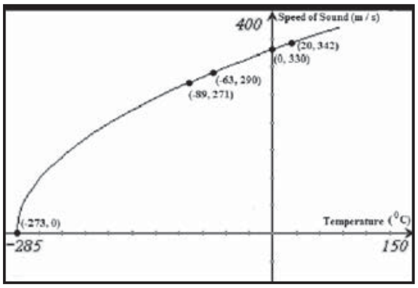
The function found on the Internet, $s = 20\sqrt{T + 273}$, is the approximate simplest form of the function in the textbook.

c) Analyse the transformations and determine the order in which they must be performed.

The graph of $s = \sqrt{T}$ is stretched vertically by a factor of about 20 and then translated about 273 units to the left. Translating 273 units to the left will be most noticeable on the graph of the function.

Are these transformations consistent with the domain and range?

d)



Are your answers to part c) confirmed by the graph?

e)

	Temperature (°C)	Approximate Speed of Sound (m/s)
i)	20	343
ii)	0	331
iii)	−63	291
iv)	−89	272

Your Turn

A company estimates its cost of production using the function $C(n) = 20\sqrt{n} + 1000$, where C represents the cost, in dollars, to produce n items.

a) Describe the transformations represented by this function as compared to $C = \sqrt{n}$.

b) Graph the function using technology. What does the shape of the graph imply about the situation?

c) Interpret the domain and range in this context.

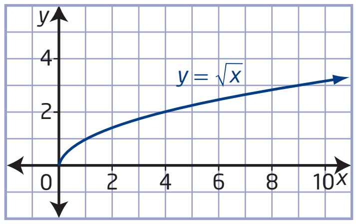
d) Use the graph to determine the expected cost to produce 12 000 items.

Did You Know?

Eureka, on Ellesmere Island, Nunavut, holds the North American record for the lowest-ever average monthly temperature, −47.9 °C in February 1979. For 18 days, the temperature stayed below −45 °C.

Key Ideas

- The base radical function is $y = \sqrt{x}$. Its graph has the following characteristics:
  - a left endpoint at (0, 0)
  - no right endpoint
  - the shape of half of a parabola
  - a domain of $\{x \mid x \geq 0, x \in \mathrm{R}\}$ and a range of $\{y \mid y \geq 0, y \in \mathrm{R}\}$



- You can graph radical functions of the form $y = a\sqrt{b(x - h)} + k$ by transforming the base function $y = \sqrt{x}$.

How does each parameter affect the graph of $y = \sqrt{x}$?

- You can analyse transformations to identify the domain and range of a radical function of the form $y = a\sqrt{b(x - h)} + k$.

Check Your Understanding

Practise

1. Graph each function using a table of values. Then, identify the domain and range.

a) $y = \sqrt{x - 1}$

b) $y = \sqrt{x + 6}$

c) $y = \sqrt{3 - x}$

d) $y = \sqrt{-2x - 5}$

2. Explain how to transform the graph of $y = \sqrt{x}$ to obtain the graph of each function. State the domain and range in each case.

a) $y = 7\sqrt{x - 9}$

b) $y = \sqrt{-x} + 8$

c) $y = -\sqrt{0.2x}$

d) $4 + y = \frac{1}{3}\sqrt{x + 6}$

3. Match each function with its graph.

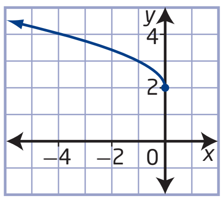
a) $y = \sqrt{x} - 2$

b) $y = \sqrt{-x} + 2$

c) $y = -\sqrt{x + 2}$

d) $y = -\sqrt{-(x - 2)}$

A



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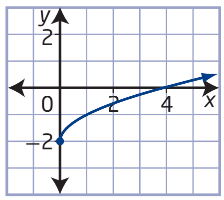

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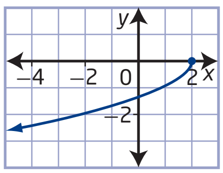

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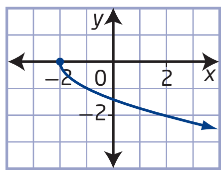


4. Write the equation of the radical function that results by applying each set of transformations to the graph of $y = \sqrt{x}$.

a) vertical stretch by a factor of 4, then horizontal translation of 6 units left

b) horizontal stretch by a factor of $\frac{1}{8}$, then vertical translation of 5 units down

c) horizontal reflection in the y-axis, then horizontal translation of 4 units right and vertical translation of 11 units up

d) vertical stretch by a factor of 0.25, vertical reflection in the x-axis, and horizontal stretch by a factor of 10

5. Sketch the graph of each function using transformations. State the domain and range of each function.

a) $f(x) = \sqrt{-x} - 3$

b) $r(x) = 3\sqrt{x + 1}$

c) $p(x) = -\sqrt{x - 2}$

d) $y - 1 = -\sqrt{-4(x - 2)}$

e) $m(x) = \sqrt{\frac{1}{2}x + 4}$

f) $y + 1 = \frac{1}{3}\sqrt{-(x + 2)}$

Apply

6. Consider the function $f(x) = \frac{1}{4}\sqrt{5x}$.

a) Identify the transformations represented by $f(x)$ as compared to $y = \sqrt{x}$.

b) Write two functions equivalent to $f(x)$: one of the form $y = a\sqrt{x}$ and the other of the form $y = \sqrt{bx}$

c) Identify the transformation(s) represented by each function you wrote in part b).

d) Use transformations to graph all three functions. How do the graphs compare?

7. a) Express the radius of a circle as a function of its area.

b) Create a table of values and a graph to illustrate the relationship that this radical function represents.

8. For an observer at a height of h feet above the surface of Earth, the approximate distance, d, in miles, to the horizon can be modelled using the radical function $d = \sqrt{1.50h}$.

a) Use the language of transformations to describe how to obtain the graph from the base square root graph.

b) Determine an approximate equivalent function of the form $d = a\sqrt{h}$ for the function. Which form of the function do you prefer, and why?

c) A lifeguard on a tower is looking out over the water with binoculars. How far can she see if her eyes are 20 ft above the level of the water? Express your answer to the nearest tenth of a mile.

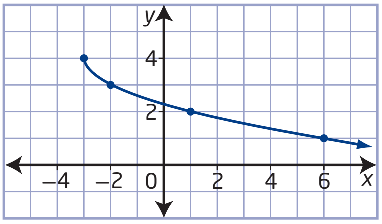
9. The function $4 - y = \sqrt{3x}$ is translated 9 units up and reflected in the x-axis.

a) Without graphing, determine the domain and range of the image function.

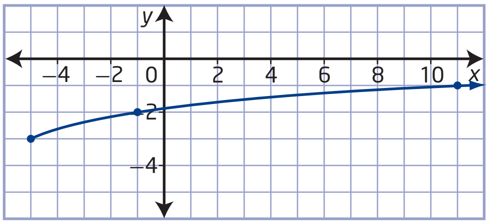
b) Compared to the base function, $y = \sqrt{x}$, by how many units and in which direction has the given function been translated horizontally? vertically?

10. For each graph, write the equation of a radical function of the form $y = a\sqrt{b(x - h)} + k$.

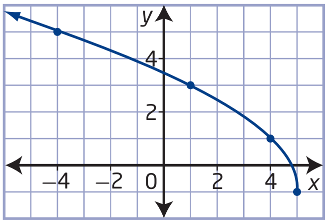
a)



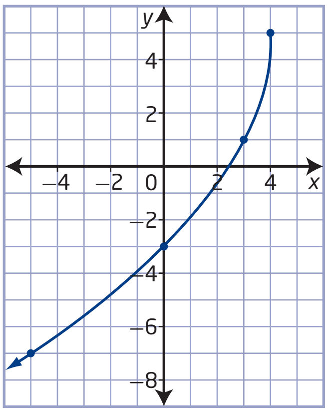
b)



c)



d)



11. Write the equation of a radical function with each domain and range.

a) $\{x \mid x \geq 6, x \in \mathrm{R}\}, \{y \mid y \geq 1, y \in \mathrm{R}\}$

b) $\{x \mid x \geq -7, x \in \mathrm{R}\}, \{y \mid y \leq -9, y \in \mathrm{R}\}$

c) $\{x \mid x \leq 4, x \in \mathrm{R}\}, \{y \mid y \geq -3, y \in \mathrm{R}\}$

d) $\{x \mid x \leq -5, x \in \mathrm{R}\}, \{y \mid y \leq 8, y \in \mathrm{R}\}$

12. Agronomists use radical functions to model and optimize corn production. One factor they analyse is how the amount of nitrogen fertilizer applied affects the crop yield. Suppose the function $Y(n) = 760\sqrt{n} + 2000$ is used to predict the yield, Y, in kilograms per hectare, of corn as a function of the amount, n, in kilograms per hectare, of nitrogen applied to the crop.

a) Use the language of transformations to compare the graph of this function to the graph of $y = \sqrt{n}$.

b) Graph the function using transformations.

c) Identify the domain and range.

d) What do the shape of the graph, the domain, and the range tell you about this situation? Are the domain and range realistic in this context? Explain.

Did You Know?

Over 6300 years ago, the Indigenous people in the area of what is now Mexico domesticated and cultivated several varieties of corn. The cultivation of corn is now global.

13. A manufacturer wants to predict the consumer interest in a new smart phone. The company uses the function $P(d) = -2\sqrt{-d} + 20$ to model the number, P, in millions, of pre-orders for the phone as a function of the number, d, of days before the phone's release date.

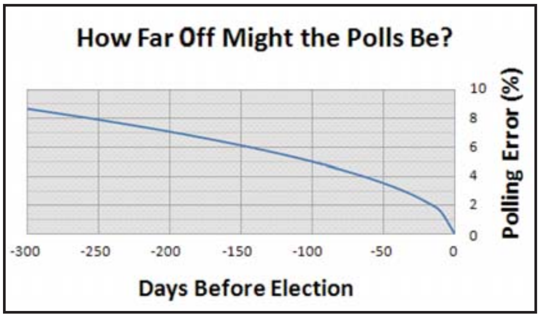
a) What are the domain and range and what do they mean in this situation?

b) Identify the transformations represented by the function as compared to $y = \sqrt{d}$.

c) Graph the function and explain what the shape of the graph indicates about the situation.

d) Determine the number of pre-orders the manufacturer can expect to have 30 days before the release date.

14. During election campaigns, campaign managers use surveys and polls to make projections about the election results. One campaign manager uses a radical function to model the possible error in polling predictions as a function of the number of days until the election, as shown in the graph.

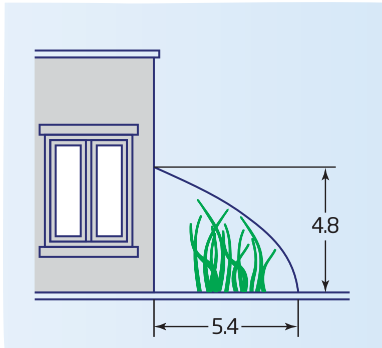


a) Explain what the graph shows about the accuracy of polls before elections.

b) Determine an equation to represent the function. Show how you developed your answer.

c) Describe the transformations that the function represents as compared to $y = \sqrt{x}$.

15. While meeting with a client, a manufacturer of custom greenhouses sketches a greenhouse in the shape of the graph of a radical function. What equation could the manufacturer use to represent the shape of the greenhouse roof?



Did You Know?

People living in the Arctic are starting to use greenhouses to grow some of their food. There are greenhouse societies in both Iqaluit, Nunavut and Inuvik, Northwest Territories that grow beans, lettuce, carrots, tomatoes, and herbs.

Web Link

To learn more about greenhouse communities in the Arctic, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

16. Determine the equation of a radical function with

a) endpoint at (2, 5) and passing through the point (6, 1)

b) endpoint at (3, −2) and an x-intercept with a value of −6

17. The Penrose method is a system for giving voting powers to members of assemblies or legislatures based on the square root of the number of people that each member represents, divided by 1000. Consider a parliament that represents the people of the world and how voting power might be given to different nations. The table shows the estimated populations of Canada and the three most populous and the three least populous countries in the world.

Country	Population
China	1 361 513 000
India	1 251 696 000
United States	325 540 000
Canada	35 100 000
Tuvalu	11 000
Nauru	10 000
Vatican City	1 000

a) Share your answers to the following two questions with a classmate and explain your thinking:
- Which countries might feel that a "one nation, one vote" system is an unfair way to allocate voting power?
- Which countries might feel that a "one person, one vote" system is unfair?

b) What percent of the voting power would each nation listed above have under a "one person, one vote" system, assuming a world population of approximately 7.302 billion?

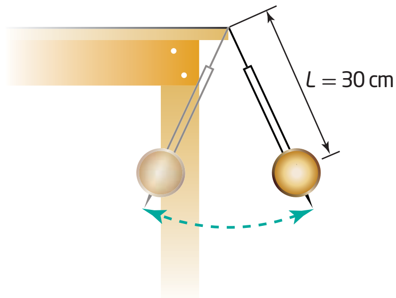
c) If x represents the population of a country and $V(x)$ represents its voting power, what function could be written to represent the Penrose method?

d) Under the Penrose method, the sum of the world voting power using the given data is approximately 765. What percent of the voting power would this system give each nation in the table?

e) Why might the Penrose method be viewed as a compromise for allocating voting power?

18. **MINI LAB** The period of a pendulum is the time for one complete swing back and forth. As long as the initial swing angle is kept relatively small, the period of a pendulum is related to its length by a radical function.

Materials
- thread
- washer or other suitable mass
- tape
- ruler
- stopwatch or timer



Step 1 Tie a length of thread to a washer or other mass. Tape the thread to the edge of a table or desk top so that the length between the pivot point and the centre of the washer is 30 cm.

Step 2 Pull the mass to one side and allow it to swing freely. Measure the total time for 10 complete swings back and forth and then divide by 10 to determine the period for this length. Record the length and period in a table.

Step 3 Repeat steps 1 and 2 using lengths of 25 cm, 20 cm, 15 cm, 10 cm, 5 cm, and 3 cm (and shorter distances if possible).

Step 4 Create a scatter plot showing period as a function of length. Draw a smooth curve through or near the points. Does it appear to be a radical function? Justify your answer.

Step 5 What approximate transformation(s) to the graph of $y = \sqrt{x}$ would produce your result? Write a radical function that approximates the graph, where T represents the period and L represents the length of the pendulum.

Extend

19. The inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2, x \geq 0$.

a) Graph both functions, and use them to explain why the restriction is necessary on the domain of the inverse function.

b) Determine the equation, including any restrictions, of the inverse of each of the following functions.

i) $g(x) = -\sqrt{x - 5}$

ii) $h(x) = \sqrt{-x} + 3$

iii) $j(x) = \sqrt{2x - 7} - 6$

20. If $f(x) = \frac{5}{8}\sqrt{-\frac{7}{12}x}$ and $g(x) = -\frac{2}{5}\sqrt{6(x + 3)} - 4$, what transformations could you apply to the graph of $f(x)$ to create the graph of $g(x)$?

Create Connections

C1 Which parameters in $y = a\sqrt{b(x - h)} + k$ affect the domain of $y = \sqrt{x}$? Which parameters affect the range? Explain, using examples.

C2 Sarah claims that any given radical function can be simplified so that there is no value of b, only a value of a. Is she correct? Explain, using examples.

C3 Compare and contrast the process of graphing a radical function using transformations with graphing a quadratic function using transformations.

C4 **MINI LAB** The Wheel of Theodorus, or Square Root Spiral, is a geometric construction that contains line segments with length equal to the square root of any whole number.

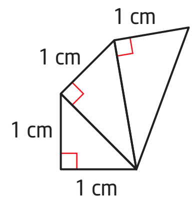
Materials

- ruler, drafting square, or other object with a right angle
- millimetre ruler

Step 1 Create an isosceles right triangle with legs that are each 1 cm long. Mark one end of the hypotenuse as point C. What is the length of the hypotenuse, expressed as a radical?

Step 2 Use the hypotenuse of the first triangle as one leg of a new right triangle. Draw a length of 1 cm as the other leg, opposite point C. What is the length of the hypotenuse of this second triangle, expressed as a radical?

Step 3 Continue to create right triangles, each time using the hypotenuse of the previous triangle as a leg of the next triangle, and a length of 1 cm as the other leg (drawn so that the 1-cm leg is opposite point C). Continue the spiral until you would overlap the initial base.



Step 4 Create a table to represent the length of the hypotenuse as a function of the triangle number (first, second, third triangle in the pattern, etc.). Express lengths both in exact radical form and in approximate decimal form.

Step 5 Write an equation to represent this function, where L represents the hypotenuse length and n represents the triangle number. Does the equation involve any transformations on the base square root function? Explain.

2.2

Square Root of a Function

Focus on...

- sketching the graph of $y = \sqrt{f(x)}$ given the graph of $y = f(x)$
- explaining strategies for graphing $y = \sqrt{f(x)}$ given the graph of $y = f(x)$
- comparing the domains and ranges of the functions $y = f(x)$ and $y = \sqrt{f(x)}$, and explaining any differences

The Pythagorean theorem is often applied by engineers. They use right triangles in the design of large domes, bridges, and other structures because the triangle is a strong support unit. For example, a truss bridge consists of triangular units of steel beams connected together to support the bridge deck.

You are already familiar with the square root operation (and its effect on given values) in the Pythagorean theorem. How does the square root operation affect the graph of the function? If you are given the graph of a function, what does the graph of the square root of that function look like?

Web Link

For more information about how triangles are fundamental to the design of domes, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Truss bridge over the Bow River in Morley, Alberta located on Chiniki First Nation territory.

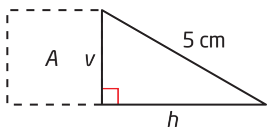
Investigate Related Functions: $y = f(x)$ and $y = \sqrt{f(x)}$

Materials

- grid paper and ruler, or graphing calculator
- dynamic geometry or graphing software (optional)

A: Right Triangles, Area, and Length

1. Draw several right triangles with a hypotenuse of 5 cm and legs of various lengths. For each triangle, label the legs as v and h.



2. a) Write an equation for the length of v as a function of h. Graph the function using an appropriate domain for the situation.

b) Compare the measured values for side v in the triangles you drew to the calculated values of v from your graph.

3. a) Draw a square on side v of each triangle. Let the area of this square be A, and write an equation for A as a function of h.

b) Graph the area function.

Reflect and Respond

4. a) How are the equations of the two functions related?

b) How do the domains of the two functions compare?

c) What is the relationship between the ranges of the two functions?

B: Compare a Function and Its Square Root

5. Consider the functions $y = 2x + 4$ and $y = \sqrt{2x + 4}$.

a) Describe the relationship between the equations for these two functions.

b) Graph the two functions, and note any connections between the two graphs.

c) Compare the values of y for the same values of x. How are they related?

6. a) Create at least two more pairs of functions that share the same relationship as those in step 5.

b) Compare the tables and graphs of each pair of functions.

Reflect and Respond

7. Consider pairs of functions where one function is the square root of the other function.

a) How do the domains compare? Explain why you think there are differences.

b) How are the values of y related for pairs of functions like these?

c) What differences occur in the ranges, and why do you think they occur?

8. How might you use the connections you have identified in this investigation as a method of graphing $y = \sqrt{f(x)}$ if you are given the graph of $y = f(x)$?

Link the Ideas

You can determine how two functions, $y = f(x)$ and $y = \sqrt{f(x)}$, are related by comparing how the values of y are calculated:

- For $y = 2x + 1$, multiply x by 2 and add 1.
- For $y = \sqrt{2x + 1}$, multiply x by 2, add 1, and take the square root.

The two functions start with the same two operations, but the function $y = \sqrt{2x + 1}$ has the additional step of taking the square root. For any value of x, the resulting value of y for $y = \sqrt{2x + 1}$ is the square root of the value of y for $y = 2x + 1$, as shown in the table.

x	$y = 2x + 1$	$y = \sqrt{2x + 1}$
0	1	1
4	9	3
12	25	5
24	49	7
⋮	⋮	⋮

The function $y = \sqrt{2x + 1}$ represents the **square root of the function** $y = 2x + 1$.

square root of a function

- the function $y = \sqrt{f(x)}$ is the square root of the function $y = f(x)$
- $y = \sqrt{f(x)}$ is only defined for $f(x) \geq 0$

Example 1

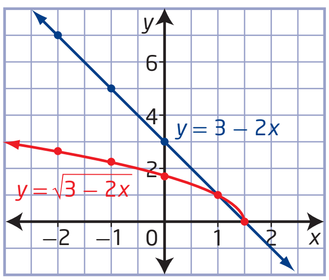
Compare Graphs of a Linear Function and the Square Root of the Function

a) Given $f(x) = 3 - 2x$, graph the functions $y = f(x)$ and $y = \sqrt{f(x)}$.
b) Compare the two functions.

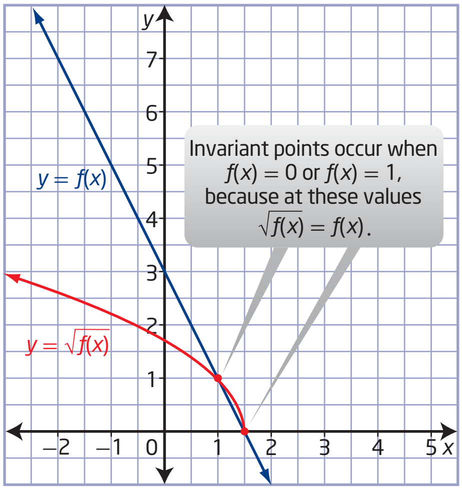
Solution

a) Use a table of values to graph $y = 3 - 2x$ and $y = \sqrt{3 - 2x}$.

x	$y = 3 - 2x$	$y = \sqrt{3 - 2x}$
–2	7	$\sqrt{7}$
–1	5	$\sqrt{5}$
0	3	$\sqrt{3}$
1	1	1
1.5	0	0



b) Compare the graphs.



Why is the graph of $y = \sqrt{f(x)}$ above the graph of $y = f(x)$ for values of y between 0 and 1? Will this always be true?

For $y = f(x)$, the domain is $\{x \mid x \in \mathrm{R}\}$ and the range is $\{y \mid y \in \mathrm{R}\}$.
For $y = \sqrt{f(x)}$, the domain is $\{x \mid x \leq 1.5, x \in \mathrm{R}\}$ and the range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

Invariant points occur at (1, 1) and (1.5, 0).

How does the domain of the graph of $y = \sqrt{f(x)}$ relate to the restrictions on the variable in the radicand? How could you determine the domain algebraically?

Your Turn

a) Given $g(x) = 3x + 6$, graph the functions $y = g(x)$ and $y = \sqrt{g(x)}$.
b) Identify the domain and range of each function and any invariant points.

Relative Locations of $y = f(x)$ and $y = \sqrt{f(x)}$

The domain of $y = \sqrt{f(x)}$ consists only of the values in the domain of $f(x)$ for which $f(x) \geq 0$.

The range of $y = \sqrt{f(x)}$ consists of the square roots of the values in the range of $y = f(x)$ for which $\sqrt{f(x)}$ is defined.

The graph of $y = \sqrt{f(x)}$ exists only where $f(x) \geq 0$. You can predict the location of $y = \sqrt{f(x)}$ relative to $y = f(x)$ using the values of $f(x)$.

Value of $f(x)$	$f(x) < 0$	$f(x) = 0$	$0 < f(x) < 1$	$f(x) = 1$	$f(x) > 1$
Relative Location of Graph of $y = \sqrt{f(x)}$	The graph of $y = \sqrt{f(x)}$ is undefined.	The graphs of $y = \sqrt{f(x)}$ and $y = f(x)$ intersect on the x-axis.	The graph of $y = \sqrt{f(x)}$ is above the graph of $y = f(x)$.	The graph of $y = \sqrt{f(x)}$ intersects the graph of $y = f(x)$.	The graph of $y = \sqrt{f(x)}$ is below the graph of $y = f(x)$.

Example 2

Compare the Domains and Ranges of $y = f(x)$ and $y = \sqrt{f(x)}$

Identify and compare the domains and ranges of the functions in each pair.

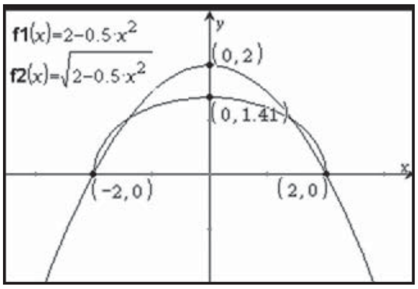
a) $y = 2 - 0.5x^2$ and $y = \sqrt{2 - 0.5x^2}$
b) $y = x^2 + 5$ and $y = \sqrt{x^2 + 5}$

Solution

a) **Method 1: Analyse Graphically**

Since the function $y = 2 - 0.5x^2$ is a quadratic function, its square root, $y = \sqrt{2 - 0.5x^2}$, cannot be expressed in the form $y = a\sqrt{b(x - h)} + k$. It cannot be graphed by transforming $y = \sqrt{x}$.

Both graphs can be created using technology. Use the *maximum* and *minimum* or equivalent features to find the coordinates of points necessary to determine the domain and range.



The graph of $y = 2 - 0.5x^2$ extends from (0, 2) down and to the left and right infinitely. Its domain is $\{x \mid x \in \mathrm{R}\}$, and its range is $\{y \mid y \leq 2, y \in \mathrm{R}\}$.

The graph of $y = \sqrt{2 - 0.5x^2}$ includes values of x from -2 to 2 inclusive, so its domain is $\{x \mid -2 \leq x \leq 2, x \in \mathrm{R}\}$. The graph covers values of y from 0 to approximately 1.41 inclusive, so its approximate range is $\{y \mid 0 \leq y \leq 1.41, y \in \mathrm{R}\}$.

To determine the exact value that 1.41 represents, you need to analyse the function algebraically.

The domain and range of $y = \sqrt{2 - 0.5x^2}$ are subsets of the domain and range of $y = 2 - 0.5x^2$.

Method 2: Analyse Key Points

Use the locations of any intercepts and the maximum value or minimum value to determine the domain and range of each function.

Function	$y = 2 - 0.5x^2$	$y = \sqrt{2 - 0.5x^2}$
x-Intercepts	-2 and 2	-2 and 2
y-Intercept	2	$\sqrt{2}$
Maximum Value	2	$\sqrt{2}$
Minimum Value	none	0

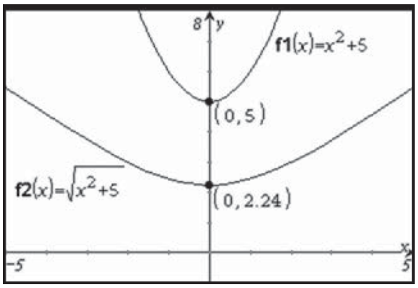
How can you justify this information algebraically?

Quadratic functions are defined for all real numbers. So, the domain of $y = 2 - 0.5x^2$ is $\{x \mid x \in R\}$. Since the maximum value is 2, the range of $y = 2 - 0.5x^2$ is $\{y \mid y \leq 2, y \in R\}$.

The locations of the x-intercepts of $y = \sqrt{2 - 0.5x^2}$ mean that the function is defined for $-2 \leq x \leq 2$. So, the domain is $\{x \mid -2 \leq x \leq 2, x \in R\}$. Since $y = \sqrt{2 - 0.5x^2}$ has a minimum value of 0 and a maximum value of $\sqrt{2}$, the range is $\{y \mid 0 \leq y \leq \sqrt{2}, y \in R\}$.

b) Method 1: Analyse Graphically

Graph the functions $y = x^2 + 5$ and $y = \sqrt{x^2 + 5}$ using technology.



Both functions extend infinitely to the left and the right, so the domain of each function is $\{x \mid x \in R\}$.

The range of $y = x^2 + 5$ is $\{y \mid y \geq 5, y \in R\}$.

The range of $y = \sqrt{x^2 + 5}$ is approximately $\{y \mid y \geq 2.24, y \in R\}$.

Method 2: Analyse Key Points

Use the locations of any intercepts and the maximum value or minimum value to determine the domain and range of each function.

Function	$y = x^2 + 5$	$y = \sqrt{x^2 + 5}$
x-Intercepts	none	none
y-Intercept	5	$\sqrt{5}$
Maximum Value	none	none
Minimum Value	5	$\sqrt{5}$

Quadratic functions are defined for all real numbers. So, the domain of $y = x^2 + 5$ is $\{x \mid x \in R\}$. Since the minimum value is 5, the range of $y = x^2 + 5$ is $\{y \mid y \geq 5, y \in R\}$.

Since $y = \sqrt{x^2 + 5}$ has no x-intercepts, the function is defined for all real numbers. So, the domain is $\{x \mid x \in R\}$. Since $y = \sqrt{x^2 + 5}$ has a minimum value of $\sqrt{5}$ and no maximum value, the range is $\{y \mid y \geq \sqrt{5}, y \in R\}$.

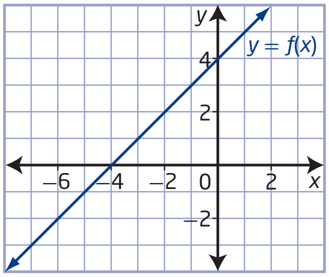
Your Turn

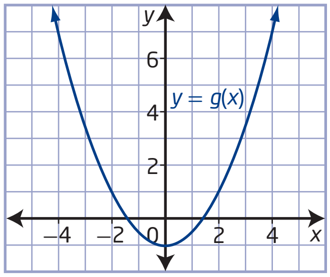
Identify and compare the domains and ranges of the functions $y = x^2 - 1$ and $y = \sqrt{x^2 - 1}$. Verify your answers.

Example 3

Graph the Square Root of a Function From the Graph of the Function

Using the graphs of $y = f(x)$ and $y = g(x)$, sketch the graphs of $y = \sqrt{f(x)}$ and $y = \sqrt{g(x)}$.





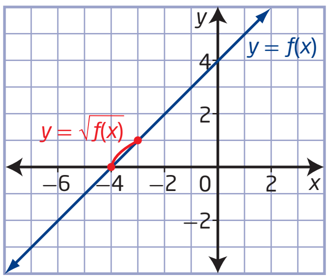
Solution

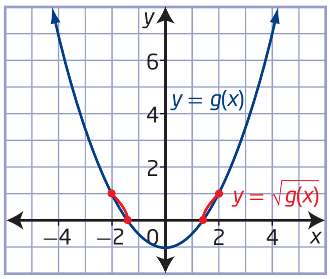
Sketch each graph by locating key points, including invariant points, and determining the image points on the graph of the square root of the function.

Step 1: Locate invariant points on $y = f(x)$ and $y = g(x)$. When graphing the square root of a function, invariant points occur at $y = 0$ and $y = 1$.

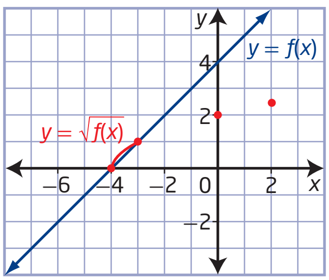
What is significant about $y = 0$ and $y = 1$? Does this apply to all graphs of functions and their square roots? Why?

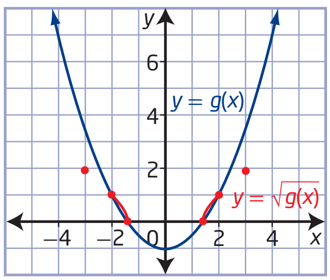
Step 2: Draw the portion of each graph between the invariant points for values of $y = f(x)$ and $y = g(x)$ that are positive but less than 1. Sketch a smooth curve *above* those of $y = f(x)$ and $y = g(x)$ in these intervals.





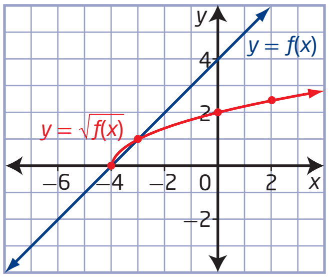
Step 3: Locate other key points on $y = f(x)$ and $y = g(x)$ where the values are greater than 1. Transform these points to locate image points on the graphs of $y = \sqrt{f(x)}$ and $y = \sqrt{g(x)}$.

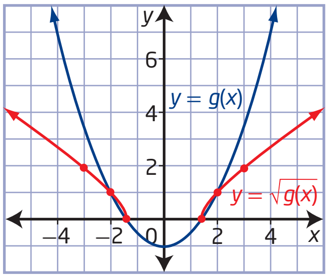




How can a value of y be mapped to a point on the square root of the function?

Step 4: Sketch smooth curves between the image points; they will be below those of $y = f(x)$ and $y = g(x)$ in the remaining intervals. Recall that graphs of $y = \sqrt{f(x)}$ and $y = \sqrt{g(x)}$ do not exist in intervals where $y = f(x)$ and $y = g(x)$ are negative (below the x-axis).

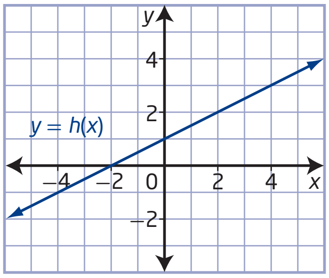




Where is the square root of a function above the original function? Where is it below? Where are they equal? Where are the endpoints on a graph of the square root of a function? Why?

Your Turn

Using the graph of $y = h(x)$, sketch the graph of $y = \sqrt{h(x)}$.



Key Ideas

- You can use values of $f(x)$ to predict values of $\sqrt{f(x)}$ and to sketch the graph of $y = \sqrt{f(x)}$.
- The key values to consider are $f(x) = 0$ and $f(x) = 1$.
- The domain of $y = \sqrt{f(x)}$ consists of all values in the domain of $f(x)$ for which $f(x) \geq 0$.
- The range of $y = \sqrt{f(x)}$ consists of the square roots of all values in the range of $f(x)$ for which $f(x)$ is defined.
- The y-coordinates of the points on the graph of $y = \sqrt{f(x)}$ are the square roots of the y-coordinates of the corresponding points on the original function $y = f(x)$.

What do you know about the graph of $y = \sqrt{f(x)}$ at $f(x) = 0$ and $f(x) = 1$? How do the graphs of $y = f(x)$ and $y = \sqrt{f(x)}$ compare on either side of these locations?

Check Your Understanding

Practise

1. Copy and complete the table.

$f(x)$	$\sqrt{f(x)}$
36	
	0.03
1	
−9	
	1.6
0	

2. For each point on the graph of $y = f(x)$, does a corresponding point on the graph of $y = \sqrt{f(x)}$ exist? If so, state the coordinates (rounded to two decimal places, if necessary).

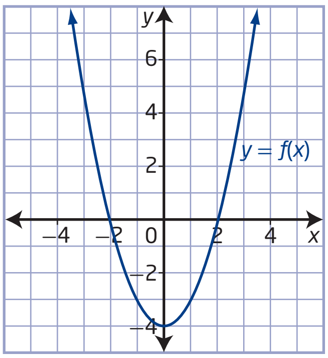
a) $(4, 12)$ **b)** $(-2, 0.4)$

c) $(10, -2)$ **d)** $(0.09, 1)$

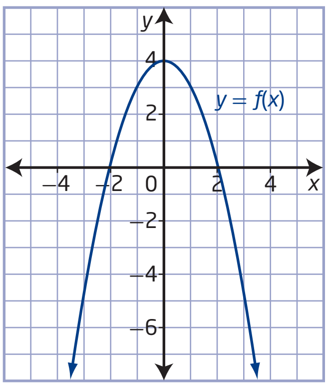
e) $(-5, 0)$ **f)** (m, n)

3. Match each graph of $y = f(x)$ to the corresponding graph of $y = \sqrt{f(x)}$.

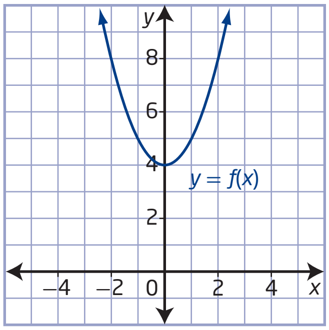
a)



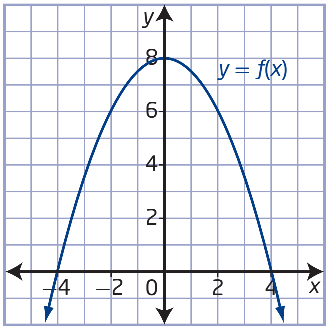
b)



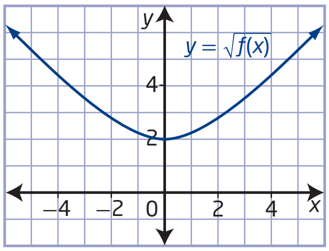
c)



d)



A



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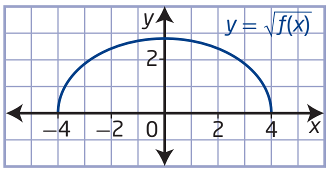
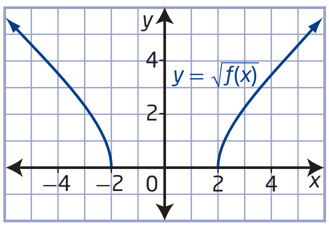
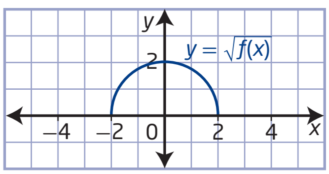


C



D



4. a) Given $f(x) = 4 - x$, graph the functions $y = f(x)$ and $y = \sqrt{f(x)}$.

b) Compare the two functions and explain how their values are related.

c) Identify the domain and range of each function, and explain any differences.

5. Determine the domains and ranges of the functions in each pair graphically and algebraically. Explain why the domains and ranges differ.

a) $y = x - 2$, $y = \sqrt{x - 2}$

b) $y = 2x + 6$, $y = \sqrt{2x + 6}$

c) $y = -x + 9$, $y = \sqrt{-x + 9}$

d) $y = -0.1x - 5$, $y = \sqrt{-0.1x - 5}$

6. Identify and compare the domains and ranges of the functions in each pair.

a) $y = x^2 - 9$ and $y = \sqrt{x^2 - 9}$

b) $y = 2 - x^2$ and $y = \sqrt{2 - x^2}$

c) $y = x^2 + 6$ and $y = \sqrt{x^2 + 6}$

d) $y = 0.5x^2 + 3$ and $y = \sqrt{0.5x^2 + 3}$

7. For each function, identify and explain any differences in the domains and ranges of $y = f(x)$ and $y = \sqrt{f(x)}$.

a) $f(x) = x^2 - 25$

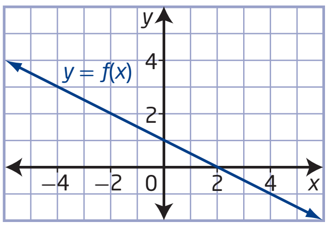
b) $f(x) = x^2 + 3$

c) $f(x) = 32 - 2x^2$

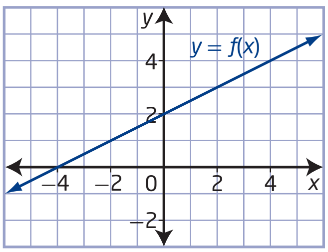
d) $f(x) = 5x^2 + 50$

8. Using each graph of $y = f(x)$, sketch the graph of $y = \sqrt{f(x)}$.

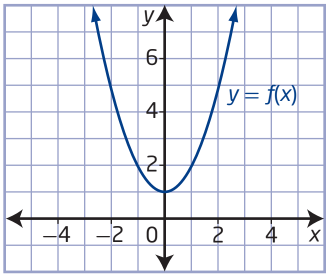
a)



b)



c)



Apply

9. a) Use technology to graph each function and identify the domain and range.

i) $f(x) = x^2 + 4$

ii) $g(x) = x^2 - 4$

iii) $h(x) = -x^2 + 4$

iv) $j(x) = -x^2 - 4$

b) Graph the square root of each function in part a) using technology.

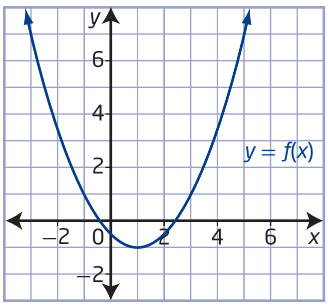
c) What do you notice about the graph of $y = \sqrt{j(x)}$? Explain this observation based on the graph of $y = j(x)$. Then, explain this observation algebraically.

d) In general, how are the domains of the functions in part a) related to the domains of the functions in part b)? How are the ranges related?

10. a) Identify the domains and ranges of $y = x^2 - 4$ and $y = \sqrt{x^2 - 4}$.

b) Why is $y = \sqrt{x^2 - 4}$ undefined over an interval? How does this affect the domain of the function?

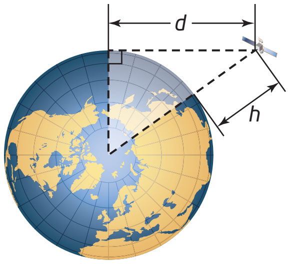
11. The graph of $y = f(x)$ is shown.



a) Sketch the graph of $y = \sqrt{f(x)}$, and explain the strategy you used.

b) State the domain and range of each function, and explain how the domains and the ranges are related.

12. For relatively small heights above Earth, a simple radical function can be used to approximate the distance to the horizon.



a) If Earth's radius is assumed to be 6378 km, determine the equation for the distance, d, in kilometres, to the horizon for an object that is at a height of h kilometres above Earth's surface.

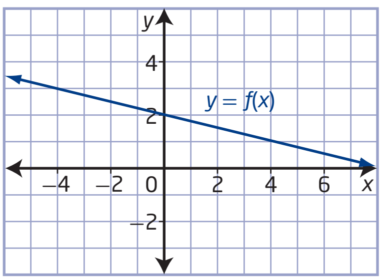
b) Identify the domain and range of the function.

c) How can you use a graph of the function to find the distance to the horizon for a satellite that is 800 km above Earth's surface?

d) If the function from part a) were just an arbitrary mathematical function rather than in this context, would the domain or range be any different? Explain.

13. a) When determining whether the graph shown represents a function or the square root of the function, Chris states, "it must be the function $y = f(x)$ because the domain consists of negative values, and the square root of a function $y = \sqrt{f(x)}$ is not defined for negative values."

Do you agree with Chris's answer? Why?



b) Describe how you would determine whether a graph shows the function or the square root of the function.

14. The main portion of an iglu (Inuit spelling of the English word igloo) is approximately hemispherical in shape.

a) For an iglu with diameter 3.6 m, determine a function that gives the vertical height, v, in metres, in terms of the horizontal distance, h, in metres, from the centre.

b) What are the domain and range of this function, and how are they related to the situation?

c) What is the height of this iglu at a point 1 m in from the bottom edge of the wall?

Did You Know?

An iglu is actually built in a spiral from blocks cut from inside the iglu floor space. Half the floor space is left as a bed platform in large iglus. This traps cold air below the sleeping area.

15. MINI LAB Investigate how the constants in radical functions affect their graphs, domains, and ranges.

Step 1 Graph the function $y = \sqrt{a^2 - x^2}$ for various values of a. If you use graphing software, you may be able to create sliders that allow you to vary the value of a and dynamically see the resulting changes in the graph.

Step 2 Describe how the value of a affects the graph of the function and its domain and range.

Step 3 Choose one value of a and write an equation for the reflection of this function in the x-axis. Graph both functions and describe the graph.

Step 4 Repeat steps 1 to 3 for the function $y = \sqrt{a^2 + x^2}$ as well as another square root of a function involving x^2.

Extend

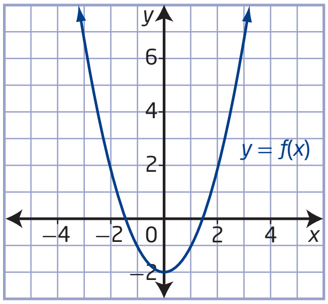
16. If $(-24, 12)$ is a point on the graph of the function $y = f(x)$, identify one point on the graph of each of the following functions.

a) $y = \sqrt{4f(x + 3)}$

b) $y = -\sqrt{f(4x)} + 12$

c) $y = -2\sqrt{f(-(x - 2)) - 4} + 6$

17. Given the graph of the function $y = f(x)$, sketch the graph of each function.



a) $y = 2\sqrt{f(x)} - 3$

b) $y = -\sqrt{2f(x - 3)}$

c) $y = \sqrt{-f(2x) + 3}$

d) $y = \sqrt{2f(-x) - 3}$

18. Explain your strategy for completing #17b).

19. Develop a formula for radius as a function of surface area for

a) a cylinder with equal diameter and height

b) a cone with height three times its diameter

Create Connections

C1 Write a summary of your strategy for graphing the function $y = \sqrt{f(x)}$ if you are given only the graph of $y = f(x)$.

C2 Explain how the relationship between the two equations $y = 16 - 4x$ and $y = \sqrt{16 - 4x}$ is connected to the relationship between their graphs.

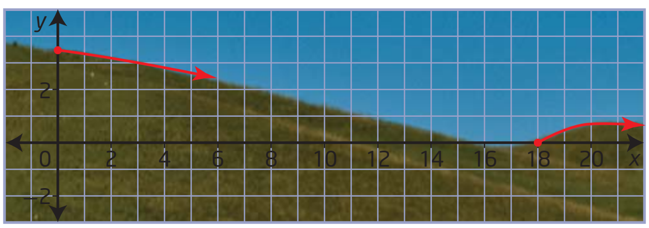
C3 Is it possible to completely graph the function $y = f(x)$ given only the graph of $y = \sqrt{f(x)}$? Discuss this with a classmate and share several examples that you create. Write a summary of your conclusions.

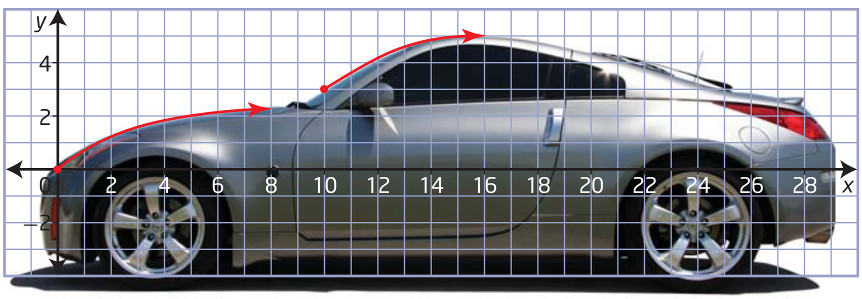
C4 **a)** Given $f(x) = (x - 1)^2 - 4$, graph the functions $y = f(x)$ and $y = \sqrt{f(x)}$.

b) Compare the two functions and explain how their values are related using several points on each graph.

Project Corner — Form Follows Function

- What radical functions are represented by the curves drawn on each image?





2.3

Solving Radical Equations Graphically

Focus on...

- relating the roots of radical equations and the *x*-intercepts of the graphs of radical functions
- determining approximate solutions of radical equations graphically

Parachutes slow the speed of falling objects by greatly increasing the drag force of the air. Manufacturers must make careful calculations to ensure that their parachutes are large enough to create enough drag force to allow parachutists to descend at a safe speed, but not so large that they are impractical. Radical equations can be used to relate the area of a parachute to the descent speed and mass of the object it carries, allowing parachute designers to ensure that their designs are reliable.

Did You Know?

French inventor Louis Sébastien-Lenormand introduced the first practical parachute in 1783.

Investigate Solving Radical Equations Graphically

Materials

- graphing calculator or graphing software

The radical equation $\sqrt{x - 4} = 5$ can be solved in several ways.

1. a) Discuss with a classmate how you might solve the equation graphically. Could you use more than one graphical method?

b) Write step-by-step instructions that explain how to use your method(s) to determine the solution to the radical equation.

c) Use your graphical method(s) to solve the equation.

2. a) Describe one method of solving the equation algebraically.

b) Use this method to determine the solution.

c) How might you verify your solution algebraically?

d) Share your method and solution with those of another pair and discuss any similarities and differences.

Reflect and Respond

3. a) How does the solution you found graphically compare with the one you found algebraically?

b) Will a graphical solution always match an algebraic solution? Discuss your answer with a classmate and explain your thoughts.

4. Do you prefer an algebraic or a graphical method for solving a radical equation like this one? Explain why.

Link the Ideas

You can solve many types of equations algebraically and graphically. Algebraic solutions sometimes produce extraneous roots, whereas graphical solutions do not produce extraneous roots. However, algebraic solutions are generally exact while graphical solutions are often approximate. You can solve equations, including radical equations, graphically by identifying the *x*-intercepts of the graph of the corresponding function.

Example 1

Relate Roots and *x*-Intercepts

a) Determine the root(s) of $\sqrt{x + 5} - 3 = 0$ algebraically.

b) Using a graph, determine the *x*-intercept(s) of the graph of $y = \sqrt{x + 5} - 3$.

c) Describe the connection between the root(s) of the equation and the *x*-intercept(s) of the graph of the function.

Solution

a) Identify any restrictions on the variable in the radical.

$$x + 5 \geq 0$$
$$x \geq -5$$

To solve a radical equation algebraically, first isolate the radical.

$$\sqrt{x + 5} - 3 = 0$$
$$\sqrt{x + 5} = 3$$
$$(\sqrt{x + 5})^2 = 3^2$$

Why do you need to square both sides?

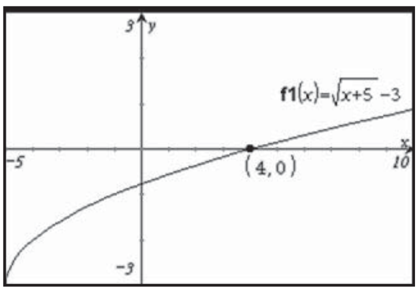
$$x + 5 = 9$$
$$x = 4$$

Is this an extraneous root? Does it meet the restrictions on the variable in the square root?

The value $x = 4$ is the root or solution to the equation.

b) To find the *x*-intercepts of the graph of $y = \sqrt{x + 5} - 3$, graph the function using technology and determine the *x*-intercepts.

The function has a single *x*-intercept at (4, 0).



c) The value $x = 4$ is the zero of the function because the value of the function is 0 when $x = 4$. The roots to a radical equation are equal to the *x*-intercepts of the graph of the corresponding radical function.

Your Turn

a) Use a graph to locate the *x*-intercept(s) of the graph of $y = \sqrt{x + 2} - 4$.

b) Algebraically determine the root(s) of the equation $\sqrt{x + 2} - 4 = 0$.

c) Describe the relationship between your findings in parts a) and b).

Example 2

Solve a Radical Equation Involving an Extraneous Solution

Solve the equation $\sqrt{x + 5} = x + 3$ algebraically and graphically.

Solution

$$\begin{aligned}
\sqrt{x + 5} &= x + 3 \\
(\sqrt{x + 5})^2 &= (x + 3)^2 \\
x + 5 &= x^2 + 6x + 9 \\
0 &= x^2 + 5x + 4 \\
0 &= (x + 4)(x + 1)
\end{aligned}$$

$x + 4 = 0$ or $x + 1 = 0$

$x = -4$ $\quad$ $x = -1$

Check:

Substitute $x = -4$ and $x = -1$ into the original equation to identify any extraneous roots.

Why do extraneous roots occur?

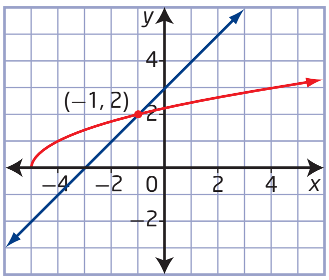
Left Side	Right Side	Left Side	Right Side
$\sqrt{x + 5}$	$x + 3$	$\sqrt{x + 5}$	$x + 3$
$= \sqrt{-4 + 5}$	$= -4 + 3$	$= \sqrt{-1 + 5}$	$= -1 + 3$
$= \sqrt{1}$	$= -1$	$= \sqrt{4}$	$= 2$
$= 1$		$= 2$	
Left Side $\neq$ Right Side		Left Side = Right Side	

The solution is $x = -1$.

Solve the equation graphically using functions to represent the two sides of the equation.

$y_1 = \sqrt{x + 5}$

$y_2 = x + 3$



The two functions intersect at the point $(-1, 2)$. The value of x at this point, $x = -1$, is the solution to the equation.

Your Turn

Solve the equation $4 - x = \sqrt{6 - x}$ graphically and algebraically.

Example 3

Approximate Solutions to Radical Equations

a) Solve the equation $\sqrt{3x^2 - 5} = x + 4$ graphically. Express your answer to the nearest tenth.

b) Verify your solution algebraically.

Solution

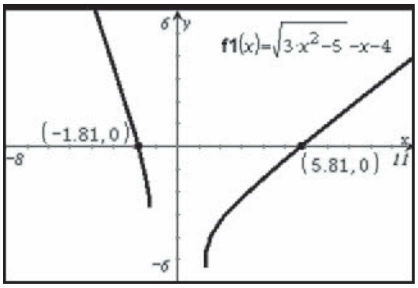
a) To determine the roots or solutions to an equation of the form $f(x) = g(x)$, identify the x-intercepts of the graph of the corresponding function, $y = f(x) - g(x)$.

Method 1: Use a Single Function

Rearrange the radical equation so that one side is equal to zero:

$$\sqrt{3x^2 - 5} = x + 4$$
$$\sqrt{3x^2 - 5} - x - 4 = 0$$

Graph the corresponding function, $y = \sqrt{3x^2 - 5} - x - 4$, and determine the x-intercepts of the graph.



The values of the x-intercepts of the graph are the same as the solutions to the original equation. Therefore, the solution is $x \approx -1.8$ and $x \approx 5.8$.

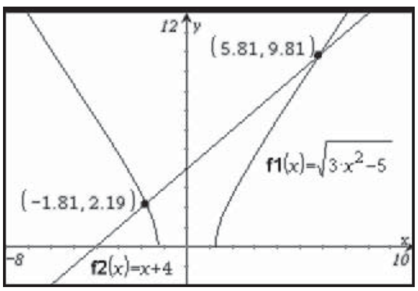
Method 2: Use a System of Two Functions

Express each side of the equation as a function:

$y_1 = \sqrt{3x^2 - 5}$

$y_2 = x + 4$

Graph these functions and determine the value of x at the point(s) of intersection, i.e., where $y_1 = y_2$.



The solution to the equation $\sqrt{3x^2 - 5} = x + 4$ is $x \approx -1.8$ and $x \approx 5.8$.

b) Identify the values of x for which the radical is defined.

$$3x^2 - 5 \geq 0$$
$$3x^2 \geq 5$$
$$x^2 \geq \frac{5}{3}$$
$$|x| \geq \sqrt{\frac{5}{3}}$$

Case 1

If $x \geq 0$, $x \geq \sqrt{\frac{5}{3}}$.

Case 2

If $x < 0$, $x < -\sqrt{\frac{5}{3}}$.

Solve for x:

$$\sqrt{3x^2 - 5} = x + 4$$
$$(\sqrt{3x^2 - 5})^2 = (x + 4)^2$$

Why do you need to square both sides?

$$3x^2 - 5 = x^2 + 8x + 16$$
$$2x^2 - 8x - 21 = 0$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(-21)}}{2(2)}$$

Why does the quadratic formula need to be used here?

$$x = \frac{8 \pm \sqrt{232}}{4}$$
$$x = \frac{8 \pm 2\sqrt{58}}{4}$$
$$x = \frac{4 \pm \sqrt{58}}{2}$$

$x = \frac{4 + \sqrt{58}}{2}$ or $x = \frac{4 - \sqrt{58}}{2}$

$x \approx 5.8$ $\quad$ $x \approx -1.8$

Do these solutions meet the restrictions on x? How can you determine whether either of the roots is extraneous?

The algebraic method gives an exact solution. The approximate solution obtained algebraically, $x \approx -1.8$ and $x \approx 5.8$, is the same as the approximate solution obtained graphically.

Your Turn

Solve the equation $x + 3 = \sqrt{12 - 2x^2}$ using two different methods.

Example 4

Solve a Problem Involving a Radical Equation

An engineer designs a roller coaster that involves a vertical drop section just below the top of the ride. She uses the equation $v = \sqrt{(v_0)^2 + 2ad}$ to model the velocity, v, in feet per second, of the ride's cars after dropping a distance, d, in feet, with an initial velocity, v_0, in feet per second, at the top of the drop, and constant acceleration, a, in feet per second squared. The design specifies that the speed of the ride's cars be 120 ft/s at the bottom of the vertical drop section. If the initial velocity of the coaster at the top of the drop is 10 ft/s and the only acceleration is due to gravity, 32 ft/s², what vertical drop distance should be used, to the nearest foot?

Did You Know?

Top Thrill Dragster is a vertical drop-launched roller coaster in Cedar Point amusement park, in Sandusky, Ohio. When it opened in 2003, it set three new records for roller coasters: tallest, fastest top speed, and steepest drop. It stands almost 130 m tall, and on a clear day riders at the top can see Canada's Pelee Island across Lake Erie.

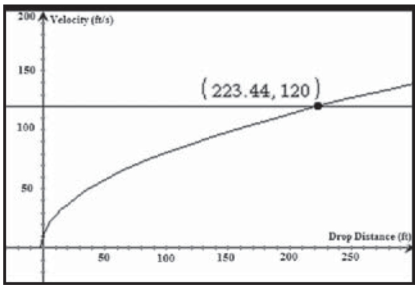
Solution

Substitute the known values into the formula. Then, graph the functions that correspond to both sides of the equation and determine the point of intersection.

$$v = \sqrt{(v_0)^2 + 2ad}$$
$$120 = \sqrt{(10)^2 + 2(32)d}$$
$$120 = \sqrt{100 + 64d}$$

What two functions do you need to graph?

The intersection point indicates that the drop distance should be approximately 223 ft to result in a velocity of 120 ft/s at the bottom of the drop.



Web Link

To see a computer animation of *Top Thrill Dragster*, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Your Turn

Determine the initial velocity required in a roller coaster design if the velocity will be 26 m/s at the bottom of a vertical drop of 34 m. (Acceleration due to gravity in SI units is 9.8 m/s².)

Key Ideas

- You can solve radical equations algebraically and graphically.
- The solutions or roots of a radical equation are equivalent to the x-intercepts of the graph of the corresponding radical function. You can use either of the following methods to solve radical equations graphically:
  - Graph the corresponding function and identify the value(s) of the x-intercept(s).
  - Graph the system of functions that corresponds to the expression on each side of the equal sign, and then identify the value(s) of x at the point(s) of intersection.

Check Your Understanding

Practise

1. Match each equation to the single function that can be used to solve it graphically. For all equations, $x \geq -4$.

a) $2 + \sqrt{x + 4} = 4$

b) $x - 4 = \sqrt{x + 4}$

c) $2 = \sqrt{x + 4} - 4$

d) $\sqrt{x + 4} + 2 = x + 6$

A $y = x - 4 - \sqrt{x + 4}$

B $y = \sqrt{x + 4} - 2$

C $y = \sqrt{x + 4} - x - 4$

D $y = \sqrt{x + 4} - 6$

2. a) Determine the root(s) of the equation $\sqrt{x + 7} - 4 = 0$ algebraically.

b) Determine the x-intercept(s) of the graph of the function $y = \sqrt{x + 7} - 4$ graphically.

c) Explain the connection between the root(s) of the equation and the x-intercept(s) of the graph of the function.

3. Determine the approximate solution to each equation graphically. Express your answers to three decimal places.

a) $\sqrt{7x - 4} = 13$

b) $9 + \sqrt{6 - 11x} = 45$

c) $\sqrt{x^2 + 2} - 5 = 0$

d) $45 - \sqrt{10 - 2x^2} = 25$

4. a) Solve the equation $2\sqrt{3x + 5} + 7 = 16$, $x \geq -\frac{5}{3}$, algebraically.

b) Show how you can use the graph of the function $y = 2\sqrt{3x + 5} - 9$, $x \geq -\frac{5}{3}$, to find the solution to the equation in part a).

5. Solve each equation graphically. Identify any restrictions on the variable.

a) $\sqrt{2x - 9} = 11$

b) $7 = \sqrt{12 - x} + 4$

c) $5 + 2\sqrt{5x + 32} = 12$

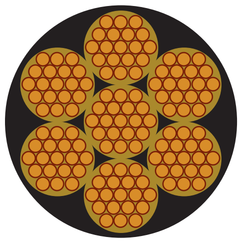
d) $5 = 13 - \sqrt{25 - 2x}$

6. Solve each equation algebraically. What are the restrictions on the variables?
   a) $\sqrt{5x^2 + 11} = x + 5$
   b) $x + 3 = \sqrt{2x^2 - 7}$
   c) $\sqrt{13 - 4x^2} = 2 - x$
   d) $x + \sqrt{-2x^2 + 9} = 3$
7. Solve each equation algebraically and graphically. Identify any restrictions on the variables.
   a) $\sqrt{8 - x} = x + 6$
   b) $4 = x + 2\sqrt{x - 7}$
   c) $\sqrt{3x^2 - 11} = x + 1$
   d) $x = \sqrt{2x^2 - 8} + 2$

Apply

8. Determine, graphically, the approximate value(s) of a in each formula if $b = 6.2$, $c = 9.7$, and $d = -12.9$. Express answers to the nearest hundredth.
   a) $c = \sqrt{ab - d}$
   b) $d + 7\sqrt{a + c} = b$
   c) $c = b - \sqrt{a^2 + d}$
   d) $\sqrt{2a^2 + c} + d = a - b$
9. Naomi says that the equation $6 + \sqrt{x + 4} = 2$ has no solutions.
   a) Show that Naomi is correct, using both a graphical and an algebraic approach.
   b) Is it possible to tell that this equation has no solutions just by examining the equation? Explain.
10. Two researchers, Greg and Yolanda, use the function $N(t) = 1.3\sqrt{t} + 4.2$ to model the number of people that might be affected by a certain medical condition in a region of 7.4 million people. In the function, N represents the number of people, in millions, affected after t years. Greg predicts that the entire population would be affected after 6 years. Yolanda believes that it would take only 1.5 years. Who is correct? Justify your answer.
11. The period, T, in seconds, of a pendulum depends on the distance, L, in metres, between the pivot and the pendulum's centre of mass. If the initial swing angle is relatively small, the period is given by the radical function $T = 2\pi\sqrt{\frac{L}{g}}$, where g represents acceleration due to gravity (approximately 9.8 m/s² on Earth). Jeremy is building a machine and needs it to have a pendulum that takes 1 s to swing from one side to the other. How long should the pendulum be, in centimetres?
12. Cables and ropes are made of several strands that contain individual wires or threads. The term "7 × 19 cable" refers to a cable with 7 strands, each containing 19 wires.

    Suppose a manufacturer uses the function $d = \sqrt{\frac{b}{30}}$ to relate the diameter, d, in millimetres, of its 7 × 19 stainless steel aircraft cable to the safe working load, b, in kilograms.



   a) Is a cable with a diameter of 6.4 mm large enough to support a mass of 1000 kg?
   b) What is the safe working load for a cable that is 10 mm in diameter?

Did You Know?

The safe working load for a cable or rope is related to its breaking strength, or minimum mass required for it to break. To ensure safety, manufacturers rate a cable's safe working load to be much less than its actual breaking strength.

13. Hazeem states that the equations $\sqrt{x^2} = 9$ and $(\sqrt{x})^2 = 9$ have the same solution. Is he correct? Justify your answer.

14. What real number is exactly one greater than its square root?

15. A parachute-manufacturing company uses the formula $d = 3.69\sqrt{\frac{m}{v^2}}$ to model the diameter, d, in metres, of its dome-shaped circular parachutes so that an object with mass, m, in kilograms, has a descent velocity, v, in metres per second, under the parachute.

a) What is the landing velocity for a 20-kg object using a parachute that is 3.2 m in diameter? Express your answer to the nearest metre per second.

b) A velocity of 2 m/s is considered safe for a parachutist to land. If the parachute has a diameter of 16 m, what is the maximum mass of the parachutist, in kilograms?

Extend

16. If the function $y = \sqrt{-3(x + c)} + c$ passes through the point $(-1, 1)$, what is the value of c? Confirm your answer graphically, and use the graph to create a similar question for the same function.

17. Heron's formula, $A = \sqrt{s(s - a)(s - b)(s - c)}$, relates the area, A, of a triangle to the lengths of the three sides, a, b, and c, and its semi-perimeter (half its perimeter), $s = \frac{a + b + c}{2}$. A triangle has an area of 900 cm^2 and one side that measures 60 cm. The other two side lengths are unknown, but one is twice the length of the other. What are the lengths of the three sides of the triangle?

Create Connections

C1 How can the graph of a function be used to find the solutions to an equation? Create an example to support your answer.

C2 The speed, in metres per second, of a tsunami travelling across the ocean is equal to the square root of the product of the depth of the water, in metres, and the acceleration due to gravity, 9.8 m/s^2.

a) Write a function for the speed of a tsunami. Define the variables you used.

b) Calculate the speed of a wave at a depth of 2500 m, and use unit analysis to show that the resulting speed has the correct units.

c) What depth of water would produce a speed of 200 m/s? Solve graphically and algebraically.

d) Which method of solving do you prefer in this case: algebraic or graphical? Do you always prefer one method over the other, or does it depend? Explain.

C3 Does every radical equation have at least one solution? How can using a graphical approach to solving equations help you answer this question? Support your answer with at least two examples.

C4 Describe two methods of identifying extraneous roots in a solution to a radical equation. Explain why extraneous roots may occur.

Chapter 2 Review

2.1 Radical Functions and Transformations, pages 62–77

1. Graph each function. Identify the domain and range, and explain how they connect to the values in a table of values and the shape of the graph.

a) $y = \sqrt{x}$

b) $y = \sqrt{3 - x}$

c) $y = \sqrt{2x + 7}$

2. What transformations can you apply to $y = \sqrt{x}$ to obtain the graph of each function? State the domain and range in each case.

a) $y = 5\sqrt{x + 20}$

b) $y = \sqrt{-2x} - 8$

c) $y = -\sqrt{\frac{1}{6}(x - 11)}$

3. Write the equation and state the domain and range of the radical function that results from each set of transformations on the graph of $y = \sqrt{x}$.

a) a horizontal stretch by a factor of 10 and a vertical translation of 12 units up

b) a vertical stretch by a factor of 2.5, a reflection in the x-axis, and a horizontal translation of 9 units left

c) a horizontal stretch by a factor of $\frac{5}{2}$, a vertical stretch by a factor of $\frac{1}{20}$, a reflection in the y-axis, and a translation of 7 units right and 3 units down

4. Sketch the graph of each function by transforming the graph of $y = \sqrt{x}$. State the domain and range of each.

a) $y = -\sqrt{x - 1} + 2$

b) $y = 3\sqrt{-x} - 4$

c) $y = \sqrt{2(x + 3)} + 1$

5. How can you use transformations to identify the domain and range of the function $y = -2\sqrt{3(x - 4)} + 9$?

6. The sales, S, in units, of a new product can be modelled as a function of the time, t, in days, since it first appears in stores using the function $S(t) = 500 + 100\sqrt{t}$.

a) Describe how to graph the function by transforming the graph of $y = \sqrt{t}$.

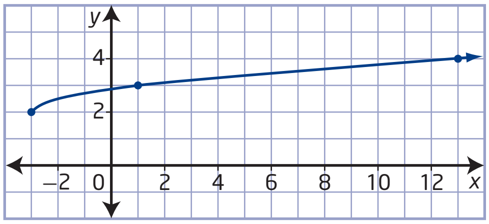
b) Graph the function and explain what the shape of the graph indicates about the situation.

c) What are the domain and range? What do they mean in this situation?

d) Predict the number of items sold after 60 days.

7. Write an equation of the form $y = a\sqrt{b(x - h)} + k$ for each graph.

a)



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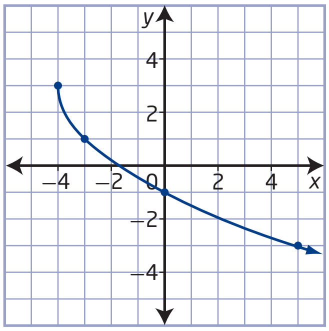

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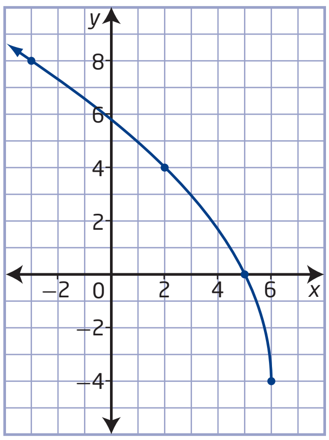


2.2 Square Root of a Function, pages 78–89

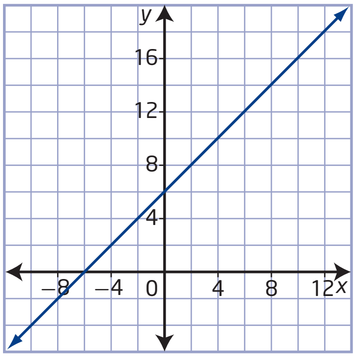
8. Identify the domains and ranges of the functions in each pair and explain any differences.

a) $y = x - 2$ and $y = \sqrt{x - 2}$

b) $y = 10 - x$ and $y = \sqrt{10 - x}$

c) $y = 4x + 11$ and $y = \sqrt{4x + 11}$

9. The graph of $y = f(x)$ is shown.



a) Graph the function $y = \sqrt{f(x)}$ and describe your strategy.

b) Explain how the graphs are related.

c) Identify the domain and range of each function and explain any differences.

10. Identify and compare the domains and ranges of the functions in each pair, and explain why they differ.

a) $y = 4 - x^2$ and $y = \sqrt{4 - x^2}$

b) $y = 2x^2 + 24$ and $y = \sqrt{2x^2 + 24}$

c) $y = x^2 - 6x$ and $y = \sqrt{x^2 - 6x}$

11. A 25-ft-long ladder leans against a wall. The height, h, in feet, of the top of the ladder above the ground is related to its distance, d, in feet, from the base of the wall.

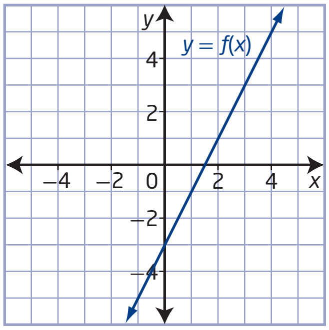
a) Write an equation to represent h as a function of d.

b) Graph the function and identify the domain and range.

c) Explain how the shape of the graph, the domain, and the range relate to the situation.

12. Using each graph of $y = f(x)$, sketch the graph of $y = \sqrt{f(x)}$.

a)



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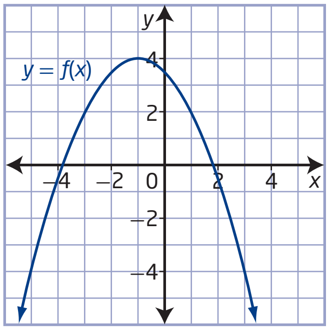

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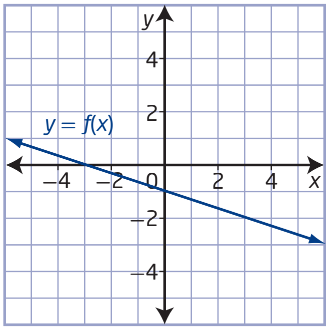


2.3 Solving Radical Equations Graphically, pages 90–98

13. a) Determine the root(s) of the equation $\sqrt{x + 3} - 7 = 0$ algebraically.

b) Use a graph to locate the x-intercept(s) of the function $f(x) = \sqrt{x + 3} - 7$.

c) Use your answers to describe the connection between the x-intercepts of the graph of a function and the roots of the corresponding equation.

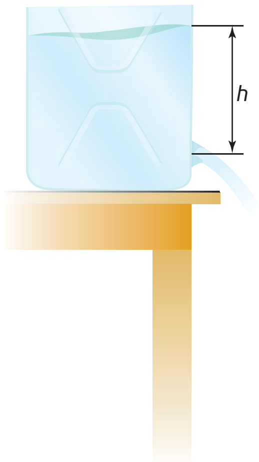
14. Determine the approximate solution to each equation graphically. Express answers to three decimal places.

a) $\sqrt{7x - 9} - 4 = 0$

b) $50 = 12 + \sqrt{8 - 12x}$

c) $\sqrt{2x^2 + 5} = 11$

15. The speed, s, in metres per second, of water flowing out of a hole near the bottom of a tank relates to the height, h, in metres, of the water above the hole by the formula $s = \sqrt{2gh}$. In the formula, g represents the acceleration due to gravity, 9.8 m/s^2. At what height is the water flowing out a speed of 9 m/s?



Did You Know?

The speed of fluid flowing out of a hole near the bottom of a tank filled to a depth, *h*, is the same as the speed an object acquires in falling freely from the height *h*. This relationship was discovered by Italian scientist Evangelista Torricelli in 1643 and is referred to as Torricelli's law.

16. Solve each equation graphically and algebraically.

a) $\sqrt{5x + 14} = 9$

b) $7 + \sqrt{8 - x} = 12$

c) $23 - 4\sqrt{2x - 10} = 12$

d) $x + 3 = \sqrt{18 - 2x^2}$

17. Atid, Carly, and Jaime use different methods to solve the radical equation $3 + \sqrt{x - 1} = x$.

Their solutions are as follows:
- Atid: $x = 2$
- Carly: $x = 5$
- Jaime: $x = 2, 5$

a) Who used an algebraic approach? Justify your answer.

b) Who used a graphical method? How do you know?

c) Who made an error in solving the equation? Justify your answer.

18. Assume that the shape of a tipi approximates a cone. The surface area, S, in square metres, of the walls of a tipi can be modelled by the function $S(r) = \pi r\sqrt{36 + r^2}$, where r represents the radius of the base, in metres.

Blackfoot Crossing, Alberta

a) If a tipi has a radius of 5.2 m, what is the minimum area of canvas required for the walls, to the nearest square metre?

b) If you use 160 m^2 of canvas to make the walls for this tipi, what radius will you use?

Chapter 2 Practice Test

Multiple Choice

For #1 to #6, choose the best answer.

1. If $f(x) = x + 1$, which point is on the graph of $y = \sqrt{f(x)}$?

A (0, 0) **B** (0, 1)

C (1, 0) **D** (1, 1)

2. Which intercepts will help you find the roots of the equation $\sqrt{2x - 5} = 4$?

A x-intercepts of the graph of the function $y = \sqrt{2x - 5} - 4$

B x-intercepts of the graph of the function $y = \sqrt{2x - 5} + 4$

C y-intercepts of the graph of the function $y = \sqrt{2x - 5} - 4$

D y-intercepts of the graph of the function $y = \sqrt{2x - 5} + 4$

3. Which function has a domain of $\{x \mid x \geq 5, x \in \mathrm{R}\}$ and a range of $\{y \mid y \geq 0, y \in \mathrm{R}\}$?

A $f(x) = \sqrt{x - 5}$

B $f(x) = \sqrt{x} - 5$

C $f(x) = \sqrt{x + 5}$

D $f(x) = \sqrt{x} + 5$

4. If $y = \sqrt{x}$ is stretched horizontally by a factor of 6, which function results?

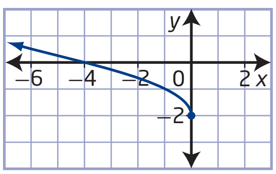
A $y = \frac{1}{6}\sqrt{x}$

B $y = 6\sqrt{x}$

C $y = \sqrt{\frac{1}{6}x}$

D $y = \sqrt{6x}$

5. Which equation represents the function shown in the graph?



A $y - 2 = -\sqrt{x}$ **B** $y + 2 = -\sqrt{x}$

C $y - 2 = \sqrt{-x}$ **D** $y + 2 = \sqrt{-x}$

6. How do the domains and ranges compare for the functions $y = \sqrt{x}$ and $y = \sqrt{5x} + 8$?

A Only the domains differ.

B Only the ranges differ.

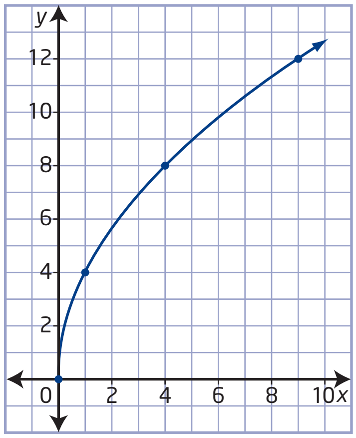
C Both the domains and ranges differ.

D Neither the domains nor the ranges differ.

Short Answer

7. Solve the equation $5 + \sqrt{9 - 13x} = 20$ graphically. Express your answer to the nearest hundredth.

8. Determine two forms of the equation that represents the function shown in the graph.



9. How are the domains and ranges of the functions $y = 7 - x$ and $y = \sqrt{7 - x}$ related? Explain why they differ.

10. If $f(x) = 8 - 2x^2$, what are the domains and ranges of $y = f(x)$ and $y = \sqrt{f(x)}$?

11. Solve the equation $\sqrt{12 - 3x^2} = x + 2$ using two different graphical methods. Show your graphs.

12. Solve the equation $4 + \sqrt{x + 1} = x$ graphically and algebraically. Express your answer to the nearest tenth.

13. The radical function $S = \sqrt{255d}$ can be used to estimate the speed, S, in kilometres per hour, of a vehicle before it brakes from the length, d, in metres, of the skid mark. The vehicle has all four wheels braking and skids to a complete stop on a dry road.

a) Use the language of transformations to describe how to create a graph of this function from a graph of the base square root function.

b) Sketch the graph of the function and use it to determine the approximate length of skid mark expected from a vehicle travelling at 100 km/h on this road.

Extended Response

14. a) How can you use transformations to graph the function $y = -\sqrt{2x} + 3$?

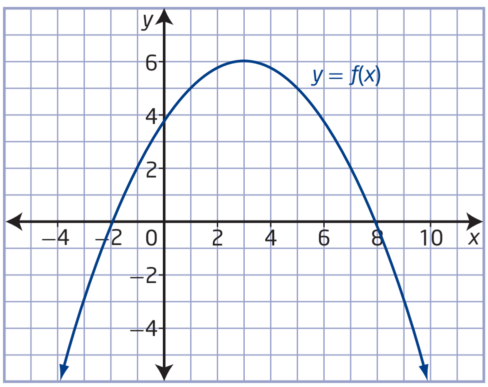
b) Sketch the graph.

c) Identify the domain and range of the function.

d) Describe how the domain and range connect to your answer to part a).

e) How can the graph be used to solve the equation $5 + \sqrt{2x} = 8$?

15. Using the graph of $y = f(x)$, sketch the graph of $y = \sqrt{f(x)}$ and explain your strategy.



16. Consider the roof of the mosque at the Canadian Islamic Centre in Edmonton, Alberta. The diameter of the base of the roof is approximately 10 m, and the vertical distance from the centre of the roof to the base is approximately 5 m.

Canadian Islamic Centre (Al-Rashid), Edmonton, Alberta

a) Determine a function of the form $y = a\sqrt{b(x - h)} + k$, where y represents the distance from the base to the roof and x represents the horizontal distance from the centre.

b) What are the domain and range of this function? How do they relate to the situation?

c) Use the function you wrote in part a) to determine, graphically, the approximate height of the roof at a point 2 m horizontally from the centre of the roof.